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LINEAR OPTIMIZATION IN $C(\Omega)$ AND PORTFOLIO INSURANCE

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Suppose that X is a subspace of $C(\Omega)$ generated by n linearly independent positive elements of $C(\Omega)$. In this article we study the problem of minimization of a positive linear functional p of X in X, under a finite number of linear inequalities. This problem does not have always a solution and if a solution exists we cannot determine it. In this article we show that if X is contained in a finite dimensional minimal lattice-subspace Y of $C(\Omega)$ (or equivalently, if X is contained in a finite dimensional minimal subspace Y of $C(\Omega)$ with a positive basis) and $m = \dim Y$, then the minimization problem has a solution and we determine the solutions by solving an equivalent linear programming problem in $\mathbb{R}^m$.

Finally note that this minimization problem has an important application in the portfolio insurance which was the motivation for the preparation of this article.

Keywords: Linear optimization; Lattice subspace; Projection basis; Portfolio insurance

Mathematics Subject Classification 2000: 90C05; 91B28

1 INTRODUCTION

In this article we will denote by Ω a compact Hausdorff topological space and by $C(\Omega)$ the space of real valued continuous functions defined on Ω. The space $C(\Omega)$ is ordered by the pointwise ordering and

$$C_+(\Omega) = \{x \in \Omega \mid x(t) \geq 0, \text{ for each } t \in \Omega\},$$

is the positive cone of $C(\Omega)$. Suppose that $x_1, x_2, \ldots, x_n$ are fixed linearly independent, positive elements of $C(\Omega)$ and

$$X = [x_1, x_2, \ldots, x_n]$$

is the subspace of $C(\Omega)$ generated by the elements x_i. We study the following minimization problem:

$$\text{Minimize } p(z) \text{ subject to } z \in X,\ z \geq 0,\ z \geq z_1, \ldots, z \geq z_r \text{ and } p_i(z) \geq \alpha_i, i = 1, 2, \ldots, l, \quad (1)$$

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where p is a positive linear functional[1] of X, z_1, z_2, ..., z_r are fixed elements of X, p_1, p_2, ..., p_l linear functionals of X and $\alpha_1, \alpha_2, \ldots, \alpha_l$ constant real numbers.

If the linear functionals p_i and the real numbers α_i are equal to zero, the problem is:

$$\text{Minimize } p(z) \text{ subject to } z \in X, z \geq 0, z \geq z_1, \ldots, z \geq z_r. \tag{2}$$

The *feasible set*

$$P = \{z \in X \mid z \geq 0, z \geq z_1, \ldots, z \geq z_r\}, \tag{3}$$

of (2) is nonempty. This holds because $X = X_+ - X_+$, therefore for each i we have $z_i = z_{i1} - z_{i2}$ with z_{i1}, $z_{i2} \in X_+$, hence $\sum_{i=1}^{r} z_{i1} \in P$. In the case where Ω is a finite set with n elements the above problem is the linear programming problem in $\mathbb{R}^n$.

In this article we solve the minimization problem in the case where the set Ω is infinite, the sum of the functions x_i is strictly positive and X is contained in a finite-dimensional minimal lattice-subspace Y of $C(\Omega)$. Then by [6,7] we have that Y has a positive basis $\{b_1, b_2, \ldots, b_m\}$ with nodes the points $t_1, t_2, \ldots, t_m$ of Ω. This implies that any inequality $x \geq y$ in X is equivalent with the finite number of real inequalities $x(t_i) \geq y(t_i)$, for each $i = 1, 2, \ldots, m$. In the sequel we construct a new basis $\{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\}$ of X which we call projection basis because its elements are projections of the elements of the positive basis of Y. This basis has the property: for any $x \in X$ the n first coordinates of x in the positive basis of Y coincide with the coordinates of x in the projection basis of X. Based on these properties of X we convert the minimization problem in an equivalent linear programming problem in $\mathbb{R}^m$.

In Theorem 13, we show that for any positive linear functional p of X, the minimization problem (2) has at least one solution and we determine the solutions of (2) by solving an equivalent linear programming problem in $\mathbb{R}^m$. In Section 6 we give some criteria by means of which we check if a finite-dimensional minimal lattice-subspace Y of $C(\Omega)$ containing X exists.

Finally note that the minimization problem does not always have a solution, Example 2, therefore the assumption that X is contained in a finite dimensional minimal lattice-subspace cannot be omitted.

The basic ideas to use lattice-subspaces in the study of the minimization problem are contained in [2] where the problem of the minimum cost portfolio insurance is studied in the case where X is a lattice-subspace. In the present article we study the minimization problem under the general assumption that X is contained in a finite dimensional minimal lattice-subspace. Also we note that a similar method but with a different proof, for the computation of the minimum cost portfolio insurance in finite dimensional spaces is used in [3].

Finally note that if the functional p is not positive, the minimization problem (2) does not have a solution. This holds because if we suppose that $p(w) < 0$ for some $w \in X_+$ then for each $z \in P$ we have $\lambda w + z \geq 0$, $z_1, \ldots, z_r$, for any positive real number λ, therefore $\lim_{\lambda \to \infty} p(\lambda w + z) = -\infty$.

[1]We will also refer p as the "price" or the "price vector" because in the economic applications the functional p is the price.

For the sake of completeness we start with the next result. We give an easy proof of it but we do not present this result as new.

THEOREM 1 *If the price vector p is a strictly positive linear functional of X, then the minimization problem* (2) *has a solution.*

Proof Since p is strictly positive, there exists a real number $a > 0$ such that $p(w) \geq a\|w\|$ for each $w \in X_+$, $w \neq 0$. This holds because if we suppose that this assertion is not true there exists a sequence $\{u_v\}$ of X_+ with $\|u_v\| = 1$ and $\lim_{v\to\infty} p(u_v) = 0$. If u_0 is the limit of a convergent subsequence of $\{u_v\}$ we have that $\|u_0\| = 1$ and $p(u_0) = 0$, contradiction because p is strictly positive. Suppose that w_0 is a constant element of P and $B = \{\psi \in P \mid p(\psi) \leq p(w_0)\}$. Then B is compact, therefore the minimization problem has at least one solution. ■

In the next example it is shown that if the price vector is not strictly positive, the minimization problem (2) does not always have a solution.

Example 2 Suppose that $\Omega = [0, 1]$ and X is the subspace of $C(\Omega)$ generated by the functions $x_1(t) = 1$, $x_2(t) = t$, $x_3(t) = t^2$. Suppose also that $z_1(t) = 1/2 - t$ and that the price p is the Dirac measure $\delta_{1/2}$ supported at $1/2$. The problem

$$\text{Minimize } p(z) \text{ subject to } z \in X, z \geq 0, z \geq z_1,$$

does not have a solution because the functions

$$w_n = w_n(t) = nt^2 - nt + \frac{n}{4} + \frac{1}{n},$$

belong to the feasible set P with $p(w_n) = 1/n$. Therefore the infimum of p in P is equal to zero which is not attained in P.

