Mathematical Finance, Vol. 0, No. 0 (xxxx 2010), 1–16

NONREPLICATION OF OPTIONS

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In this paper, we study the replication of options in security markets X with a finite number of states. Specifically, we prove that in security markets without binary vectors, for any portfolio, at most $m-3$ options can be replicated where m is the number of states. This is an essential improvement of the result of Baptista where it is proved that the set of replicated options is of measure zero. Additionally, we extend the results of Aliprantis and Tourky on the nonreplication of options by generalizing their condition that markets are strongly resolving. Our results are based on the theory of lattice-subspaces and positive bases.

KEY WORDS: replicated options, completion by options, strongly resolving markets, positive bases.

1. INTRODUCTION

The replication of options plays a crucial role in standard option pricing models because the price of a replicated option is equal to the price of the portfolio of primitive securities that replicates the option. In this paper, we examine the problem of replication of options by considering a two-period security market X with a finite number of states, denoted by m, and a finite number of primitive securities (assets) with payoffs in $\mathbb{R}^m$.

The completion $F_1(X)$ of X by options is the subspace of $\mathbb{R}^m$ generated by all options written on the elements of X and as it is natural this subspace is very important for the study of the problem of replication of options. Ross (1976) in his seminal work proved that $F_1(X)$ is the whole space $\mathbb{R}^m$ if and only if X has an efficient fund. Recall that a vector e of $\mathbb{R}^m$ is an efficient fund if e separates the states, i.e., $e(i) \neq e(j)$ for any $i \neq j$. After the paper of Ross, many authors contributed to this problem. Arditti and John (1980) proved that if X has an efficient fund, then almost any portfolio, in the sense of the Lebesgue measure of X, is an efficient fund. John (1981) studied the case where the Ross assumption for the existence of an efficient fund in X is not satisfied and the completion by options of X is a proper subspace of $\mathbb{R}^m$. John defined the notion of the maximally efficient fund and proved that $F_1(X)$ is generated by the call and put options written on a maximally efficient fund. As it is observed in Ross (1976), Green and Jarrow (1987), and

This paper was supported by the research program "K. Karatheodoris" of the National Technical University of Athens.

Manuscript received July 2008; final revision received May 2010.

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DOI: 10.1111/j.1467-9965.2010.00467.x

Brown and Ross (1991), any call and put option written on elements of X is replicated, if and only if X is a sublattice of $\mathbb{R}^m$.

Kountzakis and Polyrakis (2006) solved completely the problem of the determination of the completion by options of X by giving a method which determines a positive basis of $F_{\mathbf{1}}(X)$. This method, which is also presented in Section 3, is actually based on the theory of lattice subspaces and positive bases developed by Polyrakis (1996, 1999).

In Bajeux-Besnainou and Rochet (1996) and also in a more general framework in Baptista (2005), the results of Ross (1976) are generalized in multiperiod markets. In Galvani (2009) the results of Ross are studied in L_p spaces. In Detemple and Selden (1991), an equilibrium price analysis on a financial market in which investors trade a primitive security and options written on this security is provided. In this economy any nontrivial option is nonreplicated and it is shown that there is an interaction between the prices of the stock and the different exercise prices of the option. Nachman (1988) studies the completion of primitive security markets with options in the case where the set of states is infinite.

Aliprantis and Tourky (2002) proved that in any strongly resolving security market with $n \leq \frac{m+1}{2}$ where n is the number of primitive securities and m the number of states, any nontrivial option is nonreplicated. In Baptista (2007), the replication of options is studied in the case where the asset span X does not contain binary vectors. Baptista proves that for any $x \in X$ the set of nonreplicated exercise prices of x is a subset of the set K_x of the nontrivial exercise prices of x of full measure, or equivalently, for any $x \in X$ the set of nontrivial, replicated exercise prices of x is a subset of K_x of measure zero. Also as it is remarked in Baptista (2007), the class of markets without binary vectors is dense in $\mathbb{R}^m$, in the sense of the Lebesgue measure.

In this paper, we continue the study of Baptista (2007) and Aliprantis and Tourky (2002). First, we prove that if $\mathbf{1} \in X$, then X does not contain binary vectors if and only if for any nonconstant vector $x \in X$ at least one nontrivial option of x is nonreplicated. After this characterization of the markets without binary vectors we prove, Theorem 4.3, that in these markets, for any $x \in X$ the set K_x contains at most $k - 3$ replicated exercise prices where k is a real number $k \leq m$. Also we determine a partition of K_x consisting of $k - 3$ intervals, each of which contains at most one replicated exercise price. Since there are subsets of K_x of measure zero with infinite many elements, our theorem is an essential improvement of the result of Baptista. In the proof of this theorem, the idea that any of the above subintervals of K_x cannot contain two different replicated exercise prices is from the corresponding proof of Baptista and this is important for our proof.

In the sequel we generalize the definition and the results of strongly resolving markets of Aliprantis and Tourky (2002). Specifically, we consider the payoff matrix of primitive securities x_i with respect to the positive basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ and we define the notion of strongly resolving markets with respect to the basis $\{b_i\}$. As it is shown in Example 5.5, this new class of markets is strictly bigger than the one of strongly resolving markets. In Theorem 5.3, we extend the result of Aliprantis and Tourky (2002) for strongly resolving markets with respect to the basis $\{b_i\}$. Our proof is analogous to the excellent proof of Aliprantis–Tourky.

Finally note that, although in markets without binary vectors we do not make any assumption concerning the number of primitive securities, in strongly resolving markets we have a restrictive assumption about the number of primitive securities. Also in markets without binary vectors the number of nontrivial replicated options is at most finite, but in markets without binary vectors which satisfies the restriction on the number of primitive securities, the market replicates only trivial options.

2. THE MODEL

In this paper we study a two-period security market with a finite number of states $\Omega = \{1, 2, \ldots, m\}$ during the date 1, a finite number of primitive securities (assets) with payoffs given by the linearly independent vectors $x_1, x_2, \ldots, x_n$ of the payoff space $\mathbb{R}^m$.

A **portfolio** is a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ of $\mathbb{R}^n$ where θ_i is the number of units of security i. Then $T(\theta) = \sum_{i=1}^n \theta_i x_i \in \mathbb{R}^m$ is the **payoff of** θ. Since the operator T is one-to-one, it identifies portfolios with their payoffs. So the vectors $x_1, x_2, \ldots, x_n$ will be mentioned as **primitive securities**, the subspace

$$X = [x_1, x_2, \ldots, x_n],$$

of $\mathbb{R}^m$, generated by the vectors x_i as the **space of marketed securities** or the **asset span** and the vectors of X will be also referred as portfolios. A vector $x \in \mathbb{R}^m$ **is marketed** or x **is replicated** if it is the payoff of some portfolio θ, or equivalently if $x \in X$.

