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Coherent risk measures in general economic models and price bubbles

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A B S T R A C T

In this article we study coherent risk measures in general economic models where the set of financial positions is an ordered Banach space E and the safe asset an order unit x_0 of E. First we study some properties of risk measures. We show that the set of normalized (with respect to x_0) price systems is weak star compact and by using this result we prove a maximum attainment representation theorem which improves the one of Jaschke and Küchler (2001). Also we study how a risk measure changes under different safe assets and we show a kind of equivalence between these risk measures. In the sequel we study subspaces of E consisting of financial positions of risk greater or equal to zero and we call these subspaces unsure. We find some criteria and we give examples of these subspaces. In the last section, we combine the unsure subspaces with the theory of price-bubbles of Gilles and LeRoy (1992).

In this study we use the theory of cones (ordered spaces). This theory allows us to generalize basic results and provides new proofs and ideas in the theory of risk measures.

1. Introduction

Coherent risk measures have been introduced by Artzner et al. (1999), in financial markets in the case where the set of states Ω is finite. This theory has been extended by Delbaen (2002), in the case where Ω is infinite and the portfolio space is the space $L_\infty(\Omega, \mathcal{F}, \mathbf{P})$ or the space $L^0(\Omega, \mathcal{F}, \mathbf{P})$ of $\mathcal{F}$-measurable real valued functions, where $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. In the general frame, see the book of Föllmer and Schied (2002), it is supposed that a financial position is a real valued function $x : \Omega \to \mathbb{R}$, where $x(i)$ is the payoff of x in the state i and the set of financial positions is a closed subspace E of the space $\mathbb{R}^\Omega$ of real valued functions $x : \Omega \longrightarrow \mathbb{R}$. It is supposed that E is ordered by the pointwise ordering and also that the constant function $\mathbf{1}$, which is considered as the safe asset, belongs to E. It is easy to show that $\mathbf{1}$ is an order unit of E, i.e. for any $x \in E$ there exists a natural number n so that $-n\mathbf{1} \le x \le n\mathbf{1}$ and this property of the safe asset is crucial for the theory of risk measures. A function $\rho : E \to \mathbb{R}$, such that

(i) $x \ge y \Rightarrow \rho(x) \le \rho(y)$ (monotonicity)
(ii) $\rho(x + t\mathbf{1}) = \rho(x) - t$, for any $t \in \mathbb{R}$ (cash invariant),

is a *risk measure*. Then the function $\tilde{\rho}(x) = \rho(x) - \rho(0), x \in E$, is again a risk measure with the property $\tilde{\rho}(0) = 0$. So by this slight modification, we may suppose that any risk measure satisfies the condition $\rho(0) = 0$. If a risk measure ρ is subadditive and positively homogeneous, i.e.

(i) $\rho(x + y) \le \rho(x) + \rho(y)$, for each $x, y \in E$,

and

(ii) $\rho(\lambda x) = \lambda\rho(x)$, for each $x \in E$ and each real number $\lambda \ge 0$,

then ρ is a *coherent risk measure.* Any coherent risk measure is convex, but the converse is not true. If a convex risk measure ρ is positively homogeneous then ρ is coherent.

The set $\mathcal{A}_\rho = \{x \in E \mid \rho(x) \le 0\}$ is the set of acceptable positions. We have $E_+ \subseteq \mathcal{A}_\rho$. If the risk measure ρ is convex, the set $\mathcal{A}_\rho$ is convex and if ρ is coherent, the set $\mathcal{A}_\rho$ is a cone and the function ρ is given by the formula

$$\rho(x) = \inf\{t \in \mathbb{R} \mid \rho(x + t\mathbf{1}) \le 0\}.$$

Coherent risk measures can be defined by the converse process, starting by a cone P of E which is considered as the set of acceptable positions and suppose that E is ordered by the cone P. If $\mathbf{1}$ is an order unit of E, then the function

$$\rho(x) = \inf\{t \in \mathbb{R} \mid \rho(x + t\mathbf{1}) \in P\},$$

for any $x \in E$, is a coherent risk measure with $P \subseteq \mathcal{A}_\rho$ and if P is closed we have $P = \mathcal{A}_\rho$. Representation theorems for coherent risk measures have been proved by Delbaen (2002) and for convex risk measures by Föllmer and Schied (2002) and by Frittelli and Gianin (2002).

The theory of risk measures has been extended in the case where the space of financial positions is an ordered Banach space with order unit, where the order unit is considered as the safe asset. This overview of risk measures was started by the article of Jaschke

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and Küchler (2001), where it is supposed that E is an arbitrary ordered vector space and a representation theorem for coherent risk measures is given. G. Stoica, studies in Stoica (2006), coherent risk measures in vector lattices. In Cheridito and Li (2009), the representation of convex risk measures on Orlicz hearts i.e. on maximal subspaces of Orlicz spaces is studied. The theory of price bubbles, as it is used in this article, has been expanded by Gilles (1989) and Gilles and LeRoy (1992, 1997).

In this article we suppose that E is a Banach space ordered by a closed cone P and we consider the risk measure ρ defined on E with respect to the cone P and an order unit x_0 of E which is considered as safe asset. We start our study by the base B_{x_0} of the dual cone P^0 of P which is defined by x_0. In fact B_{x_0} is the set of normalized (with respect to the safe asset x_0) price systems. In Theorem 4, we show that the set B_{x_0} is a weak star compact subset of the topological dual E^* of E. By using this result we improve the representation theorem of Jaschke and Küchler (2001), where it is proved that $\rho(x) = \sup\{-\widehat{x}(x^*) \mid x^* \in B_{x_0}\}$, for any $x \in E$, by showing that the supremum is attained.

In Theorem 6, we show that for any $x \in E$ we have: $\rho(x) < 0$ if and only if x is an interior point of P. Note that this result is known in the case where E is a subspace of an L_∞ space and for the proof the Lipschitz continuity of the risk measure ρ is needed. In this article we generalize this result in Banach spaces without any continuity assumption of the risk measure. In Theorem 7, we study how a coherent risk measure changes under different safe assets but with the same set of acceptable positions and we show a kind of equivalence between these risk measures.

In the sequel, we study subspaces X of E with the property $\rho(x) \geq 0$ for any $x \in X$. We call these subspaces unsure because any financial position x of an unsure subspace can be approximated by a sequence of financial position of strictly positive risk. We show different criteria for unsure subspaces. For example, in Theorem 11, we show that any solid subspace X of E is unsure and in Theorem 14 we show that any subspace X of E whose positive cone $X_+ = X \cap P$ in the topology of X, does not have interior points is unsure.

It is worth noting that based on a result of Polyrakis (the lattice universality of $C[0, 1]$, (Polyrakis, 1994)) we show that anyone of the spaces $C[0, 1]$, $L_\infty[0, 1]$ and ℓ_∞, has at least so many unsure subspaces as the cardinality of the set of separable Banach lattices without order unit, see Corollary 22.

In the sequel we study the converse of the unsure subspace problem. So we start by a fixed proper subspace X of E and we study the existence of a cone P of E with an interior point x_0 so that X to be unsure with respect to the risk measure of E defined by the cone P and the safe asset x_0. In Theorems 20 and 21, we show that the answer to this problem is positive and we determine such a cone and an interior point x_0 of P.

In the last section we combine the unsure subspaces with the theory of price-bubbles of Gilles and LeRoy (1992), and we give an application in the case where the space of financial positions is the space ℓ_∞. Specifically in Theorem 24 we prove: If $E = \ell_\infty$ and from the economy we exclude the price bubbles, then any financial position $x \in \ell_\infty$ can be approximated by unsure positions of the same price, in the sense that for any $x \in \ell_\infty$ there exists a sequence $\{x_n\}$ of unsure positions (i.e. $\rho(x_n) \geq 0$ for any n) such that $\lim q(x_n) = q(x)$, for any price vector q.

For more results on risk measures but mainly for a more general perspective on this subject, we refer to the articles Acerbi and Scandolo (2008), Csoka et al. (2007), Jarrow and Purnanandam (2005). In Acerbi and Scandolo (2008), the axioms of coherence are properties of coherent risk measures, which are proved either on certain liquidity operators or on the value function related to a liquidity policy on the space of portfolios. In Csoka et al. (2007), personalized coherent risk measures are studied, being related to an incomplete markets equilibrium, through the pricing kernel which arises from the marginal rates of substitution between consumption at time-period 0 and any of the states $1, 2, \ldots, S$ calculated at the consumption vector of any consumer at the equilibrium. In Jarrow and Purnanandam (2005) a generalized risk measure notion is introduced, in order to indicate that the solvency for a firm's portfolio is the minimum quantity invested to a marketed asset (which is not necessarily cash) jointly with the original firm's portfolio to be acceptable by the regulator. This provides a motivation for the consideration of a risky asset for insuring the financial positions in a normed linear space E.

