Positive Bases in Ordered Subspaces with the Riesz Decomposition Property

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Abstract

In this article we suppose that E is an ordered Banach space the positive cone of which is defined by a countable family $\mathcal{F}=\{f_i | i \in \mathbb{N}\}$ of positive continuous linear functionals of E, i.e. $E_+ = \{x \in E \mid f_i(x) \geq 0, \text{ for each } i\}$ and we study the existence of positive (Schauder) bases in the ordered subspaces X of E with the Riesz decomposition property. So we consider the elements x of E as sequences $x = (f_i(x))$ and we develop a process of successive decompositions of a quasi-interior point of X_+ which in any step gives elements with smaller support. So we obtain elements of X_+ with minimal support and we prove that these elements define a positive basis of X which is also unconditional. In the first section of this article we study ordered normed spaces with the Riesz decomposition property.

1 Introduction and notations

The most typical examples of ordered Banach spaces E with a rich class of ordered subspaces are the universal spaces $C[0,1]$ and ℓ_∞. As it is shown in [8], Theorem 4.1[1] each separable ordered Banach space with closed and normal positive cone is

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[1]This result it is shown by a slight modification of the classical proof of the universality of $C[0,1]$.

order-isomorphic to an ordered subspace of $C[0,1]$, therefore the existence of positive bases in the separable ordered Banach spaces is equivalent with the study of positive bases in the closed ordered subspaces X of $C[0,1]$. In this article we study the general problem of the existence of positive bases in the ordered subspaces X of E, as it is formulated in the abstract, by developing a method of decompositions of a quasi-interior point of X. To develop this method we study the subspaces X of E with the *maximum support property*. In this kind of subspaces the quasi-interior points of X and in its closed principal solid subspaces, are characterized as the positive vectors of these subspaces with maximum support. We show that in this kind of subspaces the extremal points of X_+ are the nonzero elements of X_+ with minimal support and this is an important property for the study of positive bases. Also this class of subspaces is a large one. Indeed as it is shown in [7], Lemma 5.1, each Banach lattice with a positive basis is order isomorphic to a closed, ordered subspace of ℓ_∞ with the maximum support property with respect to the family $\mathcal{F}$ of the Dirac measures δ_i supported at the natural numbers i and a similar result is also true for the space $C[0,1]$, see in [8], Theorem 5.1. Therefore the class of ordered subspaces of ℓ_∞ or $C[0,1]$ with the maximum support property is a large one and contains, in the sense of an order isomorphism, the class of Banach lattices with a positive basis.

To develop our method of decompositions we study also the ordered subspaces X of E with the following property which we call *ws-property*: for any $x \in X_+$ and any $f_i \in \mathcal{F}$ the set $K = \{y \in X_+ \big| y \leq x \quad \text{and} \quad f_i(y) = 0\}$ has at least one maximal element. According to the terminology of vector optimization, X has the ws-property if and only if the set K has Pareto efficient points with respect to X_+. If E is a Banach lattice with order continuous norm or if E is a dual space, we show, Corollary 20 and 21, that the ordered subspaces of E have the ws-property. In the main result of this article, Theorem 32, we prove that the maximum support property and the ws-property are sufficient conditions for the existence of positive bases in the ordered subspaces of E with the Riesz decomposition property. As an application we show, Theorem 36, that the maximum support property and the ws-property are necessary and sufficient in order a positive biorthogonal system of an ordered Banach space E with the Riesz decomposition property to define a positive basis of E.

This article is a generalization of [7] where the same problem is studied in the lattice-subspaces of E. So in the first section of this paper we study ordered normed spaces with the Riesz decomposition property and we prove some results necessary for our method of decompositions. Specifically we study quasi-interior points and we generalize the existing results for normed lattices to ordered normed spaces with the Riesz decomposition property, Theorem 4 and 6.

Finally note that each Banach space with an unconditional basis, ordered by the positive cone of the basis, is a Banach lattice with respect to an equivalent norm. Also note that the problem of the existence of unconditional basic sequences in Banach spaces, known as the unconditional basic sequence problem, was one of the famous open problems of Functional Analysis till 1993 when W.T. Gowers and B. Maurey gave a negative answer to it, [3]. Our results give necessary conditions for the existence of unconditional basic sequences in ordered Banach spaces.

Let Y be a (partially) ordered normed space with positive cone Y_+. If $Y = Y_+ - Y_+$ the cone Y_+ is *generating* or *reproducing* and if a real number $a > 0$ exists so that $x, y \in Y_+$ with $x \leq y$ implies that $||x|| \leq a||y||$, the cone Y_+ is *normal* . Recall that a convex set P of a linear space is a *cone* if $\lambda x \in P$ for any real number $\lambda \geq 0$ and any $x \in P$ and $P \cap (-P) = \{0\}$. The set $[x, y] = \{z \in Y \mid x \leq z \leq y\}$ is the order interval xy, whenever $x, y \in Y$ with $x \leq y$. A point $x \in Y_+, x \neq 0$ is an *extremal point* of Y_+ if for any $y \in Y$ with $0 < y < x$ there exists $\lambda \in \mathbb{R}_+$ such that $y = \lambda x$. Denote by $EP(Y_+)$ the set of extremal points of Y_+. Y has the *Riesz decomposition property (RDP)* if for each $x, y_1, y_2 \in Y_+$ with $x \leq y_1 + y_2$ there exist $x_1, x_2 \in Y_+$ such that $x = x_1 + x_2$ and $0 \leq x_1 \leq y_1$, $0 \leq x_2 \leq y_2$. A subspace Z of Y is *solid* if for any $x, y \in Z$ with $x \leq y$, the order interval $[x, y] = \{z \in Y \mid x \leq z \leq y\}$ is contained in Z. We say that the cone Y_+ *gives an open decomposition* of Y or that Y_+ is *non-flat* if $U_+ - U_+$ is a neighborhood of zero, where $U_+ = U \cap Y_+$, is the positive part of the closed unit ball U of Y, or equivalently, if any $x \in Y$ has a representation $x = x_1 - x_2$ with $x_1, x_2 \in Y_+$ and $\|x_1\|, \|x_2\| \leq M\|x\|$, where M is a constant real number. A linear functional f of Y is positive if $f(x) \geq 0$ for each $x \in Y_+$ and strictly positive if $f(x) > 0$ for each $x \in Y_+, x \neq 0$. Denote by Y^* the set of continuous, linear functionals of Y and by Y^*_+ the set of positive ones. Suppose that Y is an ordered Banach space. A sequence $\{e_n\}$ of Y is a (Schauder) *basis* of Y if each $x \in Y$ has a unique expansion $x = \sum_{n=1}^{\infty} \lambda_n e_n$, with $\lambda_n \in \mathbb{R}$ for each n. If moreover $Y_+ = \{x = \sum_{n=1}^{\infty} \lambda_n e_n \mid \lambda_n \geq 0 \text{ for each } n\}$, then $\{e_n\}$ is a *positive basis* of Y. A positive basis is unique in the sense that if $\{b_n\}$ is another positive basis of Y, then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$. If $\{e_n\}$ is a positive basis of Y then, by [9], Theorem 16.3, and [4], Theorem 3.5.2. and Theorem 4.1.5, the following statements are equivalent: *(i) the basis $\{e_n\}$ is unconditional, (ii) the cone Y_+ is generating and normal, (iii) Y is a Banach lattice with respect to an equivalent norm.*

A linear operator T of Y onto an ordered normed space Z is an *order-isomorphism* of Y onto Z if T is one-to-one, T and T^{-1} are continuous and for each $x \in Y$ we have: $x \in Y_+$ if and only if $T(x) \in Z_+$. For notions and terminology on ordered spaces not defined here we refer to [4], [5], [1], [6] and [10]. For Schauder bases we refer to [9].

2 Quasi-interior points in spaces with the Riesz decomposition property

In this section we will denote by Y an ordered normed space with the Riesz decomposition property whose positive cone Y_+ is closed, normal and gives an open decomposition of Y. Then, by the Riesz-Kantorovich Theorem, the set of order bounded linear functionals Y^b of Y is an order complete linear lattice. For any $x \in Y_+$, $I_x = \cup_{n \in \mathbb{N}}[-nx,\ nx]$ is *the solid subspace of Y generated by x* and the closure of I_x is the *closed solid subspace of Y generated by x*. As we prove below the closure of I_x is again solid. Recall the following properties of an ordered Banach space W which we use in this article: (i) If W_+ is closed and generating, then W_+ gives an open decomposition of W (Krein-Smulian) and also any order bounded linear functional of W is continuous and (ii) the cone W_+ is normal if and only if $W^* = W^*_+ - W^*_+$ (M. Krein), see for example in [4], Theorems 3.5.2, 3.5.6. and 3.4.8. We start with the next obvious result.

