Positive Bases in Ordered Subspaces with the Riesz Decomposition Property

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Abstract

In this article we suppose that $E$ is an ordered Banach space the positive cone of which is defined by a countable family $\mathcal{F} = \{f_i | i \in \mathbb{N}\}$ of positive continuous linear functionals of $E$, i.e. $E_+ = \{x \in E | f_i(x) \geq 0, \text{ for each } i\}$ and we study the existence of positive (Schauder) bases in the ordered subspaces $X$ of $E$ with the Riesz decomposition property. So we consider the elements $x$ of $E$ as sequences $x = (f_i(x))$ and we develop a process of successive decompositions of a quasi-interior point of $X_+$ which in any step gives elements with smaller support. So we obtain elements of $X_+$ with minimal support and we prove that these elements define a positive basis of $X$ which is also unconditional. In the first section of this article we study ordered normed spaces with the Riesz decomposition property.

1 Introduction and notations

The most typical examples of ordered Banach spaces $E$ with a rich class of ordered subspaces are the universal spaces $C[0, 1]$ and $\ell_\infty$. As it is shown in [8], Theorem 4.1\footnote{This result is shown by a slight modification of the classical proof of the universality of $C[0, 1]$.} each separable ordered Banach space with closed and normal positive cone is

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order-isomorphic to an ordered subspace of \( C[0, 1] \), therefore the existence of positive bases in the separable ordered Banach spaces is equivalent with the study of positive bases in the closed ordered subspaces \( X \) of \( C[0, 1] \). In this article we study the general problem of the existence of positive bases in the ordered subspaces \( X \) of \( E \), as it is formulated in the abstract, by developing a method of decompositions of a quasi-interior point of \( X \). To develop this method we study the subspaces \( X \) of \( E \) with the maximum support property. In this kind of subspaces the quasi-interior points of \( X \) and in its closed principal solid subspaces, are characterized as the positive vectors of these subspaces with maximum support. We show that in this kind of subspaces the extremal points of \( X_+ \) are the nonzero elements of \( X_+ \) with minimal support and this is an important property for the study of positive bases. Also this class of subspaces is a large one. Indeed as it is shown in [7], Lemma 5.1, each Banach lattice with a positive basis is order isomorphic to a closed, ordered subspace of \( \ell_\infty \) with the maximum support property with respect to the family \( \mathcal{F} \) of the Dirac measures \( \delta_i \) supported at the natural numbers \( i \) and a similar result is also true for the space \( C[0, 1] \), see in [8], Theorem 5.1. Therefore the class of ordered subspaces of \( \ell_\infty \) or \( C[0, 1] \) with the maximum support property is a large one and contains, in the sense of an order isomorphism, the class of Banach lattices with a positive basis.

To develop our method of decompositions we study also the ordered subspaces \( X \) of \( E \) with the following property which we call \( \text{ws-property} \): for any \( x \in X_+ \) and any \( f_i \in \mathcal{F} \) the set \( K = \{ y \in X_+ | y \leq x \text{ and } f_i(y) = 0 \} \) has at least one maximal element. According to the terminology of vector optimization, \( X \) has the \( \text{ws-property} \) if and only if the set \( K \) has Pareto efficient points with respect to \( X_+ \). If \( E \) is a Banach lattice with order continuous norm or if \( E \) is a dual space, we show, Corollary 20 and 21, that the ordered subspaces of \( E \) have the \( \text{ws-property} \). In the main result of this article, Theorem 32, we prove that the maximum support property and the \( \text{ws-property} \) are sufficient conditions for the existence of positive bases in the ordered subspaces of \( E \) with the Riesz decomposition property. As an application we show, Theorem 36, that the maximum support property and the \( \text{ws-property} \) are necessary and sufficient in order a positive biorthogonal system of an ordered Banach space \( E \) with the Riesz decomposition property to define a positive basis of \( E \).

This article is a generalization of [7] where the same problem is studied in the lattice-subspaces of \( E \). So in the first section of this paper we study ordered normed spaces with the Riesz decomposition property and we prove some results necessary for our method of decompositions. Specifically we study quasi-interior points and we generalize the existing results for normed lattices to ordered normed spaces with the Riesz decomposition property, Theorem 4 and 6.

Finally note that each Banach space with an unconditional basis, ordered by the
positive cone of the basis, is a Banach lattice with respect to an equivalent norm. Also note that the problem of the existence of unconditional basic sequences in Banach spaces, known as the unconditional basic sequence problem, was one of the famous open problems of Functional Analysis till 1993 when W.T. Gowers and B. Maurey gave a negative answer to it, [3]. Our results give necessary conditions for the existence of unconditional basic sequences in ordered Banach spaces.

Let $Y$ be a (partially) ordered normed space with positive cone $Y_+$. If $Y = Y_+ - Y_+$ the cone $Y_+$ is generating or reproducing and if a real number $a > 0$ exists so that $x, y \in Y_+$ with $x \leq y$ implies that $||x|| \leq a||y||$, the cone $Y_+$ is normal. Recall that a convex set $P$ of a linear space is a cone if $\lambda x \in P$ for any real number $\lambda \geq 0$ and any $x \in P$ and $P \cap (-P) = \{0\}$. The set $[x, y] = \{z \in Y \mid x \leq z \leq y\}$ is the order interval $xy$, whenever $x, y \in Y$ with $x \leq y$. A point $x \in Y_+, x \neq 0$ is an extremal point of $Y_+$ if for any $y \in Y$ with $0 < y < x$ there exists $\lambda \in \mathbb{R}_+$ such that $y = \lambda x$. Denote by $EP(Y_+)$ the set of extremal points of $Y_+$. $Y$ has the Riesz decomposition property (RDP) if for each $x, y_1, y_2 \in Y_+$ with $x \leq y_1 + y_2$ there exist $x_1, x_2 \in Y_+$ such that $x = x_1 + x_2$ and $0 \leq x_1 \leq y_1 , 0 \leq x_2 \leq y_2$.

A subspace $Z$ of $Y$ is solid if for any $x, y \in Z$ with $x \leq y$, the order interval $[x, y] = \{z \in Y \mid x \leq z \leq y\}$ is contained in $Z$. We say that the cone $Y_+$ gives an open decomposition of $Y$ or that $Y_+$ is non-flat if $U_+ - U_+$ is a neighborhood of zero, where $U_+ = U \cap Y_+$, is the positive part of the closed unit ball $U$ of $Y$, or equivalently, if any $x \in Y$ has a representation $x = x_1 - x_2$ with $x_1, x_2 \in Y_+$ and $||x_1||, ||x_2|| \leq M||x||$, where $M$ is a constant real number. A linear functional $f$ of $Y$ is positive if $f(x) \geq 0$ for each $x \in Y_+$ and strictly positive if $f(x) > 0$ for each $x \in Y_+, x \neq 0$. Denote by $Y^*$ the set of continuous, linear functionals of $Y$ and by $Y^*_+$ the set of positive ones. Suppose that $Y$ is an ordered Banach space.

A sequence $\{e_n\}$ of $Y$ is a (Schauder) basis of $Y$ if each $x \in Y$ has a unique expansion $x = \sum_{n=1}^{\infty} \lambda_n e_n$, with $\lambda_n \in \mathbb{R}$ for each $n$. If moreover $Y_+ = \{x = \sum_{n=1}^{\infty} \lambda_n e_n \mid \lambda_n \geq 0 \text{ for each } n\}$, then $\{e_n\}$ is a positive basis of $Y$. A positive basis is unique in the sense that if $\{b_n\}$ is another positive basis of $Y$, then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$. If $\{e_n\}$ is a positive basis of $Y$ then, by [9], Theorem 16.3, and [4], Theorem 3.5.2, and Theorem 4.1.5, the following statements are equivalent: (i) the basis $\{e_n\}$ is unconditional, (ii) the cone $Y_+$ is generating and normal, (iii) $Y$ is a Banach lattice with respect to an equivalent norm.

A linear operator $T$ of $Y$ onto an ordered normed space $Z$ is an order-isomorphism of $Y$ onto $Z$ if $T$ is one-to-one, $T$ and $T^{-1}$ are continuous and for each $x \in Y$ we have: $x \in Y_+$ if and only if $T(x) \in Z_+$. For notions and terminology on ordered spaces not defined here we refer to [4], [5], [1], [6] and [10]. For Schauder bases we refer to [9].
2 Quasi-interior points in spaces with the Riesz decomposition property

In this section we will denote by $Y$ an ordered normed space with the Riesz decomposition property whose positive cone $Y_+$ is closed, normal and gives an open decomposition of $Y$. Then, by the Riesz-Kantorovich Theorem, the set of order bounded linear functionals $Y^b$ of $Y$ is an order complete linear lattice. For any $x \in Y_+$, $I_x = \bigcup_{n \in \mathbb{N}} [-nx, nx]$ is the solid subspace of $Y$ generated by $x$ and the closure of $I_x$ is the closed solid subspace of $Y$ generated by $x$. As we prove below the closure of $I_x$ is again solid. Recall the following properties of an ordered Banach space $W$ which we use in this article: (i) If $W_+$ is closed and generating, then $W_+$ gives an open decomposition of $W$ (Krein-Smulian) and any order bounded linear functional of $W$ is continuous and (ii) the cone $W_+$ is normal if and only if $W^* = W^*_+ - W^*_-$ (M. Krein), see for example in [4], Theorems 3.5.2, 3.5.6, and 3.4.8. We start with the next obvious result.

Proposition 1. Any solid subspace of $Y$ has the Riesz decomposition property.