For an introduction in ordered spaces and the notions we use here see Appendix and for a detailed study of ordered spaces we refer to Aliprantis and Burkinshaw (2006), Aliprantis and Tourky (2007) and Jameson (1970).

Note that throughout this article, for simplicity, by the term "cone" we will mean a "nontrivial cone". (A cone P of a vector space E is *nontrivial* if $P \neq \{0\}$ and $P \neq E$).

2. Risk measures in Banach spaces

Suppose that E is a Banach space ordered by the cone P and $x_0 \in P$ is an order unit of E which is considered as the safe asset. Recall that in this article by the term "cone" we mean always "nontrivial cone".

A function $\rho : E \to \mathbb{R}$ with the properties

(i) $x \geq y \Rightarrow \rho(x) \leq \rho(y)$ (monotonicity)
(ii) $\rho(x + tx_0) = \rho(x) - t$, for any $t \in \mathbb{R}$ (cash invariant),

is a risk measure on E. By a slight modification of the function ρ we may also suppose that any risk measure satisfies the condition

(iii) $\rho(0) = 0$.

The set $\mathcal{A}_\rho = \{x \in E \mid \rho(x) \leq 0\}$, is the *set of acceptable positions* of ρ. If a risk measure ρ is subadditive and positively homogeneous, then ρ is said *coherent*. Any coherent risk measure is convex, but for the converse the extra assumption that ρ is positively homogeneous is needed. If ρ is a coherent risk measure on E, then the set $\mathcal{A}_\rho$ is a cone of E which contains P and for each $x \in E$ we have

$$\rho(x) = \inf\{t \in \mathbb{R} \mid x + tx_0 \in \mathcal{A}_\rho\}.$$

As in the classical case, risk measures in Banach spaces can also be defined by a cone P of E which is considered as the set of acceptable positions and an order unit x_0 of E which is considered as safe asset. The next theorem has been formulated by Jaschke and Küchler (2001), without proof. For a proof of this result we refer to Stoica (2006, Theorem 2.1).

Theorem 1 (*Jaschke and Küchler*)**.** *Suppose that E is a Banach space ordered by the cone P. If x_0 is an order unit of E, then the function*

$$\rho(x) = \inf\{t \in \mathbb{R} \mid x + tx_0 \in P\} \quad \textit{for each } x \in E,$$

is a coherent risk measure with respect to the cone P and the vector x_0. If moreover P is closed then $\mathcal{A}_\rho = P$.

Recall the next theorem, see in Aliprantis and Tourky (2007, Theorem 2.8). By this result, if the cone P is closed, then in the theory of risk measures any order unit is an interior point of P and any interior point of P can be considered as a safe asset.

Theorem 2. *If E is a Banach space ordered by the closed cone P and $x \in P$, the following are equivalent:*

(i) *x is an order unit of E,*
(ii) *x is an algebraic interior point (internal point) of P,*
(iii) *x is an interior point of P.*

3. Results on coherent risk measures

In this section we will suppose that E is a Banach space ordered the closed cone P and ρ is the coherent risk measure defined on E with respect to the cone P and the safe asset $x_0 \in P$, according to Theorem 1 of Jaschke and Küchler. Note that, in this article, whenever we say that x_0 is a safe asset we will mean always that x_0 is an order unit of E, or equivalently an interior point of the cone P. As we have remarked in the previous section P coincides with the set of acceptable positions $\mathcal{A}_\rho$, i.e.

$$P = \{x \in E \mid \rho(x) \leq 0\}.$$

Recall again that in this article, our cones are supposed nontrivial. We will denote by P^0 the dual cone of P in E^*, i.e.

$$P^0 = \{x^* \in E^* \mid x^*(x) \geq 0 \text{ for each } x \in P\}.$$

If K is a cone of E^* then $K_0 = \{x \in E \mid x^*(x) \geq 0, \text{ for any } x^* \in K\}$ is the *dual cone of* K *in* E and K_0 is weakly closed, see in Aliprantis and Tourky (2007, Theorem 2.13), or Jameson (1970, Proposition 3.1.7). It is easy to show that if Q is a cone of E then $Q \subseteq (Q^0)_0$ and if Q is closed then $Q = (Q^0)_0$ because Q as a closed, convex set is also weakly closed. Recall that any vector x of E can be considered as a continuous linear functional of E^*, which we will denote by $\widehat{x}$. So we have

$$\widehat{x}(x^*) = x^*(x), \quad \text{for any } x^* \in E^*.$$

It is known that $\widehat{x} \in E^{**}$, $\|\widehat{x}\| = \|x\|$ and that $\widehat{x}$ is referred as the natural image of x in E^{**}. We will denote by B_{x_0} the base for the cone P^0 which is defined by x_0, i.e. the set

$$B_{x_0} = \{x^* \in P^0 \mid \widehat{x_0}(x^*) = 1\}.$$

The base B_{x_0} is the set of normalized (with respect to the safe asset x_0) price systems. In the case where the set of states is a set Ω and E is the space of all bounded, measurable functions with respect to the measure space $(\Omega, \mathcal{F})$, equipped with the supremum norm, $x_0 = \mathbf{1}$ is the constant function 1 and $P = E_+$ is the positive cone of E in the pointwise ordering, then for any $x^* \in B_{x_0}$ we can define the finitely additive probability measure μ_{x^*} such that $\mu_{x^*}(A) = x^*(\mathcal{X}_A)$ for any $A \in \mathcal{F}$, where $\mathcal{X}_A$ is the characteristic function of A. In the case where Ω is a compact and Hausdorff topological space and $E = C(\Omega)$ is the set of continuous, real valued functions on Ω, then $E^* = ca(\Omega)$ is the set of countably additive signed measures on the Borel σ-algebra $\mathcal{B}$ of Ω (Aliprantis and Border, 2006, Theorem 14.14) and B_{x_0} is the set of probability measures defined on $\mathcal{B}$.

In the next theorem we prove that B_{x_0} is a weak star compact subset of P^0. To show this, it is essential to show that the base B_{x_0} of P^0 is bounded in order to apply the theorem of Alaoglou for the weak star compactness of the unit ball of E^*. In general we cannot say that any base B_x for the cone P^0 defined by a vector x of P is bounded and therefore we cannot say that any base B_x of P^0 is weak star compact, as the next example shows. For a short note on the weak topology of E and the weak star topology of E^* see Appendix.

Example 3. Suppose that in our economy the space of financial positions E is the space c_0 of convergent to zero real sequences equipped with the supremum norm and $P = c_0^+$ is the positive cone of c_0. Then $P^0 = \ell_1^+$ is the positive cone of the space ℓ_1 of absolutely summing real sequences. Suppose that B_x is the base for P^0 defined by the vector (sequence) $x = (\frac{1}{n})$ of P. Then B_x is unbounded. Indeed, if e_n is the vector (sequence) of ℓ_1 with 1 in the n coordinate and zero everywhere else, we have that $\widehat{x}(ne_n) = 1$ for each n. Therefore $ne_n \in B_x$ and $\|ne_n\| = n$ for any n, therefore B_x is unbounded and so B_x is not weakly compact.

Theorem 4. *The set B_{x_0} is a weak star compact subset of P^0.*

Proof. According to our hypothesis, x_0 is an interior point of P. We will show that x_0, as a linear functional of E^*, is strictly positive on P, i.e. $\widehat{x_0}(y^*) > 0$ for any $y^* \in P^0$, $y^* \neq 0$. So we suppose that $\widehat{x_0}(y^*) = 0$ for some $y^* \in P^0$, $y^* \neq 0$. Since x_0 is an interior point of P, we have that $x_0 + rU \subseteq P$ for some real number $r > 0$, where by U we denote the closed unit ball of E. Hence, for each $y \in E$, we have that $x_0 + \frac{r}{2}\frac{y}{\|y\|} \in x_0 + rU \subseteq P$, therefore, by the definition of P^0, we have

$$\widehat{x_0}(y^*) + \frac{r}{2}\frac{\widehat{y}(y^*)}{\|y\|} \geq 0,$$

hence $\widehat{y}(y^*) \geq 0$ because we have supposed that $\widehat{x_0}(y^*) = 0$. So we have that $y^*(y) \geq 0$ for any $y \in E$, hence $y^*(y) = 0$ for each $y \in E$, because $y, -y \in E$. This implies that $y^* = 0$, a contradiction. Hence x_0 is strictly positive on P^0. We will show now that the base B_{x_0} is bounded.