Proposition 1. *Any solid subspace of Y has the Riesz decomposition property.*

Proposition 2. *Suppose that $x \in Y_+, x \neq 0$ and I is the closure of I_x. Then,*
(i) for any $y \in I_+$, there exists an increasing sequence $\{y_n\}$ of I_x which converges to y, with $0 \leq y_n \leq y$, for each n,
(ii) I is a solid subspace of Y,
(iii) the positive cone I_x^+ of I_x is generating,
(iv) if we suppose moreover that Y is a Banach space then each positive, continuous, linear functional of I has a positive, continuous, linear extension on Y.

Proof. Let $y \in I_+, y \neq 0$. At first we shall show that there exists a sequence $\{y'_n\}$ of $I_x \cap [0, y]$ convergent to y. Since $y \in I$, we have that $y = lim_{n \longrightarrow +\infty} t_n$ where $t_n \in [-\kappa_n x, \kappa_n x]$ and $\{\kappa_n\}$ is an increasing sequence of natural numbers. Hence $t_n - y \longrightarrow 0$, therefore by [4] Theorem 3.3.5, there exist sequences $\{w_n\}$, $\{v_n\}$ of Y_+ with $t_n - y = w_n - v_n$ and $w_n, v_n \longrightarrow 0$. Then we have that $t_n + v_n - y = w_n \geq 0$, therefore

$$y \leq t_n + v_n \leq \kappa_n x + v_n. \tag{1}$$

By the RDP we have that $y = y'_n + y''_n$ where $0 \leq y'_n \leq \kappa_n x$ and $0 \leq y''_n \leq v_n$. Since the cone Y_+ is normal and the sequence v_n converges to zero, the sequence y''_n converges also to zero, hence $y'_n \longrightarrow y$, therefore our assertion is true. So for any positive real number ε, we have $\|y - y'_n\| < \frac{\varepsilon}{2}$, for a proper n. We put $r_1 = y'_n$. Similarly there exists $r_2 \in I_x$, $r_2 \in [0, y - r_1]$ with $\|y - r_1 - r_2\| < \frac{\varepsilon}{2^2}$ and continuing this process we find a sequence $\{r_n\}$ of I_x with $r_n \in [0, y - \sum_{i=1}^{n-1} r_i]$ and $\|y - \sum_{i=1}^{n} r_i\| < \frac{\varepsilon}{2^n}$, for each n. Therefore the sequence $y_n = \sum_{i=1}^{n} r_i$ is an increasing sequence of $[0, y]$ which converges to y, therefore statement (i) is true.

For the proof of (ii) it is enough to show that $[0, y] \subseteq I_+$, for any $y \in I_+$. So we suppose that $y \in I_+$ and that $z \in [0, y]$. As in the proof of (i) we find again that y satisfies (1) and by the RDP we have that $z = z'_n + z''_n$ where $0 \leq z'_n \leq \kappa_n x$, $0 \leq z''_n \leq v_n$ and as before we have that the sequence z''_n converges also to zero. Hence $z'_n \longrightarrow z$, therefore $z \in I$ and statement (ii) is true.
Statement (iii) is obvious because for any $y \in [-nx, nx]$ we have $0 \leq y + nx \leq 2nx$, therefore $y + nx = a + b$ where $a, b \in Y_+$ with $a \leq nx, b \leq nx$, therefore $y = a - (nx - b)$.
Suppose that f is a positive, continuous linear functional of I. For any $y \in Y_+$ we put $L_y = \{z \in I_x^+ \mid z \leq y\}$. L_y is bounded because the cone Y_+ is normal. For any $y \in Y_+$ we put $g(y) = \sup\{f(z) \mid z \in L_y\}$. By the RDP and by the fact that I_x is solid we have that $L_y + L_w = L_{y+w}$. Therefore g is positively homogeneous and additive on Y_+. Hence g has a linear and positive extension on Y which we will denote again by g, i.e. $g(x) = g(x_1) - g(x_2)$ for any $x = x_1 - x_2 \in Y$ with $x_1, x_2 \in Y_+$. By [4], Corollary 3.5.6, g is continuous. By the definition of g and by the fact that I_x is solid, we have that $g(y) = f(y)$, for any $y \in I_x^+$, therefore g coincides with f on I_x because $I_x = I_x^+ - I_x^+$. Since I_x is dense in I we have that g is also equal to f on I, therefore g is an extension of f from I to Y. ∎

Definition 3. *An element $u \in Z_+$, of an ordered topological linear space Z is a quasi-interior point of Z_+ (or a quasi-interior positive element of Z) if the solid subspace $\cup_{n \in \mathbb{N}} [-nu, \ nu]$ of Z generated by u is dense in Z.*

The above definition extends the notion of the quasi-interior point (see in [1], page 259) from normed lattices to ordered topological linear spaces. It is clear that if u is a quasi-interior point of Z_+ then $f(u) > 0$ for any positive, continuous, and nonzero linear functional f of Z. In [5], page 24, the points u of an ordered Banach space Z with the property $f(u) > 0$ for any positive, continuous, nonzero linear functional f of Z are called quasi-interior points of Z_+. In Theorem 6 we show that in ordered Banach spaces with the RDP, these two definitions are equivalent. By Proposition 2 we get the following result:

Theorem 4. *An element $u \in Y_+$ is a quasi-interior point of Y_+ if and only if for each $x \in Y_+$ there exists an increasing sequence $\{x_n\}$ of I_u which converges to x, with $0 \leq x_n \leq x$, for each n.*

Proposition 5. *If u is a quasi-interior point of Y_+, then $[0, x] \cap [0, u] \neq \{0\}$, for each $x \in Y_+, x \neq 0$.*

Proof. By the above theorem there exists an increasing sequence $\{x_n\}$ of I_u with $0 < x_n \leq x$ which converges to x, therefore the proposition is true. ∎

Theorem 6. *If we suppose moreover that Y is a Banach space and $u \in Y_+$, then the following statements are equivalent:*
(i) u is a quasi-interior point of Y_+,
(ii) $f(u) > 0$, for each $f \in Y_+^*$, $f \neq 0$.

Proof. The direct is obvious because $f(u) = 0$ implies that $f = 0$ on Y. For the converse suppose that statement (ii) is true and that the closure I of I_u is a proper subspace of Y. So there exists $g \in Y^*, g \neq 0$ which is equal to zero on I. Then $|g| \in Y^*$ because Y is a Banach space and $|g|$ is positive. It is known that $|g|(y) = \sup g([-y, y])$ for any $y \in Y_+$. Since $g \neq 0$ and the positive cone of Y is generating we have that $g(y) \neq 0$, for at least one $y \in Y_+$ which implies that $|g| \neq 0$. Therefore $|g|(u) > 0$. Since $|g|(u) = \sup g([-u, u])$ we have that g is nonzero on the interval $[-u, u]$, a contradiction because g is equal to zero on I and $[-u, u] \subseteq I$. Therefore u is a quasi-interior point of Y_+ and the converse is true. ■

Proposition 7. *Suppose that Z is an ordered normed space and suppose that its positive cone Z_+ is complete. Then the following statements are equivalent:*
(i) Every $y \in Z_+, y \neq 0$, is a quasi-interior point of Z_+,
(ii) $dimZ = 1$.

Proof. Suppose that statement (i) is true. At first we shall show that the boundary ϑZ_+ of Z_+ is equal to $\{0\}$. By the Bishop-Phelps Theorem (see for example in [4] Theorem 3.8.14) the support points of Z_+ are dense in ϑZ_+. Suppose that r is a support point of Z_+ which is supported by the functional $x^* \in Z^*, x^* \neq 0$, i.e. $x^*(r) = \min\{x^*(t) \mid t \in Z_+\}$. Then $x^*(r) \leq 0$ because $0 \in Z_+$. If we suppose that x^* is not positive, there exists $a \in Z_+$ with $x^*(a) < 0$. Then x^*, restricted on the halfline defined by a, takes any negative real value, therefore $x^*(r) = -\infty$, contradiction. Therefore x^* is positive. If we suppose that $r \neq 0$, then r is a quasi-interior point of Z_+, therefore $x^*(r) > 0$, a contradiction, because as we have found before $x^*(r) \leq 0$, hence $r = 0$ and $\vartheta Z_+ = \{0\}$. We shall show now that $Z = Z_+ \cup (-Z_+)$. So we suppose that $w \in Z \setminus Z_+$ and that $y \in Z_+, y \neq 0$. Suppose also that z is a point of the line segment yw with $z \in \vartheta Z_+$. Then $z = 0$, therefore $w \in (-Z_+)$, hence $Z = Z_+ \cup (-Z_+)$. Suppose now that w is a fixed point of $Z \setminus Z_+$. As we have shown before, for any point $y \in Z_+, y \neq 0$, the line segment yw contains 0, therefore y belongs to the line defined by w and 0, hence Z_+ is a halfline and $dimZ = 1$. So (i) implies (ii). The converse is clear. ■

Definition 8. *Suppose that Z is an ordered space and $x, y \in Z_+$ with $x, y \neq 0$. If $[0, x] \cap [0, y] = \{0\}$, we will say that x, y are disjoint in Z_+ and we will write $\inf_{Z_+}\{x, y\} = 0$.*

The next result will be used later for the study of positive bases. Statement (i) is an easy consequence of the Riesz decomposition property.