2 NOTATION

Suppose that Y is a subspace of $C(\Omega)$ ordered also by the pointwise ordering. The set $Y_+ = Y \cap C_+(\Omega)$ is the positive cone of Y. A linear functional p of Y is *positive* if $p(y) \geq 0$, for each $y \in Y_+$ and *strictly positive* if $p(y) > 0$, for each $y \in Y_+$, $y \neq 0$.

If Y is a vector lattice i.e. if for each $x, y \in Y$ the supremum $\sup_Y\{x, y\}$ of $\{x, y\}$ in Y exists,[2] we will say that Y is a *lattice-subspace* of $C(\Omega)$. It is clear that

$$x \vee y \leq \sup_Y \{x, y\}$$

where $x \vee y$ is the supremum of $\{x, y\}$ in $C(\Omega)$. Suppose that Y is finite-dimensional and $\{b_1, b_2, \ldots, b_m\}$ is a basis of Y. If for each $y \in Y$ it holds: $y \in Y_+$ if and only if the coordinates of y in the basis $\{b_1, b_2, \ldots, b_m\}$ are positive, then we say that $\{b_1, b_2, \ldots, b_m\}$ is a *positive basis* of Y. The following statements are equivalent:

(i) Y has a positive basis $\{b_1, b_2, \ldots, b_m\}$.
(ii) Y is order-isomorphic[3] to $\mathbb{R}^m$.
(iii) Y is an m-dimensional lattice-subspace of $C(\Omega)$.

[2] $\sup_Y\{x, y\} = z$ if and only if $z \in Y$, $z \geq x, y$ and for each $w \in Y$, $w \geq x, y$ implies $w \geq z$.
[3] i.e. there exists a linear operator T or Y onto $\mathbb{R}^m$ with the property: $y \in Y_+$ if and only if $T(y) \in \mathbb{R}^m_+$.

The equivalence of (i) and (iii) is the most useful criterion for finite-dimensional lattice-subspaces of $C(\Omega)$. Suppose that $\{b_1, b_2, \ldots, b_m\}$ is a positive basis of Y. Then for each $x = \sum_{i=1}^m \lambda_i b_i$, $y = \sum_{i=1}^m \mu_i b_i \in Y$ we have that $\sup_Y\{x, y\} = \sum_{i=1}^m (\lambda_i \vee \mu_i) b_i$.[4] If there exist $t_1, t_2, \ldots t_m \in \Omega$ such that $b_i(t_i) > 0$ for each i and $b_i(t_j) = 0$ for each $i \neq j$, then we say that the points $t_1, t_2, \ldots, t_m$ are *nodes* for the basis $\{b_1, b_2, \ldots, b_m\}$.[5] Then it is easy to show that the expansion of any element y of Y in the basis is

$$y = \sum_{i=1}^m \frac{y(t_i)}{b_i(t_i)} b_i. \tag{4}$$

So if $x, y \in X$, the inequality $x \geq y$ is equivalent with the finite number of real inequalities $x(t_i) \geq y(t_i)$ for $i = 1, 2, \ldots, n$.

Recall that a subset C of $\mathbb{R}^l$ is a *polytope* if C is the convex hull of a finite subset of $\mathbb{R}^l$ and C is an $(m-1)$-*simplex* if it is the affine convex hull of m affinely independent vectors of $\mathbb{R}^l$. In both cases the extreme points of C are called also vertices of C. For any matrix A denote by A^T the transpose and by A^{-1} the inverse of A.

3 THE LATTICE-SUBSPACE CASE

In the case where X is a lattice-subspace the minimization problem (2) has a unique solution which is also the same for each p (price independent) but in general we cannot determine it. If we can find a positive basis of X then we can also determine the solution of the minimization problem. This method has been developed in [2] for the determination of the minimum cost portfolio insurance. In this section we use the results of [6] in order to construct a positive basis of X and also a set of nodes of this basis. The next result has been proved in [2] for $r = 1$ for the determination of the minimum cost portfolio insurance in the case where X is a lattice-subspace. For the sake of completeness we give its proof.

THEOREM 3 ([2], Theorem 3.2) *The space X is a lattice-subspace of $C(\Omega)$, if and only if for any choice of points $z_1, z_2, \ldots, z_r$ of X the minimization problem* (2) *has a price independent solution.*[6] *The solution is* $\sup_X\{z_1, z_2, \ldots, z_r, 0\}$.

Proof Suppose that X is a lattice-subspace. Then for each $z \in X$ with $z \geq z_1, z_2, \ldots, z_r, 0$ we have that $z \geq \sup_X\{z_1, z_2, \ldots, z_r, 0\}$, therefore $p(z) \geq p(\sup_X\{z_1, z_2, \ldots, z_r, 0\})$, for any positive linear functional p of X. Therefore $\sup_X\{z_1, z_2, \ldots, z_r, 0\}$ is a price independent solution. For the converse suppose that $z_1 \in X$ and that z_0 is a price independent solution of the minimization problem (2) with $r = 1$. Then if we suppose that $w \in X$ with $w \geq z_1$, 0 we have that $p(w) \geq p(z_0)$ for any positive linear functional p of X, therefore $w \geq z_0$. Hence $z_0 = \sup_X\{z_1, 0\}$ therefore X is a lattice-subspace.

Remark 4 If $\{b_1, b_2, \ldots, b_n\}$ is a positive basis of X and $z_j = \sum_{i=1}^n \lambda_{ji} b_i$, then $\sup_X\{z_1, z_2, \ldots, z_r, 0\} = \sum_{i=1}^n (\sup\{\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{ri}, 0\}) b_i$, therefore the determination

[4]for each $a, b \in \mathbb{R}$ denote also by $a \vee b$ the maximum of $\{a, b\}$.
[5]The notion of the positive basis with nodes has been defined in [6].
[6]The converse is true if the minimization problem has a price independent solution for $r = 1$ and any $z_1 \in X$.

of a positive basis of X implies also the solution of the minimization problem. Note also that if $\{b_i\}$ is a positive basis of X with nodes then the expansion of x_0 is given automatically by (4).

The function

$$\beta(t) = \frac{r(t)}{\|r(t)\|_1},$$

where $r(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $\|r(t)\|_1 = \sum_{i=1}^n |x_i(t)| = \sum_{i=1}^n x_i(t)$, for each $t \in \Omega$, is called *the basic function* (*curve*) of $x_1, x_2, \ldots, x_n$ and takes values in the simplex $\Delta_n = \{\xi \in \mathbb{R}_+^n \mid \|\xi\|_1 = 1\}$ of $\mathbb{R}_+^n$.[7]

We denote by $D(\beta)$ the domain and by $R(\beta)$ the range of β. Also we will denote by K the convex hull of the closure of the range of β, i.e.

$$K = \mathrm{co}\overline{R(\beta)}.$$

Note that if $\|r(t)\|_1 > 0$ for each $t \in \Omega$, then

$$K = \mathrm{co}R(\beta),$$

because the domain of β is the whole space Ω, therefore the range of β is compact as the continuous image of a compact set.

