Demand functions and reflexivity

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Abstract
In the theory of ordered spaces and in microeconomic theory two important notions, the notion of the base for a cone which is defined by a continuous linear functional and the notion of the budget set are equivalent. In economic theory the maximization of the preference relation of a consumer on any budget set defines the demand correspondence which at any price vector indicates the preferred vectors of goods and this is one of the fundamental notions of this theory. Contrary to the finite-dimensional economies, in the infinite-dimensional ones, the existence of the demand correspondence is not ensured. In this article we show that in reflexive spaces (and in some other classes of Banach spaces), there are only two classes of closed cones, i.e. cones whose any budget set is bounded and cones whose any budget set is unbounded. Based on this dichotomy result, we prove that in the first category of these cones the demand correspondence exists and that it is upper hemicontinuous. We prove also a characterization of reflexive spaces based on the existence of the demand correspondences.

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1. Introduction
In the theory of finite-dimensional competitive economies, budget sets corresponding to strictly positive price vectors are always bounded and therefore compact, hence any continuous preference relation \( \succeq \) attains maximum on any budget set. Therefore the demand correspondence (single valued or multivalued) of any continuous preference relation always exists and this is one of the fundamental properties of this theory. On the contrary, in the theory of infinite-dimensional competitive economies the existence of the demand correspondence is not ensured. There are known examples of preference relations \( \succeq \) defined on a closed cone \( P \) of a normed space \( X \) which attain or do not attain maximum on fixed budget sets of \( P \) but we do not know an example of an infinite-dimensional closed cone \( P \) with a continuous preference relation \( \succeq \), so that the demand correspondence of \( \succeq \) exists (i.e. \( \succeq \) attains maximum on any budget set of \( P \)). In this article we study the existence of the demand correspondence in infinite-dimensional economies. We prove the following dichotomy result for cones with respect to their budget sets (bases): If in a competitive exchange economy the commodity-price duality is the dual system \( (X, Y) \) where \( X \) is a normed space with \( \sigma(X,Y) \)-compact unit ball, then \( X \) has two classes of \( \sigma(X,Y) \)-closed cones, i.e. cones whose any budget set is norm-

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bounded and cones whose any budget set is norm-unbounded. In the sequel we show that in the first category of these cones, the demand correspondence of any \( \sigma(X, Y) \) upper semicontinuous preference relation exists and we prove also that the demand correspondence is norm to \( \sigma(X, Y) \) upper hemi-continuous. So in reflexive spaces we have two classes of closed cones, cones whose any budget set is bounded and cones whose any budget set is unbounded and in the first category of these cones the demand correspondence exists and it is norm to weak upper hemi-continuous. A similar result holds also in dual spaces. Hence we prove the existence and continuity of the demand correspondences for a large class of cones. These results are useful for applications because in general, it is easy one to check if for a fixed price vector the corresponding budget set is bounded or not.

Also in this article we prove a characterization of reflexive spaces based on the existence of the demand correspondences of linear, continuous, preferences in the cones \( P \) of \( X \) with a bounded budget set, or equivalently based on the maximization on the bases for \( P \) of the linear functionals of \( X \) whose restriction on \( P \) is continuous.

Finally we mention the work of A. Araujo [3], where it is supposed that the commodity set is a closed, convex subset \( A \) of \( \mathbb{R}^+ \), any strictly positive on \( A \) linear functional \( p \in X^* \) and any real number \( w > 0 \) it is denoted by \( C_a \) the set \( C_a = \{ x \in A \mid u(x) \geq a \} \), by \( w_p \) the real number \( \inf \{ u(y) \mid y \in C_a \} \) and by \( \phi(p, w) \) the unique solution of the maximization problem \( \max \{ \sum_{i=1}^{\infty} |x_i|^p \mid x \in \lambda_1^\infty \} \) whenever such a solution exists. In Theorem 2 of [3] it is proved: If \( A \) is bounded, \( u \) is uniformly continuous so that for some \( a \in \mathbb{R}^+ \) the set \( C_a \) has non-empty interior and for any price vector \( p \in X^* \) with \( \| p \| = 1 \), either there exists \( x \in A \) with \( p(x) < w_p \) and there exists \( \phi(p, w_p) \) or \( \inf \{ u(y) \mid y \in C_a \} \) is attained, then \( X \) is reflexive. In the same theorem it is also remarked that if \( X \) is reflexive, \( A \) is bounded and weakly closed and \( u \) is weakly continuous then \( \phi(p, w_p) \) exists for any \( (p, w_p) \). So a characterization of reflexive spaces is proved. But the results of this important article of Araujo and the results of our paper are independent.

2. Bases for cones

Let \( X \) be a normed space. Denote by \( X^* \) the norm dual of \( X \) and by \( \mathbb{R}^+ \) the set of positive real numbers \( \lambda \geq 0 \). A non-empty, convex subset \( P \) of \( X \) is a \textit{cone} (or a wedge of \( X \)) if \( \lambda P \subseteq P \) for each \( \lambda \in \mathbb{R}^+ \). If moreover \( P \cap (-P) = \{ 0 \} \) the cone \( P \) is \textit{pointed}. If \( X = P - P \), the cone \( P \) is \textit{generating}. The set \( P^0 = \{ f \in X^* \mid f(x) \geq 0 \} \) for each \( x \in P \), is the \textit{dual cone} of \( P \) in \( X^* \). Suppose that \( X \) is ordered by the cone \( P \). Then for any \( x, y \in X \) with \( x \leq y \) the set \( [x, y] = \{ z \in X \mid x \leq z \leq y \} \) is the order interval \( xy \). If \( x \in P \) so that \( X = \bigcup_{n=1}^{\infty} n[-x, x] \), \( x \) is an \textit{order unit} of \( X \). Suppose also that \( X^* \) is ordered by \( P^0 \). So a linear functional \( f \) of \( X \) is \textit{positive} (on \( P \)) if \( f(x) > 0 \) for each \( x \in P \), \textit{strictly positive} (on \( P \)) if \( f(x) > 0 \) for each \( x \in P, x \neq 0 \) and \textit{uniformly monotonic} (on \( P \)) if \( f(x) > a \| x \| \), for each \( x \in P \), where \( a \) is a constant real number \( a > 0 \). If a strictly positive linear functional exists, the cone \( P \) is pointed. A convex subset \( B \) of \( P \) is a \textit{base for the cone} \( P \) if for each \( x \in P \), \( x \neq 0 \) a unique real number \( f(x) > 0 \) exists such that \( \frac{x}{f(x)} \in B \). Then the function \( f \) is additive and positively homogeneous on \( P \) and \( f \) can be extended to a linear functional on \( P - P \) by the formula \( f(x_1 - x_2) = f(x_1) - f(x_2), x_1, x_2 \in P \), and in the sequel this linear functional can be extended to a linear functional on \( X \). So we have: \( B \) is a base for the cone \( P \) if and only if a strictly positive (not necessarily continuous) linear functional \( f \) of \( X \) exists so that, \( B = \{ x \in P \mid f(x) = 1 \} \). Then we say that the base \( B \) is defined by \( f \). If \( B \) is a base for the cone \( P \) with \( B \neq \emptyset \), \( B \) is the closed hull of \( B \), then a continuous linear functional \( f \) of \( X \) separating \( B \) and \( 0 \) exists. Then \( f \) is strictly positive and if \( B \) is bounded the base for \( P \) defined by \( f \) is also bounded. So we have:

The cone \( P \) has a base defined by a continuous linear functional \( f \) of \( X \) if and only if \( P \) has a base \( B \) with \( 0 \notin B \). If moreover the base \( B \) is bounded the base for \( P \) defined by \( f \) is bounded. Also it is known, [4, Theorem 3.8.4], that a cone \( P \) of a normed space \( X \) has a bounded base \( B \) with \( 0 \notin B \) if and only if the dual cone \( P^0 \) of \( P \) in \( X^* \) has \textit{interior points}. We give below some easy examples of bases for cones. For any real number \( 1 \leq p < +\infty \), \( \ell_p \) is the space of real sequences \( x = (x_i) \) with \( \sum_{i=1}^{\infty} |x_i|^p < +\infty \) and norm \( \| x \| = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} \). \( c_0, \ell_\infty \), are the spaces of convergent to zero, bounded, real sequences \( x = (x_i) \) with norm \( \| x \| = \sup_{i \in \mathbb{N}} |x_i| \).

Example 1. (i) The positive cone \( \ell_1^+ = \{ x = (x_i) \mid x_i \geq 0 \} \) of \( \ell_1 \) has a bounded and an unbounded base. Indeed, if \( f_i = (f_i) \in \ell_\infty \) with \( f_i > 0 \) for each \( i \), then \( B = \{ x \in \ell_1^+ \mid f(x) = 1 \} \), is a base for \( \ell_1^+ \). If a subsequence \( \{ f_{i_n} \} \) of \( \{ f_i \} \) converges to zero, then \( B \) is unbounded because \( \ell_\infty \) for each \( v \) and \( \| f_{i_n} \| = \frac{1}{f_{i_n}} \to \infty \). If \( f_i \geq a > 0 \) for each \( i \), then \( 1 = f(x) = \sum_{i=1}^{\infty} f_i x_i > a \| x \| \) for any \( x \in B \), therefore \( B \) is bounded.
(ii) Any base for the positive cone $\ell_p^+ = \{ x = (x_i) \in \ell_p \mid x_i \geq 0 \text{ for any } i \}$ of $\ell_p$ with $1 < p < +\infty$, is unbounded. Indeed the base $B = \{ x \in \ell_p^+ \mid f(x) = 1 \}$, for $\ell_p^+$ is unbounded because $\frac{e_n}{f_n} \in B$ for each $n$, where $\{e_n\}$ is the usual Schauder basis of $\ell_p^+$.

We close this introductory section by two easy results.

**Proposition 2.** Suppose that $B$ is a base for $P$ defined by the linear functional $f$. Then the base $B$ is bounded if and only if the functional $f$ is uniformly monotonic.

**Proof.** If we suppose that $\|x\| \leq M$ for each $x \in B$, then for each $x \in P$, $x \neq 0$ we have $\|\frac{x}{f(x)}\| \leq M$, therefore $\|x\| \leq Mf(x)$, for each $x \in P$, hence $f$ is uniformly monotonic. For the converse suppose that $f(x) \geq a\|x\|$ for each $x \in P$, where $a$ is a real number $a > 0$. Then for each $x \in B$ we have $1 = f(x) \geq a\|x\|$, therefore the base $B$ is bounded. \qed

**Proposition 3.** Any base for a finite-dimensional closed cone $P$ of a normed space $X$ is bounded.

**Proof.** Suppose that $B$ is a base for $P$ defined by the linear functional $f$ and $x_n \in B$ with $\|x_n\| \to \infty$. Then $f(\frac{x_n}{\|x_n\|}) \to 0$. Since the set $P \cap U_X$, where $U_X$ is the closed unit ball of $X$, is compact, a subsequence of $\{\frac{x_n}{\|x_n\|}\}$ exists which converges to an element $x_0$ of $P$. Then we have that $\|x_0\| = 1$ and $f(x_0) = 0$, contradiction because $f$ is strictly positive on $P$. \qed

### 3. A dichotomy result for cones

A **dual system** $(E, F)$ is a pair of linear spaces $E, F$ together with a bilinear map $(x, y) \to \langle x, y \rangle$ of $E \times F$ onto $\mathbb{R}$ which separates the points of $E$ and $F$. In this article we will denote $(x, y)$ by $y(x)$. Suppose that $(E, F)$ is a dual system. Then $\sigma(E, F)$, $\sigma(F, E)$ are the weak topologies of $E, F$ defined by the dual system. Suppose also that $P$ is a cone of $E$. Then $P^0 = \{ y \in Y \mid y(x) \geq 0 \text{ for each } x \in P \}$ is the dual cone of $P$ in $F$. A vector $y \in F$, is strictly positive on $P$ if $y(x) > 0$ for any $x \in P$, $x \neq 0$. Then $y$ defines the base $B = \{ x \in P \mid y(x) = 1 \}$ of $P$.