Let $x^* \in B_{x_0}$. For any $y \in U$ we have $-y \in U$, therefore $x^*(x_0 \pm ry) \geq 0$, because $x^* \in B_{x_0} \subseteq P^0$ and $x_0 + rU \subseteq P$. Therefore we have that $|x^*(y)| \leq \frac{1}{r}$. So for each $x^* \in B_{x_0}$ we have that $\|x^*\| = \sup\{|x^*(y)| \mid y \in U\} \leq \frac{1}{r}$. Hence the base B_{x_0} is bounded and by the theorem of Alaoglou, see in Megginson (1998, Theorem 2.6.18), B_{x_0} is weak star compact. □

The next theorem has been formulated by Jaschke and Küchler (2001, Theorem 2), for general vector spaces where it is proved that $\rho(x) = \sup\{-\widehat{x}(x^*) \mid x^* \in B_{x_0}\}$, for any $x \in E$. In the next theorem we improve this result by replacing the supremum by the maximum.

Theorem 5. *If E is a Banach space and ρ is a risk measure defined on E with respect to the closed cone $P \subseteq E$ and the safe asset $x_0 \in \operatorname{int} P$, then for any $x \in E$ we have:*

$$\rho(x) = \max\{-\widehat{x}(x^*) \mid x^* \in B_{x_0}\} = -\min\{\widehat{x}(x^*) \mid x^* \in B_{x_0}\}.$$

Proof. The cone P is closed therefore, as we have remarked in the beginning of this section we have $P = (P^0)_0$, hence

$$P = \{x \in E \mid x^*(x) \geq 0 \text{ for each } x^* \in P^0, x^* \neq 0\}.$$

But any $x^* \in P^0$, $x^* \neq 0$, is the positive multiple of a vector of B_{x_0}, therefore we have $P = \{x \in E \mid x^*(x) \geq 0, \text{ for each } x^* \in B_{x_0}\}$. Suppose that $x \in E$. By the definition of ρ we have, $\rho(x) = \inf\{t \in \mathbb{R} \mid x + tx_0 \in P\}$, or equivalently,

$$\rho(x) = \inf\{t \in \mathbb{R} \mid x^*(x + tx_0) \geq 0 \text{ for each } x^* \in B_{x_0}\}.$$

Since $x^* \in B_{x_0}$ we have $x^*(x_0) = 1$, therefore

$$\rho(x) = \inf\{t \in \mathbb{R} \mid t \geq -x^*(x) \text{ for each } x^* \in B_{x_0}\}. \quad (1)$$

By Theorem 4, B_{x_0} is weak star compact. Therefore $-x$, as a weak star continuous linear functional of E^* takes maximum on a vector x_0^* of B_{x_0}. Then the real number $x_0^*(-x) = -x_0^*(x)$ is the minimum value of t in (1), therefore $\rho(x) = -x_0^*(x) = \max\{-x^*(x) \mid x^* \in B_{x_0}\}$ and the theorem is true because for any subset A of $\mathbb{R}$ we have $\max(-A) = -\min(A)$. □

Before to state the next theorem note that in the case where E is a subspace of some L_∞ space then, by the Lipschitz continuity of the risk measure ρ with respect to the supremum norm of L_∞ we have that any $x \in E$ with $\rho(x) < 0$ is an interior point of the cone $P = L_\infty^+$. Indeed, if we suppose that $\rho(x) = a < 0$, then for any $y \in E$ with $\|x - y\| < -\frac{a}{2}$ we have that $|\rho(x) - \rho(y)| \leq \|x - y\| < -\frac{a}{2}$, therefore $\rho(y) < \frac{a}{2} < 0$. So the ball of center x and radius $-\frac{a}{2}$ is contained in P. This result is known, see for example in the book of Föllmer and Schied (2004, p. 162), where it is noted that the set $\mathcal{B} = \{x \in E \mid \rho(x) < 0\}$ is an open subset of P. In the next theorem we generalize this result in Banach spaces without the assumption of the Lipschitz continuity of the risk measure.

Theorem 6. *Let E be a Banach space ordered by the closed cone P and suppose that x_0 is an interior point of P. If ρ is the risk measure defined on E with respect to the cone $P \subseteq E$ and the safe asset x_0, then for any $x \in E$, the following statements are equivalent:*

(i) $\rho(x) < 0$,
(ii) *x is an interior point of P.*

Proof. (i) $\Longrightarrow$ (ii) Suppose that $\rho(x) < 0$. By Theorem 5 we have $\min\{\widehat{x}(x^*) \mid x^* \in B_{x_0}\} = -\rho(x) = a > 0$. Also x as a linear functional of E^* is strictly positive on the cone P^0. Indeed for any $x^* \in P^0$, $x^* \neq 0$ we have $\frac{x^*}{x^*(x_0)} \in B_{x_0}$, hence

$$\widehat{x}\left(\frac{x^*}{x^*(x_0)}\right) \geq a > 0, \quad (2)$$

therefore $\widehat{x}(x^*) > 0$. By Theorem 4, B_{x_0} is a bounded base for the cone P^0 and suppose that $\|x^*\| \leq M$ for any $x^* \in B_{x_0}$. We will show that the base $B_x = \{x^* \in P^0 \mid \widehat{x}(x^*) = 1\}$ is a bounded base for the cone P^0. Indeed, for any $x^* \in B_x$ we have that $\frac{x^*}{\widehat{x_0}(x^*)} \in B_{x_0}$, therefore $\|\frac{x^*}{\widehat{x_0}(x^*)}\| \leq M$, because M is a norm bound of B_{x_0}. Hence $\|x^*\| \leq M\widehat{x_0}(x^*)$.

By (2) we have $\widehat{x}(\frac{x^*}{\widehat{x_0}(x^*)}) \geq a$, hence

$$\widehat{x}(x^*) \geq a\widehat{x_0}(x^*) \geq \frac{a}{M}\|x^*\|.$$

But $\widehat{x}(x^*) = 1$ because we have supposed that $x^* \in B_x$, therefore $\|x^*\| \leq \frac{M}{a}$, and the base B_x is bounded.

We will show that x is an interior point of P. Denote by U the closed unit ball of E and by U^0 the closed unit ball of E^*. Since the base B_x is a bounded base of P^0, there exists a real number $b > 0$ such that $B_x \subseteq bU^0$. We shall show that

$$D = \{y \in E \mid |x^*(y)| \leq 1, \text{ for any } x^* \in B_x\},$$

is a neighborhood of zero in E such that $x + D \subseteq P$. We assert that $\frac{1}{b}U \subseteq D$. Indeed, for any $y \in \frac{1}{b}U$ we have that $\|y\| \leq \frac{1}{b}$, therefore for any $x^* \in B_x \subseteq bU^0$ we have that

$$|x^*(y)| \leq \|x^*\| \, \|y\| \leq b\frac{1}{b} = 1 \Rightarrow y \in D.$$

So D is a neighborhood of zero. Let $y \in D$. Then for any $x^* \in B_x$ we have

$$x^*(x + y) = 1 + x^*(y) \geq 1 + (-1) = 0,$$

because by the definition of D we have that $|x^*(y)| \leq 1$. So we have that $x + y \in (P^0)_0 = P$, therefore $x + D \subseteq P$ and x is an interior point of P.

(ii) $\Longrightarrow$ (i) Since x is an interior point of P there exists $\delta > 0$ such that $x + tx_0 \in P$ for any $t \in (-\delta, \delta)$, hence we have $\rho(x) \leq -\delta < 0$. □

In the next theorem we study risk measures for different interior points considered as safe assets but with the same set P of acceptable positions. The formula of the theorem shows a kind of equivalence between these two risk measures.