Recall that the vector space $\mathbb{R}^m = \{x = (x(1), x(2), \ldots, x(m)) | x(i) \in \mathbb{R} \text{ for each } i\}$, is ordered by the pointwise ordering, i.e., for any $x, y \in \mathbb{R}^m$ we have: $x \geq y$ if $x(i) \geq y(i)$ for each i. $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x(i) \geq 0 \text{ for each } i\}$ is the positive cone of $\mathbb{R}^m$. For any $x, y \in \mathbb{R}^m x \vee y = \big(x(1) \vee y(1), x(2) \vee y(2), \ldots, x(m) \vee y(m)\big)$ is the supremum and $x \wedge y = \big(x(1) \wedge y(1), x(2) \wedge y(2), \ldots, x(m) \wedge y(m)\big)$ is the infimum of $\{x, y\}$ in $\mathbb{R}^m$. $x^+ = x \vee 0 = \big(x(1) \vee 0, x(2) \vee 0, \ldots, x(m) \vee 0\big)$ and $x^- = (-x) \vee 0$ are the positive and the negative part of x. Note also that for any two real numbers $a, b, a \vee b$ is the supremum and $a \wedge b$ is the infimum of $\{a, b\}$. A linear subspace Z of $\mathbb{R}^m$ is a **sublattice** or a **Riesz subspace** of $\mathbb{R}^m$ if for any $x, y \in Z$, $x \vee y$ and $x \wedge y$ belong to Z. Also for any $x = (x(1), x(2), \ldots, x(m)) \in \mathbb{R}^m$, the set $supp(x) = \{i = 1, 2, \ldots, m | x(i) \neq 0\}$ is the **support** of x.

For any subset B of $\mathbb{R}^m$, the **sublattice** $S(B)$ of $\mathbb{R}^m$ generated by B is the intersection of the sublattices of $\mathbb{R}^m$ which contain B. The **riskless bond 1** is the vector of $\mathbb{R}^m$ whose every coordinate is equal to 1. Suppose that $x \in \mathbb{R}^m$ and $a \in \mathbb{R}$. The **call option** written on the vector $x \in \mathbb{R}^m$ with exercise price a is the vector $c(x, a) = (x - a\mathbf{1})^+$ of $\mathbb{R}^m$. The **put option** written on the vector $x \in \mathbb{R}^m$ with exercise price a is the vector $p(x, a) = (a\mathbf{1} - x)^+$. We have the identity $x - a\mathbf{1} = c(x, a) - p(x, a)$, which is called **put-call parity**.

If both $c(x, a) > 0$ and $p(x, a) > 0$, we say that call option $c(x, a)$ and put option $p(x, a)$ are nontrivial. In this case we say that a is a **nontrivial exercise price** of x. We denote by K_x the set of nontrivial exercise prices of x. If $c(x, a) \in X$ we say that a is a **call-replicated** exercise price of x and if $p(x, a) \in X$, we say that a is a **put-replicated** exercise price of x. If both $c(x, a)$, $p(x, a)$ are in X we say that a is a **replicated** exercise price of x. If $\mathbf{1} \in X$, we have: $c(x, a) \in X$ if and only if $p(x, a) \in X$. If the riskless bond is not contained in X it is possible only one of the call and put options to be replicated. In this paper, we do not suppose always that the riskless bond $\mathbf{1}$ belongs to X. Of course $\mathbf{1}$ belongs to the completion by options of X.

The **completion by options** of X is the subspace of $\mathbb{R}^m$ which arises inductively by adding in the market the call and put options of the marketed securities and by taking again call and put options which are added again in the market. In Kountzakis and Polyrakis (2006) a mathematical definition of the completion by options in infinite securities markets is given. Specifically in the above article, a more general study of the completion by options of the market is presented by Kountzakis and Polyrakis (2006) where the options are not taken with respect to the riskless bond $\mathbf{1}$ but with respect to risky vectors from a standard subspace U of $\mathbb{R}^m$ and the completion by options of X is denoted by $F_U(X)$. This study in Kountzakis and Polyrakis (2006) is very general and includes the case of exotic options.

In the classical case where the options are taken with respect to the riskless bond $\mathbf{1}$, the completion by options of X is denoted in Kountzakis and Polyrakis (2006) by $F_{\mathbf{1}}(X)$ and we will preserve this notation in this paper. In the above article it is proved that if the payoff space is a general vector lattice E then $F_U(X)$ is the sublattice of E generated by the set $X \cup U$. In our case where the payoff space is the space $\mathbb{R}^m$ and the call and put options are taken with respect to the riskless bond $\mathbf{1}$, the completion by options $F_{\mathbf{1}}(X)$ of X is the sublattice of $\mathbb{R}^m$ generated by the set $X \cup \{\mathbf{1}\}$. In the case where $\mathbf{1} \in X$, $F_{\mathbf{1}}(X)$ is the sublattice of $\mathbb{R}^m$ generated by the set X.

For more details on lattice-subspaces and positive bases see the Appendix. For an introduction to two-period security markets we refer to LeRoy and Werner (2001) and Lengwiler (2004).

3. DETERMINATION OF THE COMPLETION $F_{\mathbf{1}}(X)$ OF X

In this section we describe the method of determination of the completion by options of X as it is presented in Kountzakis and Polyrakis (2006). According to this method we consider the set

$$\mathcal{A} = \{x_1^+, x_1^-, x_2^+, x_2^-, \dots, x_n^+, x_n^-, \mathbf{1}\}.$$

Any maximal subset $\{y_1, y_2, \dots, y_r\}$ of linearly independent vectors of $\mathcal{A}$ is a **basic set** of the market. Note that a basic set is not necessarily unique. In general it is possible to find different basic sets of the market but all these sets have the same cardinal number r. Specifically r is the dimension of the linear subspace of $\mathbb{R}^m$ generated by $\mathcal{A}$ and a basic set is a basis of it.

THEOREM 3.1 (Kountzakis and Polyrakis 2006, theorem 11). *$F_{\mathbf{1}}(X)$ is the sublattice of $\mathbb{R}^m$ generated by a basic set $\{y_1, y_2, \dots, y_r\}$ of the market.*

After this result we use the theory of lattice-subspaces and positive bases developed by Polyrakis (1996, 1999) for the determination of $F_{\mathbf{1}}(X)$. Since $F_{\mathbf{1}}(X)$ is a sublattice of $\mathbb{R}^m$ which contains $\mathbf{1}$, we have that $F_{\mathbf{1}}(X)$ has a positive basis $\{b_1, b_2, \dots, b_\mu\}$ which is a **partition of the unit**, i.e., the vectors b_i have disjoint supports and $\sum_{i=1}^{\mu} b_i = \mathbf{1}$, see Theorem A.2 of the Appendix. This basis is unique. So we have:

THEOREM 3.2. *$F_{\mathbf{1}}(X)$ has a positive basis $\{b_1, b_2, \dots, b_\mu\}$ which is a partition of the unit.*

For the determination of the positive basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ which is a partition of the unit we follow the steps of Polyrakis algorithm, see Theorem A.5 in the Appendix, where a positive basis of the sublattice of $\mathbb{R}^m$ generated by a finite set of positive and linearly independent vectors is determined. We start by the determination of a basic set $\{y_1, y_2, \dots, y_r\}$ of the market. In the sequel we determine the **basic function** of $y_1, y_2, \dots, y_r$ which is very important for the theory of lattice-subspaces and positive bases. This function has been defined by Polyrakis (1996) and is the following:

$$\beta(i) = \left(\frac{y_1(i)}{y(i)}, \frac{y_2(i)}{y(i)}, \dots, \frac{y_r(i)}{y(i)}\right), \quad \text{for each } i = 1, 2, \dots, m, \quad \text{with } y(i) > 0,$$

where $y = y_1 + y_2 + \cdots + y_r$. This function takes values in the simplex $\Delta_r = \{\xi \in \mathbb{R}_+^r \mid \sum_{i=1}^{r} \xi_i = 1\}$ of $\mathbb{R}_+^r$.

