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RESEARCH ARTICLE

Maximal submarkets that replicate any option

Ioannis A. Polyrakis · Foivos Xanthos

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Abstract In this article we study the replication of options in security markets X with a finite number of states. Specifically, we study the existence of maximal submarkets (subspaces) Y of X so that any option written on the elements of Y is replicated by a marketed asset x of X. So inside these subspaces the pricing problem is simple because any option is priced by the replicating portfolio. Using the theory of lattice-subspaces and positive bases developed by Polyrakis (Trans Am Math Soc 348:2793–2810, 1996; 351:4183–4203, 1999), we identify the set of all maximal replicated subspaces. In particular, for any maximal replicated subspace we determine a positive basis of the subspace. Moreover we show that the union of all maximal replicated subspaces is the set of all marketed securities $x \in X$ so that any option written on x is replicated. So we determine also the set of securities with replicated options.

Keywords Security markets · Replication of options · Completion by options · Positive bases · Sublattices

JEL Classification G10 · D52 · C60

Dedicated to the memory of Charalambos D Aliprantis.

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I. A. Polyrakis (✉) · F. Xanthos
Department of Mathematics, National Technical University of Athens,
Zographou 157 80, Athens, Greece
e-mail: ypoly@math.ntua.gr

1 Introduction

The replication of options plays a crucial role in standard option pricing models because the price of a replicated option is equal to the price of the portfolio of primitive securities that replicates the option. However, this approach breaks down for options that cannot be replicated. Given the importance of option redundancy in standard option pricing models, our paper provides a characterization of submarkets where options are replicated. We consider a two-period security market X with a finite number of states, denoted by m, a finite number of primitive securities (assets) with payoffs in $\mathbb{R}^m$ and we study the existence of maximal replicated submarkets (subspaces) Y of X.

The completion $F_{\mathbf{1}}(X)$ of X by options is the subspace of $\mathbb{R}^m$ generated by all options written on the elements of X. In (1976), Ross in his seminal work , proved that $F_{\mathbf{1}}(X)$ is the whole space $\mathbb{R}^m$ if and only if X has an efficient fund. ($e \in \mathbb{R}^m$ is an efficient fund if $e(i) \neq e(j)$ for any $i \neq j$). Since then many authors contributed to this problem. In Arditti and John (1980) it is proved that if X has an efficient fund, then almost any portfolio (in the sense of the Lebesgue measure of X) is an efficient fund. In John (1981) is studied the case where the Ross assumption for the existence of efficient funds is not satisfied and the completion by options of X is a proper subspace of $\mathbb{R}^m$. He defined the notion of the maximally efficient fund and he proved that $F_{\mathbf{1}}(X)$ is generated by the call and put options written on a maximally efficient fund. As it is observed in Ross (1976),Green and Jarrow (1987),Brown and Ross (1991), in security markets any call and put option can be replicated if and only if X is a sublattice of $\mathbb{R}^m$.

In Baptista (2005) the results of Ross (1976) are extended for multiperiod security markets.

In Aliprantis and Tourky (2002) it is proved that in any strongly resolving security market with $n \leq \frac{m+1}{2}$ where n is the number of primitive securities and m the number of states, any non trivial option is non replicated.

In Baptista (2007), the replication of options is studied in the case where the asset span X doesn't contain binary vectors. Baptista shows that for any $x \in X$ the set of non replicated exercise prices of x is a subset of full measure of the set K_x of non trivial exercise prices of x.

In Galvani (2009), the results of Ross (1976) are studied in L_p spaces.

In (2006), Kountzakis and Polyrakis closed the problem of the determination of the completion by options of X by giving a complete method which determines a positive basis of $F_{\mathbf{1}}(X)$. This method is based on the theory of lattice subspaces and positive bases, developed by Polyrakis in 1996, 1999.

In the present article we use also the theory of lattices-subspaces and positive bases and we determine completely the replicated submarkets of X. We show that the union of all maximal replicated submarkets is the set of elements x of X so that any option written on x is replicated and we call this set replicated kernel of the market. Also we prove a characterization of the markets without binary vectors in terms of the replicated submarkets.

Finally note that, although there are important results on the replication of options, they cannot provide a method for the determination of the replicated options. The theory of lattice subspaces and positive bases gives to us the possibility to determine the set of securities with replicated options.

2 The model

In this article we study a two-period security market with a finite number of states $\Omega = \{1, 2, ..., m\}$ at the date 1, and a finite number of primitive securities (assets) with payoffs given by the linearly independent vectors $x_1, x_2, ..., x_n$ of the payoff space $\mathbb{R}^m$.

A **portfolio** is a vector $\theta = (\theta_1, \theta_2, ..., \theta_n)$ of $\mathbb{R}^n$ where θ_i is the number of units of the security i. Then $T(\theta) = \sum_{i=1}^{n} \theta_i x_i \in \mathbb{R}^m$ is the **payoff of** θ. Since the operator T is one-to-one, it identifies portfolios with their payoffs. So the vectors $x_1, x_2, \ldots, x_n$ will be mentioned as **primitive securities**, the subspace

$$X = [x_1, x_2, \ldots, x_n],$$

of $\mathbb{R}^m$, generated by the vectors x_i as the **space of marketed securities** or the **asset span** and the vectors of X will be also referred as portfolios. The **riskless bond 1** is the vector of $\mathbb{R}^m$ with coordinates equal to 1. In this study we assume that the riskless bond $\mathbf{1}$ is contained in X.

A vector $x \in \mathbb{R}^m$ is **marketed** or **replicated** if x is the payoff of some portfolio θ, or equivalently if $x \in X$.

Recall that $\mathbb{R}^m = \{x = (x(1), x(2), \ldots, x(m)) | x(i) \in \mathbb{R}$ for each $i\}$. For any $x = (x(1), x(2), \ldots, x(m)) \in \mathbb{R}^m$ denote by $\|x\|$ the Euclidean norm of x, $||x||_\infty = \max\{|x(i)|\ i = 1, \ldots, m\}$ is the **supremum norm** of x and the set $supp(x) = \{i = 1, 2, \ldots, m | x(i) \neq 0\}$ is the **support** of x. x is a **binary vector** if $x \neq 0,\ x \neq \mathbf{1}$ and $x(i) = 0$ or $x(i) = 1$, for any i.

The **call option** written on the vector x with exercise price a is the vector $c(x, a) = (x - a\mathbf{1})^+$ of $\mathbb{R}^m$. The **put option** written on x with exercise price a is $p(x, a) = (a\mathbf{1} - x)^+$. We have $x - a\mathbf{1} = c(x, a) - p(x, a)$.

If both $c(x, a) > 0$ and $p(x, a) > 0$, we say that the call option $c(x, a)$ and the put option $p(x, a)$ are **non trivial** and also we say that a is a **non trivial exercise price** of x. We will denote by K_x the set of non trivial exercise prices of x. If $c(x, a)$ and $p(x, a)$ belong to X we say that $c(x, a)$ and $p(x, a)$ are **replicated**.