Suppose that $E$ is a normed space. Then $(E, E^*)$ with $(x, y) = y(x)$, for any $x \in E$, $y \in E^*$, where $y(x)$ is the value of the linear functional $y$ in $x$, is a dual system. $\sigma(E, E^*)$ is the weak topology of $E$ and $\sigma(E^*, E)$ is the weak-star topology of $E^*$. Recall that in $E$, convex sets have the same closure with respect to the norm and the weak topology. Also the unit ball $U_{E^*}$ of $E^*$ is weak-star compact and if $E$ is reflexive the unit ball $U_E$ of $E$ is weakly compact. Recall also that a normed space $E$ is reflexive if the natural map $x \to \hat{x}$, $x \in E$ so that $\hat{x}(x^*) = x^*(x)$ for any $x^* \in E^*$ is onto $E^*$.

**Theorem 4.** Suppose that $(X, Y)$ is a dual system. If $X$ is a normed space, $P$ a $\sigma(X, Y)$-closed cone of $X$ so that the positive part $U_X^P = U_X \cap P$ of the closed unit ball $U_X$ of $X$ is $\sigma(X, Y)$-compact, we have: either every base for $P$ defined by a vector $y \in Y$ is bounded or every such base for $P$ is unbounded.

**Proof.** Suppose that $P$ is a $\sigma(X, Y)$-closed cone of $X$, the element $y_1 \in Y$ defines a bounded base $B$ for the cone $P$ and $y_2 \in Y$ defines an unbounded base $K$ for the cone $P$. Then $y_1$ is uniformly monotonic and suppose that $y_1(x) \geq a\|x\|$, for each $x \in P$, where $a$ is a real number $a > 0$. Also the base $B$ is $\sigma(X, Y)$-compact, therefore $y_2$ takes a minimum value $m = y_2(x_0)$ on $B$ at a point $x_0$ of $B$. Since $y_1$ is strictly positive on $P$ we have that $m > 0$. Therefore for any $x \in P$, $x \neq 0$ we have $y_2(\frac{x}{y_1(x)}) \geq m$ from where we have that $y_2$ is uniformly monotonic. We have assumed that it defines an unbounded base. \qed

**Corollary 5.** For any weak-star closed cone $P$ of the dual $X^*$ of a normed space $X$ we have: either every base for $P$ defined by a vector $x \in X$ is bounded or every such base for $P$ is unbounded.

**Corollary 6.** For any closed cone $P$ of a reflexive Banach space $X$ we have: either every base for $P$ defined by a vector $x^* \in X^*$ is bounded or every such base for $P$ is unbounded.
Example 7. (i) Any normed space $X$ has an infinite-dimensional closed cone $P$ with a bounded base defined by a continuous linear functional of $X$. Indeed, for any $x_0 \in X$, $x_0 \neq 0$, the set $P = \{ \lambda x \mid \lambda \in \mathbb{R}_+, \|x - x_0\| \leq \frac{\|x_0\|}{2} \}$ is a closed cone of $X$ and it is easy to show that any $f \in X^*$ separating 0 and $\{x \in X \mid \|x - x_0\| \leq \frac{\|x_0\|}{2} \}$ defines a bounded base for $P$. If $X$ is reflexive, by the previous result, any base for $P$ defined by an element of $X^*$ is bounded.

(ii) Suppose that $B$ is a base of $\ell_p^+$, $1 < p < \infty$, defined by a strictly positive linear functional $f \in \ell_p^*$. As we have shown in Example 1, $B$ is unbounded. Suppose that $B_\rho = \{ x \in B \mid \|x\| \leq \rho \}$ for some real number $\rho > 0$ and $P = \{ \lambda x \mid \lambda \in \mathbb{R}_+, \ x \in B_\rho \}$ is the cone generated by $B_\rho$. Since $B_\rho$ is closed and bounded we have that $P$ is closed, see also in [4, Proposition 3.8.3]. $P$ is generating because if we suppose that $x_0 \in B_\rho$ with $\|x_0\| < \rho$, then for each $x \in B \setminus B_\rho$ we have that $x = x_0 + t(x_1 - x_0)$, where $x_1$ is the point of the line segment $x_0 x$ with $\|x_1\| = \rho$ and $t$ a real number $t > 0$. Therefore $B \subseteq P - P$ and the cone $P$ is generating because any $y \in \ell_1$ is of the form $y = \lambda_1 y_1 - \lambda_2 y_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and $y_1, y_2 \in B$. $P$ has empty interior because $\ell_p^+$ does not have interior points. Also by the previous result any base for $P$ is bounded. Similarly we can define a closed cone $P$ of $L_p(\mu)$, $1 < p < +\infty$, with the same properties.

3.1. A characterization of reflexivity

Theorem 8. Suppose that $P$ is a closed cone of a normed space $X$. If

(i) any $g \in X^*$ attains maximum on any base for $P$ defined by an element of $X^*$, or

(ii) $P^0 - P^0 = X^*$ and any positive on $P$ element $g \in X^*$, attains maximum on any base for $P$ defined by an element of $X^*$,

then any base for $P$ defined by an element of $X^*$ is bounded.

Proof. Let $B$ be a base for $P$ defined by $f \in X^*$. Suppose that statement (i) is true. Then any $g \in X^*$ attains maximum on $B$ at a vector $x_g$ of $B$. Also $-g$ takes maximum on $B$ at a vector $x_{-g}$ of $B$. Then $g(x_{-g}) \leq g(x) \leq g(x_g)$ for any $x \in B$, therefore for any $x \in P$, $x \neq 0$ we have $g(x_{-g}) \leq g(x_{-g}) \leq g(x_g)$ and $g(x_{-g}) \leq g(x) \leq g(x_g)$. Since $g(x_{-g}) \leq g(x) \leq g(x_g)$ and $f$ is an order unit of $X^*$, hence $[\bigcup_{n=1}^{\infty} n[-f, f]] = X^*$. The cone $P^0$ is weak-star closed and therefore closed, hence the order interval $[-f, f]$ is closed and by the Baire’s theorem, see for example in [6, Theorem 1.3.14], $0$ is an interior point of $[-f, f]$ and therefore $f$ is an interior point of $[0, 2f]$. Hence $f$ is an interior point of $P^0$, and the base $B$ is bounded. Indeed, if we suppose that $f + V \subseteq P$ where $V$ is a closed ball of $X^*$ of center zero, for any $x \in B$ and any $h \in V$ we have $(f + h)(x) \geq 0$, therefore $h(x) \geq -1$. Hence $B$ is contained in the polar of $V$ in $X$, therefore $B$ is bounded.