The following Theorem is a criterion for lattice-subspaces and also it determines a positive basis in X.

THEOREM 5 ([6], Theorem 3.6) *The following statements are equivalent.*

(i) *X is a lattice-subspace.*
(ii) *K is an $(n-1)$-simplex.*

Suppose that statement (ii) *is true and $P_1, P_2, \ldots, P_n$ are the vertices of K. Then the following statements hold*:

(a) *If A is the $n \times n$ matrix whose the ith column is the vector P_i, for $i = 1, 2, \ldots, n$ and $b_1, b_2, \ldots, b_n$ are the functions defined by the formula*

$$(b_1(t), b_2(t), \ldots, b_n(t))^T = A^{-1}(x_1(t), x_2(t), \ldots, x_n(t))^T \tag{5}$$

then $\{b_1, b_2, \ldots, b_n\}$ is a positive basis of X.

(b) *If $\beta(t_i) = P_i$ for some i, then t_i is an i-node for the positive basis of X.*[8]

(c) *If $x_1(t) + x_2(t) + \cdots + x_n(t) > 0$, for each $t \in \Omega$, then $\{b_1, b_2, \ldots, b_n\}$ is a positive basis with nodes.*

4 MINIMAL LATTICE-SUBSPACES

Suppose that $L(X)$ is the set of lattice-subspaces of $C(\Omega)$ which contain X. If $Y \in L(X)$ and for any proper subspace Z of Y we have that $Z \notin L(X)$, then we say that Y is a minimal lattice-subspace of $C(\Omega)$ which contains X.

[7] $\|\xi\|_1 = \sum_{i=1}^n |\xi_i|$ is the ℓ_1-norm of ξ.
[8] i.e. $b_i(t_i) > 0$ and $b_j(t_i) = 0$, for each $j \neq i$.

THEOREM 6 (Existence of a minimal lattice-subspace, [7], Theorem 3.20) *The following statements hold*:

(i) *The set K is a polytope with m vertices, if and only if, an m-dimensional minimal lattice-subspace Y of $C(\Omega)$ containing X exists.*
(ii) *If an m-dimensional minimal lattice-subspace Y of $C(\Omega)$ containing X exists, then each finite dimensional minimal lattice-subspace Z of $C(\Omega)$ which contains X is of dimension m.*

THEOREM 7 (Construction of a minimal lattice-subspace, [7], Theorem 3.10) *Let the set K be a polytope with vertices $P_1, P_2, \ldots, P_m$. Suppose that the n first of them $P_1, P_2, \ldots, P_n$ are linearly independent.*[9] *Suppose also that ξ_i, $i=1,2,\ldots,m$ are positive, continuous real-valued functions defined on $D(\beta)$ such that $\sum_{i=1}^{m} \xi_i(t) = 1$ and $\beta(t) = \sum_{i=1}^{m} \xi_i(t)P_i$, for each $t \in D(\beta)$.*[10] *Suppose also that $x_{n+i}, i=1,2,\ldots,m-n$ are the functions $x_{n+i}(t) = \xi_{n+i}(t)\|r(t)\|_1$ for each $t \in D(\beta)$ and $x_{n+i}(t)=0$ if $t \notin D(\beta)$. Then the space*

$$Y = [x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m]$$

is a minimal lattice-subspace of $C(\Omega)$ containing $x_1, x_2, \ldots, x_n$ and $\dim Y = m$. *A positive basis $\{b_1, b_2, \ldots, b_m\}$ of Y is given by the formula*

$$(b_1, b_2, \ldots, b_m)^T = D^{-1}(x_1, x_2, \ldots, x_m)^T, \tag{6}$$

where D is the $m \times m$ matrix with columns the vectors

$$R_i = \frac{M_i}{\|M_i\|_1}, \quad i = 1, 2, \ldots, m, \tag{7}$$

where $M_i = (P_i, 0)$ for $i = 1, 2, \ldots, n$ and $M_i = (P_{n+i}, e_i)$ for $i = 1, 2, \ldots, l$.[11]

Remark 8 If the sum of the functions x_i, $i=1,2,\ldots,n$ is strictly positive, i.e. $\sum_{i=1}^{n} x_i(t) > 0$, for each $t \in \Omega$, then $\{b_1, b_2, \ldots, b_m\}$ is a positive basis of Y with nodes. This follows by statement (c) of Theorem 5 and the fact that the sum of the functions $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m$ which generate Y is also strictly positive.

5 THE PROJECTION BASIS

In the case where X is contained in a finite dimensional minimal lattice-subspace Y of $C(\Omega)$ we construct a new basis $\{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\}$ which we call projection basis because its elements are projections of the elements of the positive basis of Y. The positive cone of the projection basis contains the positive cone of X. This basis is very important in this article because we use it in order to convert the minimization

[9] As it is shown in [7], such an enumeration of the vertices of K always exists.
[10] Functions ξ_i with these properties always exist as it is proved in [4].
[11] $(P_i, 0)$ is the vector of $\mathbb{R}^m$ whose the n first coordinates are the coordinates of P_i and the others are zero. (P_{n+i}, e_i) is the vector of $\mathbb{R}^m$ whose the n first coordinates are the coordinates of P_{n+i}, the $n+i$ coordinate is equal to 1 and the others are zero.

problem in to a linear programming problem. Also for the sake of completeness note that, if X is a lattice-subspace then, by [5] there exists a positive projection from the sublattice generated by X onto X but the projection we use here in the projection basis is different by the previous one.

THEOREM 9 *Let Y be an m-dimensional minimal lattice-subspace of $C(\Omega)$ containing $x_1, x_2, \ldots, x_n$ constructed as in Theorem 7 and suppose that $\{b_1, b_2, \ldots, b_m\}$ is a positive basis of Y which is given by* (6). *Suppose also that $P_1, P_2, \ldots, P_m$ is the enumeration of the vertices of K which is used in Theorem* 7. *Then*

(i) $Y = X \oplus [x_{n+1}, x_{n+2}, \ldots, x_m]$.
(ii) $b_i = 2x_i$, *for each* $i = n+1, n+2, \ldots, m$.
(iii) *If* $b_i = \tilde{b}_i + b'_i$, *with* $\tilde{b}_i \in X$ *and* $b'_i \in [x_{n+1}, x_{n+2}, \ldots, x_m]$, *for each* $i = 1, 2, \ldots, n$, *then* $\{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\}$ *is a basis of X which we call projection basis and is given by the formula*

$$(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)^T = A^{-1}(x_1, x_2, \ldots, x_n)^T, \tag{8}$$

where A is the $n \times n$ matrix whose the ith column is the vector P_i, for $i = 1, 2, \ldots, n$. This basis has the property: The n first coordinates of any element x of X in the basis $\{b_1, b_2, \ldots, b_m\}$ coincide with the corresponding coordinates of x in the projection basis, i.e.