The **completion by options** of X is the subspace of $\mathbb{R}^m$ which arises inductively by adding in the market the call and put options of the marketed securities and by taking again call and put options which are added again in the market. In Kountzakis and Polyrakis (2006) the mathematical definition of the completion by options in infinite securities markets is given. Furthermore, a more general study of the completion by options of the market is presented where the options are not taken with respect to the riskless bond $\mathbf{1}$ but with respect to some risky vectors from a standard subspace U of $\mathbb{R}^m$ and the completion by options of X is denoted by $F_U(X)$. This study is very general and includes the case of exotic options. In the classical case where the options are taken with respect to the riskless bond $\mathbf{1}$, the completion by options of X is denoted in Kountzakis and Polyrakis (2006) by $F_{\mathbf{1}}(X)$ and we will preserve this notation in the present article. In Kountzakis and Polyrakis (2006) it is proved that if the payoff space is a general vector lattice E then $F_U(X)$ is the sublattice of E generated by the set $X \cup U$. In our case where the call and put options are taken with respect to the riskless bond $\mathbf{1} \in X$, the completion by options $F_{\mathbf{1}}(X)$ of X is the sublattice of $\mathbb{R}^m$ generated by X.

For more details on lattice-subspaces and positive bases see the Appendix. For an introduction to two-period security markets we refer to the book of LeRoy and Werner (2001). Also we refer to the articles Thijssen (2008) and Zurita (2008).

3 The completion by options of X

In this section we describe the method of determination of $F_{\mathbf{1}}(X)$. Recall that the security market X is generated by the linearly independent vectors $x_1, x_2, \ldots, x_n$ not necessarily positive and that $\mathbf{1} \in X$. As we have noted before, the completion by options of X is the sublattice of $\mathbb{R}^m$ generated by X. In order to use Theorem 9 for the determination of a positive basis of $F_{\mathbf{1}}(X)$ we have to determine a set of linearly independent, positive vectors $\{y_1, y_2, \ldots, y_r\}$ of $\mathbb{R}^m$ so that the sublattice generated by $\{y_1, y_2, \ldots, y_r\}$ is $F_{\mathbf{1}}(X)$. This set is called in Kountzakis and Polyrakis (2006) **basic set** of the market. In Kountzakis and Polyrakis (2006) it is proved that any maximal subset $\{y_1, y_2, \ldots, y_r\}$ of linearly independent vectors of the set

$$\mathcal{A} = \{x_1^+, x_1^-, x_2^+, x_2^-, \ldots, x_n^+, x_n^-\},$$

is a basic set of the market. Note that the vectors of this basic set are not necessarily elements of X.

In the present article the assumption that the riskless bond is contained in X implies that we can find a basic set consisting of n linearly independent and positive vectors of X as the following lemma shows. This is important for our study.

Lemma 1 *If $a = \max\{||x_i||_\infty \mid i = 1, 2, \ldots, n\}$, then at least one of the two sets of positive vectors of X,*

$$\{y_i = a\mathbf{1} - x_i, \mid i = 1, \ldots, n\} \text{ and } \{z_i = 2a\mathbf{1} - x_i, \mid i = 1, 2, \ldots, n\},$$

consists of linearly independent vectors.

Proof By the definition of a we have $-a\mathbf{1} \leq x_i \leq a\mathbf{1}$. Therefore, the vectors y_i, z_i are positive vectors of X for each i. It is easy to show that the vectors $w_i = \lambda\mathbf{1} - x_i,\ i = 1, \ldots, n$ are linearly dependent if and only if $\lambda\mathbf{1}$ is a vector of the affine hull of $x_1, x_2, \ldots, x_n$. So at least one of the families of the lemma consists of linearly independent vectors. □

Definition 1 Any set $\{y_1, y_2, \ldots, y_n\}$ of n linearly independent and positive vectors of X is a **basic set of marketed securities**.

In general we can find different basic sets of marketed securities. Such a set is a basis of X consisting of positive vectors but in general this basis is not a positive basis of X. So for any basic set of marketed securities the subspace generated by this set is the market X, therefore, the sublattice generated by this basic set is the completion of X by options. So we have:

Theorem 1 *The sublattice of $\mathbb{R}^m$ generated by a basic set $\{y_1, y_2, \ldots, y_n\}$ of marketed securities is $F_{\mathbf{1}}(X)$.*

For the determination of the positive basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ which is a partition of the unit we follow the steps of Polyrakis algorithm, see Theorem 9 in the appendix, where a positive basis of the sublattice of $\mathbb{R}^m$ generated by a finite set of positive and linearly independent vectors is determined. We start by the determination of a basic set of marketed securities $\{y_1, y_2, \ldots, y_n\}$ of the market. In the sequel we determine the **basic function** of $y_1, y_2, \ldots, y_n$. This function has been defined in Polyrakis (1996) and is the following:

$$\beta(i) = \left(\frac{y_1(i)}{y(i)}, \frac{y_2(i)}{y(i)}, \ldots, \frac{y_n(i)}{y(i)}\right), \quad \text{for each } \; i = 1, 2, \ldots, m, \quad \text{with } \; y(i) > 0,$$

where $y = y_1 + y_2 + \cdots + y_n$. This function takes values in the simplex $\Delta_n = \{\xi \in \mathbb{R}^n_+ | \sum_{i=1}^n \xi_i = 1\}$ of $\mathbb{R}^n_+$.

Denote by $R(\beta)$ the range of β and by $card R(\beta)$ the cardinal number of $R(\beta)$.

By continuing the process of the algorithm we obtain a positive basis $\{d_1, d_2, \ldots, d_\mu\}$ of $F_{\mathbf{1}}(X)$.

By a normalization of the basis $\{d_i\}$ we obtain the positive basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ which is a partition of the unit. Indeed, the vectors d_i have disjoint supports by Proposition 5. If $\mathbf{1} = \sum_{i=1}^{\mu} \rho_i d_i$ is the expansion of $\mathbf{1}$ in the basis $\{d_i\}$, for any $j \in supp(d_i)$ we have $\mathbf{1}(1) = 1 = \rho_i d_i(j)$, therefore, $d_i(j) = \frac{1}{\rho_i}$. So each d_i is constant on its support and $\rho_i = \frac{1}{\|d_i\|_\infty}$ for any i. The vectors b_i as positive multiples of the vectors d_i define a positive basis of $F_{\mathbf{1}}(X)$. This basis is a partition of the unit because $\mathbf{1} = \sum_{i=1}^{\mu} b_i$. So we have:

Theorem 2 *$F_{\mathbf{1}}(X)$ has a positive basis $\{b_1, b_2, \ldots, b_\mu\}$ which is a partition of the unit.*

Using Theorem 9, the dimension of $F_{\mathbf{1}}(X)$ is equal to the cardinal number of $R(\beta)$. So if $R(\beta)$ has n elements, then $F_{\mathbf{1}}(X) = X$ and any option is replicated. If $R(\beta)$ has m elements, then $F_{\mathbf{1}}(X) = \mathbb{R}^m$ and the options fill the whole space $\mathbb{R}^m$. This is expressed in the next result.

Theorem 3 *The dimension of $F_{\mathbf{1}}(X)$ is equal to the cardinal number of the range $R(\beta)$, therefore, we have:*

(i) $F_{\mathbf{1}}(X) = X$ *if and only if* $card R(\beta) = n$,
(ii) $F_{\mathbf{1}}(X) = \mathbb{R}^m$ *if and only if* $card R(\beta) = m$,
(iii) $F_{\mathbf{1}}(X) \subsetneq \mathbb{R}^m$ *if and only if* $card R(\beta) < m$.