Proposition 2. Suppose that $x \in Y_+, x \neq 0$ and $I$ is the closure of $I_x$. Then,

(i) for any $y \in I_+$, there exists an increasing sequence $\{y_n\}$ of $I_x$ which converges to $y$, with $0 \leq y_n \leq y$, for each $n$,

(ii) $I$ is a solid subspace of $Y$,

(iii) the positive cone $I^+_x$ of $I_x$ is generating,

(iv) if we suppose moreover that $Y$ is a Banach space then each positive, continuous, linear functional of $I$ has a positive, continuous, linear extension on $Y$.

Proof. Let $y \in I_+, y \neq 0$. At first we shall show that there exists a sequence $\{y'_n\}$ of $I_x \cap [0, y]$ convergent to $y$. Since $y \in I$, we have that $y = \lim_{\kappa_n \to +\infty} t_n$ where $t_n \in [-\kappa_n x, \kappa_n x]$ and $\{\kappa_n\}$ is an increasing sequence of natural numbers. Hence $t_n - y \to 0$, therefore by [4] Theorem 3.3.5, there exist sequences $\{w_n\}$, $\{v_n\}$ of $Y_+$ with $t_n - y = w_n - v_n$ and $w_n, v_n \to 0$. Then we have that $t_n + v_n - y = w_n \geq 0$, therefore

$$y \leq t_n + v_n \leq \kappa_n x + v_n.$$ (1)

By the RDP we have that $y = y'_n + y''_n$ where $0 \leq y'_n \leq \kappa_n x$ and $0 \leq y''_n \leq v_n$.

Since the cone $Y_+$ is normal and the sequence $v_n$ converges to zero, the sequence $y''_n$ converges also to zero, hence $y'_n \to y$, therefore our assertion is true. So for any positive real number $\varepsilon$, we have $\|y - y'_n\| < \frac{\varepsilon}{2}$, for a proper $n$. We put $r_1 = y'_n$.

Similarly there exists $r_2 \in I_x$, $r_2 \in [0, y - r_1]$ with $\|y - r_1 - r_2\| < \frac{\varepsilon}{2^n}$ and continuing this process we find a sequence $\{r_n\}$ of $I_x$ with $r_n \in [0, y - \sum_{i=1}^{n-1} r_i]$ and $\|y - \sum_{i=1}^{n} r_i\| < \frac{\varepsilon}{2^n}$, for each $n$. Therefore the sequence $y_n = \sum_{i=1}^{n} r_i$ is an increasing sequence of $[0, y]$ which converges to $y$, therefore statement (i) is true.
For the proof of \((ii)\) it is enough to show that \([0, y] \subseteq I_+\), for any \(y \in I_+\). So we suppose that \(y \in I_+\) and that \(z \in [0, y]\). As in the proof of \((i)\) we find again that \(y\) satisfies (1) and by the RDP we have that 
\[
z = z'_n + z''_n \quad \text{where} \quad 0 \leq z'_n \leq \kappa_n x, \quad 0 \leq z''_n \leq v_n, \quad \text{and as before we have that the sequence} \ z''_n \text{converges also to zero. Hence} \ z'_n \to z, \quad \text{therefore} \ z \in I \text{ and statement} \ (ii) \text{ is true.}
\]
Statement \((iii)\) is obvious because for any \(y \in [-nx, nx]\) we have \(0 \leq y + nx \leq 2nx\), therefore \(y + nx = a + b\) where \(a, b \in Y_+\) with \(a \leq nx, b \leq nx\), therefore \(y = a - (nx - b)\).
Suppose that \(f\) is a positive, continuous linear functional of \(I\). For any \(y \in Y_+\) we put \(\mathcal{L}_y = \{z \in I_x^+ | z \leq y\}\). \(\mathcal{L}_y\) is bounded because the cone \(Y_+\) is normal. For any \(y \in Y_+\) we put \(g(y) = \sup\{f(z) | z \in \mathcal{L}_y\}\). By the RDP and by the fact that \(I_x\) is solid we have that \(\mathcal{L}_y + \mathcal{L}_w = \mathcal{L}_{y+w}\). Therefore \(g\) is positively homogeneous and additive on \(Y_+\). Hence \(g\) has a linear and positive extension on \(Y\) which we will denote again by \(g\), i.e. \(g(x) = g(x_1) - g(x_2)\) for any \(x = x_1 - x_2 \in Y\) with \(x_1, x_2 \in Y_+\). By [4], Corollary 3.5.6, \(g\) is continuous. By the definition of \(g\) and by the fact that \(I_x\) is solid, we have that \(g(y) = f(y)\), for any \(y \in I_x^+\), therefore \(g\) coincides with \(f\) on \(I_x\) because \(I_x = I_x^+ - I_x^+\). Since \(I_x\) is dense in \(I\) we have that \(g\) is also equal to \(f\) on \(I\), therefore \(g\) is an extension of \(f\) from \(I\) to \(Y\).

**Definition 3.** An element \(u \in Z_+\), of an ordered topological linear space \(Z\) is a quasi-interior point of \(Z_+\) (or a quasi-interior positive element of \(Z\)) if the solid subspace \(\cup_{n \in \mathbb{N}} [-nu, nu]\) of \(Z\) generated by \(u\) is dense in \(Z\).

The above definition extends the notion of the quasi-interior point (see in [1], page 259) from normed lattices to ordered topological linear spaces. It is clear that if \(u\) is a quasi-interior point of \(Z_+\) then \(f(u) > 0\) for any positive, continuous, and nonzero linear functional \(f\) of \(Z\). In [5], page 24, the points \(u\) of an ordered Banach space \(Z\) with the property \(f(u) > 0\) for any positive, continuous, nonzero linear functional \(f\) of \(Z\) are called quasi-interior points of \(Z_+\). In Theorem 6 we show that in ordered Banach spaces with the RDP, these two definitions are equivalent.

By Proposition 2 we get the following result:

**Theorem 4.** An element \(u \in Y_+\) is a quasi-interior point of \(Y_+\) if and only if for each \(x \in Y_+\) there exists an increasing sequence \(\{x_n\}\) of \(I_u\) which converges to \(x\), with \(0 \leq x_n \leq x\), for each \(n\).

**Proposition 5.** If \(u\) is a quasi-interior point of \(Y_+\), then \([0, x] \cap [0, u] \neq \{0\}\), for each \(x \in Y_+\), \(x \neq 0\).

**Proof.** By the above theorem there exists an increasing sequence \(\{x_n\}\) of \(I_u\) with \(0 < x_n \leq x\) which converges to \(x\), therefore the proposition is true. \(\blacksquare\)
Theorem 6. If we suppose moreover that $Y$ is a Banach space and $u \in Y_+$, then the following statements are equivalent:

(i) $u$ is a quasi-interior point of $Y_+$,

(ii) $f(u) > 0$, for each $f \in Y_*^+$, $f \neq 0$.

Proof. The direct is obvious because $f(u) = 0$ implies that $f = 0$ on $Y$. For the converse suppose that statement (ii) is true and that the closure $I$ of $I_u$ is a proper subspace of $Y$. So there exists $g \in Y^*$, $g \neq 0$ which is equal to zero on $I$. Then $|g| \in Y^*$ because $Y$ is a Banach space and $|g|$ is positive. It is known that $|g|(y) = \sup g([-y, y])$ for any $y \in Y_+$. Since $g \neq 0$ and the positive cone of $Y$ is generating we have that $g(y) \neq 0$, for at least one $y \in Y_+$ which implies that $|g| \neq 0$. Therefore $|g|(u) > 0$. Since $|g|(w) = \sup g([-u, u])$ we have that $g$ is nonzero on the interval $[-u, u]$, a contradiction because $g$ is equal to zero on $I$ and $[-u, u] \subseteq I$. Therefore $u$ is a quasi-interior point of $Y_+$ and the converse is true.

Proposition 7. Suppose that $Z$ is an ordered normed space and suppose that its positive cone $Z_+$ is complete. Then the following statements are equivalent:

(i) Every $y \in Z_+, y \neq 0$, is a quasi-interior point of $Z_+$,

(ii) $\dim Z = 1$.

Proof. Suppose that statement (i) is true. At first we shall show that the boundary $\partial Z_+$ of $Z_+$ is equal to $\{0\}$. By the Bishop-Phelps Theorem (see for example in [4] Theorem 3.8.14) the support points of $Z_+$ are dense in $\partial Z_+$. Suppose that $r$ is a support point of $Z_+$ which is supported by the functional $x^* \in Z^*, x^* \neq 0$, i.e. $x^*(r) = \min\{x^*(t) \mid t \in Z_+\}$. Then $x^*(r) \leq 0$ because $0 \in Z_+$. If we suppose that $x^*$ is not positive, there exists $a \in Z_+$ with $x^*(a) < 0$. Then $x^*$, restricted on the halfline defined by $a$, takes any negative real value, therefore $x^*(r) = -\infty$, contradiction. Therefore $x^*$ is positive. If we suppose that $r \neq 0$, then $r$ is a quasi-interior point of $Z_+$, therefore $x^*(r) > 0$, a contradiction, because as we have found before $x^*(r) \leq 0$, hence $r = 0$ and $\partial Z_+ = \{0\}$. We shall show now that $Z = Z_+ \cup (-Z_+)$. So we suppose that $w \in Z \setminus Z_+$ and that $y \in Z_+, y \neq 0$. Suppose also that $z$ is a point of the line segment $yw$ with $z \in \partial Z_+$. Then $z = 0$, therefore $w \in (-Z_+)$, hence $Z = Z_+ \cup (-Z_+)$. Suppose now that $w$ is a fixed point of $Z \setminus Z_+$. As we have shown before, for any point $y \in Z_+, y \neq 0$, the line segment $yw$ contains 0, therefore $y$ belongs to the line defined by $w$ and 0, hence $Z_+$ is a halfline and $\dim Z = 1$. So (i) implies (ii). The converse is clear.

Definition 8. Suppose that $Z$ is an ordered space and $x, y \in Z_+$ with $x, y \neq 0$. If $[0, x] \cap [0, y] = \{0\}$, we will say that $x, y$ are disjoint in $Z_+$ and we will write $\inf_{Z_+}\{x, y\} = 0$. 