Denote by $R(\beta)$ the range (i.e., the set of values) of β and by $card\,R(\beta)$ the cardinal number of $R(\beta)$ (i.e., the number of the different values of β).

We continue the algorithm and we obtain a positive basis $\{d_1, d_2, \ldots, d_\mu\}$ of $F_1(X)$. The elements of this basis have disjoint supports and each d_i is constant on its support. By a normalization of the basis $\{d_i\}$ we obtain the positive basis $\{b_i\}$ of $F_1(X)$ which is a partition of the unit. This basis is very important for this paper.

By using Theorem A.5, the dimension of $F_1(X)$ is equal to the cardinal number of $R(\beta)$. So if $R(\beta)$ has n elements, then $F_1(X) = X$ and any option is replicated. If $R(\beta)$ has m elements, then $F_1(X) = \mathbb{R}^m$ and the options fill the whole space $\mathbb{R}^m$. This is expressed in the next result.

THEOREM 3.3. *The dimension of $F_1(X)$ is equal to the cardinal number of the range $R(\beta)$, therefore we have:*

(i) $F_1(X) = X$ *if and only if* $card\,R(\beta) = n$,
(ii) $F_1(X) = \mathbb{R}^m$ *if and only if* $card\,R(\beta) = m$,
(iii) $F_1(X) \subsetneq \mathbb{R}^m$ *if and only if* $card\,R(\beta) < m$.

4. MARKETS WITHOUT BINARY VECTORS

Throughout this paper we will denote by $\{b_1, b_2, \ldots, b_\mu\}$, or for simplicity by $\{b_i\}$, the positive basis of $F_1(X)$ which is a partition of the unit. For any $x = \sum_{i=1}^{\mu} \lambda_i b_i \in F_1(X)$, $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ are the coefficients of x in the basis $\{b_i\}$. We put

$$a_1 = min\{\lambda_i | i = 1, 2, \ldots, \mu\} \text{ and } \Phi_1 = \{i | \lambda_i = a_1\},$$

$$a_2 = min\{\lambda_i | \lambda_i > a_1\} \text{ and } \Phi_2 = \{i | \lambda_i = a_2\},$$

and by continuing this process we take the real numbers $a_1, a_2, \ldots, a_k$ and the subsets $\Phi_1, \Phi_2, \ldots, \Phi_k$ of $\{1, 2, \ldots, \mu\}$. The numbers $a_1, a_2, \ldots, a_k$ will be referred as the **essential coefficients** and the sets $\Phi_1, \Phi_2, \ldots, \Phi_k$ as the **essential sets of states** of x, with respect to the basis $\{b_i\}$.

The essential coefficients are in increasing order, i.e., $a_i < a_j$ for any $i < j$. Of course for the number k of the essential coefficients of x we have: $k \le \mu \le m$.

If for example $\mu = 5$ and $x = 2b_1 - 3b_2 + 2b_3 + b_4 + b_5$, then $a_1 = -3, a_2 = 1, a_3 = 2$ are the essential coefficients and $\Phi_1 = \{2\}, \Phi_2 = \{4, 5\}, \Phi_3 = \{1, 3\}$ the essential sets of states of x. We have $x = a_1 \sum_{i\in\Phi_1} b_i + a_2 \sum_{i\in\Phi_2} b_i + a_3 \sum_{i\in\Phi_1} b_i$.

PROPOSITION 4.1. *For any $x = \sum_{i=1}^{\mu} \lambda_i b_i \in F_1(X)$ we have:*

(i) $c(x, a) = \sum_{i=1}^{\mu} (\lambda_i - a)^+ b_i$ and $p(x, a) = \sum_{i=1}^{\mu} (a - \lambda_i)^+ b_i$,
(ii) if $a_1, a_2, \ldots, a_k$ are the essential coefficients of x, the interval $K_x = (a_1,\ a_k)$ is the set of nontrivial exercise prices of x.

Proof. (i): The basis $\{b_i\}$ is a partition of the unit, therefore the vectors b_i have disjoint supports and $\sum_{i=1}^{\mu} b_i = \mathbf{1}$. Therefore $b_i(j) = 1$ for any $j \in supp(b_i)$. So we have $c(x, a) = (x - a\mathbf{1})^+ = (\sum_{i=1}^{\mu} \lambda_i b_i - a \sum_{i=1}^{\mu} b_i)^+ = (\sum_{i=1}^{\mu} (\lambda_i - a) b_i)^+$. Since the basis $\{b_i\}$ is a partition of the unit, for any $j \in supp(b_i)$ we have that the j-coordinate of $c(x, a)$ in the usual basis $\{e_i\}$ of $\mathbb{R}^m$ is $(\lambda_j - a)^+$, therefore it is easy to show that $c(x, a) = \sum_{i=1}^{\mu} (\lambda_i - a)^+ b_i$. The proof for the put option is analogous.

(ii): If $a \leq a_1$ then $(a - \lambda_i)^+ = 0$ for any i, therefore $p(x, a) = 0$ because a_1 is the minimum of the coefficients λ_i of x. Therefore a is a trivial exercise price of x. If $a \geq a_k$, similarly we have that $(\lambda_i - a)^+ = 0$ for any i, therefore $c(x, a) = 0$ and a is a trivial exercise price. For any $a \in (a_1,\ a_k)$ we have that $(\lambda_i - a)^+ > 0$ for at least one i and also $(a - \lambda_j)^+ > 0$ for at least one j. Since the elements b_i of the basis are positive we have that $c(x, a) > 0$ and $p(x, a) > 0$, hence a is a nontrivial exercise price of x. □

We give below a characterization of the markets without binary vectors. We say that a vector $x \in \mathbb{R}^m$ is a **nonconstant vector** if x is not a multiple of $\mathbf{1}$, i.e., $x \neq \lambda\mathbf{1}$ for any $\lambda \in \mathbb{R}$. According to this definition, 0 is a constant vector of $\mathbb{R}^m$.