Theorem 7. *Let E be a Banach space ordered by the closed cone $P \subseteq E$, let x_1, x_2 be interior points of the cone P and suppose that a_0, b_0 are the real numbers defined by the formula: $a_0 = \min\{a \in \mathbb{R} \mid x_1 \leq ax_2\}$, $b_0 = \min\{b \in \mathbb{R} \mid x_2 \leq bx_1\}$. If ρ_1, ρ_2 are the risk measures defined on E with respect to the cone P and the safe assets x_1, x_2 respectively, then for any $x \in E$ we have:*

$$\frac{1}{a_0}\rho_2(x) \leq \rho_1(x) \leq b_0\rho_2(x).$$

Proof. We will show that a_0, b_0 exist. Since x_1, x_2 are interior points of P, we have that x_1, x_2 are order units of E. Hence there exists $a \in \mathbb{R}_+$ such that $x_1 \leq ax_2$ and suppose that $a_0 = \inf\{a \in \mathbb{R}_+ \mid x_1 \leq ax_2\}$. For any such a we have $ax_2 - x_1 \geq 0$ and it is easy to show that $a_0x_2 - x_1 \geq 0$ because the cone P is closed. Hence $a_0 = \min\{a \in \mathbb{R} \mid x_1 \leq ax_2\}$ and $a_0 \neq 0$ because if we suppose that $a_0 = 0$ we have that $x_1 = 0$, a contradiction because x_1 is an order unit of E and $E \neq \{0\}$. Similarly we have $b_0 = \min\{b \in \mathbb{R} \mid x_2 \leq bx_1\}$. By the definition of the risk measure, for any $x \in E$ we have $\rho_1(x) = \inf\{t \in \mathbb{R} \mid x + tx_1 \in P\}$. Also for $t = \rho_1(x)$ we have that $x + \rho_1(x)x_1 \in P$ because the cone is closed. But $x + \rho_1(x)x_1 \in P$ implies that $x + a_0\rho_1(x)x_2 \in P$ because $x + a_0\rho_1(x)x_2 \geq x + \rho_1(x)x_1$, therefore $\rho_2(x) \leq a_0\rho_1(x)$. Similarly we have that $\rho_1(x) \leq b_0\rho_2(x)$ and the theorem is true. □

Corollary 8. *Let E be a Banach space ordered by the closed cone $P \subseteq E$ and let x_1, x_2 be interior points of the cone P. If ρ_1, ρ_2 are the risk measures defined on E with respect to the cone P and the safe assets x_1, x_2 respectively, then for any $x \in E$ we have:*

(i) $\rho_1(x) < 0 \Longleftrightarrow \rho_2(x) < 0$,
(ii) $\rho_1(x) = 0 \Longleftrightarrow \rho_2(x) = 0$,
(iii) $\rho_1(x) > 0 \Longleftrightarrow \rho_2(x) > 0$.

Example 9. Suppose that $E = \ell_\infty$ is the space of bounded real sequences ordered by the pointwise ordering and equipped with the supremum norm and suppose that ρ is the risk measure of ℓ_∞ with respect to the cone ℓ_∞^+ and the safe asset $\mathbf{1}$.

Suppose also that the sequence $r = (r_i)$ is an interior point of ℓ_∞^+ which we consider as safe asset. Then r is a risky vector in the sense that r can have different payoffs r_i, so $x + tr$ finances x with different payoffs in the different states. Suppose that ρ_r is the risk measure with respect to ℓ_∞^+ and the safe asset r. Since r is an interior point of ℓ_∞^+, there exists $\theta > 0$ such that $r_i \geq \theta$ for each i, therefore $\mathbf{1} \leq ar$ with $a = \frac{1}{\theta}$. Similarly if $b = \sup\{r_i\}$ we have that $r \leq b\mathbf{1}$. By the above theorem and these remarks we have

$$\frac{1}{a_0}\rho_r(x) \leq \rho(x) \leq b_0\rho_r(x),$$

for any $x \in \ell^\infty$ where $a_0 = \inf\{r_i\}$ and $b_0 = \sup\{r_i\}$. So if we suppose that $r_i = 4 - \frac{1}{i}$ if i is even and $r_i = 2 + \frac{1}{i}$ if i is odd, we have

$$2\rho_r(x) \leq \rho(x) \leq 4\rho_r(x).$$

4. Unsure subspaces

In this section we will also suppose that E is a Banach space ordered by the closed cone P and ρ is the coherent risk measure defined on E with respect to the cone P and the safe asset $x_0 \in P$. Recall that whenever we say that x is a safe asset we will mean that x is an interior point of P.

In Theorem 6, we have proved that the risk measure of a financial position $x \in E$ is strictly lower than zero ($\rho(x) < 0$) if and only if x is an interior point of the cone P. This means that there exists a ball $B(x, r)$ of E of center x and radius $r > 0$ which is contained in the set of acceptable positions P of E. Since any financial position $y \in B(x, r)$ is also an interior point of P we have that $\rho(y) < 0$, therefore x "is surrounded" by financial positions of risk strictly lower than zero. On the other hand if $\rho(x) = 0$, then x is not an interior point of P therefore for any ball $B(x, r)$ there exists at least one financial position $y_r \in B(x, r)$ which does not belong to P or equivalently with $\rho(y_r) > 0$. Therefore x can be approximated, in the norm topology of E, by financial positions y_r with $\rho(y_r) > 0$. So in the sense of the above remarks we can say that the financial position x is *strictly acceptable* or *strictly safe* if $\rho(x) < 0$ and that x is not strictly safe or *unsure* if $\rho(x) \geq 0$.

In the spirit of the above remarks we give the definition of the unsure subspace as follows:

Definition 10. A subspace X of E consisting of unsure positions i.e. $\rho(x) \geq 0$ for any $x \in X$, is an *unsure subspace* of E.

We give below criteria of unsure subspaces. Recall that a subspace $X \subseteq E$ is *solid* if for any $x, y \in X$, with $x \leq y$, the order interval $[x, y] = \{z \in E \mid x \leq z \leq y\}$ is contained in X.

Theorem 11. *Let E be a Banach space and let ρ be the risk measure defined on E with respect to the closed cone $P \subseteq E$ and the safe asset x_0 and suppose that E is ordered by the cone P. Then any solid subspace X of E is unsure.*

Proof. Suppose that X is a solid subspace of E but X is not unsure. Then there exists $x \in X$ such that $\rho(x) < 0$. Then x is an interior point of P and therefore x is an order unit of E. Therefore for any $y \in E$ there exists $k \in \mathbb{R}_+$ such that $-kx \leq y \leq kx$. Hence $y \in X$ because X, as a solid subspace of E, contains all the vectors of the order interval $[-kx, kx]$ so we have that $X = E$. This is a contradiction, therefore X is an unsure subspace. □

Example 12. Suppose that $E = L_\infty[0, 1]$ and

$$X = \{x \in L_\infty[0, 1] \mid x(t) = 0, \text{ for each } t \in A\},$$

where A is a nonempty, proper subset of $[0, 1]$. Then X is a solid subspace of $L_\infty[0, 1]$ because for each $x, y \in X$, with $x \leq y$ we have: $z \in L_\infty[0, 1]$ and $x \leq z \leq y$ implies that $z(t) = 0$ for each $t \in A$ therefore $z \in X$. Recall that the equality and the pointwise ordering in $L_\infty[0, 1]$ are defined in the sense of the almost everywhere. We also suppose that $0 < \mu(A) < 1$ where $\mu(A)$ is the Lebesgue measure of A (we may suppose for example that $A = [\frac{1}{2}, 1]$ or that $A = \cup_{n:even}(\frac{1}{n+1}, \frac{1}{n})$). Then X is a proper subspace of $L_\infty[0, 1]$ and therefore an unsure subspace of $L_\infty[0, 1]$ with respect to the risk measure of $L_\infty[0, 1]$, defined by the closed cone $L_\infty^+[0, 1]$ and the safe asset $\mathbf{1}$.

As we have noted before Theorem 4, any vector $x \in E$ can be considered as a linear functional of E^*, which, is denoted by $\widehat{x}$. From the theory of Functional Analysis we know that $\widehat{x} \in E^{**}$ with $\|\widehat{x}\| = \|x\|$ and $\widehat{x}$ is referred as the natural image of x in the second dual E^{**} of E. We denote by $\widehat{E}$ the set $\widehat{E} = \{\widehat{x} \mid x \in E\}$ and we know that $\widehat{E}$ is a closed subspace of E^{**} in the norm topology of E^{**} but $\widehat{E}$ is dense in E^{**} with respect to the weak star topology of E^{**}. If $\widehat{E} = E^{**}$, the space E is called *reflexive*.

Recall also that a Banach lattice E has *order continuous norm*, if each decreasing sequence of E with infimum zero is convergent to zero.