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The next result will be used later for the study of positive bases. Statement (i) is an easy consequence of the Riesz decomposition property.

**Proposition 9.** Suppose that $Z$ is an ordered normed space with the Riesz decomposition property. Then the following statements are true:

(i) If the vectors $y_1, y_2, \ldots, y_n$ are pairwise disjoint in $Z_+$ and $x \in Z_+$ with $x \leq y_1 + y_2 + \ldots + y_n$, we have:
   (a) $x$ has a unique decomposition $x = x_1 + x_2 + \ldots + x_n$ with $0 \leq x_i \leq y_i$, for each $i = 1, 2, \ldots, n$, and
   (b) if $x \geq y_i$ for each $i = 1, 2, \ldots, n$, then $x = y_1 + y_2 + \ldots + y_n$.
   (c) if $\Phi_1, \Phi_2$ are subsets of $\{1, 2, \ldots, n\}$, $y_{\Phi_1} = \sum_{i \in \Phi_1} \lambda_i y_i$, $y_{\Phi_2} = \sum_{i \in \Phi_2} \mu_i y_i$, where $\lambda_i$ and $\mu_i$ are positive real numbers and $h \leq y_{\Phi_1}, h \leq y_{\Phi_2}$, then $h$ has a unique decomposition $h = \sum_{i \in (\Phi_1 \cap \Phi_2)} h_i$ where $0 \leq h_i \leq \min\{\lambda_i, \mu_i\} y_i$, for each $i \in \Phi_1 \cap \Phi_2$. If $\Phi_1 \cap \Phi_2 = \emptyset$ then $y_{\Phi_1}, y_{\Phi_2}$ are disjoint in $Z_+$.

(ii) If the positive cone $Z_+$ of $Z$ is normal, the vectors $y_i, i \in \mathbb{N}$ are pairwise disjoint in $Z_+$ and the sum $\sum_i y_i$ exists, then
   (a) $\inf_{x \in Z_+} \{\sum_{i=1}^{\infty} y_i, \sum_{i=n+1}^{\infty} y_i\} = 0$ for each $n$, and
   (b) each element $x$ of $Z_+$ with $0 \leq x \leq \sum_{i=1}^{\infty} y_i$ has a unique expansion $x = \sum_{i=1}^{\infty} x_i$, with $0 \leq x_i \leq y_i$ for each $i$.

**Proof.** The proof of (i) is the following: By the RDP we have that $x = x_1 + x_2 + \ldots + x_n$ with $0 \leq x_i \leq y_i$, for each $i$. Suppose that $x = x'_1 + x'_2 + \ldots + x'_n$ with $0 \leq x'_i \leq y_i$, for each $i$. Then $0 \leq x'_j \leq x_1 + x_2 + \ldots + x_n$, therefore $x'_j = x'_1 + x'_2 + \ldots + x'_n$ with $0 \leq x'_i \leq x_i \leq y_i$, for each $i$, therefore $x'_i = 0$ for each $i \neq j$ because $y_i$ and $y_j$ are disjoint. So we have that $x'_j \leq x$ and similarly $x_j \leq x'_j$, therefore $x_j = x'_j$ for each $j$, and the expansion of $x$ is unique. If we suppose that $y_j \leq x$ for each $j$, we have that $y_j = y_{jj} + y_{j2} + \ldots + y_{jn}$, with $0 \leq y_{jj} \leq x_i \leq y_i$ for each $i$, therefore $0 \leq y_{jj} \leq y_j$, hence $y_{jj} = 0$ for each $i \neq j$. So we have that $y_j = y_{jj} \leq x_j \leq y_j$, therefore $y_j = x_j$ for each $j$ and (b) is true. To prove (c) we remark that $0 \leq h \leq y_{\Phi_1}$ implies that $h = \sum_{i \in \Phi_1} h_i$ with $0 \leq h_i \leq \lambda_i y_i$ for each $i \in \Phi_1$. Since $h \leq y_{\Phi_2}$ we have that $h_i = \sum_{j \in \Phi_2} h^j_i$ with $0 \leq h^j_i \leq \mu_j y_j$, for any $j \in \Phi_2$. Since the vectors $y_i$ are disjoint we have that $h_i^j = 0$ for each $j \neq i$, therefore $h_i = h^i_i \leq \min\{\lambda_i, \mu_i\} y_i$ and (c) is true.

To prove statement (a) of (ii) we suppose that $0 \leq h \leq \sum_{i=1}^{\infty} y_i, \sum_{i=n+1}^{\infty} y_i$. Then $h = \sum_{i=1}^{\infty} h_i$, with $0 \leq h_i \leq y_i$ for each $i = 1, 2, \ldots, n$. Also $h_i \leq y_{n+1} + \sum_{i=n+2}^{\infty} y_i$, therefore $h_i = h_{n+1} + h'_{n+1}$ where $0 \leq h_{n+1} \leq y_{n+1}$ and $0 \leq h'_{n+1} \leq \sum_{i=n+2}^{\infty} y_i$. Since $y_i$ and $y_{n+1}$ are disjoint we have that $h_{n+1} = 0$, therefore $0 \leq h_i = h'_{n+1} \leq \sum_{i=n+2}^{\infty} y_i$ and by induction we have that $0 \leq h_i \leq \sum_{i=n+m}^{\infty} y_i$ for each $m \in \mathbb{N}$. Since the cone is normal and the sequence $\sum_{i=n+m}^{\infty} y_i$ converges
to zero, we have that \( h_i = 0 \), for each \( i = 1, 2, ..., n \). Therefore \( h = 0 \) and (a) is true. To prove (b) suppose that \( 0 \leq x \leq \sum_{i=1}^{n} y_i + \sum_{i=n+1}^{\infty} y_i \). Then \( x \) has a unique decomposition \( x = \sum_{i=1}^{n} x_i + x'_n \) with \( 0 \leq x_i \leq y_i \) for each \( i = 1, 2, ..., n \) and \( 0 \leq x'_n \leq \sum_{i=n+1}^{\infty} y_i \). If we suppose that \( m > n \) and \( x = \sum_{i=1}^{m} v_i + v'_m \), with \( 0 \leq v_i \leq y_i \) for \( i = 1, 2, ..., m \) and \( 0 \leq v'_m \leq \sum_{i=m+1}^{\infty} y_i \), then \( x = \sum_{i=1}^{n} v_i + (\sum_{i=n+1}^{m} v_i + v'_m) \) therefore \( x_i = v_i \) for each \( i = 1, 2, ..., n \). Hence the vectors \( x_i, i \in \mathbb{N} \) are uniquely determined and the expansion \( x = \sum_{i=1}^{\infty} x_i \), with \( 0 \leq x_i \leq y_i \) for each \( i \), of \( x \) is unique.

For a further study of the Riesz decomposition property on the space of operators between Banach lattices we refer to [2] and the references inside.

\section{Ordered subspaces}

In this section we will denote by \( E \) an infinite dimensional ordered Banach space whose positive cone \( E_+ \) is defined by a countable family \( \mathcal{F} = \{ f_i | i \in \mathbb{N} \} \), of positive, continuous linear functionals of \( E \), i.e. \( E_+ = \{ x \in E \mid f_i(x) \geq 0, \text{for each } i \} \). Also we will denote by \( X \) an ordered subspace of \( E \), i.e. \( X \) is a subspace of \( E \) ordered by the induced ordering. It is clear that \( E_+ \) is closed and that \( X_+ = X \cap E_+ \) is the positive cone of \( X \). For any \( x, y \in X \), denote by \( \sup_X \{ x, y \} \) the supremum and by \( \inf_X \{ x, y \} \) the infimum of \( \{ x, y \} \) in \( X \) whenever exist. If \( \sup_X \{ x, y \} \) and \( \inf_X \{ x, y \} \) exist for any \( x, y \in X \), we say that \( X \) is a lattice-subspace of \( E \). According to our notations, for any \( x, y \in X \) we have: \( [x, y]_X = \{ z \in X \mid x \leq z \leq y \} \), is the order interval \( xy \) in \( X \) whenever \( x \leq y \), if \( x, y \in X_+ \) with \( [0, x]_X \cap [0, y]_X = \{ 0 \} \), we say that \( x, y \) are disjoint in \( X_+ \) and we will write \( \inf_X \{ x, y \} = 0 \). Also for any \( x \in X_+, x \neq 0 \), we denote by \( I_x(X) = \bigcup_{n=1}^{\infty} [-nx, nx]_X \) the solid subspace of \( X \) generated by \( x \). The closure \( \overline{I_x(X)} \) of \( I_x(X) \) in \( X \) is the closed solid subspace of \( X \) generated by \( x \). If \( I_x(X) = X \), \( x \) is a quasi-interior point of \( X_+ \).

\subsection{The minimal and the maximum support property}

The minimal and maximum support property have been introduced in [7]. For any point \( x \in E \) we will denote by \( x(i) \) the real number \( f_i(x) \) and by \( \text{supp}(x) = \{ i \in \mathbb{N} | x(i) \neq 0 \} \), the support of \( x \) (with respect to \( \mathcal{F} \)). The set \( \text{supp}(X_+) = \bigcup_{x \in X_+} \text{supp}(x) \), is the support of \( X_+ \) (with respect to \( \mathcal{F} \)). An element \( x \) of \( X_+ \) has minimal support in \( X_+ \) (with respect to \( \mathcal{F} \)) if for any \( y \in X_+, \text{supp}(y) \subsetneq \text{supp}(x) \) implies \( y = 0 \).
Definition 10. The ordered subspace \( X \) of \( E \) has the minimal support property (with respect to \( F \)) if for each \( x \in X_+ \backslash \{0\} \) we have: \( x \) is an extremal point of \( X_+ \) if and only if \( x \) has minimal support in \( X_+ \).