4 Maximal replicated submarkets

For the study of maximal replicated submarkets the notion of the projection basis introduced in Polyrakis (2003), plays an important role. The projection basis has been introduced for a finite dimensional subspace of $C(\Omega)$ generated by a finite number of linearly independent, positive vectors of $C(\Omega)$ and the corresponding theory is contained in the Appendix. In the case where Ω is finite, then $C(\Omega)$ is an $\mathbb{R}^k$ space where k is the cardinal number of Ω. In this section, we suppose that X is generated by a basic set of marketed securities which we denote again by $\{x_1, x_2, \ldots, x_n\}$. This is always possible by Lemma 1. So we suppose that $X = [x_1, x_2, \ldots, x_n]$ is the subspace of $\mathbb{R}^m$ generated by the linearly independent, positive vectors x_i of $\mathbb{R}^m$.Suppose that Z is the sublattice of $\mathbb{R}^m$ generated by X with $dim(Z) = \mu$, β is the basic function of the vectors x_i and that $P_1, P_2, \ldots, P_n, P_{n+1}, \ldots, P_\mu$ is an enumeration of the range of β so that the first n vectors $P_1, P_2, \ldots, P_n$ are linearly independent. As we have shown above, Z is the completion $F_{\mathbf{1}}(X)$ of X.

Suppose that we follow the steps of statement (ii) of Theorem 9 in the Appendix and we determine a positive basis $\{d_1, d_2, \ldots, d_\mu\}$ of $F_{\mathbf{1}}(X)$ which is given by (4). Note that in the appendix this basis is denoted by $\{b_1, b_2, \ldots, b_\mu\}$ but here we use this notation for the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit. According to Theorem 10 in the Appendix, if

$$(\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_n)^T = A^{-1}(x_1, x_2, \ldots, x_n)^T, \tag{1}$$

where A is the $n \times n$ matrix whose ith column is the vector P_i, then $\{\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_n\}$ is a basis of X which is called in Polyrakis (2003) **projection basis** of X. This basis has the property: the n first coordinates of any element x of X in the basis $\{d_1, d_2, \ldots, d_\mu\}$ coincide with the corresponding coordinates of x in the projection basis, i.e.

$$x = \sum_{i=1}^{\mu} \lambda_i d_i \in X \implies x = \sum_{i=1}^{n} \lambda_i \widetilde{d}_i. \tag{2}$$

Suppose that $\{u_i\}$ is a positive basis of $F_{\mathbf{1}}(X)$ with $u_i = \theta_i d_i$ where $\theta_i > 0$, for any i. It is easy to show that $\{\widetilde{u}_i = \theta_i \widetilde{d}_i\}$ is a basis of X with the property: the n first coordinates of any element x of X in the basis $\{u_1, u_2, \ldots, u_\mu\}$ coincide with the corresponding coordinates of x in the basis $\{\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n\}$ of X. Indeed if $x = \sum_{i=1}^{\mu} \lambda_i u_i \in X$, we have:

$$x = \sum_{i=1}^{\mu} \lambda_i \theta_i d_i, \text{ therefore } x = \sum_{i=1}^{n} \lambda_i \theta_i \widetilde{d}_i = \sum_{i=1}^{n} \lambda_i \widetilde{u}_i.$$

Therefore, $\{\widetilde{u}_i\}$ is a projection basis of X and we will refer to it as the **projection basis of X corresponding to the basis** $\{u_i\}$**.** According to this terminology the projection basis $\{\widetilde{d}_i\}$ of X is the projection basis corresponding to the basis $\{d_i\}$.

Proposition 1 *Suppose that* $\{d_i\}$ *is the basis of* $F_{\mathbf{1}}(X)$ *given by* (4) *of Theorem* 9 *and* $\{\widetilde{d}_i\}$ *is the projection basis of* X *corresponding to the basis* $\{d_i\}$*.*

Then $\{b_i = \frac{d_i}{\|d_i\|_\infty} \mid i = 1, 2, \ldots, \mu\}$ *is the positive basis of* $F_{\mathbf{1}}(X)$ *which is a partition of the unit and* $\{\widetilde{b}_i = \frac{\widetilde{d}_i}{\|d_i\|_\infty} \mid i = 1, 2, \ldots, n\}$ *is the projection basis of* X *corresponding to the basis* $\{b_i\}$ *of* $F_{\mathbf{1}}(X)$*.*

Proof By Theorem 2 and the remarks before it, $\{b_i = \frac{d_i}{\|d_i\|_\infty}\}$ is the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit. As we have noted above, $\{\widetilde{b_i} = \frac{\widetilde{d_i}}{\|d_i\|_\infty}\}$ is the projection basis of X corresponding to the basis $\{b_i\}$. □

Definition 2 Suppose that Y is a subspace of X. If the completion by options of Y is contained in X, i.e. $F_{\mathbf{1}}(Y) \subseteq X$, we say that Y is **replicated** and if moreover for any subspace Z of X with $Y \subsetneqq Z$ we have $X \subsetneqq F_{\mathbf{1}}(Z)$, we say that Y is a **maximal replicated subspace** or **maximal replicated submarket** of X.

Proposition 2 *A subspace Y of X is a maximal replicated submarket of X if and only if Y is a maximal sublattice of $\mathbb{R}^m$ contained in X with $\mathbf{1} \in Y$.*

Proof Suppose that Y is a maximal replicated subspace of X. Then $Y \subseteq F_{\mathbf{1}}(Y) \subseteq X$. Also the subspace $Z = F_{\mathbf{1}}(Y)$ of X is a sublattice of $\mathbb{R}^m$ which contains $\mathbf{1}$ because the completion $F_{\mathbf{1}}(Y)$ of Y is the sublattice of $\mathbb{R}^m$ generated by $\{\mathbf{1}\} \cup Y$. If we take again the completion of Z we have that $Z = F_{\mathbf{1}}(Z)$, therefore, Z is a replicated subspace of X. Since Y is a maximal replicated submarket we have that $Y = Z$, therefore, Y is a sublattice of $\mathbb{R}^m$ with $\mathbf{1} \in Y$. Also Y is a maximal sublattice because for any sublattice W of $\mathbb{R}^m$ with $Y \subseteq W \subseteq X$, W is replicated, therefore, $Y = W$.

For the converse suppose that Y is a maximal sublattice of $\mathbb{R}^m$ contained in X with $\mathbf{1} \in Y$. If we suppose that Y is not a maximal replicated subspace, there exists a subspace Z of X with $Y \subsetneqq Z$ and $F_{\mathbf{1}}(Z) \subseteq X$. Then $F_{\mathbf{1}}(Z)$ is a sublattice which contains $\mathbf{1}$, a contradiction. Therefore, Y is a maximal replicated submarket. □

Definition 3 Let $\{b_i, i = 1, \ldots, \mu\}$ be a positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and let $\{\widetilde{b_i} \mid i = 1, \ldots, n\}$ be the projection basis of X corresponding to the basis $\{b_i\}$. A partition $\delta = \{\sigma_i \mid i = 1, \ldots, \kappa\}$ of $\{1, \ldots, n\}$ is **proper** (with respect to the projection basis $\{\widetilde{b_i}\}$ of X) if for any $r = 1, 2, \ldots, \kappa$, the vector $w_r = \Sigma_{i\in\sigma_r}\widetilde{b_i}$ is a binary vector with $\Sigma_{r=1}^{\kappa} w_r = \mathbf{1}$. If moreover δ is maximal, in the sense that does not exist a proper partition φ of $\{1, \ldots, n\}$ strictly finer than δ, we say that δ is a **maximal proper** partition of $\{1, \ldots, n\}$.

Recall that the partition $\varphi = \{\omega_i \mid i = 1, \ldots, \tau\}$ is **finer** than $\delta = \{\sigma_i \mid i = 1, \ldots, \kappa\}$ if any ω_i is a subset of some σ_j.