Proposition 9. *Suppose that Z is an ordered normed space with the Riesz decomposition property. Then the following statements are true:*

(i) If the vectors $y_1, y_2, ..., y_n$ are pairwise disjoint in Z_+ and $x \in Z_+$ with $x \leq y_1 + y_2 + ..., +y_n$, we have:
(a) x has a unique decomposition $x = x_1 + x_2 + ... + x_n$ with $0 \leq x_i \leq y_i$, for each $i = 1, 2, ..., n$, and
(b) if $x \geq y_i$ for each $i = 1, 2, ..., n$, then $x = y_1 + y_2 + ..., +y_n$,
(c) if Φ_1, Φ_2 are subsets of $\{1, 2, ..., n\}$, $y_{\Phi_1} = \sum_{i\in\Phi_1} \lambda_i y_i, y_{\Phi_2} = \sum_{i\in\Phi_2} \mu_i y_i,$ where λ_i and μ_i are positive real numbers and $h \leq y_{\Phi_1}, h \leq y_{\Phi_2}$ then h has a unique decomposition $h = \sum_{i\in(\Phi_1\cap\Phi_2)} h_i$ where $0 \leq h_i \leq \min\{\lambda_i, \mu_i\}y_i,$ for each $i \in \Phi_1 \cap \Phi_2$. If $\Phi_1 \cap \Phi_2 = \emptyset$ then y_{Φ_1}, y_{Φ_2} are disjoint in Z_+.

(ii) If the positive cone Z_+ of Z is normal, the vectors $y_i, i \in \mathbb{N}$ are pairwise disjoint in Z_+ and the sum $\sum_{i=1}^{\infty} y_i$ exists, then
(a) $\inf_{Z_+}\{\sum_{i=1}^{n} y_i, \sum_{i=n+1}^{\infty} y_i\} = 0$ for each n, and
(b) each element x of Z_+ with $0 \leq x \leq \sum_{i=1}^{\infty} y_i$ has a unique expansion $x = \sum_{i=1}^{\infty} x_i$, with $0 \leq x_i \leq y_i$ for each i.

Proof. The proof of (i) is the following: By the RDP we have that $x = x_1 + x_2 + ... + x_n$ with $0 \leq x_i \leq y_i$, for each i. Suppose that $x = x'_1 + x'_2 + ... + x'_n$ with $0 \leq x'_i \leq y_i$, for each i. Then $0 \leq x'_j \leq x_1 + x_2 + ... + x_n$, therefore $x'_j = x''_1 + x''_2 + ... + x''_n$ with $0 \leq x''_i \leq x_i \leq y_i$, for each i, therefore $x''_i = 0$ for each $i \neq j$ because y_i and y_j are disjoint. So we have that $x'_j \leq x_j$ and similarly $x_j \leq x'_j$, therefore $x_j = x'_j$, for each j, and the expansion of x is unique. If we suppose that $y_j \leq x$ for each j, we have that $y_j = y_{j1} + y_{j2} + ... + y_{jn}$, with $0 \leq y_{ji} \leq x_i \leq y_i$ for each i, therefore $0 \leq y_{ji} \leq y_j$, hence $y_{ji} = 0$ for each $i \neq j$. So we have that $y_j = y_{jj} \leq x_j \leq y_j$, therefore $y_j = x_j$ for each j and (b) is true. To prove (c) we remark that $0 \leq h \leq y_{\Phi_1}$ implies that $h = \sum_{i\in\Phi_1} h_i$ with $0 \leq h_i \leq \lambda_i y_i$ for each $i \in \Phi_1$. Since $h \leq y_{\Phi_2}$ we have that $h_i = \sum_{j\in\Phi_2} h_i^j$ with $0 \leq h_i^j \leq \mu_j y_j$, for any $j \in \Phi_2$. Since the vectors y_i are disjoint we have that $h_i^j = 0$ for each $j \neq i$, therefore $h_i = h_i^i \leq \min\{\lambda_i, \mu_i\}y_i$ and (c) is true.
To prove statement (a) of (ii) we suppose that $0 \leq h \leq \sum_{i=1}^{n} y_i, \sum_{i=n+1}^{\infty} y_i$. Then $h = \sum_{i=1}^{n} h_i$, with $0 \leq h_i \leq y_i$ for each $i = 1, 2, ..., n$. Also $h_i \leq y_{n+1} + \sum_{i=n+2}^{\infty} y_i$, therefore $h_i = h_{n+1} + h'_{n+1}$ where $0 \leq h_{n+1} \leq y_{n+1}$ and $0 \leq h'_{n+1} \leq \sum_{i=n+2}^{\infty} y_i$. Since y_i and y_{n+1} are disjoint we have that $h_{n+1} = 0$, therefore $0 \leq h_i = h'_{n+1} \leq \sum_{i=n+2}^{\infty} y_i$ and by induction we have that $0 \leq h_i \leq \sum_{i=n+m}^{\infty} y_i$ for each $m \in \mathbb{N}$. Since the cone is normal and the sequence $\sum_{i=n+m}^{\infty} y_i$ converges

to zero, we have that $h_i = 0$, for each $i = 1, 2, ..., n$. Therefore $h = 0$ and (a) is true. To prove (b) suppose that $0 \leq x \leq \sum_{i=1}^{n} y_i + \sum_{i=n+1}^{\infty} y_i$. Then x has a unique decomposition $x = \sum_{i=1}^{n} x_i + x'_n$ with $0 \leq x_i \leq y_i$ for each $i = 1, 2, ..., n$ and $0 \leq x'_n \leq \sum_{i=n+1}^{\infty} y_i$. If we suppose that $m > n$ and $x = \sum_{i=1}^{m} v_i + v'_m$, with $0 \leq v_i \leq y_i$ for $i = 1, 2, ..., m$ and $0 \leq v'_m \leq \sum_{i=m+1}^{\infty} y_i$, then $x = \sum_{i=1}^{n} v_i + (\sum_{i=n+1}^{m} v_i + v'_m)$ therefore $x_i = v_i$ for each $i = 1, 2, ...n$. Hence the vectors $x_i, i \in \mathbb{N}$ are uniquely determined and the expansion $x = \sum_{i=1}^{\infty} x_i$, with $0 \leq x_i \leq y_i$ for each i, of x is unique. ■

For a further study of the Riesz decomposition property on the space of operators between Banach lattices we refer to [2] and the references inside.

3 Ordered subspaces

In this section we will denote by E an infinite dimensional ordered Banach space whose positive cone E_+ is defined by a countable family $\mathcal{F}= \{f_i | i \in \mathbb{N}\}$, of positive, continuous linear functionals of E, i.e. $E_+ = \{x \in E \mid f_i(x) \geq 0, \text{for each } i\}$. Also we will denote by X an *ordered subspace* of E, i.e. X is a subspace of E ordered by the induced ordering. It is clear that E_+ is closed and that $X_+ = X \cap E_+$ is the positive cone of X. For any $x, y \in X$, denote by $\sup_X\{x, y\}$ the supremum and by $\inf_X\{x, y\}$ the infimum of $\{x, y\}$ in X whenever exist. If $\sup_X\{x, y\}$ and $\inf_X\{x, y\}$ exist for any $x, y \in X$, we say that X is a *lattice-subspace* of E. According to our notations, for any $x, y \in X$ we have: $[x, y]_X = \{z \in X \mid x \leq z \leq y\}$, is the *order interval* xy in X whenever $x \leq y$, if $x, y \in X_+$ with $[0, x]_X \cap [0, y]_X = \{0\}$, we say that x, y are *disjoint in* X_+ and we will write $\inf_{X_+}\{x, y\} = 0$. Also for any $x \in X_+, x \neq 0$, we denote by $I_x(X) = \bigcup_{n=1}^{\infty}[-nx, nx]_X$ the *solid subspace of* X *generated by* x. The closure $\overline{I_x(X)}$ of $I_x(X)$ in X is *the closed solid subspace of* X *generated by* x. If $\overline{I_x(X)} = X$, x is a quasi-interior point of X_+.

3.1 The minimal and the maximum support property

The minimal and maximum support property have been introduced in [7]. For any point $x \in E$ we will denote by $x(i)$ the real number $f_i(x)$ and by $supp(x) = \{i \in \mathbb{N} \big| x(i) \neq 0\}$, the *support of* x (with respect to $\mathcal{F}$). The set $supp(X_+) = \bigcup_{x \in X_+} supp(x)$, is the *support* of X_+ (with respect to $\mathcal{F}$). An element x of X_+ *has minimal support* in X_+ (with respect to $\mathcal{F}$) if for any $y \in X_+$, $supp(y) \subsetneqq supp(x)$ implies $y = 0$.