THEOREM 4.2. *If $\mathbf{1} \in X$, we have: X does not contain binary vectors if and only if for any nonconstant vector $x \in X$ at least one nontrivial option of x is nonreplicated.*

Proof. Suppose that X does not contain binary vectors. Suppose also that there exists a nonconstant vector $x \in X$ so that $c(x, \alpha) \in X$ for each $\alpha \in K_x$. Also for any $\alpha \notin K_x$ we have that $c(x, \alpha) = 0$ or $p(x, \alpha) = 0$, therefore $c(x, \alpha)$ and $p(x, \alpha)$ are elements of X because $x - \alpha\mathbf{1} = c(x, \alpha) - p(x, \alpha)$. Therefore if $L = [x]$ is the one-dimensional subspace generated by x, then by Kountzakis and Polyrakis (2006, theorem 21), the completion by options $F_{\mathbf{1}}(L)$ of L is the subspace generated by the set of call options written on the elements of the subspace Y of $\mathbb{R}^m$ generated by the set $L \cup \{\mathbf{1}\}$. Each vector y of Y is of the form $y = \lambda x + \xi\mathbf{1}$ therefore $c(y, a) = (y - a\mathbf{1})^+ = (\lambda x - (a - \xi)\mathbf{1})^+ = c(\lambda x, (a - \xi))$. So we have that any call option written on an element of Y is a call option written on an element of L, therefore it belongs to X as we have remarked before. So we have that $F_{\mathbf{1}}(L) \subseteq X$. Since $x, \mathbf{1} \in F_{\mathbf{1}}(L)$ we have that $F_{\mathbf{1}}(L)$ is an at least two-dimensional sublattice, therefore $F_{\mathbf{1}}(L)$ has a positive basis which is also a partition of the unit. The elements of this basis are binary vectors, and these elements belong to X, contradiction. So for any $x \in X$ at least one nontrivial option of x is nonreplicated.

For the converse suppose that for any nonconstant vector $x \in X$ at least one nontrivial option of x is nonreplicated. If we suppose that x is a binary vector of X, then it is easy to show that the essential coefficients of x in the basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ are $a_1 = 0, a_2 = 1$, therefore $K_x = (0, 1)$ and $x = \sum_{i \in \Phi_2} b_i \in X$. For any $\alpha \in (0, 1)$ we have that $c(x, \alpha) = (1 - \alpha)x \in X$ which is a contradiction. Therefore, X does not contain binary vectors. □

THEOREM 4.3. *Suppose that the asset span X does not contain binary vectors and x is a nonconstant vector of X. If $a_1, a_2, \ldots, a_k$ are the essential coefficients of x with respect to the basis $\{b_i\}$, then:*

(i) If $k = 2$, each nontrivial call option of x is nonreplicated. If $k > 2$, each of the intervals $(a_1, a_2), [a_2, a_3), \ldots, [a_{k-2}, a_{k-1})$ contains at most one call-replicated exercise price, therefore there are at most $k - 2$ call-replicated exercise prices of x.

(ii) If $k = 2$, each nontrivial put option of x is nonreplicated. If $k > 2$, each of the intervals $(a_2, a_3], \ldots, (a_{k-2}, a_{k-1}], (a_{k-1}, a_k)$ contains at most one put-replicated exercise price, therefore there are at most $k - 2$ put-replicated exercise prices of x.

(iii) If we suppose moreover that $\mathbf{1} \in X$, we have: If $k = 3$, each nontrivial option of x is nonreplicated. If $k > 3$, each of the intervals $(a_2, a_3), [a_3, a_4), \ldots, [a_{k-2}, a_{k-1})$ contains at most one replicated exercise price, therefore there are at most $k - 3$ replicated exercise prices of x.

Proof. Suppose that $x \in X$, $x \neq \lambda\mathbf{1}$, $x = \sum_{i=1}^{\mu} \lambda_i b_i$ is the expansion of x in the basis $\{b_i\}$ and suppose that $a_1, a_2, \ldots, a_k$ are the essential coefficients and $\Phi_1, \Phi_2, \ldots, \Phi_k$ are the essential sets of states of x with respect to the basis $\{b_i\}$. Since we have supposed that x is not a multiple of the riskless bond we have that x has at least two essential coefficients, hence $k \geq 2$.

Then

$$x = a_1 \sum_{i \in \Phi_1} b_i + a_2 \sum_{i \in \Phi_2} b_i + \ldots + a_k \sum_{i \in \Phi_k} b_i. \tag{4.1}$$

We put $\overline{b}_j = \sum_{i \in \Phi_j} b_i$, $j = 1, 2, \ldots, k$ and we remark that every such vector is a binary vector. Also, we have

$$x = \sum_{j=1}^{k} a_j \overline{b}_j.$$

The set of nontrivial exercise prices of x is the interval $K_x = (a_1,\ a_k)$. For any $a \in (a_1, a_k)$ we have

$$c(x, a) = \sum_{j=r+1}^{k} (a_j - a)\overline{b}_j,$$

where $r = 1$ if $a \in (a_1, a_2)$ and $r = \nu$ if $a \in [a_\nu, a_{\nu+1})$ for $\nu = 2, 3, \ldots, k-1$.

If $a \in [a_{k-1}, a_k)$ then $c(x, a) = (a_k - a)\overline{b}_k$ is a positive multiple of a binary vector, therefore $c(x, a) \notin X$. So for any $a \in [a_{k-1}, a_k)$, $c(x, a)$ is nonreplicated. This means also that if $k = 2$, i.e., if $a_1,\ a_2$ are the essential coefficients of x, then any call option of x is nonreplicated.

Suppose now that a, a' are different exercise prices belonging to the same subinterval of $(a_1,\ a_k)$, i.e., $a, a' \in (a_1, a_2)$ or $a, a' \in [a_r,\ a_{r+1})$ for some $r = 2, 3, \ldots, k-2$. Then we have

$$c(x, a) - c(x, a') = \sum_{j=r+1}^{k} ((a_j - a) - (a_j - a'))\overline{b}_j = (a' - a) \sum_{j=r+1}^{k} \overline{b}_j.$$

If we suppose that $c(x, a)$ and $c(x, a')$ belong to X we have that

$$(a' - a) \sum_{j=r+1}^{k} \overline{b}_j \in X,$$

which is a contradiction because $\sum_{j=r+1}^{k} \overline{b}_j$ is a binary vector. This implies that at most one of $c(x, a),\ c(x, a')$ belongs to X.

So any of the subintervals $(a_1, a_2), [a_2, a_3), \ldots, [a_{k-2}, a_{k-1})$ of $(a_1,\ a_k)$ contains at most one call-replicated exercise price, therefore there are at most $k-2$ call-replicated exercise prices and statement (i) is true. The proof of statement (ii) is analogous.

If we suppose that $\mathbf{1} \in X$ and at least one of $c(x, a)$, $p(x, a)$ is replicated, then both of them are replicated by put-call parity, therefore an exercise price a is call-replicated if and only if a is put-replicated. So, if $\mathbf{1} \in X$, by (i) and (ii) we have that there are at most $k-3$ replicated exercise prices because strike prices in the intervals $(a_1, a_2]$ and $[a_{k-1},\ a_k)$ are excluded. So in the case where $\mathbf{1} \in X$, each interval $(a_2,\ a_3), [a_3,\ a_4), \ldots, [a_{k-2},\ a_{k-1})$ has at most one replicated exercise price for x, therefore there are at most $k-3$ replicated exercise prices. $\square$

If the dimension of $F_{\mathbf{1}}(X)$ is at most three, then the basis $\{b_i\}$ of X has at most three elements and for any $x \in X$ the essential coefficients of x are at most three real numbers a_1, a_2, a_3 and the next corollary is obvious:

COROLLARY 4.4. *Suppose that the asset span X does not contain binary vectors.*

(i) If $dim F_{\mathbf{1}}(X) = 2$, any nontrivial option written on some element of X is nonreplicated.
(ii) If $dim F_{\mathbf{1}}(X) = 3$ and $\mathbf{1} \in X$, any nontrivial option written on some element of X is nonreplicated.