If we suppose that (ii) is true, any $g \in X^*$, positive on $P$, attains maximum on $B$ at a vector $x_g$ of $B$. Hence $0 \leq g(x) \leq g(x_g)$ for any $x \in B$, therefore $0 \leq g(x_{-g}) \leq g(x_g)$, for any $x \in P$, $x \neq 0$ so $0 \leq g \leq g(x_g)$. Since $P^0$ is generating we can show that $f$ is an order unit of $X^*$ and as in the previous case we have that $B$ is bounded. □

We use below the following characterization of reflexivity of D. Mil’man and V. Mil’man, stated in 1964, in [7]. For this result see also in [8, Theorem 2.9].

Theorem 9. A Banach space $X$ is non-reflexive if and only if the positive cone of $\ell_1$ is embeddable in $X$.

Recall that a cone $P$ of a normed space $X$ is isomorphic, or locally-isomorphic according to the terminology of [9], to a cone $Q$ of a normed space $E$ if an additive, positively homogeneous, one-to-one, map $T$ of $E$ onto $Q$ exists such that $T$ and $T^{-1}$ are continuous in the induced topologies of $P, Q$. Then we say that the cone $P$ is embeddable in $E$ or that $T$ is an isomorphism of $P$ onto $Q$. Suppose that $T$ is an isomorphism of $P$ onto $Q$. By the continuity of $T$ and $T^{-1}$ at zero, we can show that

$$
\frac{1}{B} \|x\| \leq \|Tx\| \leq A \|x\|
$$

for each $x \in P$, where

1. If $P$ is normal then $P^0 - P^0 = X^*$, by [4, Theorem 3.4.8].
2. A map $T : P \to Q$ is additive and positively homogeneous if $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for each $\lambda, \mu \in \mathbb{R}_+$ and $x, y \in P$. 

where $A = \sup \{ \| Tx \| \mid x \in P, \| x \| \leq 1 \}$ and $B = \sup \{ \| T^{-1} y \| \mid y \in Q, \| y \| \leq 1 \}$. Then $T$ can be extended to a linear, one-to-one operator of $P - P$ onto $Q - Q$ by taking $T(x_1 - x_2) = T(x_1) - T(x_2)$, for any $x_1, x_2 \in P$ but the continuity of $T$ and $T^{-1}$ is not ensured. Note that $T$ is well defined on $P - P$ because if $x_1 - x_2 = x'_1 - x'_2$ we have that $x_1 + x'_2 = x'_1 + x_2$, therefore $T(x_1 + x'_2) = T(x'_1 + x_2)$ and by the additivity of $T$ on $P$ we have that $T(x_1 - x_2) = T(x'_1) - T(x'_2)$.

**Definition 10.** A normed space $X$ has the property $(*)$ if for each closed cone $P$ of $X$ we have: either $P$ has no a bounded base defined by an element of $X^*$ or any strictly positive (on $P$) linear functional of $X$ whose restriction on $P$ is continuous in the induced topology of $P$ attains maximum on any base for $P$ which is defined by an element of $X^*$.

**Theorem 11.** A Banach space $X$ is reflexive if and only if $X$ has the property $(*)$.

**Proof.** Suppose that $X$ is reflexive and $P$ a closed cone of $X$. If $P$ has a bounded base defined by an element of $X^*$ then any base for $P$ defined by an element of $X^*$ is bounded therefore weakly compact. So any strictly positive linear functional of $X$ whose restriction on $P$ is continuous attains maximum on any base for $P$ which is defined by an element of $X^*$. Hence $X$ has property $(*)$.

For the converse suppose that $X$ has property $(*)$. Suppose that $X$ is non-reflexive. Then an isomorphism $T$ of $\ell_1^+$ onto a closed cone $P$ of $X$ exists. Suppose that $Tx = Tx^+ - Tx^-$ for any $x \in \ell_1$ is the extension of $T$ on $\ell_1$ which we denote again by $T$, where $x^+ = \sup \{ x, 0 \}$ and $x^- = \sup \{ -x, 0 \}$, are the positive and negative parts of $x$. Then $T$ is continuous because for any $x \in \ell_1$ we have

$$\| T(x) \| = \| T(x^+) - T(x^-) \| \leq \| T(x^+) \| + \| T(x^-) \| \leq A(\| x^+ \| + \| x^- \|) = A\| x \|.$$.

The positive cone of $\ell_1$ has a closed, bounded base $C$, therefore $T(C)$ is a closed, bounded base for the cone $P$, hence there exists $g \in X^*$ which separates $T(C)$ and 0. Then it easy to show that $g$ is strictly positive on $P$ and that the base $K$ for $P$ defined by $g$ is bounded.

Suppose that $T^*: X^* \rightarrow \ell_\infty$ is the adjoint of $T$, i.e. $T^*(h)(\eta) = h(T(\eta))$ for any $h \in X^*$ and $\eta \in \ell_1$. Then $T^*$ is continuous and suppose that $T^*(g) = \xi = (\xi_i)$. Then $\xi$ is strictly positive on $\ell_1^+$ and if we suppose that $D$ is the base for $\ell_1^+$ defined by $\xi$ we can show that $T(D) = K$, therefore $D$ is bounded. So $\xi_i \geq a > 0$ for any $i$ because as we have shown in Example 1, if the sequence $\{ \xi_i \}$ has a convergent to zero subsequence then $\xi$ defines an unbounded base for $\ell_1^+$. Suppose that $r \in \ell_\infty$ with $r_i = \frac{\xi_i}{a + 1}$ for any $i$. Then $r$ does not attain maximum on $D$. Indeed $r(\eta) < \xi(\eta) = 1$ for any $\eta \in D$ and also we have: $\frac{r}{2} \in D$ for any $i$ with $r(\frac{r}{2}) = \frac{\xi_i}{a + 1} \rightarrow 1$, where $\xi_i$ is the vector of $\ell_\infty$ with 1 in the $i$th place and zero in the other coordinates.