Definition 10. *The ordered subspace X of E has the minimal support property (with respect to $\mathcal{F}$) if for each $x \in X_+ \setminus \{0\}$ we have : x is an extremal point of X_+ if and only if x has minimal support in X_+.*

Proposition 11. *Suppose I is the closed solid subspace of X generated by a nonzero, positive element x of X_+. Then $supp(u) = supp(I_+)$ for any quasi-interior point u of I_+. (The converse is not always true).*

Proof. It is clear that $supp(u) \subseteq supp(I_+)$. If we suppose that $f_i(u) = 0$ for some $i \in supp(I_+)$, then f_i is equal to zero on $I_u(X)$ and therefore also on I, a contradiction because we have supposed that $i \in supp(I_+)$. Hence $f_i(u) > 0$ and $supp(u) = supp(I_+)$. By Example 15, (ii), we have that the converse is not always true. ■

Definition 12. *The ordered subspace X of E has the maximum support property (with respect to $\mathcal{F}$) if each subspace F of X which is equal to X or F is a closed solid subspace of X generated by a nonzero element of X_+ has the property: an element $x \in F_+$ is a quasi-interior point of F_+ if and only if $supp(x) = supp(F_+)$.*

Proposition 13. *If X_+ is closed and X has the maximum support property, then X_+ has quasi-interior points.*

Proof. For each $i \in supp(X_+)$ there exists $x_i \in X_+$ with $f_i(x_i) > 0$. So $u = \sum_{i \in supp(X_+)} \frac{x_i}{2^i \|x_i\|}$, is a quasi-interior point of X_+ because X has the maximum support property and $supp(u) = supp(X_+)$. ■

The proof of the next proposition is the same with the proof of Proposition 3.4 of [7]. The extra assumption here that X_+ is closed is posed in order to use Proposition 7.

Proposition 14. *If X_+ is closed and X has the maximum support property, then X has the minimal support property.*

Example 15. (i) The sequence spaces c_0 and ℓ_p for $1 \leq p < +\infty$ have the maximum support property with respect to the family $\mathcal{F}$=$\{\delta_i\}$ of the Dirac measures $\delta_i(x) = x(i)$ supported at the natural numbers i. The space ℓ_∞ of bounded real sequences does not have the maximum support property with respect to $\mathcal{F}$. Indeed the vector x with $x(i) = \frac{1}{i}$ for any i has maximum support and the closed solid subspace generated by x is c_0. ℓ_∞ has the minimal support property because the extermal points of ℓ_∞^+, as positive multiples of the vectors e_i, have minimal support.
(ii) The family $\{\delta_{r_i} \big| i \in \mathbb{N}\}$ of the Dirac measures δ_{r_i} supported at the rational numbers r_i of $[0,1]$ and also the family $\mathcal{G}$= $\{\mu_i \big| i \in \mathbb{N}\}$ of the Lebesgue measures

μ_i supported at I_i where $\{I_i\}$ is a sequence of subintervals of $[0,1]$ so that each interval (a,b) of $[0,1]$ contains at least one I_i, define the positive cone of the space $E = C[0,1]$ of continuous, real valued functions defined of $[0,1]$. E does not have the maximum support with respect to these families. Indeed, if $x \in E_+$ with $x(t_0) = 0$ for some irrational number t_0 and $x(t) > 0$ for each $t \neq t_0$, then $supp(x) = \mathbb{N}$ but x is not a quasi-interior point of E_+.

Theorem 16. *([8], Proposition 2.5.) If X is closed and X has a positive basis $\{b_n\}$, the following statements are equivalent:*
(i) X has the maximum support property with respect to $\mathcal{F}$,
(ii) there exists a sequence $\{i_n\}$ of $\mathbb{N}$ such that $f_{i_n}(b_n) > 0$ and $f_{i_n}(b_m) = 0$, for each $m \neq n$, i.e. the coefficient functionals of the basis $\{b_n\}$ can be extended on E to positive multiples of elements of $\mathcal{F}$.

The next is an example of an ordered subspace with a positive basis, without the maximum support property.

Example 17. Let $\{b_n\}$ be a sequence of l_∞ so that $b_1(4n) = \frac{1}{2^n}, b_1(4n+1) = \frac{1}{3^n}$ and $b_1(i) = 0$ in the other cases, $b_2(4n) = \frac{1}{3^n}, b_2(4n+1) = \frac{1}{2^n}$ and $b_2(i) = 0$ in the other cases and $b_n = e_{4n+2}$, for $n \geq 3$. Then $\{b_n\}$ is a positive basis of the closed subspace X of l_∞ generated by it. X does not have the maximum support property with respect to the family $\mathcal{F}$ of the Dirac measures δ_i supported at the natural numbers i. Indeed, $supp(b_1) = supp(b_2)$ therefore $\delta_i(b_1) > 0$ if and only if $\delta_i(b_2) > 0$, and by Theorem 16, X does not have the maximum support property.

3.2 The ws-property

The notion of the s-property (supremum property) has been introduced in [7]. We define here a weaker property, which we call ws-property (weak s-property) as follows:

Definition 18. *An ordered subspace X of E has the ws-property (with respect to $\mathcal{F}$) if for each $x \in X_+, x \neq 0$ and for each $i \in supp(X_+)$ the set $\{y \in [0,x]_X \mid y(i) = 0\}$ has at least one maximal element.*

If in the above definition the set $\{y \in [0,x]_X \mid y(i) = 0\}$ has a maximum element, then X has the s-property. If X has the ws-property, each solid subspace Z of X has this property. In the theory of vector optimization the maximal elements of a subset K of a normed space Z with respect to an ordering cone P of Z are the *Pareto efficient points* of K. In our case, the ws-property ensures the existence of Pareto efficient points with respect to X_+. We start with the following easy result.

Theorem 19. *Suppose that τ is a linear topology of E. If*
(i) X_+ is τ-closed,
(ii) each increasing net of X_+ order bounded in X, has a τ-convergent subnet, and
(iii) for each i the positive part $K_i^+ = \{y \in X_+ \mid f_i(y) = 0\}$ of the kernel of f_i in X is τ-closed,
then X has the ws-property.

Proof. Suppose that $x \in X_+$ and that A is a totaly ordered subset of the τ-closed set $[0, x]_X \cap K_i^+$. For each finite subset Φ of A denote by x_Φ the maximum of Φ. Then $\{x_\Phi\}$, as an increasing, order bounded net of $[0, x]_X \cap K_i^+$, is convergent to $x_0 \in [0, x]_X \cap K_i^+$ which is an upper bound of A and by Zorn's lemma the set $[0, x]_X \cap K_i^+$ has maximal elements. ■

Corollary 20. *If E is a Banach lattice with order continuous norm and X_+ is closed, then X has the ws-property.*

Proof. Each order interval of E weakly compact. Since X_+ is weakly closed, each order interval of X is weakly compact, hence X has the ws-property. ■

Corollary 21. *If E is a dual space, the functionals f_i are weak-star continuous and X_+ is weak-star closed and normal, then X has the ws-property.*

Proof. For each $x \in X_+$ the order interval $[0, x]_X$ is weak-star closed and bounded because X_+ is normal, therefore $[0, x]_X$ is weak-star compact. Hence X has the ws-property. ■

Corollary 22. *If X is closed with a positive basis, then X has the ws-property.*

Proof. By [11] Theorem 5, each order interval of X is compact. ■

Example 23. (i) The spaces c_0 and ℓ_p with $1 \leq p < +\infty$ and also the spaces $L_p^+(\mu)$ $1 \leq p < +\infty$, as Banach lattices with order continuous norm have the ws-property with respect to any countable family which defines their positive cone. Also their closed ordered subspaces have the ws-property.
(ii) By Corollary 21, ℓ_∞ and its weak-star closed ordered subspaces have the ws-property with respect to the family of the Dirac measures δ_i supported at the natural numbers i.
(iii) $C[0, 1]$ does not have the ws-property with respect to the family of the Dirac measures δ_{r_i} supported at the rational numbers r_i of $[0, 1]$. It is easy to show that the set $\{y \in C[0, 1] \big| 0 \leq y \leq x \text{ and } y(\frac{1}{2}) = 0\}$, where $x \in C_+[0, 1]$ with $x(\frac{1}{2}) > 0$, does not have maximal elements .

If P, Q, R are subcones of X_+ with $R = P + Q$ and $P \cap Q = \{0\}$, we will say that R is the *direct sum* of P, Q and we will write $P \oplus Q = R$.

Proposition 24. *Suppose that $X$is closed, X_+ is generating and normal and also that X has the Riesz decomposition property and the ws-property with respect to $\mathcal{F}$. Let $x \in X_+, x \neq 0$, $i \in supp(X_+)$ and we denote by z_i a maximal element of the set $\{y \in [0,x]_X \mid y(i) = 0\}$. Then $z'_i = x - z_i$ is a minimal element of the set $\{y \in [0,x]_X \mid y(i) = x(i)\}$. If I, J, W are the closed solid subspaces of X generated respectively by the elements x, z_i, z'_i, then*
(i) $\inf_{X_+}\{z_i, z'_i\} = 0$,
(ii) the functional f_i is equal to zero on J. If $f_i(x) > 0$ then f_i is strictly positive on W. If $f_i(x) = 0$, then $z_i = x$ and if f_i is strictly positive on I, then $z'_i = x$. If f_i is nonzero and non-strictly positive on I then $0 < z_i < x$ and $0 < z'_i < x$,
(iii) if $f_i(x) > 0$, then $I^+_{z_i}(X) \oplus I^+_{z'_i}(X) = I^+_x(X)$ and $J_+ \oplus W_+ = I_+$.