The next example is an application in a three-dimensional subspace X of $\mathbb{R}^8$ without binary vectors. We determine the completion $F_{\mathbf{1}}(X)$ and a positive basis $\{b_1, b_2, b_3, b_4\}$ which is a partition of the unit. We find a vector $x \in X$ with four essential coefficients ($k = 4$) and one replicated exercise price. So we have that the estimation $k - 3$ of the possibly replicated exercise prices in Theorem 4.3 cannot be improved.

EXAMPLE 4.5. Suppose that $x_1 = (1, 1, 2, 2, 0, 0, 0, 0)$, $x_2 = (0, 0, 0, 0, 3, 3, 4, 4)$, $x_3 = (1, 1, 1, 1, 1, 1, 1, 1)$ are the primitive securities and $X = [x_1, x_2, x_3]$ is the marketed space. It is easy to show that X does not contain binary vectors.

According to the methodology of the determination of $F_{\mathbf{1}}(X)$ we start by the determination of a basic set and we find that $\{y_1, y_2, y_3\} = \{x_1, x_2, x_3\}$ is a basic set of the market. In order to determine a positive basis of $F_{\mathbf{1}}(X)$ we follow the algorithm of Theorem A.5. So we determine the basic function β of y_1, y_2, y_3, $\beta = \frac{1}{y}(y_1, y_2, y_3)$, where y is the sum of y_i and we find that

$$\beta(1) = \beta(2) = \frac{1}{2}(1, 0, 1) = P_1, \beta(3) = \beta(4) = \frac{1}{3}(2, 0, 1) = P_2$$
$$\beta(5) = \beta(6) = \frac{1}{4}(0, 3, 1) = P_3, \beta(7) = \beta(8) = \frac{1}{5}(0, 4, 1) = P_4.$$

So we have that $card(R(\beta)) = 4$ therefore the completion $F_{\mathbf{1}}(X)$ is a four-dimensional sublattice of $\mathbb{R}^8$.

The three first vectors P_1, P_2, P_3 of $R(\beta)$ are linearly independent, so we preserve the enumeration of $R(\beta)$. According to the algorithm, $I_4 = \beta^{-1}(P_4) = \{7, 8\}$ and we define the new vector $y_4 = (0, 0, 0, 0, 0, 0, 5, 5)$. We determine the basic function $\gamma = \frac{1}{y'}(y_1, y_2, y_3, y_4)$ where y' is the sum of these vectors. We find that

$$\gamma(1) = \gamma(2) = \frac{1}{2}(1, 0, 1, 0) = P_1', \gamma(3) = \gamma(4) = \frac{1}{3}(2, 0, 1, 0) = P_2'$$
$$\gamma(5) = \gamma(6) = \frac{1}{4}(0, 3, 1, 0) = P_3', \gamma(7) = \gamma(8) = \frac{1}{10}(0, 4, 1, 5) = P_4'.$$

A positive basis of $F_{\mathbf{1}}(X)$ is given by the formula $(d_1, d_2, d_3, d_4)^T = A^{-1}(y_1, y_2, y_3, y_4)^T$ where A is the matrix whose columns are the vectors $P_i', i = 1, \ldots, 4$. We find that the vectors

$$d_1 = (2, 2, 0, 0, 0, 0, 0, 0), d_2 = (0, 0, 3, 3, 0, 0, 0, 0),$$
$$d_3 = (0, 0, 0, 0, 4, 4, 0, 0), d_4 = (0, 0, 0, 0, 0, 0, 10, 10),$$

define a positive basis of $F_{\mathbf{1}}(X)$. By a normalization of this basis we have that the vectors

$$b_1 = (1,1,0,0,0,0,0,0), b_2 = (0,0,1,1,0,0,0,0),$$
$$b_3 = (0,0,0,0,1,1,0,0), b_4 = (0,0,0,0,0,0,1,1),$$

define the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit.

Consider the portfolio $x = -x_1 + x_2 = (-1,-1,-2,-2,3,3,4,4)$. The expansion of x in the basis $\{b_i\}$ is $x = -b_1 - 2b_2 + 3b_3 + 4b_4$ and according to the above theorem $a_1 = -2, a_2 = -1, a_3 = 3, a_4 = 4$ are the essential coefficients of x. For any $\alpha \in (-2,-1]$ or $\alpha \in [3,4)$, any option is nonreplicated. By statement (iii) of Theorem 4.2, x has at most one replicated exercise price $\alpha \in (-1,3)$. If we suppose that $a \in (-1,3)$ is a replicated exercise price, we have

$$c(x,\alpha) = (3-\alpha)b_3 + (4-\alpha)b_4 \in X.$$

Then

$$c(x,\alpha) = \rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 = \rho_1(b_1 + 2b_2) + \rho_2(3b_3 + 4b_4) + \rho_3(b_1 + b_2 + b_3 + b_4).$$

We find that $\rho_1 = \rho_3 = 0, 3\rho_2 = 3-\alpha, 4\rho_2 = 4-\alpha$, therefore $\alpha = 0$. Indeed, $c(x,0) = 3b_3 + 4b_4 = x_2 \in X$ and $p(x,0) = b_1 + 2b_2 = x_1 \in X$.

5. STRONGLY RESOLVING MARKETS

The replication of options in strongly resolving markets has been studied by Aliprantis and Tourky (2002). If we expand the vectors x_i in the usual basis $\{e_1,\dots,e_m\}$ of $\mathbb{R}^m$ we have the $m \times n$ matrix

$$A(x_i,e_i) = \begin{bmatrix} x_1(1) & x_2(1) & \dots & x_n(1) \\ x_1(2) & x_2(2) & \dots & x_n(2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_1(m) & x_2(m) & \dots & x_n(m) \end{bmatrix},$$

which is the **payoff matrix** of vectors x_i. The notion of a strongly resolving market is defined in the above article as follows. If any $n \times n$ submatrix of $A(x_i,e_i)$ is nonsingular, the asset span $X = [x_1,\dots,x_n]$ (or the market) is called strongly resolving. As we have noted in the introduction, Aliprantis and Tourky proved that if $\mathbf{1} \in X, n \le \frac{m+1}{2}$ and the asset span is strongly resolving, any nontrivial option written on some element of X is nonreplicated.

In this paper, we extend the definition of strongly resolving markets by taking the payoff matrix of the payoff vectors in the basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$. So if $\{b_1,\dots,b_\mu\}$ is the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit, we expand each x_i in this basis and suppose that $x_i = \sum_{j=1}^{\mu} x_i^b(j)b_j$. The $\mu \times n$ matrix

$$A(x_i, b_i) = \begin{bmatrix} x_1^b(1) & x_2^b(1) & \dots & x_n^b(1) \\ x_1^b(2) & x_2^b(2) & \dots & x_n^b(2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_1^b(\mu) & x_2^b(\mu) & \dots & x_n^b(\mu) \end{bmatrix},$$

is the payoff matrix of the basic securities x_i in the basis $\{b_i\}$.