Remark 1 For any proper partition δ of $\{1, \ldots, n\}$, the vectors w_r in the above definition have disjoint supports. This holds because the vectors w_r are binary vectors with sum equal to $\mathbf{1}$.

Proposition 3 *For any proper partition δ of $\{1, 2, \ldots, n\}$ there exists a maximal proper partition of $\{1, 2, \ldots, n\}$ finer than δ. At least one maximal proper partition of $\{1, 2, \ldots, n\}$ exists.*

Proof Suppose that $\{\sigma_1, \ldots, \sigma_k\}$ is a proper partition of $\{1, 2, \ldots, n\}$. We say that a subset σ of $\{1, \ldots, n\}$ is an atom, if does not exist a nonempty set $\omega \subsetneqq \sigma$, so that $\sum_{i\in\omega}\widetilde{b_i}$ and $\sum_{i\in\sigma\setminus\omega}\widetilde{b_i}$ are binary vectors. Each σ_i is a finite set, therefore, we can decompose any σ_i in a maximum set of atoms. By this procedure we take a maximal proper partition of $\{1, 2, \ldots, n\}$ finer than δ.

Finally note that the whole set $\{1, 2, \ldots, n\}$ is a proper partition, because $\sum_{i=1}^{n}\widetilde{b_i} = \mathbf{1}$. Therefore, a maximal proper partition exists. □

Theorem 4 *Suppose that $\delta = \{\sigma_i | i = 1, \ldots, k\}$ is a proper partition of $\{1, \ldots, n\}$ and $w_r = \sum_{i \in \sigma_r} \widetilde{b}_i$ for $r = 1, \ldots, k$. Then the space $Y = [w_1, \ldots, w_k]$ is a sublattice of $\mathbb{R}^m$ which is contained in X and $\{w_1, w_2, \ldots, w_k\}$ is a positive basis of Y which is a partition of the unit. We will refer to Y as the* **sublattice of $\mathbb{R}^m$ generated by** *δ and we will denote Y by Y_δ. For any proper partition φ of $\{1, \ldots, n\}$ strictly finer than δ, Y_δ is a proper subspace of Y_φ.*

Proof Suppose that $\delta = \{\sigma_i | i = 1, \ldots, k\}$ is a proper partition of $\{1, \ldots, n\}$. Then $w_r = \sum_{i \in \sigma_r} \widetilde{b}_i$ is a binary vector for any r. The vectors w_r have disjoint supports and also define a partition of the unit. So $Y_\delta = [w_1, \ldots, w_k]$ is a sublattice of $\mathbb{R}^m$ which is contained in X with $\mathbf{1} \in Y_\delta$ and $\{w_1, w_2, \ldots, w_k\}$ is a positive basis of Y_δ which is partition of the unit. If we suppose that $\varphi = \{\omega_i \mid i = 1, \ldots, \tau\}$ is a proper partition of $\{1, \ldots, n\}$ finer than δ we have: Each σ_r is decomposed in a finite number of atoms $\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_\nu}$, therefore, $w_r = v_{r_1} + v_{r_2} + \cdots + v_{r_\nu}$ where $v_{r_j} = \sum_{i \in \omega_{r_j}} \widetilde{b}_i$. If Y_φ is the sublattice generated by the vectors $v_j = \sum_{i \in \omega_j} \widetilde{b}_i$, $j = 1, 2, \ldots, \tau$, we have $w_j \in Y_\varphi$ for any $j = 1, 2, \ldots, k$, therefore, $Y_\delta \subseteq Y_\varphi$. Since φ is strictly finer than δ, at least one w_r is decomposed in more than one vector v_j, therefore, Y_δ is a proper subspace of Y_φ. □

Theorem 5 *Let $\{b_i, i = 1, \ldots, \mu\}$ be the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and let $\{\widetilde{b}_i \mid i = 1, \ldots, n\}$ be the projection basis of X corresponding to the basis $\{b_i\}$.*

Suppose Y is a sublattice of $\mathbb{R}^m$ contained in X with $\mathbf{1} \in Y$. If $\{w_1, w_2, \ldots, w_k\}$ is the positive basis of Y which is a partition of the unit, then a proper partition $\delta_Y = \{\sigma_i \mid i = 1, \ldots, \kappa\}$ of $\{1, \ldots, n\}$ exists so that $w_r = \sum_{i \in \sigma_r} \widetilde{b}_i$ for any r, therefore, Y is the sublattice generated by the partition δ_Y. The partition δ_Y is unique.

For any sublattice Z of $\mathbb{R}^m$ contained in X with $Y \subsetneqq Z$, the partition δ_Z is strictly finer than δ_Y.

Proof The positive basis $\{w_1, w_2, \ldots, w_k\}$ of Y is a partition of the unit, hence $supp(w_i) \cap supp(w_j) = \emptyset$ for any $i \neq j$ and $\sum_{r=1}^k w_r = \mathbf{1}$. Also the vectors w_r are binary vectors. Suppose that $w_r = \sum_{i \in \Phi_r} b_i$ where $\Phi_r \subseteq \{1, 2, \ldots, \mu\}$, is the expansion of w_r in the basis $\{b_i\}$. It is easy to show that $\{\Phi_r \mid r = 1, 2, \ldots, k\}$ is a partition of $\{1, 2, \ldots, \mu\}$. Since $\{\widetilde{b}_i\}$ is the projection basis of X corresponding to the basis $\{b_i\}$, we have that $w_r = \sum_{i \in \sigma_r} \widetilde{b}_i$, where $\sigma_r = \Phi_r \cap \{1, 2, \ldots, n\}$. Any w_r, as a vector of the positive basis of Y, is nonzero therefore, $\sigma_r \neq \emptyset$ for any $r = 1, 2, \ldots, k$. Also any w_r is a binary vector, therefore, $\delta = \{\sigma_r | r = 1, 2, \ldots, k\}$ is a proper partition of $\{1, 2, \ldots, n\}$. By Theorem 4, Y is the sublattice of $\mathbb{R}^m$ generated by the partition δ. Also the partition δ is unique because the expansion $w_r = \sum_{i \in \Phi_r} \widetilde{b}_i$ is unique and we denote this partition by δ_Y.

Suppose that Z is a sublattice of $\mathbb{R}^m$ such that $Y \subsetneqq Z \subseteq X$. We will show that the partition δ_Z is strictly finer than δ_Y. Suppose that $\{z_1, \ldots, z_\lambda\}$ is the positive basis of Z which is partition of the unit. Suppose also that $\delta_Z = \{\omega_1, \omega_2, \ldots, \omega_\lambda\}$. Then $z_r = \sum_{i \in \omega_r} \widetilde{b}_i$ for any $r = 1, 2, \ldots, \lambda$. Since Y is a proper subspace of Z, we have that $\lambda > k$ and also that any w_r can be expanded in the basis $\{z_i\}$ of Z. Suppose that $w_r = \sum_{i \in \Psi_r} z_i$ where $\Psi_r \subseteq \{1, \ldots, \lambda\}$, is the expansion of w_r in the basis $\{z_i\}$. Then

$\{\Psi_r \mid r = 1, \ldots, k\}$ is a partition of $\{1, \ldots, \lambda\}$. So we have

$$w_r = \sum_{i \in \Psi_r} z_i = \sum_{i \in \Psi_r} \left(\sum_{j \in \omega_i} \widetilde{b}_j \right),$$

therefore, $\sigma_r = \cup_{i \in \Psi_r} \omega_i$. Therefore, any ω_i is included in some σ_r. Furthermore, this inclusion is proper for at least one i because Y is a proper subspace of Z, therefore, δ_Z is strictly finer than δ_Y. □

Theorem 6 *Let $\{b_i, i = 1, \ldots, \mu\}$ be the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and let $\{\widetilde{b_i} \mid i = 1, \ldots, n\}$ be the projection basis of X corresponding to the basis $\{b_i\}$. If Y is a subspace of X, the following are equivalent:*

(i) *Y is a maximal replicated subspace of X,*
(ii) *there exists a maximal proper partition $\delta = \{\sigma_i \mid i = 1, \ldots, \kappa\}$ of $\{1, \ldots, n\}$ so that Y is the sublattice of $\mathbb{R}^m$ generated by δ.*

The set of maximal replicated submarkets of X is nonempty.