Proof. Suppose that $z \in A = \{y \in [0,x]_X \mid y(i) = x(i)\}$ with $z'_i > z$. Then $x - z > z_i$ and $f_i(x - z) = 0$, which contradicts the definition of z_i. Therefore z'_i is a minimal element of A.
(i) Let $h \in X$ with $0 < h \leq z_i, z'_i$. Then $0 \leq h(i) \leq z_i(i) = 0$, hence $h(i) = 0$. So we have that $h + z_i \leq x$ and $(h + z_i)(i) = 0$, a contradiction. Therefore $h = 0$ and $\inf_{X_+}\{z_i, z'_i\} = 0$.
(ii) Since $z_i(i) = 0$, f_i is equal to zero on I_{z_i} and therefore also on J. Suppose that $f_i(x) > 0$. Then $z_i < x$, hence $z'_i > 0$ and $W_+ \neq \{0\}$. Suppose that $w \in W_+, w > 0$ with $w(i) = 0$. Then by Theorem 4, w is the limit of an increasing sequence of elements of $I^+_{z'_i}(X)$, therefore $y(i) = 0$ for at least one $y \in X$ with $0 < y \leq z'_i$. Then $y + z_i \leq x$ and $(y + z_i)(i) = 0$, a contradiction, therefore f_i is strictly positive on W. If we suppose that $f_i(x) = 0$, then by the definition of z_i we have that $z_i = x$ and if we suppose that f_i is strictly positive on I we have that $z_i = 0$, therefore $z'_i = x$. Suppose now that f_i is nonzero and also non strictly positive on I. Then $x(i) > 0$ and also $v(i) = 0$, for at least one nonzero point v of I_+. Since v is the limit of an increasing sequence of elements of $I^+_x(X)$, we have that $y(i) = 0$ for at least one nonzero element $y \in [0,x]_X$. This implies that $z_i > 0$ because if we suppose that $z_i = 0$ we have that $z_i < y$, which contradicts the definition of z_i. Hence $0 < z_i$. Also $z_i < x$ because $x(i) > 0$. So we have $0 < z_i < x$ and $0 < z'_i < x$.
(iii) Let $f_i(x) > 0$. Suppose that $h \in J_+ \cap W_+$. Then $h \in J_+$ therefore $h(i) = 0$. Since the functional f_i is strictly positive on W we have that $h = 0$, therefore $J_+ \cap W_+ = \{0\}$. Suppose that $y \in [0,x]_X$. Then $y \leq z_i + z'_i$ and by the RDP we have that $y = y_1 + y_2$ with $y_1 \in [0, z_i]_X$ and $y_2 \in [0, z'_i]_X$. By the above remarks we have that the first assertion of (iii) is true. Suppose now that $y \in I_+$. By Theorem 4, y is the limit of an increasing sequence y_n of $I^+_x(X)$, with $y_n \leq y$ for each n. Hence $y_{n+1} - y_n \in I^+_x(X)$, therefore $y_{n+1} - y_n \leq k_n x = k_n(z_i + z'_i)$, and by the RDP we have that $y_{n+1} - y_n = a_{n+1} + b_{n+1}$ with $a_{n+1} \in I^+_{z_i}(X)$ and

$b_{n+1} \in I^+_{z'_i}(X)$. If $y_1 = a_1 + b_1$ with $a_1 \in I^+_{z_i}(X)$ and $b_1 \in I^+_{z'_i}(X)$, we have that $y_n = (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n)$. If $s_n = a_1 + a_2 + ... + a_n$ and $r_n = b_1 + b_2 + ... + b_n$ we have that $s_{n+1} - s_n = a_{n+1} \leq y_{n+1} - y_n$, therefore the sequence $\{s_n\}$ is convergent because $\{y_n\}$ is convergent and the cone X_+ is normal. Similarly we have that $\{r_n\}$ is convergent therefore $y = y' + y''$ with $y' \in J_+$ and $y'' \in W_+$. Hence $I_+ = J_+ \oplus W_+$. ■

Definition 25. *Let X be a closed ordered subspace of E as in the previous proposition, and suppose that x is a nonzero element of X_+ and $f_i \in \mathcal{F}$. If f_i is nonzero and non-strictly positive on $I_x(X)$ and $x = x_1 + x_2$ where x_1 is a maximal element of the set $\{y \in [0, x]_X \mid y(i) = 0\}$, then we will say that $x = x_1 + x_2$ is a decomposition of x with respect to f_i (or with respect to i) and also that x is decomposed with respect to f_i in the elements x_1, x_2. If f_i is equal to zero on $I_x(X)$ or if f_i is strictly positive on $I_x(X)$, we will say that x is not decomposed with respect to f_i (or with respect to i).*

3.3 Existence of positive bases

In what follows we will denote by X a closed, ordered subspace of E so that: (i) X has the Riesz decomposition property, (ii) the positive cone X_+ of X is closed, normal and generating and (iii) X has the maximum support property and the ws-property with respect to $\mathcal{F}$. As we have noted in the beginning of the previous section, (i) and (ii) imply that X_+ gives an open decomposition of X and that X^* is an order complete linear lattice. We will also denote by M the following subset of $\mathbb{N}$: $M = \{i \in supp(X_+) \mid \; f_i$ is non strictly positive on $X\}$. Therefore for each $x \in X_+, x \neq 0$ we have that $x(i) > 0$, for each $i \in supp(X_+) \setminus M$. Also $M \neq \varnothing$ because if we suppose that $M = \varnothing$, we have that $supp(x) = supp(X_+)$ for each $x \in X_+, x \neq 0$, therefore dim$X = 1$ by Proposition 7. In order to prove the existence of extremal points of X_+ we develop a process of successive decompositions of a quasi-interior point of X_+. So we suppose that u is a quasi-interior point of X_+ (such a point exists by Proposition 13) and we decompose u as follows:

Step 1: We put $i_1 = \min M$ and we decompose u with respect to i_1 in the elements x_1, x_2. Then $u = x_1 + x_2$ and $\inf_{X_+}\{x_1, x_2\} = 0$. Also f_{i_1} is equal to zero on I_1 and strictly positive on I_2 where I_1, I_2 are the closed solid subspaces of X generated by x_1, x_2 respectively. The set $m_1 = \{x_1, x_2\}$ is the *front* and the natural number i_1 is the *index* of the first decomposition.

Step $\nu + 1$: Suppose that we have accomplished the νth step and suppose that m_ν is the front and i_ν the index of the νth decomposition. Then at least one of the elements of m_ν is decomposed with respect to an $i \in M$. Indeed if we suppose that any element x of m_ν is not decomposed with respect to any $i \in M$ then for any $i \in M$, f_i is strictly positive or equal to zero on the closed solid subspace I of X generated by x and it is easy to show that $supp(y) = supp(I_+)$ for any $y \in I_+$, $y \neq 0$ therefore y is a quasi-interior point of I. Hence $dim I = 1$ and X is finite-dimensional because m_ν is finite. We put $i_{\nu+1} = \min\{i \in M \mid$ at least one element of m_ν is decomposed with respect to $i\}$. Then $i_{\nu+1} > i_\nu$ and we decompose with respect to $i_{\nu+1}$ the elements of m_ν which allow such a decomposition. We denote by $m_{\nu+1}$ the set which contains the elements of m_ν which are not decomposed with respect to $i_{\nu+1}$ and also the elements that arise from the decomposition of the elements of m_ν with respect to $i_{\nu+1}$. The set $m_{\nu+1}$ is *the front* and $i_{\nu+1}$ is *the index* of the $(\nu+1)th$ decomposition. The set $\delta(u) = \cup_{\nu=0}^{\infty} m_\nu$ where $m_0 = \{u\}$, is the *tree of decompositions* of u.

Proposition 26. *In the above process of decompositions of u we have:*
(i) the sequence of indices of decompositions $\{i_\nu\}$ is strictly increasing,
(ii) for each $i \in M$ with $i \leq i_\nu$ and for each $x \in m_\nu$, x is not decomposed with respect to i, therefore f_i is strictly positive or equal to zero on $I = \overline{I_x(X)}$,
(iii) the elements of m_ν are nonzero with sum equal to u. Also $\inf_{X_+}\{x, y\} = 0$, for each $x, y \in m_\nu$, with $x \neq y$,
(iv) $\inf_{X_+}\{x, u - x\} = 0$, for each $x \in \delta(u)$.