DEFINITION 5.1. If any $n \times n$ submatrix of $A(x_i, b_i)$ is nonsingular, the market X is strongly resolving with respect to the basis $\{b_i\}$.

In the next theorem we prove that if $F_1(X) \neq \mathbb{R}^m$, the market cannot be strongly resolving. So if the market is strongly resolving, then $F_1(X) = \mathbb{R}^m$ and the two definitions coincide because the basis $\{b_i\}$ of $F_1(X)$ is the usual basis $\{e_i\}$ of $\mathbb{R}^m$ and therefore $A(x_i, b_i) = A(x_i, e_i)$. As we show in Example 5.5, it is possible for a market to be strongly resolving with respect to the basis $\{b_i\}$ but not strongly resolving. Therefore, our notion of a market being strongly resolving with respect to a basis generalizes the notion of a market being strongly resolving of Aliprantis and Tourky (2002).

THEOREM 5.2. *If $n \geq 2$ and the completion by options $F_1(X)$ of X is a proper subspace of $\mathbb{R}^m$, then the market is not strongly resolving.*

Proof. The assumption that $F_1(X) \neq \mathbb{R}^m$, implies $\mu < m$, where $\{b_1, b_2, \dots, b_\mu\}$ is the positive basis of $F_1(X)$ which is also a partition of the unit. Since $\mu < m$, the support of at least one of the elements of the basis is not a singleton. So we may suppose that $i_1, i_2 \in supp(b_r)$ for some r. For any x_i we have $x_i = \sum_{j=1}^{\mu} x_i^b(j)b_j$, therefore $x_i(i_1) = x_i^b(r)b_r(i_1) = x_i^b(r)$ because $b_r(i_1) = 1$. Similarly $x_i(i_2) = x_i^b(r)$, therefore for any vector x_i we have $x_i(i_1) = x_i(i_2)$. This implies that the i_1 and i_2-row of the matrix $A(x_i,\ e_i)$ coincide and the theorem is true. □

For any $x \in F_1(X)$ we expand x in the basis $\{b_i\}$ of $F_1(X)$ which is a partition of the unit and suppose that $x = \sum_{i=1}^{\mu} \lambda_i b_i$. Then $supp_b(x) = \{i | \lambda_i \neq 0\}$ is the support of x and $zeros_b(x) = \{i | \lambda_i = 0\}$ is the set of zeros of x with respect to the basis $\{b_i\}$. Also $\#supp_b(x)$ and $\#zeros_b(x)$ is the cardinal number of the sets $supp_b(x)$ and $zeros_b(x)$.

THEOREM 5.3. *Suppose that the riskless bond* **1** *is contained in X. If the market is strongly resolving with respect to the basis $\{b_i\}$ and $n \leq \frac{\mu+1}{2}$ then any nontrivial option written on some element of X is nonreplicated.*

Proof. Let $x = \sum_{i=1}^{\mu} \lambda_i b_i \in X$ and suppose that $y = c(x, \alpha) = \sum_{i=1}^{\mu} (\lambda_i - \alpha)^+ b_i$ is a nontrivial call option. Then $y > 0$ and also the corresponding put option $z = p(x, \alpha) = \sum_{i=1}^{\mu} (\alpha - \lambda_i)^+ b_i$ is also greater than zero, $z > 0$. Let

$$\#supp_b(y) = \beta, \#zeros_b(y) = \gamma, \#supp_b(z) = \beta', \#zeros_b(z) = \gamma'.$$

We shall show that

$$max\{\gamma, \gamma'\} \geq \frac{\mu}{2}.$$

It is clear that $i \in supp_b(y) \Rightarrow i \in zeros_b(z)$ and $i \in supp_b(z) \Rightarrow i \in zeros_b(y)$, therefore $\gamma' \geq \beta$ and $\gamma \geq \beta'$.

Also $\beta + \gamma = \beta' + \gamma' = \mu$. If $\beta \geq \gamma$ then $\beta \geq \frac{\mu}{2}$, therefore $\gamma' \geq \frac{\mu}{2}$. If $\gamma \geq \beta$ then $\gamma \geq \frac{\mu}{2}$ and the assertion is true. Since the risklesss bond belongs to X we have that both y, z are replicated or not. Suppose that y, z are replicated. Then as we have proved above at least one of them, for example the call option y has a number of zero coordinates in the basis $\{b_i\}$ greater or equal to $\frac{\mu}{2}$, i.e., $\gamma \geq \frac{\mu}{2}$.

Since $y \in X$, it can be expanded in the basis $\{x_1, \dots, x_n\}$ of X and suppose that $y = \sum_{i=1}^{n} \rho_i x_i$. Then we have

$$\begin{bmatrix} (\lambda_1 - \alpha)^+ \\ (\lambda_2 - \alpha)^+ \\ \cdot \\ \cdot \\ \cdot \\ (\lambda_\mu - \alpha)^+ \end{bmatrix} = \begin{bmatrix} x_1^b(1) & x_2^b(1) & \dots & x_n^b(1) \\ x_1^b(2) & x_2^b(2) & \dots & x_n^b(2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_1^b(\mu) & x_2^b(\mu) & \dots & x_n^b(\mu) \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \rho_n \end{bmatrix}. \tag{5.1}$$

By our assumption that $n \leq \frac{\mu+1}{2}$ we have that $n \leq \frac{\mu}{2} + \frac{1}{2} \leq \gamma + \frac{1}{2}$, therefore we have that $n \leq \gamma$ because n, γ are natural numbers. Therefore at least n coordinates of y in the basis $\{b_i\}$ are equal to zero and suppose that $(\lambda_{i_1} - \alpha)^+ = (\lambda_{i_2} - \alpha)^+ = \dots = (\lambda_{i_n} - \alpha)^+ = 0$. Then

$$\begin{bmatrix} x_1^b(i_1) & x_2^b(i_1) & \dots & x_n^b(i_1) \\ x_1^b(i_2) & x_2^b(i_2) & \dots & x_n^b(i_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_1^b(i_n) & x_2^b(i_n) & \dots & x_n^b(i_n) \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \rho_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \tag{5.2}$$

where $(\rho_1, \rho_2, \dots, \rho_n) \neq (0, 0, \dots, 0)$ because ρ_i are the coordinates of y in the basis $\{x_i\}$ and $y > 0$. This is a contradiction because the matrix of the system is nonsingular. So none of y, z belong to X and the theorem is true. □

Suppose that $x \in X$ and suppose that $a_1, a_2, \dots, a_k$ and $\Phi_1, \Phi_2, \dots, \Phi_k$ are the essential coefficients and the essential sets of states of x in the basis $\{b_i\}$. For any $r = 1, 2, \dots, k$ we define $c_x(r) = card(\Phi_1 \cup \dots \cup \Phi_r)$ and $p_x(r) = card(\Phi_{r+1} \cup \dots \cup \Phi_k)$, i.e., $c_x(r)$ and $p_x(r)$ are the cardinal numbers of $\Phi_1 \cup \dots \cup \Phi_r$ and $\Phi_{r+1} \cup \dots \cup \Phi_k$.