Let $\phi$ be the linear functional of $P - P$ defined by the formula: $\phi(T(\eta)) = r(\eta)$, for any $\eta \in \ell_1$. Then $\phi$ has a linear extension on the whole space $X$, $\phi$ is strictly positive on $P$ and also its restriction on $P$ is continuous with respect to the induced topology of $P$ because $T$ is an isomorphism of $\ell_1^+$ onto $P$.

If we suppose that $\phi$ takes maximum on $K$ at a point $T(t)$, we have that $t \in D$ and also that $r$ takes maximum on $D$ at the point $t$, a contradiction. Hence $\phi$ does not attain maximum on $K$. This contradicts our assumption that $X$ has property $(*)$, therefore $X$ is reflexive. $\square$

4. Demand functions and reflexivity

In a competitive exchange economy we usually suppose that the commodity-price duality is the dual system $\langle X, X^* \rangle$, i.e. the commodity space is a normed space $X$ and the topological dual $X^*$ of $X$ is the price space. We suppose also that the consumption set is a cone $P$ of $X$ and that $X$ is ordered by the cone $P$. Then any strictly positive (on $P$) and continuous linear functional $f$ of $X$ is a price vector. For any price vector $f$ and for any real number (wealth level) $w > 0$, the set $B_w(f) = \{ x \in P \mid f(x) \leq w \}$ is the budget set corresponding to $f$ and $w$ and the set $L = \{ x \in P \mid f(x) = w \}$ is the budget line of $B_w(f)$. Of course $L$ is the base for the cone $P$ defined by the continuous, linear functional $g = \frac{f}{w}$. Therefore any budget set defines a base for the cone $P$ (the budget line) which is defined by a continuous linear functional. Conversely to any base $K$ for $P$ defined by an element $f \in X^*$, corresponds the budget set $B_1(f) = \{ x \in P \mid f(x) \leq 1 \}$ whose budget line is the base $K$. So there exists an one-to-one correspondence
between budget sets of $P$ and bases for $P$ which are defined by continuous linear functionals and in the sense of this correspondence, we may identify budget sets of $P$ with bases for $P$ defined by continuous linear functionals.

In general in competitive economies the commodity-price duality is expressed by a dual system $(E, F)$ where $E$, $F$ are linear topological vector spaces and the consumption set is a closed cone $P$ of $E$. Any $f \in P^0$, where $P^0$ is the dual cone of $P$ in $F$, strictly positive on $P$, is a price vector. For any price vector $f$ and any real number $w > 0$, $B_{w}(f) = \{x \in P \mid f(x) \leq w\}$ is the budget set for $f$, $w$ and, as above, an one-to-one correspondence between the budget sets of $P$ and bases for $P$ defined by elements of $Y$, exists.

We suppose also that $E$, $F$ are ordered by the cones $P$, $P^0$. Suppose that $\succsim$ is a preference relation of $P$, i.e. $\succsim$ is reflexive, complete and transitive. $\succsim$ is upper semicontinuous if for each $x \in P$ the set $\{y \in P \mid y \succsim x\}$ is closed in the induced topology of $P$, $\succsim$ is lower semicontinuous if for each $x \in P$ the set $\{y \in P \mid x \succsim y\}$ is closed in the induced topology of $P$ and $\succsim$ is continuous if it is lower and upper semicontinuous. If for any $x, y \in P$, $x \succsim y$ (i.e. $y - x \in P$) implies $x \succ y$, $\succsim$ is strictly monotone.

For any $p \in P^0$, strictly positive on $P$ and any real number $w > 0$ denote by $x(p, w)$ the demand set of $\succsim$, i.e. $x(p, w) = \{x \in B_{w}(p) \mid x \succsim y \text{ for any } y \in B_{w}(p)\}$. If for any $p \in P^0$ strictly positive on $P$ and any $w > 0$ the set $x(p, w)$ is non-empty, then $(p, w) \rightarrow x(p, w)$ is the demand correspondence of $\succsim$ and we say that the demand correspondence of $\succsim$ exists. If for any $p$ and $w$ the set $x(p, w)$ is a singleton, the demand correspondence defines the demand function of $\succsim$. If $p(x) = w$ for any $p$ and $w$, and any $x \in x(p, w)$, the demand correspondence satisfies the Walras’ Law.

If $E$ is a normed space, $\succsim$ is locally non-satiated if for any $x \in P$ and any real number $\epsilon > 0$ there exists $y \in P$ so that $y \succsim x$ and $\|y - x\| < \epsilon$. Recall that if $\succsim$ is strictly monotone, or if $\succsim$ has an extremely desirable bundle (i.e. there exists $u \in P$ so that $x + \lambda u \succsim x$ for any $x \in P$ and any real number $\lambda > 0$) then $\succsim$ is locally non-satiated.

In many cases $E$ is a Banach lattice see for example in [5], or [2] and $F$ is a closed ideal of $E^\ast$. In the case where $E$ is the dual $X^\ast$ of a Banach lattice $X$ with order continuous norm, the initial space $X$ as an ideal of $X^{**}$ can be considered as the price space and the dual system $(X^\ast, X)$ as the pair of the commodity-price duality.

We give below some known properties of the demand correspondences. The proof of (i) and (ii) is quite analogous with the proof in finite-dimensional economies. For the proof of (iii) see in [2, Theorem 1.2.2].

**Theorem 12.** Suppose that in a competitive exchange economy the commodity-price duality is the dual system $(E, F)$, the consumption set is a cone $P$ of $E$ and suppose that $\succsim$ is a preference relation defined on $P$.

(i) If $E$ is a normed space, $F \subseteq E^\ast$ and $\succsim$ is locally non-satiated then $p(x) = w$ for any $x \in x(p, w)$.