Proposition 11. Suppose \( I \) is the closed solid subspace of \( X \) generated by a nonzero, positive element \( x \) of \( X_+ \). Then \( \text{supp}(u) = \text{supp}(I_+) \) for any quasi-interior point \( u \) of \( I_+ \). (The converse is not always true).

Proof. It is clear that \( \text{supp}(u) \subseteq \text{supp}(I_+) \). If we suppose that \( f_i(u) = 0 \) for some \( i \in \text{supp}(I_+) \), then \( f_i \) is equal to zero on \( I_u(X) \) and therefore also on \( I \), a contradiction because we have supposed that \( i \in \text{supp}(I_+) \). Hence \( f_i(u) > 0 \) and \( \text{supp}(u) = \text{supp}(I_+) \). By Example 15, (ii), we have that the converse is not always true. \( \blacksquare \)

Definition 12. The ordered subspace \( X \) of \( E \) has the maximum support property (with respect to \( F \)) if each subspace \( F \) of \( X \) which is equal to \( X \) or \( F \) is a closed solid subspace of \( X \) generated by a nonzero element of \( X_+ \) has the property: an element \( x \in F_+ \) is a quasi-interior point of \( F_+ \) if and only if \( \text{supp}(x) = \text{supp}(F_+) \).

Proposition 13. If \( X_+ \) is closed and \( X \) has the maximum support property, then \( X_+ \) has quasi-interior points.

Proof. For each \( i \in \text{supp}(X_+) \) there exists \( x_i \in X_+ \) with \( f_i(x_i) > 0 \). So \( u = \sum_{i \in \text{supp}(X_+)} \frac{f_i}{\|x_i\|} x_i \) is a quasi-interior point of \( X_+ \) because \( X \) has the maximum support property and \( \text{supp}(u) = \text{supp}(X_+) \). \( \blacksquare \)

The proof of the next proposition is the same with the proof of Proposition 3.4 of [7]. The extra assumption here that \( X_+ \) is closed is posed in order to use Proposition 7.

Proposition 14. If \( X_+ \) is closed and \( X \) has the maximum support property, then \( X \) has the minimal support property.

Example 15. (i) The sequence spaces \( c_0 \) and \( \ell_p \) for \( 1 \leq p < +\infty \) have the maximum support property with respect to the family \( \mathcal{F} = \{\delta_i\} \) of the Dirac measures \( \delta_i(x) = x(i) \) supported at the natural numbers \( i \). The space \( \ell_\infty \) of bounded real sequences does not have the maximum support property with respect to \( \mathcal{F} \). Indeed the vector \( x \) with \( x(i) = \frac{1}{i} \) for any \( i \) has maximum support and the closed solid subspace generated by \( x \) is \( c_0 \). \( \ell_\infty \) has the minimal support property because the extremal points of \( \ell_\infty^+ \), as positive multiples of the vectors \( e_i \), have minimal support.

(ii) The family \( \{\delta_{r_i} \mid i \in \mathbb{N}\} \) of the Dirac measures \( \delta_{r_i} \) supported at the rational numbers \( r_i \) of \([0, 1]\) and also the family \( \mathcal{G} = \{\mu_i \mid i \in \mathbb{N}\} \) of the Lebesgue measures
μ, supported at λi, where \( \{I_i\} \) is a sequence of subintervals of \( [0, 1] \) so that each interval \((a, b)\) of \( [0, 1] \) contains at least one \( I_i \), define the positive cone of the space \( E = C[0, 1] \) of continuous, real valued functions defined of \( [0, 1] \). \( E \) does not have the maximum support with respect to these families. Indeed, if \( x \in E_+ \) with \( x(t_0) = 0 \) for some irrational number \( t_0 \) and \( x(t) > 0 \) for each \( t \neq t_0 \), then \( \text{supp}(x) = \mathbb{N} \) but \( x \) is not a quasi-interior point of \( E_+ \).

**Theorem 16.** ([8], Proposition 2.5.) If \( X \) is closed and \( X \) has a positive basis \( \{b_n\} \), the following statements are equivalent:

(i) \( X \) has the maximum support property with respect to \( F \),
(ii) there exists a sequence \( \{i_n\} \) of \( \mathbb{N} \) such that \( f_{i_n}(b_{i_n}) > 0 \) and \( f_{i_m}(b_{i_m}) = 0 \), for each \( m \neq n \), i.e. the coefficient functionals of the basis \( \{b_n\} \) can be extended on \( E \) to positive multiples of elements of \( F \).

The next is an example of an ordered subspace with a positive basis, without the maximum support property.

**Example 17.** Let \( \{b_n\} \) be a sequence of \( l_\infty \) so that \( b_1(4n) = \frac{1}{2^n}, b_1(4n + 1) = \frac{1}{3^n} \) and \( b_1(i) = 0 \) in the other cases, \( b_2(4n) = \frac{1}{2^n}, b_2(4n + 1) = \frac{1}{3^n} \) and \( b_2(i) = 0 \) in the other cases and \( b_n = e_{4n+2}, \) for \( n \geq 3 \). Then \( \{b_n\} \) is a positive basis of the closed subspace \( X \) of \( l_\infty \) generated by it. \( X \) does not have the maximum support property with respect to the family \( F \) of the Dirac measures \( \delta_i \) supported at the natural numbers \( i \). Indeed, \( \text{supp}(b_1) = \text{supp}(b_2) \) therefore \( \delta_i(b_1) > 0 \) if and only if \( \delta_i(b_2) > 0 \), and by Theorem 16, \( X \) does not have the maximum support property.

### 3.2 The ws-property

The notion of the s-property (supremum property) has been introduced in [7]. We define here a weaker property, which we call ws-property (weak s-property) as follows:

**Definition 18.** An ordered subspace \( X \) of \( E \) has the ws-property (with respect to \( F \)) if for each \( x \in X_+, x \neq 0 \) and for each \( i \in \text{supp}(X_+) \) the set \( \{y \in [0, x]_X \mid y(i) = 0\} \) has at least one maximal element.

If in the above definition the set \( \{y \in [0, x]_X \mid y(i) = 0\} \) has a maximum element, then \( X \) has the s-property. If \( X \) has the ws-property, each solid subspace \( Z \) of \( X \) has this property. In the theory of vector optimization the maximal elements of a subset \( K \) of a normed space \( Z \) with respect to an ordering cone \( P \) of \( Z \) are the Pareto efficient points of \( K \). In our case, the ws-property ensures the existence of Pareto efficient points with respect to \( X_+ \). We start with the following easy result.
**Theorem 19.** Suppose that \( \tau \) is a linear topology of \( E \). If

(i) \( X_+ \) is \( \tau \)-closed,

(ii) each increasing net of \( X_+ \) order bounded in \( X \), has a \( \tau \)-convergent subnet, and

(iii) for each \( i \) the positive part \( K_i^+ = \{ y \in X_+ \mid f_i(y) = 0 \} \) of the kernel of \( f_i \) in \( X \) is \( \tau \)-closed,

then \( X \) has the ws-property.

**Proof.** Suppose that \( x \in X_+ \) and that \( A \) is a totally ordered subset of the \( \tau \)-closed set \([0, x]_X \cap K_i^+\). For each finite subset \( \Phi \) of \( A \) denote by \( x_\Phi \) the maximum of \( \Phi \). Then \( \{ x_\Phi \} \), as an increasing, order bounded net of \([0, x]_X \cap K_i^+\), is convergent to \( x_0 \in [0, x]_X \cap K_i^+ \) which is an upper bound of \( A \) and by Zorn’s lemma the set \([0, x]_X \cap K_i^+\) has maximal elements.

**Corollary 20.** If \( E \) is a Banach lattice with order continuous norm and \( X_+ \) is closed, then \( X \) has the ws-property.

**Proof.** Each order interval of \( E \) weakly compact. Since \( X_+ \) is weakly closed, each order interval of \( X \) is weakly compact, hence \( X \) has the ws-property.

**Corollary 21.** If \( E \) is a dual space, the functionals \( f_i \) are weak-star continuous and \( X_+ \) is weak-star closed and normal, then \( X \) has the ws-property.

**Proof.** For each \( x \in X_+ \) the order interval \([0, x]_X\) is weak-star closed and bounded because \( X_+ \) is normal, therefore \([0, x]_X\) is weak-star compact. Hence \( X \) has the ws-property.

**Corollary 22.** If \( X \) is closed with a positive basis, then \( X \) has the ws-property.

**Proof.** By [11] Theorem 5, each order interval of \( X \) is compact.

**Example 23.** (i) The spaces \( c_0 \) and \( \ell_p \) with \( 1 \leq p < +\infty \) and also the spaces \( L_p^+(\mu) \) \( 1 \leq p < +\infty \), as Banach lattices with order continuous norm have the ws-property with respect to any countable family which defines their positive cone. Also their closed ordered subspaces have the ws-property.

(ii) By Corollary 21, \( \ell_\infty \) and its weak-star closed ordered subspaces have the ws-property with respect to the family of the Dirac measures \( \delta_i \) supported at the natural numbers \( i \).

(iii) \( C[0, 1] \) does not have the ws-property with respect to the family of the Dirac measures \( \delta_{r_i} \) supported at the rational numbers \( r_i \) of \( [0, 1] \). It is easy to show that the set \( \{ y \in C[0, 1] \mid 0 \leq y \leq x \text{ and } y(\frac{1}{2}) = 0 \} \), where \( x \in C_+[0, 1] \) with \( x(\frac{1}{2}) > 0 \), does not have maximal elements.