$$x = \sum_{i=1}^{m} \lambda_i b_i \in X, \quad \textit{implies } x = \sum_{i=1}^{n} \lambda_i \tilde{b}_i.$$

Proof Since the elements x_i, $i = 1, 2, \ldots, m$, are linearly independent, the space Y is the direct sum

$$Y = X \oplus [x_{n+1}, x_{n+2}, \ldots, x_m].$$

The basis $\{b_1, b_2, \ldots, b_m\}$ of Y is given by the formula

$$(b_1, b_2, \ldots, b_m)^T = D^{-1}(x_1, x_2, \ldots, x_m)^T,$$

where D is the matrix with columns the vectors R_i, $i = 1, 2, \ldots, m$ of (7). Therefore

$$D = \begin{bmatrix} A & B \\ 0 & \frac{1}{2}I \end{bmatrix}$$

where A is the $n \times n$ matrix whose the ith column is the vector P_i, B is the $n \times (m-n)$ matrix with columns the vectors $(1/2)P_i, i = n+1, n+2, \ldots, m$[12] 0 is the $(m-n) \times n$ zero matrix and I the $(m-n) \times (m-n)$ unit matrix. Hence

$$D^{-1} = \begin{bmatrix} A^{-1} & C \\ 0 & 2I \end{bmatrix}$$

[12]note that $\|M_i\|_1 = 2$, for each $i = n+1, n+2, \ldots, m$.

where $c_{ij} = -2a_i \cdot b_j$ and $a_i \cdot b_j$ is the dot product of the ith row a_i of A^{-1} and the jth column b_j of B. Therefore for each $i = 1, 2, \ldots, n$ we have:

$$b_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=n+1}^{m} c_{ij}x_j = \tilde{b}_i + b'_i,$$

where $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ is the ith row of the matrix A^{-1}, $\tilde{b}_i = \sum_{j=1}^{n} a_{ij}x_j$ and $b'_i = \sum_{j=n+1}^{m} c_{ij}x_j$. It is clear that

$$(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)^T = A^{-1}(x_1, x_2, \ldots, x_n)^T.$$

Also $\{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n]$ is a basis of X because the matrix A is invertible and the functions x_i are linearly independent. By the form of D^{-1} we have also that $b_i = 2x_i$, for each $i = n+1, \ldots, m$.

Suppose that $x \in X$ and $x = \sum_{i=1}^{m} \lambda_i b_i$ is the expansion of x in the basis $\{b_i\}$. Then we have:

$$x = \sum_{i=1}^{n} \lambda_i \tilde{b}_i + \sum_{i=1}^{n} \lambda_i b'_i + \sum_{i=n+1}^{m} \lambda_i b_i,$$

therefore

$$x - \sum_{i=1}^{n} \lambda_i \tilde{b}_i = \sum_{i=1}^{n} \lambda_i b'_i + \sum_{i=n+1}^{m} \lambda_i b_i,$$

hence

$$x = \sum_{i=1}^{n} \lambda_i \tilde{b}_i$$

and

$$\sum_{i=1}^{n} \lambda_i b'_i + \sum_{i=n+1}^{m} \lambda_i b_i = 0.$$

■

6 CRITERIA

In general it is difficult to study if K is a polytope or not and if it is it is also difficult to determine its vertices. In this section we give two criteria. The derivative criterion below says that if K a polytope and $\beta(t_0)$ is a vertex of K then the derivative of the restriction of β at any curve of Ω having t_0 as an interior point is equal to zero. We start with the general case where the restriction of β in the curves of Ω is not differentiable.

THEOREM 10 *Let K be a polytope and let $\beta(t_0)$ be a vertex of K. Suppose that $\{a_r\}$ is a sequence of real numbers convergent to zero with $a_{2r} > 0$ and $a_{2r+1} < 0$ for each r and suppose also that $\{t_r\}$ is a sequence of Ω. If $\lim_{r\to\infty}(\beta(t_r) - \beta(t_0))/a_r = \ell$, $\ell \in \mathbb{R}^n$, then $\ell = 0$.*

Proof Let $\lim_{r\to\infty}(\beta(t_r) - \beta(t_0))/a_r = \ell \neq 0$. Then there exists r_0 such that $\beta(t_r) \neq \beta(t_0)$ for each $r > r_0$. Hence

$$\lim_{r\to\infty} \frac{\beta(t_{2r}) - \beta(t_0)}{\|\beta(t_{2r}) - \beta(t_0)\|} = \lim_{r\to\infty} \frac{\beta(t_{2r}) - \beta(t_0)}{a_{2r}} \cdot \lim_{r\to\infty} \frac{1}{\|(\beta(t_{2r}) - \beta(t_0))/a_{2r}\|} = \frac{\ell}{\|\ell\|},$$

and similarly

$$\lim_{r\to\infty} \frac{\beta(t_{2r+1}) - \beta(t_0)}{\|\beta(t_{2r+1}) - \beta(t_0)\|} = -\frac{\ell}{\|\ell\|},$$

because

$$\left\| \frac{\beta(t_{2r+1}) - \beta(t_0)}{a_{2r+1}} \right\| = -\frac{\|\beta(t_{2r+1}) - \beta(t_0)\|}{a_{2r+1}}.$$

Since $\beta(t_0)$ is a vertex of K it is easy to show that there exists a real number $\rho > 0$ such that

$$\beta(t_0) + \rho \frac{\xi - \beta(t_0)}{\|\xi - \beta(t_0)\|} \in K, \quad \text{for each } \xi \in K, \xi \neq \beta(t_0),$$

therefore

$$\lim_{r\to\infty} \left(\beta(t_0) + \rho \frac{\beta(t_{2r}) - \beta(t_0)}{\|\beta(t_{2r}) - \beta(t_0)\|} \right) = \beta(t_0) + \rho \frac{\ell}{\|\ell\|} = z_1 \in K$$

and

$$\lim_{r\to\infty} \left(\beta(t_0) + \rho \frac{\beta(t_{2r+1}) - \beta(t_0)}{\|\beta(t_{2r+1}) - \beta(t_0)\|} \right) = \beta(t_0) - \rho \frac{\ell}{\|\ell\|} = z_2 \in K.$$

Hence $\beta(t_0) = (1/2)(z_1 + z_2)$, contradiction. Therefore $\ell = 0$. ■

COROLLARY 1 (The derivative criterion) *Let K be a polytope and let $\beta(t_0)$ be a vertex of K. Suppose that σ is a function defined on the real interval $(-\epsilon, \epsilon)$ with values in Ω, $\sigma(0) = t_0$ and suppose that $\varphi = \beta_0\sigma$ is the composition of σ, β. Then*

$$\varphi'(0) = 0,$$

whenever the derivative $\varphi'(0)$ of φ at the point 0 exists.