(ii) if $P$ is $\sigma(E, F)$-closed and $\succsim$ is $\sigma(E, F)$ upper semicontinuous, then the demand set $x(p, w)$ is $\sigma(E, F)$-closed, and

(iii) if for some topology $\tau$ of $P$, $\succsim$ is $\tau$-upper semicontinuous and the budget set $B_{w}(p)$ is $\tau$-compact, then $x(p, w) \neq \emptyset$.

**Proof.** (i) Suppose that $\succsim$ is locally non-satiated and $x \in x(p, w)$. If we suppose that $p(x) < w$, there exists a ball $U$ of $E$ of center $x$ and radius $\rho$ which is contained in the open halfspace $H_\rho = \{z \in E \mid p(z) < w\}$. Since $\succsim$ is locally non-satiated, there exists $y \in P \cap U$ so that $y \succsim x$, a contradiction because $y$ is an element of $B_{w}(p)$, therefore (i) is true.

(ii) Suppose that $(z_a)_{a \in A}$ is a net of $x(p, w)$ which converges to $z_0$ in the $\sigma(E, F)$-topology of $E$. Then $z_0 \in B_{w}(p)$ because the budget set is $\sigma(E, F)$-closed. Also for any $x \in B_{w}(p)$ we have that $z_a \succsim x$, and by the upper semicontinuity of $\succsim$, $z_0 \succsim x$, therefore $x(p, w)$ is closed.

The next is one of the main results of this article. In fact this is the equivalence formulation of our dichotomy result for cones, in competitive economies.

**Theorem 13.** Suppose that in a competitive exchange economy the commodity-price duality is the dual system $(X, Y)$ where $X$ is a normed space and suppose also that the consumption set is a $\sigma(X, Y)$-closed cone $P$ of $X$. If the positive part $U_Y^+ = U_X \cup P$ of the unit ball $U_X$ of $X$ is $\sigma(X, Y)$-compact and $P$ has a norm-bounded budget set, then for any $\sigma(X, Y)$ upper semicontinuous preference relation $\succsim$ of $P$, the demand correspondence of $\succsim$ exists.
Proof. The cone $P$ has a bounded budget set, therefore it has a bounded base defined by a vector $y \in Y$. By the dichotomy result, any base for $P$ defined by a vector of $Y$ is bounded, therefore any budget set of $P$ is bounded. But any budget set of $P$ is $\sigma(X,Y)$-closed, therefore any budget set is $\sigma(X,Y)$-compact. Hence any $\sigma(X,Y)$ upper semicontinuous preference relation $\succ$ of $P$ attains maximum on any budget set of $P$, therefore the demand correspondence of $\succ$ exists. □

Corollary 14. Suppose that in a competitive exchange economy the commodity-price duality is the dual system $\langle X^*, X \rangle$ where $X$ is a normed space and $X^*$ its dual. If the consumption set is a weak-star closed cone $P$ of $X^*$ and $P$ has a bounded budget set, then the demand correspondence of any weak-star upper semicontinuous preference relation of $P$ exists.

Proof. Consider the dual system $\langle E, F \rangle$ with $E = X^*$, $F = X$. Then the $\sigma(E, F)$ topology of $E$ is the weak-star topology of $X^*$ and the corollary is true. □

Corollary 15. Suppose that in a competitive exchange economy the commodity-price duality is the dual system $\langle X, X^* \rangle$ where $X$ is a reflexive Banach space and the consumption set is a closed cone $P$ of $X$. If $P$ has a bounded budget set, then for any weakly upper semicontinuous preference relation $\succ$ of $P$, the demand correspondence of $\succ$ exists.

4.1. Continuity of the demand correspondences

A correspondence (multivalued function) $\varphi$ from a topological space $F$ into the subsets of a topological space $G$ is upper hemicontinuous at a point $x \in F$ if for any open neighborhood $V$ of $\varphi(x)$ the upper inverse $^3 \varphi^u(V)$ of $V$ is a neighborhood of $x$. $\varphi$ is upper hemicontinuous if it is upper hemicontinuous at any point of $F$. By the Closed Graph Theorem, see for example in [1, Theorem 16.12], we have: if the range space $G$ of $\varphi$ is compact and Hausdorff and $\varphi$ has closed graph, then $\varphi$ is upper hemicontinuous.

Definition 16. Suppose that $\succ$ is a preference relation defined on a cone $P$ of a linear topological space $E$. If for each $x, y \in P$ the set \{\lambda x \mid \lambda \in \mathbb{R}_+ \text{ so that } y \succ \lambda x\} is closed the preference relation $\succ$ is radially lower semicontinuous. If for each $x, y \in P$ the set \{\lambda x \mid \lambda \in \mathbb{R}_+ \text{ so that } \lambda x \succ y\} is closed then $\succ$ is radially upper semicontinuous. If $\succ$ is radially lower and radially upper semicontinuous then $\succ$ is radially continuous.

Of course any lower semicontinuous preference relation is radially lower semicontinuous but the converse is not always true.

Theorem 17. Suppose that in a competitive exchange economy the commodity-price duality is the dual system $\langle X, Y \rangle$ where $X$ is a normed space and $Y$ a subspace$^4$ of $X^*$ and suppose also that the consumption set is a $\sigma(X,Y)$-closed cone $P$ of $X$. If the positive part $U_X^+ = U_X \cap P$ of the unit ball $U_X$ of $X$ is $\sigma(X,Y)$-compact and the cone $P$ has a norm-bounded budget set, then for any $\sigma(X,Y)$ upper semicontinuous and $\sigma(X,Y)$ radially lower semicontinuous preference relation $\succ$ of $P$, the demand correspondence of $\succ$ is norm to $\sigma(X,Y)$, upper hemicontinuous.

Proof. By Theorem 13, the demand correspondence of $\succ$ exists. $Y$ is ordered by the dual cone $P^0 = \{q \in Y \mid q(x) \geq 0 \text{ for any } x \in P\}$ of $P$ in $Y$ and suppose that $D = \{q \in Y \mid q(x) > 0 \text{ for any } x \in P, x \neq 0\}$ is the set of strictly positive on $P$ vectors of $Y$. We shall show first that for any $q \in D$ and each $\lambda \in (0, 1)$ the order interval $[(1-\lambda)q, (1+\lambda)q]$ is a neighborhood of $q$ contained in $D$.