If \( P, Q, R \) are subcones of \( X_+ \) with \( R = P + Q \) and \( P \cap Q = \{ 0 \} \), we will say that \( R \) is the direct sum of \( P, Q \) and we will write \( P \oplus Q = R \).
Proposition 24. Suppose that $X$ is closed, $X_+$ is generating and normal and also that $X$ has the Riesz decomposition property and the ws-property with respect to $\mathcal{F}$. Let $x \in X_+, x \neq 0, i \in \text{supp}(X_+)$ and we denote by $z_i$ a maximal element of the set $\{y \in [0,x]_X \mid y(i) = 0\}$. Then $z_i' = x - z_i$ is a minimal element of the set $\{y \in [0,x]_X \mid y(i) = x(i)\}$. If $I,J,W$ are the closed solid subspaces of $X$ generated respectively by the elements $x, z_i, z_i', then$

(i) \inf_{x_+} \{z_i, z_i'\} = 0,$

(ii) the functional $f_i$ is equal to zero on $J$. If $f_i(x) > 0$ then $f_i$ is strictly positive on $W$. If $f_i(x) = 0$, then $z_i = x$ and if $f_i$ is strictly positive on $I$, then $z_i' = x$. If $f_i$ is nonzero and non-strictly positive on $I$ then $0 < z_i < x$ and $0 < z_i' < x$,

(iii) if $f_i(x) > 0$, then $I^+_i(X) \oplus I^+_i(X) = I^+_i(X)$ and $J_+ \oplus W_+ = I_+.$

Proof. Suppose that $z \in A = \{y \in [0,x]_X \mid y(i) = x(i)\}$ with $z_i' > z$. Then $x - z > z_i$ and $f_i(x-z) = 0$, which contradicts the definition of $z_i$. Therefore $z_i'$ is a minimal element of $A$.

(i) Let $h \in X$ with $0 < h \leq z_i, z_i'$. Then $0 \leq h(i) \leq z_i(i) = 0$, hence $h(i) = 0$. So we have that $h + z_i \leq x$ and $(h + z_i)(i) = 0$, a contradiction. Therefore $h = 0$ and $\inf_{x_+} \{z_i, z_i'\} = 0$.

(ii) Since $z_i(i) = 0$, $f_i$ is equal to zero on $I_{z_i}$ and therefore also on $J$. Suppose that $f_i(x) > 0$. Then $z_i < x$, hence $z_i' > 0$ and $W_+ \neq \{0\}$. Suppose that $w \in W_+, w > 0$ with $w(i) = 0$. Then by Theorem 4, $w$ is the limit of an increasing sequence of elements of $I^+_z(X)$, therefore $y(i) = 0$ for at least one $y \in X$ with $0 < y \leq z_i'$. Then $y + z_i \leq x$ and $(y + z_i)(i) = 0$, a contradiction, therefore $f_i$ is strictly positive on $W$. If we suppose that $f_i(x) = 0$, then by the definition of $z_i$ we have that $z_i = x$ and if we suppose that $f_i$ is strictly positive on $I$ we have that $z_i = 0$, therefore $z_i' = x$. Suppose now that $f_i$ is nonzero and also non strictly positive on $I$. Then $x(i) > 0$ and also $v(i) = 0$, for at least one nonzero point $v$ of $I_+$. Since $v$ is the limit of an increasing sequence of elements of $I^*_z(X)$, we have that $y(i) = 0$ for at least one nonzero element $y \in [0,x]_X$. This implies that $z_i > 0$ because if we suppose that $z_i = 0$ we have that $z_i < y$, which contradicts the definition of $z_i$. Hence $0 < z_i$. Also $z_i < x$ because $x(i) > 0$. So we have $0 < z_i < x$ and $0 < z_i' < x$.

(iii) Let $f_i(x) > 0$. Suppose that $h \in J_+ \cap W_+$. Then $h \in J_+$ therefore $h(i) = 0$. Since the functional $f_i$ is strictly positive on $W$ we have that $h = 0$, therefore $J_+ \cap W_+ = \{0\}$. Suppose that $y \in [0,x]_X$. Then $y \leq z_i + z_i'$ and by the RDP we have that $y = y_1 + y_2$ with $y_1 \in [0,z_i]_X$ and $y_2 \in [0,z_i']_X$. By the above remarks we have that the first assertion of (iii) is true. Suppose now that $y \in I_+$. By Theorem 4, $y$ is the limit of an increasing sequence $y_n$ of $I^+_z(X)$, with $y_n \leq y$ for each $n$. Hence $y_{n+1} - y_n \in I^+_z(X)$, therefore $y_{n+1} - y_n \leq k_n x = k_n(z_i + z_i')$, and by the RDP we have that $y_{n+1} - y_n = a_{n+1} + b_{n+1}$ with $a_{n+1} \in I^+_z(X)$ and
Let $M \ni x = x_1 + x_2$ where $x_1$ is a maximal element of the set $\{ y \in [0, x]_X \mid y(i) = 0\}$, then we will say that $x = x_1 + x_2$ is a decomposition of $x$ with respect to $f_i$ (or with respect to $i$) and also that $x$ is decomposed with respect to $f_i$ in the elements $x_1, x_2$. If $f_i$ is equal to zero on $I_x(X)$ or if $f_i$ is strictly positive on $I_x(X)$, we will say that $x$ is not decomposed with respect to $f_i$ (or with respect to $i$).

### 3.3 Existence of positive bases

In what follows we will denote by $X$ a closed, ordered subspace of $E$ as in the previous proposition, and suppose that $x$ is a nonzero element of $X_+$ and $f_i \in F$. If $f_i$ is nonzero and non-strictly positive on $I_x(X)$ and $x = x_1 + x_2$ where $x_1$ is a maximal element of the set $\{ y \in [0, x]_X \mid y(i) = 0\}$, then we will say that $x = x_1 + x_2$ is a decomposition of $x$ with respect to $f_i$ (or with respect to $i$) and also that $x$ is decomposed with respect to $f_i$ in the elements $x_1, x_2$. If $f_i$ is equal to zero on $I_x(X)$ or if $f_i$ is strictly positive on $I_x(X)$, we will say that $x$ is not decomposed with respect to $f_i$ (or with respect to $i$).

We will also denote by $M$ the following subset of $\mathbb{N}$: $M = \{ i \in \text{supp}(X_+) \mid f_i \text{ is non strictly positive on } X \}$. Therefore for each $x \in X_+, x \neq 0$ we have that $x(i) > 0$ for each $i \in \text{supp}(X_+) \setminus M$. Also $M \neq \emptyset$ because if we suppose that $M = \emptyset$, we have that $\text{supp}(x) = \text{supp}(X_+)$ for each $x \in X_+, x \neq 0$, therefore $\dim X = 1$ by Proposition 7. In order to prove the existence of extremal points of $X_+$ we develop a process of successive decompositions of a quasi-interior point of $X_+$. So we suppose that $u$ is a quasi-interior point of $X_+$ (such a point exists by Proposition 13) and we decompose $u$ as follows:

**Step 1**: We put $i_1 = \min M$ and we decompose $u$ with respect to $i_1$ in the elements $x_1, x_2$. Then $u = x_1 + x_2$ and $\inf_{X_+} \{ x_1, x_2 \} = 0$. Also $f_{i_1}$ is equal to zero on $I_1$ and strictly positive on $I_2$ where $I_1, I_2$ are the closed solid subspaces of $X$ generated by $x_1, x_2$ respectively. The set $m_1 = \{ x_1, x_2 \}$ is the front and the natural number $i_1$ is the index of the first decomposition.

**Step $\nu + 1$**: Suppose that we have accomplished the $\nu$th step and suppose that $m_\nu$ is the front and $i_\nu$ the index of the $\nu$th decomposition. Then at least one of the elements of $m_\nu$ is decomposed with respect to an $i \in M$. Indeed if we sup-
pose that any element $x$ of $m_\nu$ is not decomposed with respect to any $i \in M$ then for any $i \in M$, $f_i$ is strictly positive or equal to zero on the closed solid subspace $I$ of $X$ generated by $x$ and it is easy to show that $\text{supp}(y) = \text{supp}(I_+)$ for any $y \in I_+$, $y \neq 0$ therefore $y$ is a quasi-interior point of $I$. Hence $\dim I = 1$ and $X$ is finite-dimensional because $m_\nu$ is finite. We put $i_{\nu+1} = \min\{i \in M \mid \text{at least one element of } m_\nu \text{ is decomposed with respect to } i\}$. Then $i_{\nu+1} > i_\nu$ and we decompose with respect to $i_{\nu+1}$ the elements of $m_\nu$ which allow such a decomposition. We denote by $m_{\nu+1}$ the set which contains the elements of $m_\nu$ which are not decomposed with respect to $i_{\nu+1}$ and also the elements that arise from the decomposition of the elements of $m_\nu$ with respect to $i_{\nu+1}$. The set $m_{\nu+1}$ is the front and $i_{\nu+1}$ is the index of the $(\nu + 1)th$ decomposition. The set $\delta(u) = \cup_{\nu=0}^\infty m_\nu$, where $m_0 = \{u\}$, is the tree of decompositions of $u$.

**Proposition 26.** In the above process of decompositions of $u$ we have:
(i) the sequence of indices of decompositions $\{i_\nu\}$ is strictly increasing,
(ii) for each $i \in M$ with $i \leq i_\nu$, for each $x \in m_\nu$, $x$ is not decomposed with respect to $i$, therefore $f_i$ is strictly positive or equal to zero on $I = I_+(X)$,
(iii) the elements of $m_\nu$ are nonzero with sum equal to $u$. Also $\inf_{X_+}\{x,y\} = 0$, for each $x, y \in m_\nu$, with $x \neq y$,
(iv) $\inf_{X_+}\{x,u-x\} = 0$, for each $x \in \delta(u)$.