THEOREM 11 (The Wronskian criterion, [6], Corollary 3.3) *Let Ω be the closed interval $[a, b]$ of $\mathbb{R}$ and $\dim X > 2$. Suppose that (c, d) is an open interval of $\mathbb{R}$ which contains $[a, b]$.*

If the functions x_i have continuous derivatives up to the nth order in (c, d) and the Wronskian[13] *of the functions x_i is nonzero for any point of (c, d), then X is not a lattice-subspace of $C[a, b]$.*

Remark 12 In the case where the assumptions of the derivative criterion are satisfied and the sum of the functions x_i is strictly positive, we study if K is a polytope or not as follows:

Since the sum of the functions x_i is strictly positive, the domain of β is the set Ω, therefore the range of β as the continuous image of a compact set, is closed. So we have that $K = \mathrm{co}R(\beta)$, therefore each extreme point of K is the image of an element of Ω. In the simplest case where $\Omega = [a, b]$ is a real interval we have: if the Wronskian of the functions x_i is nonzero in an open interval which contains $[a, b]$, then X itself is not a lattice-subspace. If we suppose that K is a polytope and that $\beta(t_0)$ is a vertex of K, then $\beta'(t_0) = 0$ or t_0 is not an interior point of $[a, b]$. So the vertices of K belong to the set

$$G = \{\beta(t) \mid t \text{ is a root of the equation } \beta'(t) = 0 \text{ or } t = a \text{ or } t = b\}.$$

If Ω is a convex subset of $\mathbb{R}^l$ the situation is analogous but more complicated. So if we suppose that K is a polytope and $\beta(t_0)$ is a vertex of K we have: if t_0 is an interior point of Ω, then the partial derivatives of β at the point t_0 are equal to zero and if t_0 belongs to the boundary $\vartheta(\Omega)$ of Ω, then the derivative at t_0 of the restriction of β at any differentiable curve of $\vartheta(\Omega)$ having t_0 as an interior point is equal to zero. Hence the points t_0 of Ω whose images $\beta(t_0)$ are vertices of K can be obtained as solutions of a system of equations or are extreme points of Ω which cannot be interior points of a differentiable curve of Ω.

If for example Ω is the square $[0, 1] \times [0, 1]$ of $\mathbb{R}^2$, then the vertices of K are of the form $\beta(t_0)$, where t_0 is:

(i) a root of the system of equations $D_1\beta(t) = 0$, $D_2\beta(t) = 0$[14] (if t_0 is an interior point of Ω), or
(ii) a root of an equation $D_i\beta(t) = 0$ (if t_0 is an interior point of an edge), or
(iii) t_0 is a vertex of the square.

If Ω is the circle of $\mathbb{R}^2$ with centre 0 and radius 1, the restriction of β on the boundary of Ω is $\sigma(\vartheta) = \beta(\cos\vartheta, \sin\vartheta)$, therefore the vertices of K are of the form $\beta(t_0)$, where t_0 is a root of the system $D_1\beta(t) = 0$, $D_2\beta(t) = 0$, or $t_0 = \sigma(\vartheta_0)$ with $\sigma'(\vartheta_0) = 0$.

7 THE EQUIVALENT LINEAR PROGRAMMING PROBLEM

In this section we suppose that Y is a minimal lattice subspace of $C(\Omega)$ containing X and that $\{b_1, b_2, \ldots, b_m\}$ is a positive basis of Y constructed as in the Theorem 7. Also suppose that $\{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\}$ is a projection basis of X. We also assume that the sum of the functions x_i is strictly positive, i.e. $\sum_{i=1}^{n} x_i(t) > 0$, for each $t \in \Omega$. Then we may suppose that $\{b_1, b_2, \ldots, b_m\}$ is a positive basis of Y with nodes the

[13] i.e. then $n \times n$ determinant which ith row is constituted of the $(i-1)$th derivatives of the functions x_i.
[14] $D_i\beta(t)$ is the ith partial derivative of β.

points $t_1, t_2, \ldots, t_m$ of Ω, Remark 8. Therefore the expansion of any element z of Y in this basis is of the form

$$z = \sum_{i=1}^{m} \frac{z(t_i)}{b_i(t_i)} b_i.$$

It is clear that z is positive if and only if

$$z(t_k) \geq 0, \quad \text{for each } k = 1, 2, \ldots, m. \tag{9}$$

Also we remark that an element z of $C(\Omega)$ belongs to X if and only if it is of the form $z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i$. So the set

$$P = \{z \in X \mid z \geq 0, z \geq z_1, \ldots, z \geq z_r\},$$

is of the form:

$$P = \left\{ z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i \mid z \geq 0, z \geq z_1, \ldots, z \geq z_r \right\}$$

or equivalently

$$P = \left\{ z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i \mid z(t_k) \geq \sigma_k, k = 1, 2, \ldots, m \right\},$$

where $\sigma_k = \max\{0, z_1(t_k), z_2(t_k), \ldots, z_r(t_k)\}$. Therefore for each

$$z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i \in P,$$

we have

$$B(\lambda_1, \lambda_2, \ldots, \lambda_n)^T \geq (\sigma_1, \sigma_2, \ldots, \sigma_m)^T,$$

where B is the $m \times n$ matrix with columns the vectors

$$(\tilde{b}_i(t_1), \tilde{b}_i(t_2), \ldots, \tilde{b}_i(t_m)), \quad i = 1, 2, \ldots, n,$$

hence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \hat{P}$, where

$$\hat{P} = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid B(\lambda_1, \lambda_2, \ldots, \lambda_n)^T \geq (\sigma_1, \sigma_2, \ldots, \sigma_m)^T\}.$$

Conversely if we suppose that $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \hat{P}$ and $z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i$ we have that $z(t_k) \geq \sigma_k$ for any k, therefore $z \geq 0, z \geq z_1, \ldots, z \geq z_r$, hence $z \in P$. So we have that

$$(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \hat{P} \quad \text{if and only if } z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i \in P. \tag{10}$$

Also $\hat{P} \subseteq \mathbb{R}_+^n$ because if we suppose that $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \hat{P}$, $z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i$ and that $z = \sum_{i=1}^{m} \mu_i b_i$ is the expansion of z in the basis $\{b_i\}$ of Y we have that $\mu_i \geq 0$ for each i because z is a positive element of X. By the statement (iii) of Theorem 9, we have that $\lambda_i = \mu_i$ for each $i = 1, 2, \ldots, n$, therefore $\lambda_i \geq 0$ for each i.