By our dichotomy result, $q$ defines a bounded base $K$ for $P$ and suppose that $K$ is contained in the closed ball of $X$ of center zero and radius $\delta$. Then for any $h \in Y$ with $\|h\| \leq \frac{1}{\delta}$ we have that $|h(x)| \leq 1$ for any $x \in K$ therefore $(q+h)(\frac{x}{q(x)}) \geq 0$, for any $x \in P, x \neq 0$, hence $q + h \geq 0$ for any $h \in \frac{1}{\delta} U_Y$, where $U_Y$ is the closed unit ball of $Y$.

---

3 The upper inverse $^3 \varphi^u(V)$ of $V$ is the set of vectors $x$ of $F$ so that $\varphi(x) \subseteq V$.
4 According to the definition of the dual system, $Y$ separates the points of $X$. 
So \( q \geq h \) because \( U_Y \) is symmetric, therefore \( \frac{1}{\delta} U_Y \) is contained in the order interval \([-q, q]\). So for any \( \varepsilon > 0 \) we have

\[
q + \varepsilon U_Y = q + \varepsilon \frac{1}{\delta} U_Y \subseteq q + \varepsilon [ -q, q ] = \left( (1 - \varepsilon \delta)q, (1 + \varepsilon \delta)q \right).
\]

For any \( 0 < \lambda = \varepsilon \delta < 1 \) or equivalently for any \( 0 < \varepsilon < \frac{1}{\delta} \) the order interval \([ (1 - \lambda)q, (1 + \lambda)q ]\) is contained in \( D \subseteq P^0 \) because any \( p \in Y \) with \( p \geq (1 - \lambda)q \) is strictly positive.

Suppose that \( \omega > 0, \lambda \in (0, 1) \) are constant real numbers and \( q \) a fixed element of \( D \). We shall show that the family of budget sets \( B_w(z) = \{ x \in P \mid z(x) \leq w \}, \, z \in \{ (1 - \lambda)q, (1 + \lambda)q \} \) and \( w \in (0, \omega) \) is uniformly bounded. Indeed, for any \( x \in B_w(z) \) we have

\[
(1 - \lambda)q(x) \leq z(x) \leq w \leq \omega.
\]

Since \( q \) defines a bounded base for \( P \) it is uniformly monotonic, hence there exists a real number \( \theta > 0 \) so that \( q(y) \geq \theta \| y \| \), for any \( y \in P \). Thus \( (1 - \lambda)q(x) \geq (1 - \lambda)\theta \| x \| \) therefore

\[
\| x \| \leq \frac{\omega}{(1 - \lambda)\theta}.
\]

We shall show now that the demand correspondence \( x(p, w) \) has closed graph in the \((\text{norm}, \sigma(X, Y))\) topology of \( F \times X \), where \( F = D \times (0, +\infty) \) equipped with the topology induced by the norm \( \| (p, w) \| = \| p \| + |w| \) of \( Y \times \mathbb{R} \).

So we suppose that the net \( ((p^a, w_a), x(p^a, w_a)) \) is convergent to the point \( ((p, w), x^o) \) of \( F \times X \), in the \((\text{norm}, \sigma(X, Y))\) topology of \( F \times X \), where \( x(p^a, w_a) \in \mathbf{x}(p^a, w_a) \) for any \( a \). So we have that \( p^a \to p \) in the norm topology of \( Y \), \( w_a \to w > 0 \) in \( \mathbb{R} \) and that \( x(p^a, w_a) \to x^o \) in the \( \sigma(X, Y) \) topology of \( X \). We shall show that \( x^o \in \mathbf{x}(p, w) \), i.e. \( x^o \in B_w(p) \) and \( x^o \equiv b \) for any \( b \in B_w(p) \).

Suppose that \( \delta \) is a norm bound of the base for \( P \) defined by \( p \). For any \( 0 < \varepsilon < \frac{1}{\delta} \), or equivalently \( \lambda = \varepsilon \delta < \frac{1}{\delta} \), there exists \( a_0 \) so that for any \( a \succ a_0 \), where \( \succ \) is the ordering of the index set of the net, we have:

\[
p^a \in p + \varepsilon U_Y \subseteq \left[ (1 - \lambda)p, (1 + \lambda)p \right], \quad \| p(x(p^a, w_a)) - p(x^o) \| < \varepsilon \quad \text{and} \quad w_a \leq 2w.
\]

If \( \theta \) is a constant of the uniform monotonicity of \( p \), then by (1) and the above inequalities we have:

\[
\| p^a(x(p^a, w_a)) - p(x^o) \| \leq \left( 4w \theta + 1 \right) \varepsilon,
\]

for any \( a \succ a_0 \). Hence \( p^a(x(p^a, w_a)) \to p(x^o) \). Also \( p(x^o) \leq w \), because \( p^a(x(p^a, w_a)) \leq w_a \to w \), hence \( x^o \in B_w(p) \).