Proof. Statements (i),(ii) and (iii) are obvious. To prove (iv) we suppose that $x \in m_\nu$ for some ν and suppose that $m_\nu = \{x, y_1, y_2, ..., y_k\}$. Since the elements of m_ν are pairwise disjoint in X_+ with sum equal to u we have that $u - x = \sum_{i=1}^{k} y_i$ and (iv) is true by Proposition 9. ■

For any $x \in m_\nu$ with $\nu \geq 1$ it is easy to show that there exists a unique vector $y \in m_{\nu-1}$ with $y \geq x$. Also for any $x \in m_\nu$ there exists at least one $y \in m_{\nu+1}$ with $x \geq y$. So if we suppose that $x, y \in \delta(u)$ with $x \in m_\nu$, $y \in m_{\nu+\mu}$ and $y \leq x$, we will say that x is the *presuccessor* of y in m_ν, or that y is a *successor* of x in $m_{\nu+\mu}$. If moreover $y \in m_{\nu+1}$ we will say that x is the *first presuccessor* of y or that y is a *first successor* of x.

Proposition 27. *The following are true:*
(i) for any $x \in m_\nu$ the sum of the successors of x in $m_{\nu+\mu}$ is equal to x,
(ii) if y is a successor of x with $x > y$ and I is the closed solid subspace of X generated by x, then $\inf_{X_+}\{y, x - y\} = 0$ and y is not a quasi-interior point of I_+,
(iii) for each $x \in \delta(u)$ and each $i \in M \cap supp(x)$, there exists a successor y of x such that the functional f_i is strictly positive on the closed solid subspace I of X generated by y.

Proof. (i) Any element of $\delta(u)$ is the sum of its first successors, therefore the proposition is true for $\mu = 1$ and continuing, we have that the proposition is true for any μ.
(ii) Since $x-y \leq u-y$ and $\inf_{X_+}\{y, u-y\} = 0$ we have that $\inf_{X_+}\{y, x-y\} = 0$, therefore y is not a quasi-interior point of I_+ by Proposition 5.
(iii) Suppose that $x \in m_\kappa$. Since the sequence $\{i_\nu\}$ is strictly increasing, there exists $\nu \in \mathbb{N}$ with $\nu > \kappa$ and $i \leq i_\nu$. Then f_i is strictly positive or equal to zero on any closed solid subspace of X generated by an element of m_ν. But $x = \sum_{j=1}^{r} x_j$ where $x_1, ..., x_r$ are the successors of x in m_ν and $f_i(x) > 0$ because $i \in supp(x)$, therefore f_i is strictly positive on at least one of the closed solid subspaces of X generated by $x_1, ..., x_r$ and the proposition is true. ■

If $x \in \delta(u)$ and $x \in m_\nu$ for each $\nu \geq \nu_0$, then we will say that *the process of decomposition stops* at the point x of $\delta(u)$. In other words, the process of decomposition stops at x if there exists $\nu_0 \in \mathbb{N}$ so that $x \in m_{\nu_0}$ and for each $i \in M$ with $i > i_{\nu_0}$, the functional f_i is strictly positive or equal to zero on the closed solid subspace I of X generated by x. Then for each $i \in M$ with $i \leq i_{\nu_0}$, f_i is strictly positive or equal to zero on I, Proposition 26, therefore $supp(z) = supp(I_+)$ for any $z \in I_+, z \neq 0$, hence any nonzero vector of I_+ is a quasi-interior point of I which implies that $dim(I) = 1$. So x is an extremal point of X_+ and we have proved the following:

Proposition 28. *If the process of decomposition of u stops at an element $x_0 \in \delta(u)$ then x_0 is an extremal point of X_+.*

A sequence $\{x_\nu\}$ of $\delta(u)$ is a *branch of* $\delta(u)$, if $x_\nu > x_{\nu+1}$ for each $\nu \in \mathbb{N}$.

Proposition 29. *Each branch of $\delta(u)$ converges to zero.*

Proof. It is enough to show that any branch $\{x_\nu\}$ of $\delta(u)$ with $x_0 = u$ converges to zero. Let $z_\nu = x_{\nu-1} - x_\nu,$ for each $\nu \geq 1$. Then for each ν, μ, we have

$$u = z_1 + z_2 + ... + z_\nu + x_\nu \text{ and } x_\nu = z_{\nu+1} + ... + z_{\nu+\mu} + x_{\nu+\mu}. \tag{2}$$

The vectors $z_1, z_2, ..., z_\nu, x_\nu$ are pairwise disjoint in X_+. Indeed, $\inf_{X_+}\{x_\nu, u - x_\nu\} = 0$, hence $\inf_{X_+}\{x_\nu, \sum_{i=1}^{\nu} z_i\} = 0$, therefore $\inf_{X_+}\{x_\nu, z_i\} = 0$ for each $i \leq \nu$, because $z_i \leq \sum_{j=1}^{\nu} z_j$. Suppose that $j > i$. Then $z_j \leq x_i$ and $\inf_{X_+}\{z_i, x_i\} = 0$, therefore $\inf_{X_+}\{z_j, z_i\} = 0$. Hence $\inf_{X_+}\{z_j, z_i\} = 0$ for any $i \neq j$. Let $u_0 = \sum_{\nu=1}^{\infty} \frac{z_\nu}{2^\nu}$. We shall show that $supp(u_0) = supp(X_+)$. For each $i \in supp(X_+) \setminus M$ we have that $x(i) > 0$ for each $x \in X_+, x \neq 0$, hence $i \in supp(u_0)$. Suppose that $i \in M$ and that x_ν is decomposed at the κ_νth decomposition. Since the sequence $\{i_{\kappa_\nu}\}$ is strictly increasing, there exists

$\mu \in \mathbb{N}$ with $i < i_{\kappa_\mu}$. By statement (ii) of Proposition 26, f_i is strictly positive or equal to zero on $I = \overline{I_{x_\mu}(X)}$. We shall show that in both cases $i \in supp(u_0)$. If f_i is strictly positive on I we have $z_{\mu+1}(i) > 0$ because $0 < z_{\mu+1} < x_\mu$ therefore $i \in supp(u_0)$. If f_i is equal to zero on I then $x_\mu(i) = 0$, therefore $f_i(z_1 + ... + z_\mu) = f_i(z_1 + ... + z_\mu + x_\mu) = f_i(u) > 0$, hence $f_i(z_j) > 0$ for at least one j, therefore $i \in supp(u_0)$. Therefore $supp(X_+) = supp(u_0)$ and u_0 is a quasi-interior point of X. By Theorem 4, there exists an increasing sequence $\phi_n \in [0,u]_X \cap [0, r_n u_0]_X$, where $\{r_n\}$ is a strictly increasing sequence of natural numbers with $\lim_{n\longrightarrow\infty} \phi_n = u$. Let $h_\nu = \sum_{\mu=1}^{\infty} r_\nu \frac{z_{r_\nu+\mu}}{2^{r_\nu+\mu}}$. Since $0 \leq \phi_n \leq r_n u_0$ we have $0 \leq \phi_n \leq r_n z_1 + ... + r_n z_{r_n} + h_n$ and by Proposition 9, ϕ_n has a unique decomposition $\phi_n = \phi_n^1 + ... + \phi_n^{r_n} + H_n$ with $0 \leq \phi_n^i \leq r_n z_i$ for each i and $0 \leq H_n \leq h_n$. The last inequality implies that $\lim_{n\longrightarrow\infty} H_n = 0$ because $\lim_{n\longrightarrow\infty} h_n = 0$ and the cone X_+ is normal. Also we have $0 \leq \phi_n^i \leq u, r_n z_i$ for $i = 1, 2, ..., r_n$, therefore $\phi_n^i = a_1 + a_2 + ... + a_{r_n} + b_n$ with $0 \leq a_j \leq z_j$, for each j and $0 \leq b_n \leq x_{r_n}$. Since the vectors $z_1, z_2, ..., z_{r_n}, x_{r_n}$ are pairwise disjoint in X_+ we have that $\phi_n^i = a_i$, therefore $0 \leq \phi_n^i \leq z_i$ for each $i = 1, 2, ..., r_n$. Since $H_n \leq u$, we have that $H_n = \gamma_1 + \gamma_2 + ... + \gamma_{r_n} + c_n$ with $0 \leq \gamma_j \leq z_j$, for each $j = 1, 2, ..., r_n$ and $0 \leq c_n \leq x_{r_n}$. Since $H_n \leq h_n$ we have also that $\gamma_j \leq h_n$, for each j. Since the vectors z_j, $j = 1, 2, ..., r_n$ and h_n are pairwise disjoint in X_+ we have $\gamma_j = 0$ for each $j = 1, 2, ..., r_n$, hence $H_n = c_n$, therefore $H_n \leq x_{r_n}$. So we have that $\lim_{n\longrightarrow\infty} \big(u - (\phi_n^1 + ... + \phi_n^{r_n} + H_n)\big) = 0$, therefore

$$\lim_{n\longrightarrow\infty} \big[(z_1 - \phi_n^1) + (z_2 - \phi_n^2) + ... + (z_{r_n} - \phi_n^{r_n}) + (x_{r_n} - H_n)\big] = 0.$$