So we obtain the following *linear programming problem* which is equivalent to the minimization problem (1).[15]

$$\text{Minimize} \sum_{i=1}^{n} \lambda_i \, p(\tilde{b}_i) \tag{11}$$

subject to

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n,$$

$$B(\lambda_1, \lambda_2, \ldots, \lambda_n)^T \geq (\sigma_1, \sigma_2, \ldots, \sigma_m)^T,$$

and

$$\sum_{i=1}^{n} \lambda_i \, p_j(\tilde{b}_i) \geq \alpha_j, \quad j = 1, 2, \ldots, l.$$

Also the minimization problem (2) is equivalent with the linear programming problem:

$$\text{Minimize} \sum_{i=1}^{n} \lambda_i \, p(\tilde{b}_i) \tag{12}$$

subject to

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n$$

and

$$B(\lambda_1, \lambda_2, \ldots, \lambda_n)^T \geq (\sigma_1, \sigma_2, \ldots, \sigma_m)^T.$$

THEOREM 13 *Suppose that Y is an m-dimensional minimal lattice-subspace of $C(\Omega)$ which contains X and that the sum of the functions x_i is strictly positive. Then the minimization problem* (1) *is equivalent with the linear programming problem* (11). *The minimi-*

[15] in the sense that $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a solution of the linear programming problem, if and only if, $z_0 = \sum_{i=1}^{n} \lambda_i \tilde{b}_i$ is a solution of the minimization problem.

zation problem (2) *has at least one solution. The solutions of* (2) *are determined by the linear programming problem* (12).

Proof Suppose that T is an order-isomorphism of Y onto $\mathbb{R}^m$ with $T(\sum_{i=1}^m \lambda_i b_i) = \sum_{i=1}^m \lambda_i e_i$ where $\{b_i\}$ is a positive basis of Y and $\{e_i\}$ the usual basis of $\mathbb{R}^m$. Suppose that $T^* : \mathbb{R}^m \to Y^*$ is the adjoint of T with $T^*(y) = \hat{p}$, where $\hat{p}$ is an extension of p on Y. Then $p(z) = y \cdot (Tz)$ for each $z \in X$ and also $z \geq 0, z_1, z_2, \ldots, z_r$ if and only if $T(z) \geq T(0), T(z_1), T(z_2), \ldots, T(z_r)$. So the minimization problem is equivalent with the following:

Minimize $y \cdot (Tz)$ subject to $z \in X$ and $T(z) \geq 0, T(z_1), T(z_2), \ldots, T(z_r)$. Since p is positive on X we have that the functional y is positive on $T(X)$, therefore $y \cdot (Tz) \geq 0$ for each $z \in X_+$. The feasible set of the new minimization problem is the set $T(P)$ which is a polyhedral set of $\mathbb{R}^m$. This holds because if we suppose that $f_1, f_2, \ldots, f_m$ are the coefficient functionals of the basis $\{b_i\}$ of Y[16] and $f_k = T^*(y_k)$ we have that $T(P) = \{T(z) \mid z \in X$ with $y_k \cdot (Tz) \geq \rho_k$ for each $k = 1, \ldots, m\}$, where $\rho_k = \max\{0, y_k \cdot (Tz_1), y_k \cdot (Tz_2), \ldots, y_k \cdot (Tz_r)\}$. Therefore $T(P) = G + C$, is the sum of a polytope G and a finitely generated convex cone C, [8], Theorem 4.1.3. The functional y is positive on the cone C because if we suppose that $y \cdot c < 0$ for some $c \in C$ then for each $g \in G$ we have the $\lim_{\lambda \to \infty} y \cdot (g + \lambda c) = -\infty$. This is a contradiction because $g + \lambda c \in T(P)$ and y is positive on $T(X)$. Hence y is positive on the cone C, therefore y takes minimum in $T(P)$ which is obtained in some point of G. Therefore the minimization problem has a solution. ■

Remark 14 The minimization problem of Example 2 does not have a solution although p is positive. In Example 17 below we show that a finite-dimensional minimal lattice-subspace containing the subspace X of Example 2 does not exist. Therefore in the previous Theorem the assumption that X is contained in a finite-dimensional minimal lattice-subspace of $C(\Omega)$ cannot be omitted.

8 THE ALGORITHM

In order to study the minimization problem (1) we study if a finite dimensional minimal lattice subspace Y of $C(\Omega)$ containing X exists and in the sequal we determine a positive basis of Y. To this end we follow the next algorithm:

(i) We determine the basic function β of the functions x_i.
(ii) We study if K is a polytope or not and (if it is) we determine its vertices. We find an enumeration $P_1, P_2, \ldots, P_n, P_{n+1}, \ldots, P_m$ of the vertices of K such that the n first of them to be linearly independent.
(iii) By Theorem 7 we determine a minimal lattice-subspace Y containing X and a positive basis $\{b_1, b_2, \ldots, b_m\}$ of X. Also we determine a set of nodes $\{t_1, t_2, \ldots, t_m\}$ of the basis $\{b_i\}$.
If X is a lattice-subspace and the price vector p is positive, then $\sup_X\{0, z_1, z_2, \ldots, z_r\}$ is the solution of the minimization problem (2).
(iv) By Theorem 9 determine a projection basis $\{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\}$ of X.
(v) We determine the matrix B whose columns are the vectors

[16] i.e. $f_k(\sum_{i=1}^m \lambda_i b_i) = \lambda_k$, for each $k = 1, 2, \ldots, m$.

$$(\tilde{b}_i(t_1), \tilde{b}_i(t_2), \ldots, \tilde{b}_i(t_m)), \quad i = 1, 2, \ldots, n,$$

and we compute the real numbers $p(\tilde{b}_i), i = 1, 2, \ldots, n$ and $\sigma_k = \max\{0, z_1(t_k), z_2(t_k), \ldots, z_r(t_k)\}$, $k = 1, 2, \ldots, m$ and also the numbers $p_j\ (\tilde{b}_i)$, $j = 1, 2, \ldots, l$, $i = 1, 2, \ldots, n$.

(vi) If $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a solution of the linear programming problem (11) then

$$z = \sum_{i=1}^{n} \lambda_i \tilde{b}_i,$$

is a solution of the minimization problem.

9 MINIMUM COST PORTFOLIO INSURANCE

In this section we apply the previous results in the portfolio insurance. Our model is based on a new method of comparing portfolios, the portfolio dominance ordering. This ordering has been introduced in [1] and compares portfolios not in the pointwise ordering of the portfolio space but by means of the ordering of their payoffs. This type of ordering enables us to use the order structure of the payoff space and also the theory of lattice-subspaces. In [2] the minimum cost portfolio insurance is studied in the case where the asset span X is a lattice-subspace of $\mathbb{R}^l$ and it is also indicated that the method of the computation of the minium cost insurance is also applicable if X is a lattice-subspace of an infinite dimensional space. Here we study the same problem under the general assumption that the asset span X is contained in a finite dimensional minimal lattice-subspace of $C(\Omega)$.