**Proof.** Statements (i),(ii) and (iii) are obvious. To prove (iv) we suppose that $x \in m_\nu$ for some $\nu$ and suppose that $m_\nu = \{x,y_1,y_2,\ldots,y_k\}$. Since the elements of $m_\nu$ are pairwise disjoint in $X_+$ with sum equal to $u$ we have that $u-x = \sum_{i=1}^k y_i$ and (iv) is true by Proposition 9.

For any $x \in m_\nu$ with $\nu \geq 1$ it is easy to show that there exists a unique vector $y \in m_{\nu-1}$ with $y \geq x$. Also for any $x \in m_\nu$ there exists at least one $y \in m_{\nu+1}$ with $x \geq y$. So if we suppose that $x,y \in \delta(u)$ with $x \in m_\nu$, $y \in m_{\nu+1}$ and $y \leq x$, we will say that $x$ is the presuccessor of $y$ in $m_\nu$, or that $y$ is a successor of $x$ in $m_{\nu+1}$. If moreover $y \in m_{\nu+1}$ we will say that $x$ is the first presuccessor of $y$ or that $y$ is a first successor of $x$.

**Proposition 27.** The following are true:
(i) for any $x \in m_\nu$ the sum of the successors of $x$ in $m_{\nu+\mu}$ is equal to $x$,
(ii) if $y$ is a successor of $x$ with $x > y$ and $I$ is the closed solid subspace of $X$ generated by $x$, then $\inf_{X_+}\{y,x-y\} = 0$ and $y$ is not a quasi-interior point of $I_+$,
(iii) for each $x \in \delta(u)$ and each $i \in M \cap \text{supp}(x)$, there exists a successor $y$ of $x$ such that the functional $f_i$ is strictly positive on the closed solid subspace $I$ of $X$ generated by $y$. 

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Proof. (i) Any element of \( \delta(u) \) is the sum of its first successors, therefore the proposition is true for \( \mu = 1 \) and continuing, we have that the proposition is true for any \( \mu \).

(ii) Since \( x - y \leq u - y \) and \( \inf X_+ \{ y, u - y \} = 0 \) we have that \( \inf X_+ \{ y, x - y \} = 0 \), therefore \( y \) is not a quasi-interior point of \( I_+ \) by Proposition 5.

(iii) Suppose that \( x \in m_\kappa \). Since the sequence \( \{ i_\nu \} \) is strictly increasing, there exists \( \nu \in \mathbb{N} \) with \( \nu > \kappa \) and \( i \leq i_\nu \). Then \( f_i \) is strictly positive or equal to zero on any closed solid subspace of \( X \) generated by an element of \( m_\nu \). But \( x = \sum_{j=1}^\nu x_j \)

where \( x_1, \ldots, x_r \) are the successors of \( x \) in \( m_\nu \) and \( f_i(x) > 0 \) because \( i \in \text{supp}(x) \), therefore \( f_i \) is strictly positive on at least one of the closed solid subspaces of \( X \) generated by \( x_1, \ldots, x_r \) and the proposition is true.

If \( x \in \delta(u) \) and \( x \in m_\nu \) for each \( \nu \geq \nu_0 \), then we will say that the process of decomposition stops at the point \( x \) of \( \delta(u) \). In other words, the process of decomposition stops at \( x \) if there exists \( \nu_0 \in \mathbb{N} \) so that \( x \in m_{\nu_0} \) and for each \( i \in M \) with \( i > \nu_0 \), the functional \( f_i \) is strictly positive or equal to zero on the closed solid subspace \( I \) of \( X \) generated by \( x \). Then for each \( i \in M \) with \( i \leq \nu_0 \), \( f_i \) is strictly positive or equal to zero on \( I \), Proposition 26, therefore \( \text{supp}(z) = \text{supp}(I_+) \) for any \( z \in I_+ \), \( z \neq 0 \), hence any nonzero vector of \( I_+ \) is a quasi-interior point of \( I \) which implies that \( \dim(I) = 1 \). So \( x \) is an extremal point of \( X_+ \) and we have proved the following:

**Proposition 28.** If the process of decomposition of \( u \) stops at an element \( x_0 \in \delta(u) \) then \( x_0 \) is an extremal point of \( X_+ \).

A sequence \( \{ x_\nu \} \) of \( \delta(u) \) is a branch of \( \delta(u) \), if \( x_\nu > x_{\nu+1} \) for each \( \nu \in \mathbb{N} \).

**Proposition 29.** Each branch of \( \delta(u) \) converges to zero.

Proof. It is enough to show that any branch \( \{ x_\nu \} \) of \( \delta(u) \) with \( x_0 = u \) converges to zero. Let \( z_\nu = x_{\nu-1} - x_\nu \), for each \( \nu \geq 1 \). Then for each \( \nu, \mu \), we have

\[
u = z_1 + z_2 + \ldots + z_\nu + x_\nu \quad \text{and} \quad x_\nu = z_{\nu+1} + \ldots + z_{\nu+\mu} + x_{\nu+\mu}.
\]

The vectors \( z_1, z_2, \ldots, z_\nu, x_\nu \) are pairwise disjoint in \( X_+ \). Indeed, \( \inf X_+ \{ x_\nu, u - x_\nu \} = 0 \), hence \( \inf X_+ \{ x_\nu, \sum_{i=1}^\nu z_i \} = 0 \), therefore \( \inf X_+ \{ x_\nu, z_i \} = 0 \) for each \( i \leq \nu \), because \( z_i \leq \sum_{j=1}^\nu z_j \). Suppose that \( j > i \). Then \( z_j \leq x_i \) and \( \inf X_+ \{ z_i, x_i \} = 0 \), therefore \( \inf X_+ \{ z_j, z_i \} = 0 \). Hence \( \inf X_+ \{ z_j, z_i \} = 0 \) for any \( i \neq j \). Let \( u_0 = \sum_{\nu=1}^\infty \frac{z_\nu}{\nu^2} \). We shall show that \( \text{supp}(u_0) = \text{supp}(X_+) \).

For each \( i \in \text{supp}(X_+) \setminus M \) we have that \( x(i) > 0 \) for each \( x \in X_+, x \neq 0 \), hence \( i \in \text{supp}(u_0) \). Suppose that \( i \in M \) and that \( x_\nu \) is decomposed at the \( \kappa_\nu \)-th decomposition. Since the sequence \( \{ i_{\kappa_\nu} \} \) is strictly increasing, there exists
\( \mu \in \mathbb{N} \) with \( i < i_{\nu}. \) By statement \((ii)\) of Proposition 26, \( f_i \) is strictly positive or equal to zero on \( I = T_{x_i}(X). \) We shall show that in both cases \( i \in \text{supp}(u_0). \)

If \( f_i \) is strictly positive on \( I \) we have \( z_{\mu+1}(i) > 0 \) because \( 0 < z_{\mu+1} < x_\mu \) therefore \( i \in \text{supp}(u_0). \) If \( f_i \) is equal to zero on \( I \) then \( x_{\mu}(i) = 0, \) therefore \( f_i(z_1 + \ldots + z_{\mu}) = f_i(z_1 + \ldots + z_{\mu} + x_\mu) = f_i(u) > 0, \) hence \( f_i(z_j) > 0 \) for at least one \( j, \) therefore \( i \in \text{supp}(u_0). \) Therefore \( \text{supp}(X_+) = \text{supp}(u_0) \) and \( u_0 \) is a quasi-interior point of \( X. \) By Theorem 4, there exists an increasing sequence \( \phi_n \in [0, u] \cap [0, r, u_0], \) where \( \{r_n\} \) is a strictly increasing sequence of natural numbers with \( \lim_{n \to \infty} \phi_n = u. \) Let \( h_\nu = \sum_{n=1}^{\infty} r_n \frac{z_{r_n+\mu}}{z_{r_n+\mu}}. \) Since \( 0 \leq \phi_n \leq r_n u_0 \) we have \( 0 \leq \phi_n \leq r_n z_1 + \ldots + r_n z_{r_n} + h_n \) and by Proposition 9, \( \phi_n \) has a unique decomposition \( \phi_n = \phi_n^1 + \ldots + \phi_n^r + H_n \) with \( 0 \leq \phi_n^i \leq r_n z_i \) for each \( i \) and \( 0 \leq H_n \leq h_n. \) The last inequality implies that \( \lim_{n \to \infty} H_n = 0 \) because \( \lim_{n \to \infty} h_n = 0 \) and the cone \( X_+ \) is normal. Also we have \( 0 \leq \phi_n^i \leq u, r_n z_i \) for \( i = 1, 2, \ldots, r_n, \) therefore \( \phi_n^i = a_1 + a_2 + \ldots + a_{r_n} + b_n \) with \( 0 \leq a_j \leq z_j, \) for each \( j \) and \( 0 \leq b_n \leq x_{r_n}. \) Since the vectors \( z_1, z_2, \ldots, z_{r_n}, x_{r_n} \) are pairwise disjoint in \( X_+ \) we have that \( \phi_n^i = a_1, \) therefore \( 0 \leq \phi_n^i \leq z_i \) for each \( i = 1, 2, \ldots, r_n. \) Since \( H_n \leq u, \) we have that \( H_n = \gamma_1 + \gamma_2 + \ldots + \gamma_{r_n} + c_n \) with \( 0 \leq \gamma_j \leq z_j \) for each \( j = 1, 2, \ldots, r_n \) and \( 0 \leq c_n \leq x_{r_n}. \) Since \( H_n \leq h_n \) we have also that \( \gamma_j \leq h_n \) for each \( j. \) Since the vectors \( z_j, j = 1, 2, \ldots, r_n \) and \( h_n \) are pairwise disjoint in \( X_+ \) we have \( \gamma_j = 0 \) for each \( j = 1, 2, \ldots, r_n, \) hence \( H_n = c_n, \) therefore \( H_n \leq x_{r_n}. \) So we have that \( \lim_{n \to \infty} (u - (\phi_n^1 + \ldots + \phi_n^r + H_n)) = 0, \) therefore

\[
\lim_{n \to \infty} \left[ (z_1 - \phi_n^1) + (z_2 - \phi_n^2) + \ldots + (z_{r_n} - \phi_n^{r_n}) + (x_{r_n} - H_n) \right] = 0.
\]

Since the members in the above limit are positive and the cone of \( X_+ \) is normal we have that \( \lim_{n \to \infty} (x_{r_n} - H_n) = 0. \) As we have shown above \( \lim H_n = 0 \) therefore \( \lim x_{r_n} = 0. \) Since the sequence \( \{x_n\} \) is decreasing it converges to zero and the proposition is true.