Corollary 13. *Suppose that E is a non reflexive Banach lattice with order continuous norm, x_0^{**} is an interior point of E_+^{**} and ρ is the risk measure defined on E^{**} with respect to the cone E_+^{**} and the safe asset x_0^{**}. Then $\widehat{E}$ is an unsure subspace of E^{**} with respect to the risk measure ρ.*

Proof. Suppose that E^{**} is ordered by the cone E_+^{**}. By Aliprantis and Burkinshaw (2006, Theorem 4.9), $\widehat{E}$ is a solid subspace of E^{**} and $\widehat{E} \subsetneq E^{**}$ because E is non reflexive. Therefore $\widehat{E}$ is an unsure subspace of E^{**}. □

Suppose that X is a subspace of E, $A \subseteq X$ and $x \in A$. x is an interior point of A in the topology of X if there exists a ball $B(x, r)$ of E of center x and radius r so that $X \cap B(x, r) \subseteq A$. An interior point of A in the topology of X is not necessarily an interior point of A in the topology of E. For example, if $E = \mathbb{R}^2$, then $x = (1, 0)$ is an interior point of the positive cone $X_+ = \{(t, 0) \mid t \in \mathbb{R}_+\}$ of $X = \{(t, 0) \mid t \in \mathbb{R}\}$, in the topology of X, but x is not an interior point of X_+ in the topology of E. In the next theorem the set $Q = P \cap X$ is a cone which in some cases can be the trivial cone $\{0\}$.

Theorem 14. *Suppose that E is Banach space ordered by the closed cone P, x_0 is an interior point of P and suppose that ρ is the risk measure defined on E with respect to the cone P and the safe asset x_0. If X is a closed subspace of E such that the set $Q = P \cap X$, in the topology of X, does not have interior points, then X is an unsure subspace of E with respect to ρ.*

Proof. Suppose that $\rho(x) < 0$ for some $x \in X$. Then x is an interior point of P, therefore there exists a ball $D = \{y \in E \mid \|x - y\| \leq \delta\}$ of E of center x and radius $\delta > 0$ which is contained in P. So for any $z \in X$ with $\|x - z\| < \delta$ we have that $z \in P$ therefore $z \in Q$. Hence x is an interior point of the set $Q = P \cap X$, a contradiction. Hence X is an unsure subspace. □

Theorem 15. *Let E be a Banach space and let ρ be the risk measure defined on E with respect to the closed cone $P \subseteq E$ and the safe asset $x_0 \in \operatorname{int} P$. Suppose that X is a subspace of E, F is a subspace of the algebraic dual E' of E and consider the dual pair $\langle X, F\rangle$ with $\langle x, x'\rangle = x'(x)$, for any $x \in X$, $x' \in F$ and consider also the cone*

$$P_F^0 = \{x' \in F \mid x'(x) \geq 0 \text{ for any } x \in P\}.$$

If $P_F^0 \neq \{0\}$ and the vector 0 of E' belongs to the $\sigma(F, X)$-closure of the base

$$B_{x_0}^F = \{x' \in P_F^0 \mid \widehat{x_0}(x') = 1\},$$

of the cone P_F^0, then X is an unsure subspace of E.

Proof. Suppose that 0 belongs to the $\sigma(F, X)$-closure of $B_{x_0}^F$. Then there exists a net $(x_a')_{a\in A}$ of $B_{x_0}^F$ such that $x_a'(x) \longrightarrow 0$ for any $x \in X$. Suppose that $x \in X$. Then

$$\rho(x + \rho(x)x_0) = 0,$$

hence $x + \rho(x)x_0 \in P$, therefore we have

$$x_a'(x + \rho(x)x_0) = x_a'(x) + \rho(x)x_a'(x_0) \geq 0,$$

therefore

$$\rho(x)x_a'(x_0) \geq -x_a'(x), \quad \text{for any } a.$$

By our assumptions that x_a' is a net of $B_{x_0}^F$ we have $x_a'(x_0) = 1$, therefore

$$\rho(x) \geq -x_a'(x), \quad \text{for any } a,$$

and by taking limits we have $\rho(x) \geq 0$. □

In the above theorem we may also suppose that $F \subseteq E^*$. In the case where $F = E^*$, we have:

Corollary 16. *Let E be a Banach space and let ρ be the risk measure defined on E with respect to the closed cone $P \subseteq E$ and the safe asset $x_0 \in \operatorname{int} P$. Suppose that X is a subspace of E. If 0 belongs to the $\sigma(E^*, X)$-closure of the base B_{x_0} for the cone P^0, then X is an unsure subspace of E.*

Example 17. The space $X = c_0$, of convergent to zero real sequences, is an unsure subspace of the space $E = \ell_\infty$ of bounded real sequences with respect to the cone $P = \ell_\infty^+$ and the safe asset $x_0 = \mathbf{1}$. We can show this directly because for any $x \in c_0$ we have that $x + t\mathbf{1} \geq 0$ implies $t \geq 0$. Indeed, since the sequence $x = (x(i))$ converges to zero, if we suppose that $x + t\mathbf{1} \geq 0$ we have that $x(i) + t \geq 0$, for each i and by taking limits we have that $t \geq 0$, therefore $\rho(x) \geq 0$ and c_0 is an unsure subspace of ℓ_∞. By Corollary 8, c_0 is also an unsure subspace of E with respect to any risk measure ρ_r of E having as safe asset an interior point r of ℓ_∞^+.

Also we can show this by applying our criteria of unsure subspaces as follows:

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(i) By Theorem 11, because c_0 is a solid subspace of ℓ_∞. Indeed, for any $x, y \in c_0$ and any $z \in \ell_\infty$ with $x \le z \le y$ we have that $z \in c_0$ because z is dominated between two zero sequences, hence $[x, y] \subseteq c_0$.

(ii) By Theorem 14, because one can show that c_0^+, in the topology of c_0 does not have interior points. Indeed if we suppose that $x = (x(i))$ is an interior point of c_0^+, there exists $\varepsilon > 0$ such that the ball of c_0 of center x and radius ε is contained in c_0^+. Since the sequence x converges to zero we have that $0 \le x(j) < \frac{\varepsilon}{4}$ for at least one j. If $y \in c_0$ such that $y(i) = x(i)$ for any $i \ne j$ and $y(j) = -\frac{\varepsilon}{4}$, then $\|x - y\| < \varepsilon$ and $y \notin c_0^+$, a contradiction.

(iii) By Theorem 15, as follows: It is known that the dual E^* of ℓ_∞ is the direct sum $E^* = F \oplus G$ where $F = \ell_1$ and $G = \ell_1^d$ is the disjoint component of ℓ_1 (see Appendix). Consider the dual system $\langle X, F\rangle$, where $X = c_0$.

Then $P_F^0 = \{f \in F \mid f(x) \ge 0 \text{ for each } x \in P\} = \ell_1^+$ and

$$B_{x_0}^F = \left\{ f \in \ell_1^+ \mid \sum_{i=1}^{\infty} f_i = 1 \right\},$$

is the base for the cone P_F^0 defined by x_0. It is easy to show that 0 belongs to the $\sigma(F, X)$-closure of B_{x_0}. Indeed, the sequence $\{e_n\}$, where e_n is the vector of ℓ_1^+ with 1 in the n coordinate and zero everywhere else, is a sequence of $B_{x_0}^F$ which converges to zero in the $\sigma(F, X)$ topology because for each $x \in X$ we have that $\widehat{x}(e_n) = x(n) \longrightarrow 0$.

We have proved above some criteria for unsure subspaces and we have given some examples. It is worth noting that, by applying the result of Polyrakis (1994, Theorem 4.1) that *any separable Banach lattice X is order isomorphic to a lattice-subspace Z of $C[0, 1]$* we can show that the space $C[0, 1]$ has at least so many unsure subspaces as the cardinality of the set of separable Banach lattices without order unit. Since $C[0, 1]$ is a closed sublattice of $L_\infty[0, 1]$ we have that any separable Banach lattice X is order isomorphic to a lattice-subspace Z of $L_\infty[0, 1]$, therefore $L_\infty[0, 1]$ has at least so many unsure subspaces as the cardinality of the set of separable Banach lattices without order unit. An analogous result is also true for the subspaces of ℓ_∞ because $C[0, 1]$ is order isomorphic to a closed sublattice of ℓ_∞. Indeed, if $\{r_i\}$ is the sequence of rational numbers of the real interval $[0, 1]$, it is easy to show that the function $T(x) = (x(r_i))$, $x \in C[0, 1]$, is an order isometry of $C[0, 1]$ into ℓ_∞ and its image W is a sublattice of ℓ_∞. To emphasize this fact we state the next theorem. Of course the safe asset of the theorem can be the constant function **1**.