9.1 The Portfolio Dominance Ordering

The model of security markets we study here is extended over two periods, the period 0 and the period 1 and is the following: suppose that there are n securities labeled by the natural numbers $1, 2, \ldots, n$, acquired the period 0. Securities are described by their payoffs at date 1. The *payoff* of the ith security is in general a positive element x_i of an ordered space E which is called *payoff space*. It is also supposed that the payoffs $x_1, x_2, \ldots, x_n$ are linearly independent vectors (*nonredundant securities*) of E. In our model we suppose that E is the space of real valued continuous functions $C(\Omega)$ defined on a compact, Hausdorff topological space Ω, therefore the payoff vectors x_i are linearly independent, positive elements of $C(\Omega)$. We suppose that the elements of the set Ω expresses all the possible states of the world during the period 1 and that the value $x_i(t)$ of x_i at the point t is the payoff of the security i in the state t, therefore the function x_i is the profile of the ith security at the period 1. A portfolio is a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ of $\mathbb{R}^n$ where θ_i is the number of shares of the ith security. The space $\mathbb{R}^n$ is called *portfolio space*. The *payoff of portfolio* θ is

$$R(\theta) = \sum_{i=1}^{n} \theta_i x_i \in C(\Omega).$$

The operator R is one-to-one and it is called *payoff operator*. The pointwise ordering of $C(\Omega)$, induces the partial ordering $\geq_R$ in the portfolio space $\mathbb{R}^n$ which we call *the portfolio dominance ordering* and is defined as follows: For each $\theta, \varphi \in \mathbb{R}^n$ we have

$$\theta \geq_R \varphi, \text{ if and only if } R(\theta) \geq R(\varphi).$$

So the portfolio θ is better than the portfolio φ is and only if at any state t of the world in the period 1, the payoff $\theta(t)$ of the portfolio θ is greater or equal than the payoff $\varphi(t)$ of the portfolio φ. According to the pointwise ordering of the portfolio space, $\theta \geq \varphi$ if and only if $\theta_i \geq \varphi_i$, for each $i = 1, 2, \ldots, n$. It is clear that $\theta \geq \varphi$ implies $\theta \geq_R \varphi$ but the converse is not always true. The positive cone

$$C = \{\theta \in \mathbb{R}^n \mid R(\theta) \geq_R 0\}$$

of the portfolio space in the portfolio dominance ordering contains the positive cone $\mathbb{R}^n_+$ of $\mathbb{R}^n$. The range of the payoff operator, i.e. the subspace

$$X = [x_1, x_2, \ldots, x_n]$$

of $C(\Omega)$, is the set of the payoffs of the portfolios and is called the *asset span* of securities or the *space of marketed securities*. The positive cone $X_+ = X \cap C(\Omega)$ of X is the image of the positive cone C of the portfolio space i.e. $X_+ = R(C)$. Any vector $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$, where q_i is the price of security i, is called *security price vector* and the dot product $q \cdot \theta = \sum_{i=1}^n q_i\theta_i$, is the value of security θ at price q. In this section for any price vector q we will suppose that $q \cdot \theta \geq 0$ for any portfolio θ with positive payoff, therefore for any security price vector q we will suppose that it is positive in the portfolio dominance ordering (i.e. $q \cdot \theta \geq 0$, for any $\theta \in C$).

9.2 Portfolio Insurance

Suppose that θ, κ are fixed portfolios. The insurance of the portfolio θ in the floor κ is stated at the date 0 and the payments will take place at the date 1. Any portfolio η with payoff better than the payoff of θ, κ, at any state of t in the date 1 is an approval insurance premium. The company has the obligation "to pay" only if the payoff of θ at the state t in the date 1 is lower to the payoff of κ, i.e. if $R(\theta)(t) < R(\kappa)(t)$. Suppose that q is the vector of security prices in the date 1 and also that the company has the obligation to pay. Then the company faces the following problem of the determination of the minimum cost insurance:

$$\text{Minimize } q \cdot \eta, \text{ subject to } \eta \in \mathbb{R}^n \text{ and } \eta \geq_R \theta,\ \eta \geq_R \kappa. \tag{13}$$

Any solution of this problem will be referred as a *minimum cost insurance of the portfolio θ at the floor κ and in the price q*.

If $y_1 = R(\theta)$, $y_2 = R(\kappa)$ and p is the linear functional of X with $p(x_i) = q_i$ for each $i = 1, 2, \ldots, n$, we have: $p(R\theta) = q \cdot \theta$, for any θ. Since the price q is positive in the portfolio dominance ordering and $X_+ = R(C)$ we have that p is a positive linear functional of X. So the above minimization problem is equivalent with the following:

$$\text{Minimize } p(w) \text{ subject to } w \in X \text{ and } w \geq y_1,\ w \geq y_2,$$

where p is a positive linear functional of X.

We put $z = w - y_2$, $z_1 = y_1 - y_2$ and we obtain the equivalent minimization problem:

$$\text{Minimize } p(z) \text{ subject to } z \in X \text{ and } z \geq z_1,\ z \geq 0. \tag{14}$$

It is clear that z_0 is a solution of (14), if and only if $\eta_0 = R^{-1}(z_0 + y_2)$ is a solution of (13).

THEOREM 15 *If the payoff space X is contained in a finite-dimensional minimal lattice-subspace Y of $C(\Omega)$ and the sum of the payoff vectors x_i is strictly positive, then a minimum cost insurance of the portfolio θ at the floor κ and in the price q exists*[17] *and it is determined by solving the minimization problem* (14).

10 EXAMPLES

Example 16 Let $\Omega = [0, 2]$, $x_1(t) = t^2 - 2t + 2$, $x_2(t) = -t^3 + 2t^2 - t + 2$, $x_3(t) = t^3 - 3t^2 + 3t$ and X be the subspace of $C[0, 2]$ generated by x_1, x_2, x_3. We study the problem

$$\text{Minimize } p(z) \text{ subject to } z \in X \text{ and } z \geq x_0,\ z \geq 0,$$

where $x_0 = -3t^3 + 7t^2 - 10t + 8$ and p is positive.

The basic function of x_1, x_2, x_3 is

$$\beta(t) = \frac{1}{4}(x_1(t), x_2(t), x_3(t)).$$

Suppose that K is a polytope. Since the derivative of β is equal to zero only in the point 1 of $(0, 2)$, in accordance with the Remark 12, we have that $\{P_1 = \beta(0), P_2 = \beta(1), P_3 = \beta(2)\}$ is the set of the possible vertices of K. It is easy to show that each $\beta(t)$ is a convex combination of the vectors P_i, therefore K is a simplex with vertices P_1, P_2, P_3 and X is a lattice subspace. A positive basis of X is given by (5). After the computations we have that

$$b_1(t) = 2(t-1)^2(2-t), \quad b_2(t) = 4t(2-t), \quad b_3(t) = 2t(t-1)^2,$$

is a positive basis of X.

The points $t_1 = 0$, $t_2 = 1$ and $t_3 = 2$ are nodes of the basis, therefore the expansion of x_0 is

$$x_0 = \frac{x_0(0)}{b_1(0)} b_1 + \frac{x_0(1)}{b_2(1)} b_2 + \frac{x_0(2)}{b_3(2)} b_3 = 2b_1 + \frac{1}{2} b_2 - 2b_3,$$

[17]We have assumed that any security price vector is positive in the portfolio dominance ordering.

therefore $\sup_X\{x_0, 0\} = 2b_1 + (1/2)b_2$ is the solution of the minimization problem.