We shall show now that \( \succ \) takes maximum on \( B_w(p) \) at \( x^o \). For any \( b \in B_w(p), b \neq 0 \) we have that \( p(tb) < 0 \) because \( p \) is strictly positive on \( P \), therefore for any \( t \in (0, 1) \) we have \( p(tb) < w \). Consider a constant \( t \in (0, 1) \). We assert that an index \( a_1 \) exists so that \( p^a(tb) < w_a \) for any \( a \succ a_1 \). Indeed if we suppose that this assertion is not true, for any index \( a \) there exists an index \( r_a \) with \( r_a \succ a \), so that \( p^{r_a}(tb) \geq w_{r_a} \). Then \( \{ p^{r_a} \} \) is a subnet of \( \{ p^a \} \) convergent to \( p \), therefore \( p(tb) \geq w \), a contradiction. Therefore our assertion is true, hence \( tb \in B_w(p^a) \) for any \( a \succ a_1 \), therefore \( x(p^a, w_a) \equiv tb \) for any \( a \succ a_1 \). Since the subnet \( (x(p^a, w_a))_{a \succ a_1} \) is \( \sigma(X, Y) \)-convergent to \( x^o \) and the set \( \{ y \in P \mid y \equiv tb \} \) is \( \sigma(X, Y) \)-closed, we have that \( x^o \equiv tb \). Therefore \( x^o \equiv tb \) for any \( t \in (0, 1) \). By our assumption that \( \succ \) is \( \sigma(X, Y) \) radially lower semicontinuous we have that \( x^o \equiv b \) and \( x^o \equiv 0 \), therefore \( x^o \equiv b \), for any \( b \in B_w(p) \). Hence \( x^o \in \mathbf{x}(p, w) \). So the demand correspondence has closed graph in the \((\text{norm}, \sigma(X, Y))\) topology of \( F \times X \).

Suppose that \( p^o \in D \) and \( w^o > 0 \). We shall show that the demand correspondence is upper hemicontinuous at \( (p^o, w^o) \).

As we have shown before, \( \left[ \frac{1}{2} p^o, \frac{3}{2} p^o \right] \) is a neighborhood of \( p^o \) contained in \( D \), therefore the set \( U = \left[ \frac{1}{2} p^o, \frac{3}{2} p^o \right] \times \left( 0, \infty \right) \) is a neighborhood of \( \left( p^o, w^o \right) \) contained in \( D \times (0, +\infty) \). Also we have shown in (1) that the budget sets \( B_w(z), (z, w) \in U \) are uniformly bounded, therefore the demand correspondence, restricted on \( U \) takes values in a positive multiple of \( U^+ \), therefore in a \( \sigma(X, Y) \)-compact, Hausdorff topological space. By the closed graph theorem the demand correspondence is norm to \( \sigma(X, Y) \) upper hemicontinous on \( U \). Since \( (p^o, w^o) \) is an interior point of \( U \),
the demand correspondence is upper hemicontinuous at \((p^0, w^0)\) and therefore upper hemicontinuous at any point of \(F\). □

**Corollary 18.** Suppose that in a competitive exchange economy the commodity-price duality is the dual system \(<X^*, X>\) where \(X\) is a normed space and \(X^*\) its dual, the consumption set is a weak-star closed cone \(P\) of \(X^*\) and suppose also that \(P\) has a bounded budget set. If \(\succeq\) is a weak-star upper semicontinuous and weak-star radially lower semicontinuous preference relation of \(P\), then the demand correspondence of \(\succeq\) is norm to weak-star upper hemicontinuous.

**Corollary 19.** Suppose that in a competitive exchange economy the commodity-price duality is the dual system \(<X, X^*>\) where \(X\) is a reflexive Banach space and the consumption set is a closed cone \(P\) of \(X\). If \(P\) has a bounded budget set, then for any weakly upper semicontinuous and radially lower semicontinuous preference relation \(\succeq\) of \(P\) the demand correspondence of \(\succeq\) is norm to weak upper hemicontinuous.

### 4.2. Linear preferences

Suppose that \(X\) is a normed space, \(P\) a closed cone of \(X\) and \(\succeq\) is a **linear** preference relation of \(P\), i.e. \(\succeq\) is defined by a linear functional (utility function) \(f\) of \(X\). So for any \(x, y \in P\) we have: \(x \succeq y\) if and only if \(f(x) \geq f(y)\). Then \(\succeq\) is continuous on \(P\) if and only if the restriction \(f\vert_P\) of \(f\) on \(P\) is continuous (in the induced topology of \(P\)). So for the continuity of \(\succeq\) it is not needed the continuity of \(f\) on the whole space \(X\). If \(\succeq\) is also defined by the linear functional \(g\), then \(f(x) \geq f(y)\) if and only if \(g(x) \geq g(y)\) and it is easy to show that \(f\vert_P = g\vert_P\). If \(f\vert_P\) has a continuous linear extension on \(X\), then \(\succeq\) is defined by a vector of \(X^*\) and we say that \(\succeq\) has a linear, continuous extension on \(X\).

**Theorem 20.** Suppose that in a competitive exchange economy the commodity-price duality is the dual system \(<X, X^*>\) where \(X\) is a normed space, \(X^*\) its dual, and suppose that the consumption set is a closed cone \(P\) of \(X\). If the dual cone \(P^0\) of \(P\) in \(X^*\) is generating, i.e. \(P^0 - P^0 = X^*\) and for any preference relation \(\succeq\) of \(P\) which is defined by a positive on \(P\) vector \(g \in X^*\) the demand correspondence of \(\succeq\) exists, then any budget set of \(P\) is bounded.

**Proof.** Suppose that \(B\) is a base for \(P\) defined by \(f \in X^*\). By Theorem 8, it is enough to show that any \(g \in P^0\) attains maximum on \(B\). By our assumption, the preference relation \(\succeq\) of \(P\) defined by \(g\) attains maximum on the budget set \(B_1(f)\) of \(P\) at \(x_0\). If \(g(z_0) > 0\) for some \(z_0 \in P\), then \(\succeq\) is locally non-satiated, therefore \(x_0 \in B\) and \(g\) attains maximum on \(B\) at \(x_0\). □

**Theorem 21.** Suppose that in a competitive exchange economy the commodity-price duality is the dual system \(<X, X^*>\) where \(X\) is a Banach space and \(X^*\) its dual. The space \(X\) is reflexive if and only if \(X\) has the property: for any closed cone (consumption set) \(P\) of \(X\) with a bounded budget set and for any strictly monotone, linear, continuous preference relation \(\succeq\) of \(P\), the demand correspondence of \(\succeq\) exists.

**Proof.** Our assertion that for any strictly monotone, linear, continuous preference relation \(\succeq\) of \(P\), the demand correspondence of \(\succeq\) exists is equivalent with the following: any strictly positive linear functional of \(X\) whose restriction on \(P\) is continuous attains maximum on any base for \(P\) which is defined by a continuous linear functional. By Theorem 11, the result is true. □

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**References**