Since the members in the above limit are positive and the cone of X_+ is normal we have that $\lim_{n\longrightarrow\infty}(x_{r_n} - H_n) = 0$. As we have shown above $\lim H_n = 0$ therefore $\lim x_{r_n} = 0$. Since the sequence $\{x_n\}$ is decreasing it converges to zero and the proposition is true. ∎

Proposition 30. *For each $x \in \delta(u)$ at least one of the successors of x is an extremal point of X_+.*

Proof. Let $x \in \delta(u)$. If at least one of the successors x' of x does not belong to a branch of $\delta(u)$, then the process of decompositions stops after a finite number of steps at any successor of x', therefore any successor of x' is an extremal point of X_+ dominated by x and the proposition is true. So we suppose that any successor of x belongs to a branch of $\delta(u)$. Also we may suppose that $x < u$ because in the case where $x = u$, it is enough to show the proposition for one of its successors. Let I be the closed solid subspace of X generated by x and suppose that $L = \{i \in supp(x) \mid f_i$ is not strictly positive on $I_x(X)\}$. Then $L \subseteq M$. Also $supp(x) = supp(I_+)$. If L is finite, then after a finite number of steps the decomposition stops at any successor of x and the proposition is true. So we suppose that the set L is infinite. Let $j_1 = \min L$. Then by statement (iii) of Proposition 27, there exists $x_1 \in \delta(u)$ such that $x_1 \leq x$ and f_{j_1} is strictly positive on $\overline{I_{x_1}(X)}$. Since x_1 is an element of a branch of $\delta(u)$ dominated by x and any such a branch of $\delta(u)$ converges to zero, we may suppose that there exists an element $y_1 \in \delta(u)$ such that $y_1 < x_1 \leq x$, $\|y_1\| \leq 2^{-1}\varepsilon$, where ε is a constant real number with $0 < \varepsilon < \|x\|$. Note also that f_{j_1} is strictly positive on $\overline{I_{y_1}(X)}$ because it is strictly positive on $\overline{I_{x_1}(X)}$ and $0 < y_1 < x_1$. By Proposition 27 we have that $\inf_{X_+}\{y_1, x - y_1\} = 0$ hence $y_1$is not a quasi-interior point of I. Therefore $supp(y_1) \neq supp(I_+)$, hence there exists at least one $i \in supp(I_+)$ with $i \notin supp(y_1)$, therefore there exists $i \in L$ with $y_1(i) = 0$. We put $j_2 = \min\{i \in L : y_1(i) = 0\}$. Then $j_1 < j_2$ and as before we can find a vector $y_2 \in \delta(u)$ so that $y_2 < x$, $\|y_2\| \leq 2^{-2}\varepsilon$ and f_{j_2} is strictly positive on $\overline{I_{y_2}(X)}$. Then $\inf_{X_+}\{y_1, y_2\} = 0$, because for any $h \in X$ with $0 \leq h \leq y_1, y_2$ we have that $0 \leq h(j_2) \leq y_1(j_2) = 0$, therefore $h = 0$ because f_{j_2} is strictly positive on $\overline{I_{y_2}(X)}$. By the way we have selected y_2 (as a sufficiently small member of a branch which converges to zero) we may also suppose that $y_1 \in m_{\nu_1}$ and $y_2 \in m_{\nu_2}$ with $\nu_1 < \nu_2$. We may also suppose that ν_2 is sufficiently large so that m_{ν_2}, except the successors of x and the element y_2, contains at least one extra element so we may suppose that $m_{\nu_2} = \{y_2, a_1, a_2, ..., a_k, b_1, b_2, ..., b_r, c_1, c_2, ..., c_l\}$, where $a_1, a_2, ..., a_k$ are the successors of y_1 and $y_2, a_1, a_2, ..., a_k, b_1, b_2, ..., b_r$ are the successors of x. We put $s_1 = y_1, s_2 = y_1 + y_2$. Then $s_1 < x$ and $s_2 < x$. The first inequality is obvious and the second holds because x is the sum of its successors in m_{ν_2}. Also $s_1(j_1) > 0$ and by the definition of j_2, we have that $s_2(i) > 0$ for each $i \in L$ with $i \leq j_2$. By Proposition 9, $\inf_{X_+}\{s_i, x - s_i\} = 0$, for each $i = 1, 2$ because the successors of x in m_{ν_2} are pairwise disjoint. Since $\inf_{X_+}\{s_2, x - s_2\} = 0$ we have that s_2 is not quasi-interior point of I_+, hence there exists $i \in L$ with $s_2(i) = 0$. Let $j_3 = \min\{i \in L : s_2(i) = 0\}$. Then $j_2 < j_3$ and as before we can find $y_3 \in m_{\nu_3}$ such that $\nu_2 < \nu_3$, $\|y_3\| \leq 2^{-3}\varepsilon$, f_{j_3} is strictly positive on $\overline{I_{y_3}(X)}$ and the set of the successors of x in m_{ν_3} contains the successors of y_1, the successors of y_2, the element y_3 and at least one extra element. As before we can show that $\inf_{X_+}\{y_1, y_3\} = \inf_{X_+}\{y_2, y_3\} = 0$. We put $s_3 = s_2 + y_3$. Continuing this process we obtain a sequence $\{j_\nu\}$ of L and the sequences $\{y_\nu\}$, $\{s_\nu\}$ of X_+ such that $s_1 = y_1, s_\nu = s_{\nu-1} + y_\nu$, for each $\nu = 2, 3, ...$, with the following properties:

(i) $0 < s_\nu < s_{\nu+1} < x$,

(ii) $\|s_{\nu+1} - s_\nu\| = \|y_{\nu+1}\| \leq 2^{-\nu-1}\varepsilon$ and $y_\nu \in m_{k_\nu}$ with $k_\nu < k_{\nu+1}$, for each ν,

(iii) $\inf_{X_+}\{s_\nu, x - s_\nu\} = 0$ for each ν,

(iv) $\{j_\nu\}$ is a strictly increasing sequence of L and for each $i \in L$ with $i < j_{\nu+1}$

we have $s_\nu(i) > 0$.
By (ii), the sequence $\{s_\nu\}$ is Cauchy and suppose that $s = lim_{\nu \longrightarrow \infty} s_\nu$. Then $0 \leq s_\nu \leq s \leq x$, for each ν. Since $\|s_\nu\| \leq \sum_{i=1}^{\nu} \|y_i\| \leq \varepsilon < \|x\|$, we have that $s < x$. Also by (iv) and by the fact that the sequence $\{s_\nu\}$ is increasing we have that $s(i) > 0$ for each $i \in L$, therefore $supp(s) = supp(I_+)$. Hence s is a quasi-interior point of I_+. We will show that $\inf_{X_+}\{s, x - s\} = 0$. To this end we suppose that $0 \leq h \leq s, x - s$. Since $s = s_\nu + (s - s_\nu)$ we have that $h = h_\nu + h'_\nu$ with $0 \leq h_\nu \leq s_\nu, 0 \leq h'_\nu \leq (s - s_\nu)$. Since the cone is normal and $\lim(s - s_\nu) = 0$ we have that $\lim h'_\nu = 0$, therefore $h = \lim h_\nu$. Since $0 \leq h_\nu \leq s_\nu$ and $h_\nu \leq x - s \leq x - s_\nu$ we have that $h_\nu = 0$ for each ν, by (iii). Therefore $h = 0$, hence $\inf_{X_+}\{s, x - s\} = 0$. Since I is solid we have also that $\inf_{I_+}\{s, x - s\} = 0$ which contradicts the fact that s is a quasi-interior point of I_+, Proposition 5. Hence at least one of the successors x' of x does not belong to a branch of $\delta(u)$, therefore at least one of the successors x_0 of x is an extremal point of X_+ and the proposition is true. ■

Proposition 31. *Any extremal point x_0 of X_+ is a positive multiple of a unique element of $\delta(u)$.*

Proof. Suppose that x_0 is an extremal point of X_+. By proposition 5 and by the fact that x_0 is an extremal point of X_+ a real number $r > 0$ exists with $rx_0 \leq u$. Hence $r \leq a\frac{\|u\|}{\|x_0\|}$, where a is the constant of the normal cone X_+. Therefore $sup\{r \in \mathbb{R}_+ : rx_0 \leq u\} = \lambda \in \mathbb{R}_+$, with $\lambda > 0$. Let $z_0 = \lambda x_0$. Then $0 < z_0 \leq u$. Since $u = \sum_{z \in m_\nu} z$ and the elements of m_ν are pairwise disjoint, there exists a unique $y_\nu \in m_\nu$ so that $z_0 \leq y_\nu$. Then $\inf_{X_+}\{z_0, x\} = 0$ for each $x \in m_\nu,\ x \neq y_\nu$. Also $y_\nu \geq y_{\nu+1} \geq z_0$ for each ν. Since each branch of $\delta(u)$ converges to zero, the process of decompositions stops at a point y_μ which is an extremal point of X_+ with $z_0 \leq y_\mu$. Hence $y_\mu = \lambda' x_0$. Also $y_\mu \leq u$ and by the definition of λ we have that $\lambda' \leq \lambda$, therefore $y_\mu \leq z_0$ which implies that $y_\mu = z_0$ and $z_0 \in \delta(u)$. If we suppose that $z'_0 = kx_0 \in \delta(u)$, then $kx_0 \leq u$, therefore $k \leq \lambda$ and $z'_0 \leq z_0$. Hence z'_0 is a successor of z_0. If we suppose that $z'_0 < z_0$ we get a contradiction because z_0 as an extremal point of X_+ is undecomposable. Therefore $z'_0 = z_0$ and the proposition is true. ■

In our main result below we prove that X has a positive basis. For the sake of completeness we repeat the standard assumptions for E and X. In the next results the positive basis of X is also unconditional because X_+ is generating and normal.