PROPOSITION 5.4. *Suppose that the security market X is strongly resolving with respect to the basis $\{b_i\}$ of $F_1(X)$, $x \in X$ and $a_1, a_2, \dots, a_k$ are the essential coefficients of x with respect to the basis $\{b_i\}$.*

(i) If $c_x(r) \geq n$, the interval $[a_r, a_{r+1})$ does not contain call-replicated exercise prices of x,

(ii) If $p_x(r) \geq n$, the interval $(a_r, a_{r+1}]$ does not contain put-replicated exercise prices of x,

(iii) if $\mathbf{1} \in X$ and $max\{c_x(r), p_x(r)\} \geq n$, the interval $[a_r, a_{r+1}]$ does not contain replicated exercise prices of x.

Proof. Suppose that $c_x(r) \geq n$ and that $a \in [a_r,\ a_{r+1})$ is a call-replicated exercise price. Then

$$y = c(x, a) = \sum_{j=r+1}^{k} (a_j - a)\overline{b}_j \in X,$$

where $\overline{b}_j = \sum_{i \in \Phi_j} b_i$ for any $j = 1, 2, \dots, k$.

We expanded y in the basis $\{x_1, \dots, x_n\}$ of X and suppose that $y = \sum_{i=1}^{n} \lambda_i x_i$ and suppose also that $x_i = \sum_{j=1}^{\mu} x_i^b(j) b_j$. By our hypothesis we have that $\#zeros_b(y) = c_x(r) \geq n$, therefore at least n of the coordinates ξ_i of y in the basis $\{b_i\}$ are equal to zero and suppose that $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n}$ are n such coordinates. This leads to the system

$$\begin{bmatrix} x_1^b(i_1) & x_2^b(i_1) & \dots & x_n^b(i_1) \\ x_1^b(i_2) & x_2^b(i_2) & \dots & x_n^b(i_2) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ x_1^b(i_n) & x_2^b(i_n) & \dots & x_n^b(i_n) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ . \\ . \\ . \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}.$$

The fact that X is strongly resolving implies that the system has the unique solution $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, therefore $y = c(x, a) = 0$, which is a contradiction because a is a nontrivial exercise price of x. Hence $[a_r, a_{r+1})$ does not contain call-replicated exercise prices and statement (i) is true.

The proof of statement (ii) is analogous and (iii) follows by (i) and (ii) and by the fact that $\mathbf{1} \in X$. □

EXAMPLE 5.5. Let $x_1 = (4, 4, 3, 3, 2, 2, 1, 1)$, $x_2 = (1, 1, 2, 2, 3, 3, 4, 4)$, and $X = [x_1, x_2]$. $\mathbf{1}$ is contained in X because $x_1 + x_2 = 5\mathbf{1}$. The payoff matrix

$$A(x_i, e_i) = \begin{bmatrix} 4 & 1 \\ 4 & 1 \\ 3 & 2 \\ 3 & 2 \\ 2 & 3 \\ 2 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix},$$

has singular 2×2 submatrices, therefore the market is not strongly resolving.

In order to apply our theorem, we determine the positive basis of $F_1(X)$ which is a partition of the unit and we find that the vectors

$$b_1 = (1, 1, 0, 0, 0, 0, 0, 0), b_2 = (0, 0, 1, 1, 0, 0, 0, 0),$$
$$b_3 = (0, 0, 0, 0, 1, 1, 0, 0), b_4 = (0, 0, 0, 0, 0, 0, 1, 1),$$

define this basis. We expand the vectors x_i in the basis $\{b_i\}$ and we find that

$$A(x_i, b_i) = \begin{bmatrix} 4 & 1 \\ 3 & 2 \\ 2 & 3 \\ 1 & 4 \end{bmatrix},$$

is the payoff matrix with respect to this basis and we remark that the market is strongly resolving with respect to the basis $\{b_i\}$. Since $n = 2 \leq \frac{\mu+1}{2} = \frac{5}{2}$ we have that any nontrivial option written on elements of X is nonreplicated.

APPENDIX: LATTICE-SUBSPACES AND POSITIVE BASES IN $C(\Omega)$

In this section we give the basic mathematical notions and results, which are needed for this paper. $C(\Omega)$ is the space of real valued functions defined on a compact Hausdorff topological space Ω. $C(\Omega)$ is ordered by the pointwise ordering, i.e., for any $x, y \in C(\Omega)$ we have: $x \geq y$ if and only if $x(t) \geq y(t)$ for each $t \in \Omega$. $C_+(\Omega) = \{x \in C(\Omega) \big| x(t) \geq 0$ for each $t \in \Omega\}$ is the positive cone of $C(\Omega)$. Recall that if the set Ω is finite, for example if $\Omega = \{1, 2, \ldots, m\}$, then $C(\Omega)$ is the vector space $\mathbb{R}^m$, therefore the results presented below hold also for the space $\mathbb{R}^m$ which we use in this paper. But we present the results in $C(\Omega)$ as they are formulated in Polyrakis (1996, 1999). The results of these articles are presented below.

The space $C(\Omega)$, ordered by the pointwise ordering is a vector lattice, i.e., for any $x, y \in C(\Omega)$ the supremum $x \vee y$ and the infimum $x \wedge y$ of $\{x, y\}$ in $C(\Omega)$ exists. Suppose that L is an **ordered subspace** of $C(\Omega)$, i.e., L is a linear subspace of $C(\Omega)$ ordered again by the pointwise ordering. Then $L_+ = C_+(\Omega) \cap L$ is the positive cone of L. If L is a vector lattice, i.e., if for any $x, y \in L$ the supremum $sup_L\{x, y\}$ and the infimum $inf_L\{x, y\}$ of $\{x, y\}$ in L exist, then L is a **lattice-subspace** of $C(\Omega)$. Then we have

$$sup_L\{x, y\} \geq x \vee y \geq x \wedge y \geq inf_L\{x, y\}.$$

If for any $x, y \in L$, $x \vee y \in L$ and $x \wedge y \in L$, L is a **sublattice** of $C(\Omega)$. It is clear that any sublattice of $C(\Omega)$ is a lattice-subspace but the converse is not true. In general an ordered subspace L of $C(\Omega)$ is not a lattice-subspace and also a lattice-subspace is not always a sublattice.

For any subset B of $C(\Omega)$, the intersection of all sublattices of $C(\Omega)$ which contain B is a sublattice of $C(\Omega)$ and it is the minimum sublattice of $C(\Omega)$ which contains B. This subspace is the **sublattice of** $C(\Omega)$ generated by B.

Suppose that L is finite-dimensional. A basis $\{b_1, b_2, \ldots, b_r\}$ of L is a **positive basis** of L if $L_+ = \{x = \sum_{i=1}^{r} \lambda_i b_i \mid \lambda_i \in \mathbb{R}_+$for each $i\}$. In other words, the basis $\{b_i\}$ of L is positive if for any $x \in L$ we have: $x(t) \geq 0$ for any $t \in \Omega$ if and only if the coefficients λ_i

of x in the basis $\{b_i\}$ are positive. Although L has infinitely many bases the existence of a positive basis of L is not always ensured.