Theorem 18. *Suppose that E is one of the spaces $C[0, 1]$, ℓ_∞, or $L_\infty[0, 1]$, ordered by the pointwise ordering and suppose that x_0 is an interior point of E_+. If ρ is the risk measure defined on E with respect to the positive cone E_+ of E and the safe asset x_0, then E has at least so many infinite dimensional unsure subspaces, with respect to ρ, as the cardinality of the set of infinite dimensional, separable Banach lattices without order unit.*

Proof. Suppose X is an infinite dimensional, separable Banach lattice without order unit. Then by the discussion above X is order isomorphic to an infinite dimensional closed lattice-subspace Z of E and suppose that T is an order isomorphism of X onto Z. If we suppose that Z is not an unsure subspace we have $\rho(y_0) < 0$ for some $y_0 \in Z$. Then, by Theorem 6, y_0 is an interior point of E_+. Therefore y_0 is an interior point of $Z_+ = E_+ \cap Z$ in the topology of X. So we have that y_0 is an order unit of Z, therefore

$$Z = \bigcup_{n \in \mathbf{N}} [-ny_0, ny_0]_Z,$$

where

$$[-ny_0, ny_0]_Z = \{x \in Z \mid -ny_0 \le x \le ny_0\},$$

is the order interval in Z, defined by $-ny_0$ and ny_0. Since T is an order isomorphism we have that

$$X = \bigcup_{n \in \mathbf{N}} T^{-1}([-ny_0, ny_0]_Z) = \bigcup_{n \in \mathbf{N}} ([-nT^{-1}(y_0), nT^{-1}(y_0)]),$$

therefore $T^{-1}(y_0)$ is an order unit of X, a contradiction. Hence Z is an unsure subspace of E. □

As an application of the above theorem we give below an example of an unsure subspace of $L_\infty[0, 1]$. Of course the existence of the subspace is ensured by the theorem and the subspace cannot be completely determined.

Example 19. Suppose that $X = L_1[0, 1]$ is the Banach lattice of absolutely integrable real sequences with respect to the Lebesgue measure μ. Note that $L_1[0, 1]$ itself cannot be considered as a subset of $L_\infty[0, 1]$ because there are vectors of $L_1[0, 1]$ which are not essentially bounded. By Theorem 4.1 of Polyrakis (1994) and the remarks above, there exists an order isomorphism T of $L_1[0, 1]$ onto a closed lattice-subspace Z of $L_\infty[0, 1]$. The cone $L_1^+[0, 1]$ does not have interior points because if we suppose that x is an interior point of $L_1^+[0, 1]$ we have a contradiction as follows: There exists a ball $B(x, \epsilon)$ of center x and radius ϵ which is contained in $L_1^+[0, 1]$. We select a measurable subset A of $[0, 1]$ so that $0 < \int_A |x(t)| d\mu < \frac{\epsilon}{2}$ and we define the vector y of $L_1[0, 1]$ so that $y(t) = x(t)$ for any $t \notin A$ and $y(t) = -x(t)$ for $t \in A$. Then $y \in B(x, \epsilon)$ but $y \notin L_1^+[0, 1]$. So $L_1[0, 1]$ does not have interior points and therefore does not have order units. By the above theorem we have that Z is an unsure subspace of $L_\infty[0, 1]$ with respect to the risk measure defined by the cone $L_\infty^+[0, 1]$ and the safe asset **1** or any other interior point of $L_\infty^+[0, 1]$.

In the next theorems we start by a fixed subspace $X \subsetneqq E$ of E. Our aim is to define a risk measure ρ on E so that the subspace X to be unsure with respect to the risk measure ρ. We show that such a risk measure always exists. Recall that if A is a closed subset of E and $x \notin A$, then

$$d = \inf\{\|x - y\| \mid y \in A\} > 0,$$

is *the distance of x from A.*

Theorem 20. *Let E be a Banach space and let be $X \subsetneqq E$ be a closed subspace of E. Suppose that $x_0 \in E \setminus X$ and D is the closed ball of E of center x_0 and radius δ, where $0 < \delta < d$ and d is the distance of x_0 from X. If P is the cone of E generated by D and if ρ is the risk measure defined on E with respect to the cone P and the safe asset x_0, then P is closed and X is an unsure subspace of E with respect to the risk measure ρ. Specifically we have: $\rho(x) > 0$, for any $x \in X, x \ne 0$.*

Proof. Since $0 \in X$ we have that $d \le \|x_0 - 0\| = \|x_0\|$, therefore $0 \notin D$ because we have supposed that $0 < \delta < d$. Also P is closed because it is generated by a closed and bounded set D. For any $x \in X, x \ne 0$ we have that $x \notin P$ because if we suppose that $x \in P$, then $x = ty$ for some $y \in D$ and $t > 0$. So we have that $y \in X$ and $\|x_0 - y\| < d$, a contradiction. Hence $x \notin P$, therefore $\rho(x) > 0$. □

Theorem 21. *Let E be a Banach space and let $X \subseteq E$ be a closed subspace of E. Suppose that $x_0 \in E \setminus X$, such that $d < \|x_0\|$, where d is the distance of x_0 from X. If D is the closed ball of E of center x_0 and radius d, P is the cone of E generated by D, and ρ is the risk measure defined on E with respect to the cone P and the safe asset x_0, then X is an unsure subspace of E with respect ρ.*

Proof. Suppose that $\rho(x) < 0$ for some $x \in X$. Then by the definition of the risk measure, there exists a real number $t > 0$ such that $x - tx_0 \in P$. Therefore $x - tx_0 = \lambda y$ for some $y \in D$ and $\lambda \ge 0$. But

$y \in D$, therefore $y = x_0 + z$ for some $z \in E$ with $\|z\| \leq d$. So we have $x - tx_0 = \lambda(x_0 + z)$. Hence

$$\frac{x}{\lambda + t} = x_0 + \frac{\lambda}{\lambda + t} z,$$

with $\|\frac{\lambda}{\lambda+t} z\| < \|z\| \leq d$ because $\frac{\lambda}{\lambda+t} < 1$.

Therefore we have that $w = \frac{x}{\lambda+t} \in X$ with $\|x_0 - w\| < d$ which contradicts the fact that d is the distance of x_0 from X. Therefore X is an unsure subspace. □

Corollary 22. *Let E be a non reflexive Banach space and $x_0^{**} \in E^{**} \setminus \widehat{E}$, such that $d < \|x_0^{**}\|$, where d is the distance of x_0^{**} from $\widehat{E}$. Suppose that $0 < \delta \leq d$ and that P is the cone of E^{**} generated by D, where D is the closed ball of E^{**} of center x_0^{**} and radius δ. If ρ is the risk measure defined on E^{**} with respect to the cone P and the safe asset x_0^{**}, then $\widehat{E}$ is an unsure subspace of E^{**} with respect to ρ. Specifically if $\delta < d$, we have that $\rho(\widehat{x}) > 0$ for any $x \in E$, $x \neq 0$.*

5. Price-bubbles and unsure subspaces

In this section we follow the definition of Gilles and LeRoy (1992), for price-bubbles. In Gilles and LeRoy (1992), $L_\infty(\mu) = L_\infty(M, \mathcal{M}, \mu)$ is the commodity space, where M is a set, $\mathcal{M}$ is a σ-algebra of M, μ is a σ-finite measure on $\mathcal{M}$ and $L_\infty^*(\mu)$ is the price space. From the pages 327–328 of Gilles and LeRoy (1992), we quote the following text where the definition of the price bubble is given: *Our problem now is to formulate the general principle underlying the separation of a price system into fundamental and bubble when the price system is characterized as a general continuous linear functional on $L_\infty(\mu)$. For any positive and continuous linear function u of $L_\infty(\mu)$ the fundamental f can be defined by*

$$f(x) = \int_M p(s)x(s)\mu(ds),$$

where p is a maximal element of the set

$$D = \left\{ q \in L_1(\mu) \,\middle|\, \int_M q(s)x(s)\mu(ds) \leq u(x), \text{ for any positive } x \in L_\infty(\mu) \right\}$$

and the bubble b is defined by the formula

$$b(x) = u(x) - f(x), \quad x \in L_\infty(\mu).$$

In Gilles and LeRoy (1992), it is noted that p is the maximum of D, therefore p is unique. Note that $L_\infty^*(\mu)$, as the dual of $L_\infty(\mu)$, is ordered by the ordering: $h \geq g$, where $h, g \in L_\infty^*(\mu)$, if and only if $h(x) \geq g(x)$ for any $x \in L_\infty^+(\mu)$. So if we consider $L_1(\mu)$ as a subspace of $L_\infty^*(\mu)$ the above definition is the following: For any price system $u \in (L_\infty^*(\mu))_+$ we consider the set

$$D = \{q \in L_1(\mu) \mid q(x) \leq u(x) \text{ for any } x \in L_\infty^+(\mu), \text{ or equivalently } q \leq u\}$$

and we consider the maximum point p of D. Then by the definition of f we have that $f(x) = p(x)$ for any $x \in L_\infty(\mu)$, therefore $f = p$ and $b = u - p$. So according to the definition of Gilles and LeRoy, the price system u is decomposed in a fundamental p and a bubble b, i.e.