Theorem 32. *Suppose that E is an ordered Banach space and that E_+ is defined by the family $\mathcal{F} = \{f_i | i \in \mathbb{N}\}$ of E^*_+. Let X be a closed ordered subspace of E with the Riesz decomposition property and suppose that X_+ is normal and generating.*

If X has the maximum support property and the ws-property with respect to $\mathcal{F}$, then X has a positive basis.

Proof. Let B be the set of extremal points of X_+ with norm 1. By proposition 30, $B \neq \varnothing$ and by the previous proposition the map $T : B \longrightarrow \delta(u)$ so that $T(x) = \lambda x \in \delta(u)$ is one-to-one. Since $\delta(u)$ is countable B is also countable. Suppose that $B = \{u_i : i \in \mathbb{N}\}$ and $b_i = \lambda_i u_i \in \delta(u)$. Let $u_0 = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$. For each $i \in M$ there exists $z \in \delta(u)$ so that f_i is strictly positive on $I = \overline{I_z(X)}$. By Proposition 30, $z \geq b_j$ for at least one j, therefore $b_j(i) > 0$. Hence $supp(u_0) = suppX_+$ and u_0 is a quasi-interior point of X_+. Let $x \in X_+$. Then there exists an increasing sequence $x_n \in [0, x] \cap [0, k_n u_0]$ where the sequence k_n is strictly increasing with $\lim_{n \longrightarrow \infty} x_n = x$. Since $0 \leq x_n \leq k_n u_0$, each x_n has a unique expression $x_n = \sum_{i=1}^{\infty} \sigma_{ni} u_i$, with $\sigma_{ni} \in \mathbb{R}_+$, by Proposition 9. The sequence $\{\sigma_{ni} \mid n \in \mathbb{N}\}$ is increasing. Indeed, for each $m > n$ we take again the expansion $x_m - x_n = \sum_{i=1}^{\infty} a_i u_i$ and we have that $\sigma_{mi} = \sigma_{ni} + a_i \geq \sigma_{ni}$. Suppose that $\sigma_i = lim_{n \longrightarrow \infty} \sigma_{ni}$. Then $0 \leq \sigma_i u_i \leq x$ because $0 \leq \sigma_{ni} u_i \leq x$, for each i. For each $m \in \mathbb{N}$ we have $\sum_{i=1}^{m} \sigma_{ni} u_i \leq x_n \leq x$ and by taking limits as $n \longrightarrow +\infty$ we have that $\sum_{i=1}^{m} \sigma_i u_i \leq x$. Since the sequence $\{x_n\}$ converges to x there exists a strictly increasing sequence m_n of natural numbers so that the sequence $y_n = \sum_{i=1}^{m_n} \sigma_{ni} u_i$ converges to x. Then $\sum_{i=1}^{m_n} \sigma_{ni} u_i \leq \sum_{i=1}^{m_n} \sigma_i u_i \leq x$, from where we get that $x = \sum_{i=1}^{\infty} \sigma_i u_i$. Let $\overline{u}_j = \sum_{i \neq j} \frac{b_i}{2^i}$. Then $\overline{u}_j$ is not a quasi-interior point of X_+, because $\inf_{X_+}\{b_j, \overline{u}_j\} = 0$, Proposition 9. Therefore $supp(\overline{u}_j)$ is a proper subset of $supp(X_+)$, hence there exists $k_j \in M$ with $f_{k_j}(\overline{u}_j) = 0$. Therefore $f_{k_j}(u_i) = 0$, for each $i \neq j$. Also $f_{k_j}(u_j) > 0$ because $f_{k_j}(u_0) > 0$. Let $g_j = \frac{f_{k_j}}{f_{k_j}(u_j)}$. Then for each $x \in X_+$ we have $g_j(x) = \sigma_j$, therefore $x = \sum_{i=1}^{\infty} g_i(x) u_i$. Since the cone X_+ is generating we have that $x = \sum_{i=1}^{\infty} g_i(x) u_i$ for each $x \in X$ and this expansion is unique. Therefore $\{u_n\}$ is a positive basis of X. ■

By the previous result and Corollary 20 and 21 we have:

Corollary 33. *Suppose that E is a Banach lattice with order continuous norm and suppose that E_+ is defined by a countable family $\mathcal{F}$ of E_+^*. Let X be a closed ordered subspace of E with the Riesz decomposition property and generating positive cone X_+. If X has the maximum support property with respect to $\mathcal{F}$, then X has a positive basis.*

Corollary 34. *Let E be an ordered Banach space whose positive cone is defined by the family $\mathcal{F}$= $\{f_i | i \in \mathbb{N}\}$ of E_+^*. Suppose also that E is a dual space and that the functionals f_i are weak-star continuous. If X is a closed ordered subspace of E with the Riesz decomposition property, X_+ is weak-star closed, normal and* *generating and X has the maximum support property with respect to $\mathcal{F}$, then X has a positive basis.*

Remark 35. In the special case where $E = \ell_\infty$ and X is a weak-star closed ordered subspace of ℓ_∞ with the RDP and generating positive cone X_+ we have: If X has the maximum support property with respect to the family of the Dirac measures δ_i supported at the natural numbers i, then X has a positive basis.

4 Biorthogonal systems

The results of the previous section can be applied to the problem: *under what conditions a biorthogonal system defines a positive basis?* So in this section we suppose that E is an ordered Banach space with a *positive biorthogonal system* $\{(e_i, f_i)\big| i \in \mathbb{N}\}$, i.e. $e_i \in E$ and $f_i \in E_+^*$ for each i so that $f_i(e_i) = 1$, $f_i(e_j) = 0$, for each $j \neq i$ and the family $\mathcal{F} = \{f_i | i \in \mathbb{N}\}$ defines the positive cone of E. In the next results the positive basis of E is also unconditional.

Theorem 36. *Suppose that E is an ordered Banach space with a positive biorthogonal system $\{(e_i, f_i)\big| i \in \mathbb{N}\}$. If E_+ is normal and generating and E has the Riesz decomposition property, the following statements are equivalent:*
(i) The sequence $\{e_i\}$ of the biorthogonal system is a positive basis of E,
(ii) E has the maximum support property and the ws-property with respect to the family $\mathcal{F}$= $\{f_i\big| i \in \mathbb{N}\}$.

Proof. Suppose that $\{e_i\}$ is a positive basis of E. Since $\{(e_i, f_i)\}$ is a positive biorthogonal system of E we have that $f_i(e_i) = 1$ and $f_i(e_j) = 0$, for each $j \neq i$ therefore, by Theorem 16, E has the maximum support property with respect to $\mathcal{F}$. Since $\{e_i\}$ is a positive basis of E, by Corollary 22, E has the ws-property, therefore (i) implies (ii). Suppose now that statement (ii) is true. Then E has a positive basis $\{b_n\}$. Since E has the maximum support property with respect to $\mathcal{F}$, E has the minimal support property, therefore an element x_0 of E_+ is an extremal point of E_+ if and only if x_0 has minimal support in E_+. Therefore the extremal points of E_+ are the positive multiples of the elements e_n ($supp(e_n) = \{n\}$). Since the elements of the positive basis define the extremal rays of E_+ we have that the basis $\{b_n\}$ coincides, in the sense of a scalar multiple and a proper enumeration, with the sequence $\{e_n\}$. ■

Corollary 37. *Suppose that E is an ordered Banach space with a positive biorthogonal system $\{(e_i, f_i)\big| i \in \mathbb{N}\}$ and suppose also that E has the Riesz decomposition property. If*
(a) E is a Banach lattice with order continuous norm, or

(b) E is a dual space, the positive cone E_+ of E is weak-star closed, normal and generating and the functionals f_i are weak-star continuous,
then the following statements are equivalent:
(i) The sequence $\{e_i\}$ of the biorthogonal system is a positive basis of E,
(ii) E has the maximum support property with respect to the family $\{f_i \big| i \in \mathbb{N}\}$.

Remark 38. According to the Corrolary, the sequence $\{e_i\}$ of the usual biorthogonal system $\{e_i, \delta_i\}$ of ℓ_∞ is not a positive basis of ℓ_∞ because it does not have the maximum support property with respect to the family $\{\delta_i\}$, Example 15.

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