Proof (i) $\Longrightarrow$ (ii): Suppose that Y is a maximal replicated subspace of X. Then, by Theorem 5, a proper partition δ_Y of $\{1, \ldots, n\}$ exists, so that Y is the sublattice of $\mathbb{R}^m$ generated by δ_Y. Also δ_Y is maximal because if we suppose that φ is a proper partition of $\{1, \ldots, n\}$ strictly finer than δ and Z is the sublattice of $\mathbb{R}^m$ generated by φ, then by Theorem 4, we have that $Y \subsetneqq Z \subseteq X$. But Z, as a sublattice which contains $\mathbf{1}$ is replicated. This is a contradiction, therefore, δ_Y is maximal.

(ii) $\Longrightarrow$ (i): Suppose that Y is the sublattice of $\mathbb{R}^m$ generated by the maximal proper partition δ of $\{1, \ldots, n\}$. Then $\mathbf{1} \in Y$. If we suppose that W is a replicated subspace of X which contains Y as a proper subspace we have $Y \subsetneqq W \subseteq F_{\mathbf{1}}(W) \subseteq X$. Also $Z = F_{\mathbf{1}}(W)$ is a sublattice. By Theorem 5, the partition δ_Z which generates Z is strictly finer than δ, a contradiction. Therefore, Y is a maximal replicated subspace of X.

By Proposition 3, a maximal proper partition δ of $\{1, \ldots, n\}$ exists. This partition generates a maximal replicated submarket. □

Definition 4 The union of all maximal replicated subspaces of the market is the **replicated kernel of the market.**

Proposition 4 *The replicated kernel of the market is the set of all $x \in X$ so that any option written on x is replicated.*

Proof Denote by $\mathcal{R}$ the set of all vectors x of X so that any option written on x is replicated and by $\mathcal{F}$, the family of maximal replicated submarkets of X. Let $y \in Y$ and $Y \in \mathcal{F}$. Then Y is a sublattice contained in X with $\mathbf{1} \in Y$. If $[y]$ is the subspace generated by y, then the completion by options of $[y]$ is the sublattice generated by $[y] \cup \mathbf{1}$, so it is contained in Y, therefore, $F_{\mathbf{1}}([y]) \subseteq Y \subseteq X$. Hence any option written on y is replicated, therefore, $y \in \mathcal{R}$. So we have that $\cup_{Y \in \mathcal{F}} Y \subseteq \mathcal{R}$.

For the converse suppose that $y \in \mathcal{R}$. Then $Y = F_{\mathbf{1}}([y])$ is a sublattice of $\mathbb{R}^m$ contained in X, with $\mathbf{1} \in Y$. By Theorem 5, Y is generated by a proper partition δ_Y of $\{1, \ldots, n\}$ and by Proposition 3, δ_Y is contained in a maximal proper partition δ of $\{1, \ldots, n\}$. Therefore, Y is contained in the maximal replicated submarket Y_δ of X. So $y \in \cup_{Y \in \mathcal{F}} Y$ and the theorem is true. □

It is clear that the one-dimensional subspace $[\mathbf{1}]$ is a replicated sublattice which is contained in any replicated submarket Y of X. Also the options written on the elements of $[\mathbf{1}]$ are trivial. So we will refer to $[\mathbf{1}]$ as the **trivial replicated submarket** (subspace) of X. Any replicated subspace $Y \neq [\mathbf{1}]$ of X is a **non trivial** replicated submarket. We give now a characterization of the markets without binary vectors in terms of replicated subspaces. The direct follows also by Baptista (2007), Theorem 3, but we prove it by using the results of this article.

Theorem 7 *The market X does not contain binary vectors if and only if X does not have non trivial maximal replicated submarkets.*

Proof Suppose that X does not contain binary vectors. If Y is a non trivial maximal replicated submarket of X then Y is a sublattice of $\mathbb{R}^m$ which contains $[\mathbf{1}]$ as a proper subspace. If $\{z_1, z_2, \ldots, z_r\}$ is the positive basis of Y which is a partition of the unit, then $r \geq 2$, therefore, each z_i is a binary vector. This is a contradiction, hence X does not have non trivial maximal replicated submarkets.

For the converse suppose that X does not have non trivial maximal replicated submarkets. If we suppose that y is a binary vector of X, then $z = \mathbf{1} - y$ is also a binary vector and y, z have disjoint supports. It is easy to show that $\{y, z\}$ is a positive basis of the subspace Y of X generated by the vectors y, z. By proposition 5 in the Appendix, Y is a sublattice of $\mathbb{R}^m$. By Theorem 5, Y is generated by a proper partition δ_Y of $\{1, 2, \ldots, n\}$. By Proposition 3, there exists a maximal proper partition δ finer than δ_Y. By Theorem 6, a maximal replicated submarket Z of X which contains Y exists. Since $Y \neq [\mathbf{1}]$, Z is non trivial. This is a contradiction, therefore, X does not have binary vectors. □

5 Examples

In the next examples we determine the maximal replicated subspaces and the replicated kernel of the market. In doing so, we take the following steps:

(1) We determine a basic set $\{y_1, y_2, \ldots, y_n\}$ of marketed securities. We determine the basic function β of the vectors y_i and the range $R(\beta)$ of β. According to Theorem 9, we determine a positive basis $\{d_1, \ldots, d_\mu\}$ of $F_{\mathbf{1}}(X)$.
(2) According to Theorem 10, we determine the projection basis $\{\widetilde{d}_1, \ldots, \widetilde{d}_n\}$ of X.
(3) We determine the positive basis $\{b_1, \ldots, b_\mu\}$ of $F_{\mathbf{1}}(X)$ which is a partition of the unit and the corresponding projection basis $\{\widetilde{b}_1, \ldots, \widetilde{b}_n\}$ of X. The elements of these bases are given by the formula: $b_i = \frac{d_i}{||d_i||_\infty}$, $i = 1, 2, \ldots, \mu$ and $\widetilde{b}_i = \frac{\widetilde{d}_i}{||d_i||_\infty}$, $i = 1, 2, \ldots, n$.
(4) We determine the maximal proper partitions of $\{1, \ldots, n\}$. For any maximal proper partition δ we determine the maximal replicated submarket generated

by δ. The union of all maximal replicated submarkets is the replicated kernel of the market.