$$u = p + b.$$

For this approach in Gilles and LeRoy (1992), the Yosida–Hewitt decomposition of signed measures of bounded variation in a countably additive and a purely additive part is used. In this article we use an analogous decomposition from the theory of Banach lattices. Especially we use the fact that the second dual E^{**} of a KB space E is the direct sum

$$E^{**} = E \oplus E^d,$$

where, for simplicity, by E is denoted the natural image $\widehat{E}$ of E in E^{**}. In the above formula, E is a projection band in E^{**} and E^d is the disjoint component of E in E^{**}. Recall that reflexive Banach lattices and AL-spaces are standard examples of KB spaces. The space $E = L_1(\mu)$ is an AL-space, therefore its second dual is decomposed as follows

$$L_\infty^*(\mu) = L_1(\mu) \oplus L_1^d(\mu).$$

So any $u \in L_\infty^*(\mu)$ has a unique decomposition $u = u_1 + u_2$ with $u_1 \in L_1(\mu)$ and $u_2 \in L_1^d(\mu)$ and if u is positive we have

$$u_1 = \sup\{y \in L_1(\mu) \mid y \leq u\},$$

see in Aliprantis and Burkinshaw (2006, Theorem 1.43, p. 36). Therefore for positive u, $u_1 = p$ is the fundamental and $u_2 = b$ is the bubble of the definition of Gilles and LeRoy. So if we exclude the bubbles the price space is $L_1(\mu)$.

In the next remark we give an intuitional explanation of the above definition for price-bubbles in the case of ℓ_∞.

Remark 23. Consider an economy with commodity–price duality $\langle E, E^* \rangle$ where $E = \ell_\infty$. Then $E^* = X^{**}$ where $X = \ell_1$. Therefore

$$\ell_\infty^* = \ell_1 \oplus \ell_1^d,$$

because ℓ_1 is a KB space. So any $u \in (\ell_\infty^*)_+$ has a unique decomposition $u = u_1 + u_2$ where $u_1 \in \ell_1^+$ is the fundamental and $u_2 \in (\ell_1^d)^+$, is the bubble. It is known that any $g \in \ell_1^d$ is a limit functional of ℓ_∞ i.e. there exists a real number k_g such that for any $x \in c$ we have

$$g(x) = k_g \lim_{n\to\infty} x(n),$$

where c is the set of convergent real sequences and $k_g = g(\mathbf{1})$, see in Aliprantis and Border (2006, Lemma 16.30). Hence, if $\mathbf{1}$ is the constant sequence 1 and $\mathbf{1}_n$ is the sequence with $\mathbf{1}_n(i) = 0$, for $i = 1, \ldots, n$ and $\mathbf{1}_n(i) = 1$ for any $i > n$, we have that

$$g(\mathbf{1}) = g(\mathbf{1}_n) = k_g, \quad \text{for any } n \in \mathbb{N}.$$

For any fixed state i, we have that any financial position $\mathbf{1}_n$ with $n > i$ has payoff 0 at the state i, but k_g is the price (payoff) of any financial position $\mathbf{1}_n$ corresponding to the price system g. From these remarks we have that in order the price system g to finance a financial position x by k_g it is only needed the promise that in the outer future the payoffs of x will be equal to one. This property of ℓ_1^d justifies by a natural way the definition of price-bubbles of Gilles and LeRoy which in the case of ℓ_∞, identifies the price-bubbles with the component ℓ_1^d of ℓ_∞^*.

Theorem 24. *Suppose that the commodity–price duality is the dual pair $\langle \ell_\infty, \ell_1 \rangle$ and ρ is the coherent risk measure defined on ℓ_∞ with respect to the positive cone ℓ_∞^+ of ℓ_∞ and the interior point x_0 of ℓ_∞^+. Then any financial position can be approximated by unsure positions of the same price, in the sense that for any $x \in \ell_\infty$, there exists a sequence $\{x_n\}$ of c_0 such that $\rho(x_n) \geq 0$ for any n and $\lim q(x_n) = q(x)$ for any price system $q \in \ell_1$.*

Proof. Let $x \in \ell_\infty$. Since ℓ_∞ is the second dual of c_0, the space c_0 is weak star dense in ℓ_∞. Also the weak star topology of ℓ_∞ is metrizable because ℓ_∞ is the dual of a separable space (the space ℓ_1) therefore there exists a sequence $\{x_n\}$ of c_0 such that for any $q \in \ell_1$ we have $q(x_n) \to q(x)$. But as we have noted before (Example 17) c_0 is an unsure subspace of ℓ_∞, therefore $\rho(x_n) \geq 0$ for any n. So we have proved that for any financial position $x \in \ell_\infty$ there exists a sequence $\{x_n\}$ of unsure positions such that $\lim q(x_n) = q(x)$ for any price system $q \in \ell_1^+$ and the theorem is true. □

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Remark 25. The above theorem can be formulated in more general cases. If for example E is an ordered Banach space and if we suppose that the commodity–price duality is the dual pair $\langle E^{**}, E^{*}\rangle$ we have the following: The space $\widehat{E}$ is weak star dense in E^{**}, i.e. for any $x^{**} \in E^{**}$, there exists a net $(x_a)_{a\in A}$ of E such that $\lim_a q(x_a) = x^{**}(q)$ for any $q \in E^{*}$. Note that in the case where the dual E^{*} of E is not separable, then we cannot say that the weak star topology of E^{**} is metrizable therefore any vector of E^{**} is interpolated by a net of E^{**}, not by a sequence. So in the case where $\widehat{E}$ is an unsure subspace of E^{**}, then any financial position $x^{**} \in E^{**}$, can be approximated by unsure positions of the same price. Recall that if $\widehat{E}$ is a solid subspace of E^{**}, or if E_+ does not have interior points, then $\widehat{E}$ is an unsure subspace of E^{**}.

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Appendix. Partially ordered linear spaces

We give some notions and results from the theory of cones and partially ordered linear spaces which are needed in this article. Let E be a linear space. A nonempty, convex subset P of E is a *cone* if $\lambda x \in P$, for any $\lambda \in \mathbb{R}_+$ and $x \in P$, where $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$. A cone P of E with the property $P \cap (-P) = \{0\}$, is *pointed* and P is *nontrivial* if $\{0\} \subsetneq P \subsetneq E$. If P is a cone of E, then we can define the binary relation $\geq$ of E so that for any $x, y \in E$ we have: $x \geq y \Longleftrightarrow x - y \in P$. This binary relation satisfies the properties

(i) $x \geq x$, for any $x \in E$ (reflexivity),
(ii) $x \geq y$ and $y \geq z$ implies $x \geq z$, for any $x, y, z \in E$ (transitivity)
(iii) for any $x, y \in E$, $x \geq y$ implies $\lambda x \geq \lambda y$ for any $\lambda \in \mathbb{R}_+$ and $x + w \geq y + w$ for any $w \in E$ (compatibility with the linear structure of E),

and we say that E *is ordered by the cone* P or that $\geq$ is *the ordering of* E *defined by the cone* P. This relation is *antisymmetric* i.e. $\geq$ satisfies the property

(iv) $x \geq y$ and $y \geq x$ implies $x = y$, for any $x, y \in E$,

if and only if the cone P is pointing. If $\geq$ is a binary relation of E which satisfies (i), (ii), (iii) and (iv), then we say that $\geq$ is a *partial linear ordering* of E or that $(E, \geq)$ (or simply E) is a *partially ordered linear space*.

Let E be ordered by the cone P. For any $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E \mid x \leq z \leq y\}$ is an *order interval* of E defined by x, y.

A vector $e \in E_+$ is an *order unit* of E if $E = \cup_{n=1}^{\infty}[-ne, ne]$. If E is a normed linear space, then every interior point of P is an order unit of E (Aliprantis and Tourky, 2007, Lemma 2.5). If E is a Banach space and E_+ is closed, then the converse is also true, i.e. every order unit of E_+ is an interior point of E_+ (Aliprantis and Tourky, 2007, Theorem 2.8).

Denote by E' the algebraic and by E^{*} the topological dual of E, i.e. E' is the set of linear and E^{*} the set of continuous, linear functionals of E.

A linear functional $f \in E'$ is *positive* (on P) if $f(x) \geq 0$ for any $x \in P$ and f is *strictly positive* (on P) if $f(x) > 0$ for any $x \in P \setminus \{0\}$. If a strictly positive linear functional (of P) exists, then P is pointed.