**Proposition 30.** For each \( x \in \delta(u) \) at least one of the successors of \( x \) is an extremal point of \( X_+. \)

**Proof.** Let \( x \in \delta(u). \) If at least one of the successors \( x' \) of \( x \) does not belong to a branch of \( \delta(u), \) then the process of decompositions stops after a finite number of steps at any successor of \( x', \) therefore any successor of \( x' \) is an extremal point of \( X_+ \) dominated by \( x \) and the proposition is true. So we suppose that any successor of \( x \) belongs to a branch of \( \delta(u). \) Also we may suppose that \( x < u \) because in the case where \( x = u, \) it is enough to show the proposition for one of its successors. Let \( I \) be the closed solid subspace of \( X \) generated by \( x \) and suppose that \( L = \{i \in \text{supp}(x) \mid f_i \) is not strictly positive on \( I_x(X)\}. \) Then \( L \subseteq M. \) Also \( \text{supp}(x) = \text{supp}(I_x). \) If \( L \) is finite, then after a finite number of steps the decomposition stops
at any successor of \( x \) and the proposition is true. So we suppose that the set \( L \) is infinite. Let \( j_1 = \min L \). Then by statement \((iii)\) of Proposition 27, there exists \( x_1 \in \delta(u) \) such that \( x_1 \leq x \) and \( f_{j_1} \) is strictly positive on \( T_{x_1}(X) \). Since \( x_1 \) is an element of a branch of \( \delta(u) \) dominated by \( x \) and any such a branch of \( \delta(u) \) converges to zero, we may suppose that there exists an element \( y_1 \in \delta(u) \) such that \( y_1 < x_1 \leq x, \|y_1\| \leq 2^{-1}\varepsilon \), where \( \varepsilon \) is a constant real number with \( 0 < \varepsilon < \|x\| \).

Note also that \( f_{j_1} \) is strictly positive on \( T_{y_1}(X) \) because it is strictly positive on \( T_{x_1}(X) \) and \( 0 < y_1 < x_1 \). By Proposition 27 we have that \( \inf_{X_+} \{y_1, x - y_1\} = 0 \) hence \( y_1 \) is not a quasi-interior point of \( I \). Therefore \( \text{supp}(y_1) \neq \text{supp}(I_+) \), hence there exists at least one \( i \in \text{supp}(I_+) \) with \( i \notin \text{supp}(y_1) \), therefore there exists \( i \in L \) with \( y_1(i) = 0 \). We put \( j_2 = \min \{i \in L : y_1(i) = 0\} \). Then \( j_1 < j_2 \) and as before we can find a vector \( y_2 \in \delta(u) \) so that \( y_2 < x, \|y_2\| \leq 2^{-2}\varepsilon \) and \( f_{j_2} \) is strictly positive on \( T_{y_2}(X) \). Then \( \inf_{X_+} \{y_1, y_2\} = 0 \), because for any \( h \in X \) with \( 0 \leq h \leq y_1, y_2 \) we have that \( 0 \leq h\|y_2\| \leq y_1(j_2) = 0 \), therefore \( h = 0 \) because \( f_{j_2} \) is strictly positive on \( T_{y_2}(X) \). By the way we have selected \( y_2 \) (as a sufficiently small member of a branch which converges to zero) we may also suppose that \( y_1 \in m_{v_1} \) and \( y_2 \in m_{v_2} \) with \( v_1 < v_2 \). We may also suppose that \( v_2 \) is sufficiently large so that \( m_{v_2} \), except the successors of \( x \) and the element \( y_2 \), contains at least one extra element so we may suppose that \( m_{v_2} = \{y_2, a_1, a_2, ..., a_k, b_1, b_2, ..., b_r, c_1, c_2, ..., c_l\} \), where \( a_1, a_2, ..., a_k \) are the successors of \( y_1 \) and \( y_2, a_1, a_2, ..., a_k, b_1, b_2, ..., b_r \) are the successors of \( x \). We put \( s_1 = y_1, s_2 = y_1 + y_2 \). Then \( s_1 < x_1 \) and \( s_2 < x \). The first inequality is obvious and the second holds because \( x \) is the sum of its successors in \( m_{v_2} \). Also \( s_1(j_1) > 0 \) and by the definition of \( j_2 \), we have that \( s_2(i) > 0 \) for each \( i \in L \) with \( i \leq j_2 \). By Proposition 9, \( \inf_{X_+} \{s_i, x - s_i\} = 0 \), for each \( i = 1, 2 \) because the successors of \( x \) in \( m_{v_2} \) are pairwise disjoint. Since \( \inf_{X_+} \{s_2, x - s_2\} = 0 \) we have that \( s_2 \) is not quasi-interior point of \( I_+ \), hence there exists \( i \in L \) with \( s_2(i) = 0 \). Let \( j_3 = \min \{i \in L : s_2(i) = 0\} \). Then \( j_2 < j_3 \) and as before we can find \( y_3 \in m_{v_2} \) such that \( v_2 < v_3 \), \( \|y_3\| \leq 2^{-3}\varepsilon \). \( f_{j_3} \) is strictly positive on \( T_{y_3}(X) \) and the set of the successors of \( x \) in \( m_{v_2} \) contains the successors of \( y_1 \), the successors of \( y_2 \), the element \( y_3 \) and at least one extra element. As before we can show that \( \inf_{X_+} \{y_1, y_3\} = \inf_{X_+} \{y_2, y_3\} = 0 \). We put \( s_3 = s_2 + y_3 \). Continuing this process we obtain a sequence \( \{j_\nu\} \) of \( L \) and the sequences \( \{y_\nu\}, \{s_\nu\} \) of \( X_+ \) such that \( s_1 = y_1, s_\nu = s_{\nu - 1} + y_\nu \), for each \( \nu = 2, 3, ..., \) with the following properties:

(i) \( 0 < s_\nu < s_{\nu + 1} < x \),
(ii) \( \|s_{\nu + 1} - s_\nu\| = \|y_{\nu + 1}\| \leq 2^{-\nu - 1}\varepsilon \) and \( y_\nu \in m_{k_\nu} \) with \( k_\nu < k_{\nu + 1} \), for each \( \nu \),
(iii) \( \inf_{X_+} \{s_\nu, x - s_\nu\} = 0 \) for each \( \nu \),
(iv) \( \{j_\nu\} \) is a strictly increasing sequence of \( L \) and for each \( i \in L \) with \( i < j_{\nu + 1} \)
we have $s_\nu(i) > 0$.

By (ii), the sequence $\{s_\nu\}$ is Cauchy and suppose that $s = \lim_{\nu \to \infty} s_\nu$. Then $0 \leq s_\nu \leq s \leq x$, for each $\nu$. Since $\|s_\nu\| \leq \sum_{i=1}^{\nu} \|y_i\| \leq \varepsilon < \|x\|$, we have that $s < x$. Also by (iv) and by the fact that the sequence $\{s_\nu\}$ is increasing we have that $s(i) > 0$ for each $i \in I$, therefore $\text{supp}(s) = \text{supp}(I_+)$. Hence $s$ is a quasi-interior point of $I_+$. We will show that $\inf_{x_+} \{s, x - s\} = 0$. To this end we suppose that $0 \leq h \leq s, x - s$. Since $s = s_\nu + (s - s_\nu)$ we have that $h = h_\nu + h'_\nu$ with $0 \leq h_\nu \leq s_\nu, 0 \leq h'_\nu \leq (s - s_\nu)$. Since the cone is normal and $\lim(s - s_\nu) = 0$ we have that $\lim h'_\nu = 0$, therefore $h = \lim h_\nu$. Since $0 \leq h_\nu \leq s_\nu$ and $h_\nu \leq x - s \leq x - s_\nu$ we have that $h_\nu = 0$ for each $\nu$, by (iii).

Therefore $h = 0$, hence $\inf_{x_+} \{s, x - s\} = 0$. Since $I$ is solid we have also that $\text{interior}_{I_+} \{s, x - s\} = 0$ which contradicts the fact that $s$ is a quasi-interior point of $I_+$, Proposition 5. Hence at least one of the successors $x'$ of $x$ does not belong to a branch of $\delta(u)$, therefore at least one of the successors $x_0$ of $x$ is an extremal point of $X_+$ and the proposition is true.

\textbf{Proposition 31.} Any extremal point $x_0$ of $X_+$ is a positive multiple of a unique element of $\delta(u)$.