Example 17 The subspace X of the Example 2 is not a lattice-subspace of $C[0,1]$ and also it is not contained in a finite-dimensional minimal lattice-subspace Y of $C[0,1]$. This holds because if we suppose that K is a polytope then K has at least three vertices $\beta(t_1)$, $\beta(t_2)$, $\beta(t_3)$. Then at least one of t_i is an interior point of $(0,1)$, therefore the derivative of β at this point is equal to zero. This is a contradiction because the basic function is $\beta(t) = (1/(1+t+t^2))(1, t, t^2)$ and its derivative is nonzero for each $t \in (0,1)$.

Example 18 Let $\Omega = [0,3]$, $x_1(t) = -t^5 + 8t^4 - 26t^3 + 44t^2 - 37t + 12$, $x_2(t) = -t^5 + 8t^4 - 24t^3 + 35t^2 - 28t + 12$, $x_3(t) = t^5 - 7t^4 + 18t^3 - 19t^2 + 7t$, and X be the subspace of $C[0,3]$ generated by x_1, x_2, x_3. We study the problem

$$\text{Minimize } p(z) \text{ subject to } z \in X \text{ and } z \geq x_0,\ z \geq 0,$$

where $x_0 = 2x_1 + x_2 - x_3 = -4t^5 + 31t^4 - 94t^3 + 142t^2 - 109t + 36$ and the price vector is $p(w) = \int_0^3 tw(t)\,dt$ for each $w \in C(\Omega)$.

The basic function is

$$\beta(t) = \frac{1}{s(t)}(x_1(t), x_2(t), x_3(t)),$$

where $s(t) = -t^5 + 9t^4 - 32t^3 + 60t^2 - 58t + 24$ is the sum of the functions x_i.

Using "Mmathematica" we find that the roots of the derivative of β in the interval $(0,3)$ are the numbers 1 and 2, therefore the set of the possible vertices of K is

$$\left\{\beta(0) = \left(\frac{1}{2}, \frac{1}{2}, 0\right),\ \beta(1) = (0,1,0),\ \beta(2) = \left(\frac{1}{2}, 0, \frac{1}{2}\right),\ \beta(3) = (0,0,1)\right\}.$$

Let

$$P_1 = (0,1,0), \quad P_2 = (0,0,1), \quad P_3 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \quad P_4 = \left(\frac{1}{2}, 0, \frac{1}{2}\right).$$

We can show that each $\beta(t)$ is a convex combination of the vectors P_i. Especially we have $\beta(t) = \sum_{i=1}^4 \xi_i(t)P_i$ with $\xi_1(t) = (-t^4 + 7t^3 - 16t^2 + 12t)/s(t)$, $\xi_2(t) = (t^5 - 6t^4 + 13t^3 - 12t^2 + 4t)/s(t)$, $\xi_3(t) = 2(-t^5 + 9t^4 - 31t^3 + 51t^2 - 40t + 12)/s(t)$, $\xi_4(t) = 2(-t^4 + 5t^3 - 7t^2 + 3t)/s(t)$.

By Theorem 7, $Y = [x_1, x_2, x_3, x_4]$, where $x_4 = \xi_4(t)s(t) = 2(-t^4 + 5t^3 - 7t^2 + 3t)$, is a minimal lattice-subspace of $C(\Omega)$ which contains X. By the same theorem a positive basis of Y is given by (6), where D is the 4×4 matrix with columns the vectors R_i of (7). Therefore

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

hence $b_1 = -t^4 + 7t^3 - 16t^2 + 12t$, $b_2 = t^5 - 6t^4 + 13t^3 - 12t^2 + 4t$, $b_3 = 2(-t^5 + 9t^4 - 31t^3 + 51t^2 - 40t + 12)$, $b_4 = 4(-t^4 + 5t^3 - 7t^2 + 3t)$ is a positive basis of Y and it is easy to see that $t_1 = 1$, $t_2 = 3$, $t_3 = 0$, $t_4 = 2$ are nodes for this basis. The projection basis of X is

$$(\tilde{b}_1(t), \tilde{b}_2(t), \tilde{b}_3(t))^T = A^{-1}(x_1(t), x_2(t), x_3(t))^T,$$

where A is the 3×3 matrix with columns the vectors P_1, P_2, P_3. Therefore

$$\begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

hence

$$\tilde{b}_1 = x_2 - x_1, \quad \tilde{b}_2 = x_3, \quad \tilde{b}_3 = 2x_1.$$

The matrix B with columns the vectors $(\tilde{b}_i(t_1), \tilde{b}_i(t_2), \tilde{b}_i(t_3), \tilde{b}_i(t_4))$ is

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 24 \\ -2 & 2 & 4 \end{bmatrix}.$$

Also $p(\tilde{b}_1) = -81/20$, $p(\tilde{b}_2) = 2097/140$, $p(\tilde{b}_3) = 558/35$. The numbers $\sigma_k = x_0(t_k) \vee 0$ are $\sigma_1 = 2$, $\sigma_2 = 0$, $\sigma_3 = 36$, $\sigma_4 = 2$. The equivalent linear programming problem is:

Minimize $-(81/20)\lambda_1 + (2097/140)\lambda_2 + (558/35)\lambda_3$ subject to

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 24 \\ -2 & 2 & 4 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \geq \begin{pmatrix} 2 \\ 0 \\ 36 \\ 2 \end{pmatrix}.$$

It is easy to find that $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = 3/2$ is the solution of the this problem, therefore $z_0 = 2\tilde{b}_1 + 0\tilde{b}_2 + (3/2)\tilde{b}_3 = -3t^5 + 24t^4 - 74t^3 + 114t^2 - 93t + 36$ is the solution of the minimization problem.

References

[1] C.D. Aliprantis, D. Brown, I. Polyrakis and J. Werner (1998). Portfolio dominance and optimality in infinite security markets. *Journal of Mathematical Economics*, **30**, 347–366.
[2] C.D. Aliprantis, D. Brown and J. Werner (2000). Minimum-cost portfolio insurance. *J. Economic Dynamics and Control*, **24**, 1703–1719.
[3] C.D. Aliprantis, I. Polyrakis and R. Tourky (2002). The cheapest hedge. *Journal of Mathematical Economics*, **37**, 269–295.
[4] J.A. Kalman (1961). *Continuity and Convexity of Projections and Barycentric Coordinates in Convex Polyhedra*, pp. 1017–1022.
[5] S. Miyajima (1983). Structure of Banach quasi-sublattices. *Hokkaido Math. J.*, **11**, 83–91.
[6] I.A. Polyrakis (1996). Finite-dimentional lattice-subspaces of $C(\Omega)$ and curves of $\mathbb{R}^n$. *Trans. American Math. Soc.*, **384**, 2793–2810.
[7] I.A. Polyrakis (1999). Minimal lattice-subspaces. *Trans. American Math. Soc.*, **351**, 4183–4203.
[8] R. Webster (1994). *Convexity*. Oxford University Press.