A convex set $B \subseteq P$ is a *base* for the cone P if a strictly positive linear functional $f \in E'$ exists such that $B = \{x \in P \mid f(x) = 1\}$. In this case the base B is denoted by B_f and we say that B is the base for P defined by f. $P^0 = \{x^{*} \in E^{*} | x^{*}(x) \geq 0 \text{ for any } x \in P\}$ is the *dual cone of* P *in* E^{*}. For any strictly positive $x^{*} \in E^{*}$ we have: The base $B_{x^{*}} = \{x \in P | x^{*}(x) = 1\}$ of P is bounded if an only if x^{*} is an interior point of P^0 (Jameson, 1970, Theorem 3.8.4). If K is a cone of E^{*} then $K_0 = \{x \in E | x^{*}(x) \geq 0, \text{ for any } x^{*} \in K\}$ is the *dual cone of* K *in* E and K_0 is weakly closed, see in Aliprantis and Tourky (2007, Theorem 2.13), or Jameson (1970, Proposition 3.1.7). Recall that the dual cone P^0 of P is weak star closed in E^{*}. So for the cone P of E we have: $P \subseteq (P^0)_0$ and if P is closed then $P = (P^0)_0$ because P, as a convex set, is also weakly closed.

Let E be a partially ordered vector space. If for each $x, y \in E$ the supremum and the infimum of the set $\{x, y\}$ exist in E, then E is a *vector lattice* or a *Riesz space*. Following the classical notation we write $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. Let E be a vector lattice. For any $x \in E$ we denote by $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$ the positive and negative part of x respectively and with $|x| = x \vee (-x)$ the absolute value of x. Suppose that X is a subspace of E. If for any $x, y \in X$, $x \vee y \in X$ and $x \wedge y \in X$, then X is a *sublattice* or a *Riesz subspace* of E. X is a *lattice-subspace* of E if for any $x, y \in X$ the supremum, $\sup_X\{x, y\}$ of $\{x, y\}$ in X and the infimum $\inf_X\{x, y\}$ of $\{x, y\}$ in X exist. Recall that $\sup_X\{x, y\}$ is the minimum of the set of upper bounds of the set $\{x, y\}$ which belong (the upper bounds) to X but $x \vee y$ is the minimum of the set of upper bounds of the set $\{x, y\}$ which belong to E. Note also that every sublattice of E is a lattice-subspace of E but the converse is not true. If X is a lattice-subspace of E we have:

$$\inf_X\{x, y\} \leq x \wedge y \quad \text{and} \quad x \vee y \leq \sup_X\{x, y\}.$$

If X is a sublattice of E and for each $x, y \in E$ we have: $y \in X$ and $|x| \leq |y|$ implies $x \in X$, then X is an *ideal* of E. An ideal X of E is a *band* of E if for each $D \subseteq X$ such that $\sup(D)$ exists in E we have that $\sup(D) \in X$. For any subset D of E we denote by D^d the disjoint complement of D in E, i.e.

$$D^d = \{x \in E \mid |x| \wedge |y| = 0 \text{ for each } y \in D\}.$$

A band B of E is a *projection band* if $E = B \oplus B^d$. If B is a projection band then every element $x \in E$ has a unique decomposition $x = x_1 + x_2$, with $x_1 \in B$ and $x_2 \in B^d$. Then any $x \in E_+$ is decomposed in x_1, x_2 where

$$x_1 = \sup\{y \in B \mid 0 \leq y \leq x\}$$

and the map $P_B : E \rightarrow B$ with $P_B(x) = x_1$ for each $x \in E$ is a projection of E onto B which is called the band projection of B (Aliprantis and Burkinshaw, 2006, Theorem 1.43). Note that, for each $x \in E_+$, x_1 is also given by the formula

$$x_1 = \sup\{y \in B \mid y \leq x\},$$

because B is a band of E.

An ordered Banach space E is a *Banach lattice* if E is a vector lattice and for each $x, y \in E$ we have: $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A Banach lattice E is an *AL-space* if for each $x, y \in E_+$, $x \wedge y = 0$ implies that $\|x + y\| = \|x\| + \|y\|$. A Banach lattice E is a *KB space* if every increasing positive and bounded sequence of E_+ is norm convergent. Aliprantis and Burkinshaw (2006, p. 232). Reflexive Banach lattices and AL-spaces are standard examples of KB spaces. If E is a KB space, then E is a projection band in E^{**}, Aliprantis and Burkinshaw (2006, Theorem 4.60) and by Aliprantis and Burkinshaw (2006, Theorem 1.43), we have

$$E^{**} = E \oplus E^d.$$

Note also that there are KB spaces which are dual spaces. For example the space ℓ_1 is a dual space. Also the AL-space of finite Borel measures defined on a compact, metrizable topological space K, is the topological dual of the space of continuous real valued functions $C(K)$ defined on K, Aliprantis and Border (2006, Theorem 14.15).

Suppose that X, Y are partially ordered Banach spaces. A linear operator $T : X \longrightarrow Y$ is an *isomorphism* of X onto Y, if T is one to one and onto and T, T^{-1} are continuous. If moreover $\|T(x)\| = \|x\|$, for any $x \in X$, T is an isometric isomorphism of X onto Y. It is known that *any separable Banach space is isometric with a closed subspace Z of $C[0, 1]$*. Since an isomorphism "identifies" the spaces the space $C[0, 1]$ of continuous real valued functions on $[0, 1]$ is a *universal Banach space* because it contains any separable Banach space.

If X, Y are ordered Banach spaces, a linear operator $T : X \longrightarrow Y$ is an *order isomorphism* of X onto Y, if T is an isomorphism of X onto Y and for each $x \in X$ we have: $x \in X_+$ if and only if $T(x) \in Y_+$. Then the spaces X and Y are *order isomorphic* and their topological and order structure "are identified".

It is known (Polyrakis, 1994, Theorem 4.1) that *any separable Banach lattice X is order isomorphic to a closed lattice-subspace Z* of $C[0, 1]$, therefore $C[0, 1]$ is also a *universal Banach lattice*. So in the sense of this result, the class of closed lattice-subspaces of $C[0, 1]$ represents the class of all separable Banach lattices.

Weak topologies: A dual system $\langle E, F\rangle$ is a pair of vector spaces together with bilinear form $\langle\cdot,\cdot\rangle : E \times F \rightarrow \mathbb{R}$ such that: $\langle x, x'\rangle = 0$ for any $x' \in F$ implies $x = 0$ (F separates the points of E) and $\langle x, x'\rangle = 0$ for any $x \in E$ implies $x' = 0$ (E separates the points of F).

The $\sigma(E, F)$*-topology* of E, or the *weak topology* of E with respect to the dual system $\langle E, F\rangle$, is the linear topology of E whose a base of neighborhoods of zero is consisting of the sets

$$V_{x'_1,x'_2,\ldots,x'_n,\epsilon} = \{x \in E : |\langle x, x'_i\rangle| < \epsilon, \text{ for any } i = 1, 2, \ldots, n\},$$

for any finite set of vectors x'_i of F and any $\epsilon > 0$. Similarly, the linear topology of F with a neighborhood base of zero consisting of the sets

$$V_{x_1,x_2,\ldots,x_n,\epsilon} = \{x' \in F : |\langle x_i, x'\rangle| < \epsilon, \text{ for any } i = 1, 2, \ldots, n\},$$

for any finite set of vectors x_i of E and any $\epsilon > 0$, is the $\sigma(F, E)$-topology of F, or the *weak topology* of F with respect to the dual system $\langle E, F\rangle$.

A net $(x_a)_{a\in A}$ of E converges to $x \in E$ in the $\sigma(E, F)$-topology of E if

$$\lim_a \langle x_a, x'\rangle = \langle x, x'\rangle, \quad \text{for any } x' \in F.$$

Analogously, a net $(x'_a)_{a\in A}$ of F converges to $x' \in F$ in the $\sigma(F, E)$-topology of F if $\lim_a \langle x, x'_a\rangle = \langle x, x'\rangle$, for any $x \in E$.

If E is a normed space and $F = E^*$ is the topological dual of E and the bilinear form is $\langle x, x^*\rangle = x^*(x)$ for any $x \in E, x^* \in E^*$, then the $\sigma(E, E^*)$-topology of E is referred as the *weak topology* of E and the $\sigma(E^*, E)$-topology of E^* as the *weak star topology* of E^*. For more details on the weak topologies see Aliprantis and Border (2006) or Megginson (1998).

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