Example 1 Suppose that

$$x_1 = (1, 0, -1, 0, 0),\ \ x_2 = (1, -1, 0, 2, 2), x_3 = (0, -1, -2, -1, -1),$$

are the primitive securities and $X = [x_1, x_2, x_3]$ is the marketed space. We remark that $\mathbf{1} = x_1 - x_3 \in X$. Since $\max\{||x_i||_\infty \mid i = 1, 2, 3\} = 2$, by using Lemma 1 we find that the vectors

$$\begin{aligned} &y_1 = 2\mathbf{1} - x_1 = (1, 2, 3, 2, 2), y_2 = 2\mathbf{1} - x_2 = (1, 3, 2, 0, 0), \\ &y_3 = 2\mathbf{1} - x_3 = (2, 3, 4, 3, 3) \end{aligned}$$

define a basic set of marketed securities. The basic function of the vectors y_i is

$$\beta(i) = \frac{1}{y(i)}(y_1(i), y_2(i), y_3(i)), \quad \text{for } i = 1, \ldots, 5,$$

where $y = \sum_{i=1}^{3} y_i$. We find that

$$\begin{aligned} &\beta(1) = \frac{1}{4}(1, 1, 2) = P_1, \beta(2) = \frac{1}{8}(2, 3, 3) = P_2, \beta(3) = \frac{1}{9}(3, 2, 4) = P_3, \\ &\beta(4) = \beta(5) = \frac{1}{5}(2, 0, 3) = P_4. \end{aligned}$$

So $card(R(\beta)) = 4$, therefore, $F_{\mathbf{1}}(X)$ is a four-dimensional subspace of $\mathbb{R}^5$. The three first vectors P_1, P_2, P_3 of $R(\beta)$ are linearly independent, so we preserve the enumeration of $R(\beta)$. According to Theorem 9, $I_4 = \beta^{-1}(P_4) = \{4, 5\}$ and we define the new vector $y_4 = (0, 0, 0, 5, 5)$. The basic function of the vectors y_1, y_2, y_3, y_4 is

$$\gamma(i) = \frac{1}{y'(i)}(y_1(i), y_2(i), y_3(i), y_4(i)), \quad \text{for } i = 1, \ldots, 5,$$

where $y' = \sum_{i=1}^{4} y_i$. We we find that

$$\begin{aligned} &\gamma(1) = \frac{1}{4}(1, 1, 2, 0) = P_1', \gamma(2) = \frac{1}{8}(2, 3, 3, 0) = P_2', \gamma(3) = \frac{1}{9}(3, 2, 4, 0) = P_3', \\ &\gamma(4) = \gamma(5) = \frac{1}{10}(2, 0, 3, 5) = P_4'. \end{aligned}$$

A positive basis of $F_{\mathbf{1}}(X)$ is given by the formula $(d_1, d_2, d_3, d_4)^T = D^{-1}(y_1, y_2, y_3, y_4)^T$ where D is the matrix whose columns are the vectors P_i', $i = 1, \ldots, 4$ and we find that

$$\{d_1 = (4, 0, 0, 0, 0), d_2 = (0, 8, 0, 0, 0), d_3 = (0, 0, 9, 0, 0), d_4 = (0, 0, 0, 10, 10)\},$$

is a positive basis of $F_{\mathbf{1}}(X)$. According to Theorem 10, the projection basis of X is given by the formula $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)^T = A^{-1}(y_1, y_2, y_3)^T$ where A is the matrix whose columns are the vectors P_i, $i = 1, 2, 3$ and we find that

$$\{\tilde{d}_1 = (4, 0, 0, 4, 4), \tilde{d}_2 = (0, 8, 0, -8, -8), \tilde{d}_3 = (0, 0, 9, 9, 9)\},$$

is the projection basis of X. We find that

$$\left\{ b_1 = \frac{1}{4}d_1 = (1, 0, 0, 0, 0), b_2 = \frac{1}{8}d_2 = (0, 1, 0, 0, 0), b_3 = \frac{1}{9}d_3 = (0, 0, 1, 0, 0), \right.$$
$$\left. b_4 = \frac{1}{10}d_4 = (0, 0, 0, 1, 1) \right\}$$

is the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and that

$$\left\{\tilde{b}_1 = \frac{1}{4}\tilde{d}_1 = (1, 0, 0, 1, 1), \tilde{b}_2 = \frac{1}{8}\tilde{d}_2 = (0, 1, 0, -1, -1), \tilde{b}_3 = \frac{1}{9}\tilde{d}_3 = (0, 0, 1, 1, 1)\right\},$$

is the corresponding projection basis of X. We are looking now for the maximal proper partitions of $\{1, 2, 3\}$. We see that $q_1 = \tilde{b}_1 = (1, 0, 0, 1, 1)$ and $q_2 = \tilde{b}_2 + \tilde{b}_3 = (0, 1, 1, 0, 0)$ are binary vectors, therefore, $\{\{1\}, \{2, 3\}\}$ is a maximal proper partition of $\{1, 2, 3\}$. Also $r_1 = \tilde{b}_1 + \tilde{b}_2 = (1, 1, 0, 0, 0)$ and $r_2 = \tilde{b}_3 = (0, 0, 1, 1, 1)$ are binary vectors, therefore, $\{\{1, 2\}, \{3\}\}$ is also a maximal proper partition.

So the subspaces $Y_1 = [(1, 0, 0, 1, 1), (0, 1, 1, 0, 0)]$ and $Y_2 = [(1, 1, 0, 0, 0), (0, 0, 1, 1, 1)]$ are the maximal replicated subspaces. The set $Y_1 \cup Y_2$ is the replicated kernel of the market.

Example 2 Suppose that

$$x_1 = (1, 1, 1, 1, 2, 1), x_2 = (2, 3, 1, 1, 1, 1),$$
$$x_3 = (2, 2, 2, 1, 3, 1), x_4 = (1, 1, 1, 2, 0, 2),$$

are the primitive securities and $X = [x_1, x_2, x_3, x_4]$ is the marketed space. We remark that

$$\mathbf{1} = \frac{x_3 + x_4}{3} \in X.$$

Since the vectors x_i are positive, $\{x_1, x_2, x_3, x_4\}$ is a basic set of marketed securities. The basic function of the vectors x_1, x_2, x_3, x_4 is

$$\beta(i) = \frac{1}{x(i)}(x_1(i), x_2(i), x_3(i), x_4(i)), \quad \text{for } i = 1, \ldots, 6,$$

where $x = \sum_{i=1}^{4} x_i$. We find that

$$\beta(1) = \frac{1}{6}(1, 2, 2, 1) = P_1, \beta(2) = \frac{1}{7}(1, 3, 2, 1) = P_2, \beta(3) = \frac{1}{5}(1, 1, 2, 1) = P_3,$$
$$\beta(4) = \beta(6) = \frac{1}{5}(1, 1, 1, 2) = P_4, \beta(5) = \frac{1}{6}(2, 1, 3, 0) = P_5.$$

We remark that $card(R(\beta)) = 5$, therefore, $F_{\mathbf{1}}(X)$ is a five-dimensional subspace of $\mathbb{R}^6$. The first four P_i are linearly dependent so we consider the new the new enumeration $R(\beta) = \{P_5, P_2, P_3, P_4, P_1\}$ of $R(\beta)$. According to Theorem 9, $I_5 = \beta^{-1}(P_1) = \{1\}$ and we define the new vector $x_5 = (6, 0, 0, 0, 0, 0)$. The basic function of the vectors $x_1, \ldots, x_5$ is

$$\gamma(i) = \frac{1}{x'(i)}(x_1(i), x_2(i), x_3(i), x_4(i), x_5(i)), \quad \text{for } i = 1, \ldots, 6,$$

where $x' = \sum_{i=1}^{5} x_i$. We find that

$$\gamma(1) = \frac{1}{12}(1, 2, 2, 1, 6) = P_1', \gamma(2) = \frac{1}{7}(1, 3, 2, 1, 0) = P_2', \gamma(3) = \frac{1}{5}(1, 1, 2, 1, 0) = P_3',$$
$$\gamma(4) = \gamma(6) = \frac{1}{5}(1, 1, 1, 2, 0) = P_4', \gamma(5) = \frac{1}{6}(2, 1, 3, 0, 0) = P_5'.$$