Suppose that $\{b_1, b_2, \dots, b_r\}$ is a positive basis of L. Then it is easy to show that for any $x = \sum_{i=1}^r \lambda_i b_i$, $y = \sum_{i=1}^r \mu_i b_i \in L$ we have $x \geq y$ if and only if $\lambda_i \geq \mu_i$ for each i. This property implies that $\sup_L\{x, y\} = \sum_{i=1}^r (\lambda_i \vee \mu_i) b_i$ and $\inf_L\{x, y\} = \sum_{i=1}^r (\lambda_i \wedge \mu_i) b_i$, therefore L is a lattice-subspace. The converse is also true. One can prove it directly or by using the Choquet–Kendall theorem, see in Polyrakis (1999), proposition 1.1. So we have the following:

THEOREM A.1. *A finite-dimensional ordered subspace L of $C(\Omega)$ is a lattice-subspace if and only if L has a positive basis.*

Also each vector b_i of the positive basis of L is an extremal point of L_+. (A vector $x_0 \in L_+$, $x_0 \neq 0$ is an extremal point of L_+ if for any $x \in L$, $0 \leq x \leq x_0$ implies $x = \lambda x_0$ for some real number λ.) This property implies that a positive basis of L is unique in the sense of positive multiples.

THEOREM A.2 (Polyrakis 1999, proposition 2.2). *A finite-dimensional ordered subspace L of $C(\Omega)$ is a sublattice of $C(\Omega)$ if and only if L has a positive basis $\{b_1, b_2, \dots, b_r\}$ with the property: $b_i^{-1}(0, +\infty) \cap b_j^{-1}(0, +\infty) = \emptyset$ for any $i \neq j$.*

As an application of the above result we have:

THEOREM A.3. *Suppose that L is a sublattice of $\mathbb{R}^m$. If the constant vector $\mathbf{1} = (1, 1, \dots, 1)$ is an element of L, then L has a positive basis $\{b_1, b_2, \dots, b_r\}$ which is a **partition of the unit**, i.e., the vectors b_i have disjoint supports and $\mathbf{1} = \sum_{i=1}^r b_i$. This basis is unique.*

Indeed, by Theorem A.2, L has a positive basis $\{d_1, d_2, \dots, d_r\}$ with disjoint supports. Since $\mathbf{1} \in L$ we have $\mathbf{1} = \sum_{i=1}^r \lambda_i d_i$ and for each $j \in supp(d_i)$ we have $1 = \mathbf{1}(j) = \lambda_i d_i(j)$, therefore $d_i(j) = \frac{1}{\lambda_i}$ for any $j \in supp(d_i)$. So each d_i is constant on its support, therefore the basis $\{b_i = \lambda_i d_i\}$ is a positive basis of L which is a partition of the unit.

We suppose now that $z_1, z_2, \dots, z_r$ are fixed, linearly independent, positive vectors of $C(\Omega)$ and that

$$L = [z_1, z_2, \dots, z_r],$$

is the subspace of $C(\Omega)$ generated by the vectors z_i. We study the problem: under what conditions L is a lattice-subspace or a sublattice of $C(\Omega)$? In the case where L fails to be a lattice-subspace we study if L is contained in a finite-dimensional minimal lattice-subspace of $C(\Omega)$ or if the sublattice generated by L is finite-dimensional.

The function

$$\beta(t) = \left(\frac{z_1(t)}{z(t)}, \frac{z_2(t)}{z(t)}, \dots, \frac{z_r(t)}{z(t)} \right), \text{ for each } t \in \Omega, \text{ with } z(t) > 0,$$

where $z = z_1 + z_2 + \cdots + z_r$, is the **basic function** of $z_1, z_2, \dots, z_r$. This function is very important for the study of lattice-subspaces and positive bases and has been defined in Polyrakis (1996). The set $R(\beta) = \{\beta(t) \big| t \in \Omega \text{ with } z(t) > 0\}$, is the range of β and the cardinal number $card\, R(\beta)$ of $R(\beta)$ is the number of the (different) elements of $R(\beta)$. Under the above notations we have, see in Polyrakis (1999) theorem 3.6:

THEOREM A.4 (Polyrakis). *L is a sublattice of $C(\Omega)$ if and only if card $R(\beta) = r$. If $R(\beta) = \{P_1, P_2, \ldots, P_r\}$, a positive basis $\{b_1, b_2, \ldots, b_r\}$ of L is given by the formula:*

$$(b_1, b_2, \ldots, b_r)^T = A^{-1}(z_1, z_2, \ldots, z_r)^T, \tag{A.1}$$

where A is the $r \times r$ matrix whose the i^{th} column is the vector P_i, for each $i = 1, 2, \ldots, r$, and $(b_1, b_2, \ldots, b_r)^T$, $(z_1, z_2, \ldots, z_r)^T$ are the matrices with rows the vectors $b_1, b_2, \ldots, b_r, z_1, z_2, \ldots, z_r$.

A.1. The Algorithm for the Sublattice Generated by L

The next result gives an algorithm for the construction of the sublattice Z of $C(\Omega)$ generated by a finite set $\{z_1, z_2, \ldots, z_r\}$ of linearly independent and positive vectors, in the case where Z is finite-dimensional. In this case a positive basis of Z is determined. As in the previous theorem, β is the basic function of $z_1, z_2, \ldots, z_r$. Statement (d) determines the positive basis of Z. In fact (d) is an application of the previous theorem for the determination of a positive basis of Z. For more details, see in Polyrakis (1999), Theorem 3.7.

THEOREM A.5 (Polyrakis). *Let Z be the sublattice of $C(\Omega)$ generated by $\{z_1, z_2, \ldots, z_r\}$ and let $\mu \in \mathbb{N}$. Then the statements (i) and (ii) are equivalent:*

(i) $dim(Z) = \mu$.
(ii) $R(\beta) = \{P_1, P_2, \ldots, P_\mu\}$.

If statement (ii) is true then Z is constructed as follows:

(a) Enumerate $R(\beta)$ so that its r first vectors to be linearly independent (such an enumeration always exists). Denote again by P_i, $i = 1, 2, \ldots, \mu$ the new enumeration and we put $I_{r+k} = \{t \in \Omega \,|\, \beta(t) = P_{r+k}\}$, for each $k = 1, 2, \ldots, \mu - r$.
(b) Define the vectors z_{r+k}, $k = 1, 2, \ldots, \mu - r$ as follows:

$$z_{r+k}(i) = z(i) \text{ if } i \in I_{r+k} \text{ and } z_{r+k}(i) = 0 \text{ if } i \notin I_{r+k},$$

where $z = z_1 + z_2 + \cdots + z_r$ is the sum of the vectors z_i.
(c) $Z = [z_1, z_2, \ldots, z_r, z_{r+1}, \ldots, z_\mu]$.
(d) A positive basis $\{b_1, b_2, \ldots, b_\mu\}$ of Z is constructed as follows: Consider the basic function γ of $z_1, z_2, \ldots, z_r, z_{r+1}, \ldots, z_\mu$ and suppose that $\{P'_1, P'_2, \ldots, P'_\mu\}$ is the range of γ (the range of γ has exactly μ points). Then

$$(b_1, b_2, \ldots, b_\mu)^T = D^{-1}(z_1, z_2, \ldots, z_\mu)^T,$$

where D is the $\mu \times \mu$ matrix with columns the vectors $P'_1, P'_2, \ldots, P'_\mu$.

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