\textbf{Proof.} Suppose that $x_0$ is an extremal point of $X_+$. By proposition 5 and by the fact that $x_0$ is an extremal point of $X_+$, a real number $r > 0$ exists with $rx_0 \leq u$. Hence $r \leq a \frac{\|u\|}{\|x_0\|}$, where $a$ is the constant of the normal cone $X_+$. Therefore $\text{sup} \{r \in \mathbb{R}_+ : rx \leq u\} = \lambda \in \mathbb{R}_+$, with $\lambda > 0$. Let $z_0 = \lambda x_0$. Then $0 < z_0 \leq u$. Since $u = \sum z \in m_\nu z$ and the elements of $m_\nu$ are pairwise disjoint, there exists a unique $y_\nu \in m_\nu$ so that $z_0 \leq y_\nu$. Then $\inf_{x_+} \{z_0, x\} = 0$ for each $x \in m_\nu$, $x \neq y_\nu$. Also $y_\nu \geq y_{\nu+1} \geq z_0$ for each $\nu$. Since each branch of $\delta(u)$ converges to zero, the process of decompositions stops at a point $y_\mu$ which is an extremal point of $X_+$ with $z_0 \leq y_\mu$. Hence $y_\mu = \lambda x_0$. Also $y_\mu \leq u$ and by the definition of $\lambda$ we have that $\lambda' \leq \lambda$, therefore $y_\mu \leq z_0$ which implies that $y_\mu = z_0$ and $z_0 \in \delta(u)$. If we suppose that $z_0' = kx_0 \in \delta(u)$, then $kx_0 \leq u$, therefore $k \leq \lambda$ and $z_0' \leq z_0$. Hence $z_0'$ is a successor of $z_0$. If we suppose that $z_0' < z_0$ we get a contradiction because $z_0$ as an extremal point of $X_+$ is undecomposable. Therefore $z_0' = z_0$ and the proposition is true.

In our main result below we prove that $X$ has a positive basis. For the sake of completeness we repeat the standard assumptions for $E$ and $X$. In the next results the positive basis of $X$ is also unconditional because $X_+$ is generating and normal.

\textbf{Theorem 32.} Suppose that $E$ is an ordered Banach space and that $E_+$ is defined by the family $F = \{f_i | i \in \mathbb{N}\}$ of $E_+^*$. Let $X$ be a closed ordered subspace of $E$ with the Riesz decomposition property and suppose that $X_+$ is normal and generating.
If $X$ has the maximum support property and the ws-property with respect to $\mathcal{F}$, then $X$ has a positive basis.

Proof. Let $B$ be the set of extremal points of $X_+$ with norm 1. By proposition 30, $B \neq \emptyset$ and by the previous proposition the map $T : B \to \delta(u)$ so that $T(x) = \lambda x \in \delta(u)$ is one-to-one. Since $\delta(u)$ is countable $B$ is also countable. Suppose that $B = \{u_i : i \in \mathbb{N}\}$ and $b_i = \lambda_i u_i \in \delta(u)$. Let $u_0 = \sum_{i=1}^{\infty} b_i/n_i$. For each $i \in M$ there exists $z \in \delta(u)$ so that $f_i$ is strictly positive on $I = I_z(X)$. By Proposition 30, $z \geq b_j$ for at least one $j$, therefore $b_j(i) > 0$. Hence supp$(u_0) = suppX_+$ and $u_0$ is a quasi-interior point of $X_+$. Let $x \in X_+$. Then there exists an increasing sequence $x_n \in [0,x] \cap [0,k_nu_0]$ where the sequence $k_n$ is strictly increasing with $\lim_{n \to \infty} x_n = x$. Since $0 \leq x_n \leq k_nu_0$, each $x_n$ has a unique expression $x_n = \sum_{i=1}^{\infty} \sigma_{ni} u_i$, with $\sigma_{ni} \in \mathbb{R}_+$, by Proposition 9. The sequence $\{\sigma_{ni} | n \in \mathbb{N}\}$ is increasing. Indeed, for each $m > n$ we take again the expansion $x_m - x_n = \sum_{i=1}^{\infty} a_i u_i$ and we have that $\sigma_{mi} = \sigma_{ni} + a_i \geq \sigma_{ni}$. Suppose that $\sigma_i = \lim_{n \to \infty} \sigma_{ni}$. Then $0 \leq \sigma_i u_i \leq x$ because $0 \leq \sigma_{ni} u_i \leq x$, for each $i$. For each $m \in \mathbb{N}$ we have $\sum_{i=1}^{m} \sigma_{ni} u_i \leq x_n \leq x$ and by taking limits as $n \to +\infty$ we have that $\sum_{i=1}^{m} \sigma_{ni} u_i \leq x$. Since the sequence $\{x_n\}$ converges to $x$ there exists a strictly increasing sequence $m_n$ of natural numbers so that the sequence $y_n = \sum_{i=1}^{m_n} \sigma_{ni} u_i$ converges to $x$. Then $\sum_{i=1}^{m} \sigma_{ni} u_i \leq \sum_{i=1}^{m_n} \sigma_{ni} u_i \leq x$, from where we get that $x = \sum_{i=1}^{\infty} \sigma_{ni} u_i$. Let $\pi_j = \sum_{i \neq j} b_i/n_i$. Then $\pi_j$ is a quasi-interior point of $X_+$, because $\inf_{X_+} \{b_j, \pi_j\} = 0$, Proposition 9. Therefore supp$(\pi_j)$ is a proper subset of supp$(X_+)$, hence there exists $k_j \in M$ with $f_{k_j}(\pi_j) = 0$. Therefore $f_{k_j}(u_i) = 0$, for each $i \neq j$. Also $f_{k_j}(u_j) > 0$ because $f_{k_j}(u_0) > 0$. Let $g_j = f_{k_j}(u_j). Then for each $x \in X_+$ we have $g_j(x) = \sigma_j$, therefore $x = \sum_{i=1}^{\infty} g_i(x) u_i$. Since the cone $X_+$ is generating we have that $x = \sum_{i=1}^{\infty} g_i(x) u_i$ for each $x \in X$ and this expansion is unique. Therefore $\{u_n\}$ is a positive basis of $X$.

By the previous result and Corollary 20 and 21 we have:

Corollary 33. Suppose that $E$ is a Banach lattice with order continuous norm and suppose that $E_+$ is defined by a countable family $\mathcal{F}$ of $E_+^*$. Let $X$ be a closed ordered subspace of $E$ with the Riesz decomposition property and generating positive cone $X_+$. If $X$ has the maximum support property with respect to $\mathcal{F}$, then $X$ has a positive basis.

Corollary 34. Let $E$ be an ordered Banach space whose positive cone is defined by the family $\mathcal{F} = \{f_i | i \in \mathbb{N}\}$ of $E_+^*$. Suppose also that $E$ is a dual space and that the functionals $f_i$ are weak-star continuous. If $X$ is a closed ordered subspace of $E$ with the Riesz decomposition property, $X_+$ is weak-star closed, normal and
generating and $X$ has the maximum support property with respect to $\mathcal{F}$, then $X$ has a positive basis.

**Remark 35.** In the special case where $E = \ell_\infty$ and $X$ is a weak-star closed ordered subspace of $\ell_\infty$ with the RDP and generating positive cone $X_+$ we have: If $X$ has the maximum support property with respect to the family of the Dirac measures $\delta_i$ supported at the natural numbers $i$, then $X$ has a positive basis.

4 Biorthogonal systems

The results of the previous section can be applied to the problem: under what conditions a biorthogonal system defines a positive basis? So in this section we suppose that $E$ is an ordered Banach space with a positive biorthogonal system $\{(e_i, f_i) | i \in \mathbb{N}\}$, i.e. $e_i \in E$ and $f_i \in E_+^*$ for each $i$ so that $f_i(e_i) = 1$, $f_i(e_j) = 0$, for each $j \neq i$ and the family $\mathcal{F} = \{f_i | i \in \mathbb{N}\}$ defines the positive cone of $E$. In the next results the positive basis of $E$ is also unconditional.

**Theorem 36.** Suppose that $E$ is an ordered Banach space with a positive biorthogonal system $\{(e_i, f_i) | i \in \mathbb{N}\}$. If $E_+$ is normal and generating and $E$ has the Riesz decomposition property, the following statements are equivalent:

(i) The sequence $\{e_i\}$ of the biorthogonal system is a positive basis of $E$,
(ii) $E$ has the maximum support property and the ws-property with respect to the family $\mathcal{F} = \{f_i | i \in \mathbb{N}\}$.

**Proof.** Suppose that $\{e_i\}$ is a positive basis of $E$. Since $\{(e_i, f_i)\}$ is a positive biorthogonal system of $E$ we have that $f_i(e_i) = 1$ and $f_i(e_j) = 0$, for each $j \neq i$ therefore, by Theorem 16, $E$ has the maximum support property with respect to $\mathcal{F}$. Since $\{e_i\}$ is a positive basis of $E$, by Corollary 22, $E$ has the ws-property, therefore (i) implies (ii). Suppose now that statement (ii) is true. Then $E$ has a positive basis $\{b_n\}$. Since $E$ has the maximum support property with respect to $\mathcal{F}$, $E$ has the minimal support property, therefore an element $x_0$ of $E_+$ is an extremal point of $E_+$ if and only if $x_0$ has minimal support in $E_+$. Therefore the extremal points of $E_+$ are the positive multiples of the elements $e_n$ (supp$(e_n) = \{n\}$). Since the elements of the positive basis define the extremal rays of $E_+$ we have that the basis $\{b_n\}$ coincides, in the sense of a scalar multiple and a proper enumeration, with the sequence $\{e_n\}$. ■

**Corollary 37.** Suppose that $E$ is an ordered Banach space with a positive biorthogonal system $\{(e_i, f_i) | i \in \mathbb{N}\}$ and suppose also that $E$ has the Riesz decomposition property. If (a) $E$ is a Banach lattice with order continuous norm, or
(b) $E$ is a dual space, the positive cone $E_+\text{ of } E$ is weak-star closed, normal and generating and the functionals $f_i\text{ are weak-star continuous,}$ then the following statements are equivalent:

(i) The sequence $\{e_i\}$ of the biorthogonal system is a positive basis of $E$,

(ii) $E$ has the maximum support property with respect to the family $\{f_i| i \in \mathbb{N}\}$.

**Remark 38.** According to the Corollary, the sequence $\{e_i\}$ of the usual biorthogonal system $\{e_i, \delta_i\}$ of $\ell_\infty$ is not a positive basis of $\ell_\infty$ because it does not have the maximum support property with respect to the family $\{\delta_i\}$, Example 15.

**References**