A positive basis of $F_{\mathbf{1}}(X)$ is given by the formula

$$(d_1, d_2, d_3, d_4, d_5)^T = D^{-1}(x_1, x_2, x_3, x_4, x_5)^T,$$

where D is the matrix whose columns are the vectors $P_5', P_2', P_3', P_4', P_1'$ and we find that

$$\{d_1 = (0, 0, 0, 0, 6, 0), d_2 = (0, 7, 0, 0, 0, 0), d_3 = (0, 0, 5, 0, 0, 0),$$
$$d_4 = (0, 0, 0, 5, 0, 5), d_5 = (12, 0, 0, 0, 0, 0)\},$$

is a positive basis of $F_{\mathbf{1}}(X)$. According to Theorem 10, the projection basis of X is given by the formula $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4)^T = A^{-1}(x_1, x_2, x_3, x_4)^T$ where A is the matrix whose columns are the vectors P_5, P_2, P_3, P_4 and we find that

$$\{\tilde{d}_1 = (0, 0, 0, 0, 6, 0), \tilde{d}_2 = (3.5, 7, 0, 0, 0, 0), \tilde{d}_3 = (2.5, 0, 5, 0, 0, 0),$$
$$\tilde{d}_4 = (0, 0, 0, 5, 0, 5)\},$$

is the projection basis of X. We find that

$$\left\{ b_1 = \frac{1}{6}d_1 = (0, 0, 0, 0, 1, 0), b_2 = \frac{1}{7}d_2 = (0, 1, 0, 0, 0, 0), b_3 = \frac{1}{5}d_3 = (0, 0, 1, 0, 0, 0), \right.$$
$$\left. b_4 = \frac{1}{5}d_4 = (0, 0, 0, 1, 0, 1), b_5 = \frac{1}{12}d_5 = (1, 0, 0, 0, 0, 0) \right\},$$

is the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and

$$\left\{ \tilde{b}_1 = \frac{1}{6}\tilde{d}_1 = (0,0,0,0,1,0), \tilde{b}_2 = \frac{1}{7}\tilde{d}_2 = (0.5,1,0,0,0,0), \tilde{b}_3 = \frac{1}{5}\tilde{d}_3 = (0.5,0,1,0,0,0), \right.$$
$$\left. \tilde{b}_4 = \frac{1}{5}\tilde{d}_4 = (0,0,0,1,0,1) \right\},$$

is the corresponding projection basis of X. We remark that $r_1 = \tilde{b}_1 = (0,0,0,0,1,0)$ and $r_2 = \tilde{b}_2 + \tilde{b}_3 = (1,1,1,0,0,0)$, $r_3 = \tilde{b}_4 = (0,0,0,1,0,1)$, are binary vectors. These vectors define the maximal proper partition $\{\{1\},\{2,3\},\{4\}\}$ and we remark that this partition is unique. So the market has only one maximal replicated submarket which is the subspace $Y = [(0,0,0,0,1,0),(1,1,1,0,0,0),(0,0,0,1,0,1)]$ of X. Note also that Y is the replicated kernel of the market.

6 Appendix

A real vector space E is an **ordered vector space** if E is endowed with a reflexive, antisymmetric and transitive partial order relation $\geq$ which is compatible with the linear structure of E, i.e. $x \geq y$ implies $x+z \geq y+z$ and $ax \geq ay$, for any $z \in E$ and $a \in \mathbb{R}_+$. The set $E_+ = \{x \in E | x \geq 0\}$ is the **positive cone** of E. Any subspace Z of E ordered by the induced ordering is an **ordered subspace** of E. Then $Z_+ = Z \cap E_+$ is the positive cone of Z. E is a **Riesz space** or a **vector lattice** if for any $x, y \in E$ the supremum $x \vee y$ and the infimum $x \wedge y$ of the set $\{x, y\}$ in E exist. Suppose that E is a Riesz space. For any $x \in E$, $x^+ = x \vee 0$ is the **positive part**, $x^- = (-x) \vee 0$ is the **negative part** and $|x| = x \vee (-x)$ is the absolute value of x. We have $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Suppose that Z is an ordered subspace of E. Z is a **sublattice** of E if for every $x, y \in Z$, $x \vee y, x \wedge y \in Z$. If for any $x, y \in Z$ the supremum $x \vee_Z y$ and the infimum $x \wedge_Z y$ of $\{x, y\}$ in Z exist then Z is a **lattice-subspace** of E. Then we have $x \vee_Z y \geq x \vee y \geq x \wedge y \geq x \wedge_Z y$. Any sublattice of E is a lattice-subspace but the converse is not always true.

If B is a nonempty subset of E, **the sublattice of** E **generated by** B is the intersection of all sublattices of E which contain B. We can show that the sublattice of E generated by B is the minimum sublattice of E which contains B.

6.1 Lattice-subspaces and positive bases in $C(\Omega)$

We present here the basic mathematical notions and results of the articles Polyrakis (1996, 1999) which are needed for this study.

Suppose that $E = C(\Omega)$ is the space of real valued functions defined on a compact Hausdorff topological space Ω. Recall that if the set Ω is finite, i.e. $\Omega = \{1, 2, \ldots, m\}$, then $C(\Omega)$ is the space $\mathbb{R}^m$.

The space $C(\Omega)$ is ordered by the pointwise ordering i.e. for any $x, y \in C(\Omega)$ we have $x \geq y$ if and only if $x(t) \geq y(t)$ for each $t \in \Omega$ and $C_+(\Omega) = \{x \in C(\Omega) \big| x(t) \geq 0 \text{ for each } t \in \Omega\}$ is the positive cone of $C(\Omega)$.

Suppose that Z is a finite-dimensional **ordered subspace** of $C(\Omega)$. The set $\{b_1, b_2, \ldots, b_r\}$ is a **positive basis** of Z if it is a basis of Z and $Z_+ = \{x = \sum_{i=1}^r \lambda_i b_i \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\}$. Although Z has infinitely many bases, the existence of a positive basis of Z is not always ensured. For a finite-dimensional ordered subspace Z of $C(\Omega)$ we have: Z has a positive basis if and only if Z is a lattice-subspace of $C(\Omega)$.

Proposition 5 (Polyrakis 1996, Proposition 2.2) *An ordered subspace Z of $C(\Omega)$ with a positive basis $\{b_1, b_2, \ldots, b_r\}$ is a sublattice of $C(\Omega)$ if and only if $b_i^{-1}(0, +\infty) \cap b_j^{-1}(0, +\infty) = \emptyset$ for any $i \neq j$.*

As a consequence of this result we have the following which is very useful in the theory of options.

Proposition 6 *Suppose that Z is a finite dimensional sublattice of $\mathbb{R}^m$. If the constant function $\mathbf{1}$ is an element of Z, then Z has a positive basis $\{b_1, b_2, \ldots, b_r\}$ which is a* **partition of the unit***, i. e. $\mathbf{1} = \sum_{i=1}^r b_i$ and for each vector b_i we have: $b_i(t) = 1$, for each $t \in \Omega$ with $b_i(t) > 0$.*

Suppose now that $z_1, z_2, \ldots, z_r$ are fixed, linearly independent, positive vectors of $C(\Omega)$. The function

$$\beta(t) = \left(\frac{z_1(t)}{z(t)}, \frac{z_2(t)}{z(t)}, \ldots, \frac{z_r(t)}{z(t)}\right), \text{ for each } \ t \in \Omega, \text{ with } z(t) > 0,$$

where $z = z_1 + z_2 + \cdots + z_r$, is the **basic function** of $z_1, z_2, \ldots, z_r$. Denote by $R(\beta)$ the range of β and by $card R(\beta)$ the cardinal number of $R(\beta)$. These notations are used in the next two results.

Theorem 8 (Polyrakis 1999, Theorem 3.6) *If Z is the subspace of $C(\Omega)$ generated by the vectors $z_1, z_2, \ldots, z_r$ and β the basic function of $z_1, z_2, \ldots, z_r$, we have: Z is a sublattice of $C(\Omega)$ if and only if $card R(\beta) = r$.*

If $R(\beta) = \{P_1, P_2, \ldots, P_r\}$, a positive basis $\{b_1, b_2, \ldots, b_r\}$ of Z is given by the formula:

$$(b_1, b_2, \ldots, b_r)^T = A^{-1}(z_1, z_2, \ldots, z_r)^T, \tag{3}$$

where A is the $r \times r$ matrix whose the ith column is the vector P_i, for each $i = 1, 2, \ldots, r$, $(b_1, b_2, \ldots, b_r)^T$, $(z_1, z_2, \ldots, z_r)^T$ are the matrices with rows the vectors $b_1, b_2, \ldots, b_r$ and $z_1, z_2, \ldots, z_r$.

The next result gives an algorithm for the determination of the sublattice generated by a finite set of linearly independent and positive vectors.

Theorem 9 (Polyrakis 1999, Theorem 3.7) *Let Z be the sublattice of $C(\Omega)$ generated by the set $\{z_1, z_2, \ldots, z_r\}$ and let $\mu \in \mathbb{N}$. If β is the basic function of $z_1, z_2, \ldots, z_r$, statements* (i) *and* (ii) *are equivalent*:

(i) $dim(Z) = \mu$.
(ii) $R(\beta) = \{P_1, P_2, \ldots, P_\mu\}$.

If statement (ii) *is true,* Z *is constructed as follows:*

(a) *Enumerate* $R(\beta)$ *so that its* r *first vectors to be linearly independent (as it is shown in Polyrakis (1999), such an enumeration always exists). Denote again by* $P_i,\ i = 1, 2, \ldots, \mu$ *the new enumeration and we put* $I_{r+k} = \{t \in \Omega \big| \beta(t) = P_{r+k}\}$*, for each* $k = 1, 2, \ldots, \mu - r$.

(b) *Define the vectors* z_{r+k}*,* $k = 1, 2, \ldots, \mu - r$ *as follows:*

$$z_{r+k}(i) = z(i) \text{ if } i \in I_{r+k} \text{ and } z_{r+k}(i) = 0 \text{ if } i \notin I_{r+k},$$

where $z = z_1 + z_2 + \cdots + z_r$.

(c) $Z = [z_1, z_2, \ldots, z_r, z_{r+1}, \ldots, z_\mu]$.

(d) *A positive basis* $\{b_1, b_2, \ldots, b_\mu\}$ *of* Z *is constructed as follows:*
Consider the basic function γ *of* $z_1, z_2, \ldots, z_r, z_{r+1}, \ldots, z_\mu$ *and suppose that*

$$\{P'_1, P'_2, \ldots, P'_\mu\}$$

is the range of γ *(the range of* γ *has exactly* μ *points). Then*

$$(b_1, b_2, \ldots, b_\mu)^T = D^{-1}(z_1, z_2, \ldots, z_\mu)^T, \tag{4}$$

where D *is the* $\mu \times \mu$ *matrix with columns the vectors* $P'_1, P'_2, \ldots, P'_\mu$.

6.2 Projection bases

The notion of the projection basis has been defined in Polyrakis (2003) for a finite dimensional subspace L of $C(\Omega)$ generated by the linearly independent positive vectors

$y_1, y_2, \ldots, y_r$ of $C(\Omega)$ which is contained (the subspace L) in a finite dimensional lattice-subspace W of $C(\Omega)$. This basis, $\{\widetilde{b_1}, \widetilde{b_2}, \ldots, \widetilde{b_r}\}$, is called projection basis because its elements are projections of the elements of the positive basis of the lattice-subspace W. In the case where the sublattice Z of $C(\Omega)$ generated by L is finite dimensional, a projection basis of L can be defined analogously by taking the projections of the positive basis of Z in L. The proof of the next result is exactly analogous with the one of Theorem 9 of Polyrakis (2003), so we give it without proof and we do not consider this result as new.

Theorem 10 (Polyrakis 2003, Theorem 9) *Let* Z *be the sublattice of* $C(\Omega)$ *generated by the set* $\{z_1, z_2, \ldots, z_r\}$ *of linearly independent and positive vectors of* $C(\Omega)$ *and suppose that* $dim(Z) = \mu$*. Suppose that we follow the steps and the same notations of statement* (ii) *of Theorem 9 for the determination of a positive basis* $\{b_1, b_2, \ldots, b_\mu\}$ *of* Z *which is given by* (4)*. So we suppose that* β *is the basic function of the vectors* $z_1, z_2, \ldots, z_r$ *and* $P_1, P_2, \ldots, P_r, P_{r+1}, \ldots, P_\mu$ *is an enumeration of*

$R(\beta)$ so that the vectors $P_1, P_2, \ldots, P_r$ are linearly independent and suppose also that $z_{r+1}, z_{r+2}, \ldots, z_\mu$ are the new vectors constructed in (b) of Theorem 9.

If $L = [z_1, z_2, \ldots, z_r]$ is the subspace of $C(\Omega)$ generated by the vectors $z_1, z_2, \ldots, z_r$ we have:

(i) $Z = L \oplus [z_{r+1}, z_{r+2}, \ldots, z_\mu]$,

(ii) $\{b_{r+1}, b_{r+2}, \ldots, b_\mu\} = \{2z_{r+1}, 2z_{r+2}, \ldots, 2z_\mu\}$,

(iii) *If $b_i = \widetilde{b_i} + b_i'$, with $\widetilde{b_i} \in L$ and $b_i' \in [z_{r+1}, z_{r+2}, \ldots, z_\mu]$, for each $i = 1, 2, \ldots, r$, then $\{\widetilde{b_1}, \widetilde{b_2}, \ldots, \widetilde{b_r}\}$ is a basis of L which we call* **projection basis** *of L and is given by the formula*

$$(\widetilde{b_1}, \widetilde{b_2}, \ldots, \widetilde{b_r})^T = A^{-1}(z_1, z_2, \ldots, z_r)^T, \tag{5}$$

where A is the $n \times n$ matrix whose the ith column is the vector P_i, for $i = 1, 2, \ldots, r$. This basis has the property: The r first coordinates of any element x of L in the basis $\{b_1, b_2, \ldots, b_\mu\}$ coincide with the corresponding coordinates of x in the projection basis, i.e.

$$x = \sum_{i=1}^{\mu} \lambda_i b_i \in L \implies x = \sum_{i=1}^{r} \lambda_i \widetilde{b_i}. \tag{6}$$

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