Cones Locally Isomorphic to the Positive Cone of $l_1(\Gamma)$

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ABSTRACT

We give necessary and sufficient conditions in order for an infinite-dimensional, closed cone P of a Banach space X to be locally isomorphic to the positive cone $l_1^+(\Gamma)$ of $l_1(\Gamma)$.

1. INTRODUCTION

In [4, Theorem 4.1], it is proved that if an infinite-dimensional Banach space X, ordered by the closed generating cone P, has the Riesz decomposition property, then X is order-isomorphic to $l_1(\Gamma)$ iff P has a closed, bounded base with the Krein-Milman property. In the same paper the concept of the continuous projection property (CPP) is introduced for an ordered Banach space, by means of which the existence of strongly exposed points in a base for the positive cone of X is studied.

In the present paper (Theorem 4.1) we prove that if P is an infinite-dimensional closed cone of a Banach space X and the cone P has the CPP, then the cone P is locally isomorphic to $l_1^+(\Gamma)$ iff P has a closed, bounded base with the Krein-Milman property. As an application we obtain a generalization of [4, Theorem 4.1] by replacing the Riesz decomposition property with the CPP (Corollary 4.1). Also, in Proposition 4.1, we give various conditions in order for an infinite-dimensional closed cone P with the Radon-Nikodym property to be locally isomorphic to $l_1^+(\Gamma)$. Finally, we note that the hypotheses of Theorem 4.1 do not assure the closedness of the subspace $Y = P - P$ of X, and therefore that we cannot conclude, by virtue of [4, Theorem 4.1], that Y is order-isomorphic to $l_1(\Gamma)$.

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2. NOTATION

Let X be a Banach space. We denote by X^* the topological dual of X, and for each convex subset K of X we denote by ep(K) [respectively, by sep(K)] the set of extreme [respectively, strongly exposed] points of K. A point x of K is a strongly exposed point of K iff there exists $g \in X^*$ such that $g(x) > g(y) \ \forall y \in K \setminus \{x\}$ and for each sequence (x_ν) of K, $\lim_{\nu \to \infty} g(x_\nu) = g(x)$ implies $\lim_{\nu \to \infty} x_\nu = x$. For each $A \subseteq X$ we denote by $\bar{A}$ the closure of A and by $\overline{\text{co}}A$ the closed convex hull of A.

Let X be ordered by the cone P. A point x of P is an extremal point of P iff for each $y \in X$, $0 < y < x$ implies $y = \lambda x$ where $\lambda \in \mathbb{R}_+$. We denote by EP(P) the set of extremal points of P. We say that the cone P is generating if $X = P - P$. A linear functional f of X is strictly positive if $f(x) > 0$ for each $x \in P \setminus \{0\}$. A subset B of P is a base for P if there exists a strictly positive linear functional f of X such that $B = \{x \in P \mid f(x) = 1\}$. Then we say that the base B for P is defined by the functional f. Each base for P is convex. The cone P is well based if P has a bounded base B and $0 \notin \bar{B}$.

For notions not defined here, see [1] and [2].

3. CONES WITH THE CONTINUOUS PROJECTION PROPERTY

Let X be a normed space ordered by the cone P. In order to study the existence of strongly exposed points in a base B for the cone P, in [4], the continuous projection property for the space X has been defined as follows: We say that an extremal point x_0 of P has continuous projection if there exists a linear continuous, positive projection P_{x_0} of X onto $[x_0]$ such that $P_{x_0}(x) \leqslant x \ \forall x \in P$. We say that X has the continuous projection property (CPP) if $x_0 \in \text{EP}(P)$ implies that x_0 has continuous projection.

If X has the CPP, then for each $x_0 \in \text{EP}(P)$ we denote by P_{x_0} the continuous positive projection of X onto $[x_0]$, and by $\rho(x_0, \cdot)$ the continuous linear functional of X defined by the formula

$$P_{x_0}(x) = \rho(x_0, x)x_0 \qquad \forall x \in X.$$

Let $Y = P - P$. If X has the CPP, then Y, ordered by the cone P, has the CPP. If Y has the CPP, then for each $x_0 \in \text{EP}(P)$ there exists a continuous positive projection, $P_{x_0}(x) = \rho(x_0, x)x_0$, of x_0 defined on Y. If $\rho'(x_0, \cdot)$ is a Hahn-Banach extension of $\rho(x_0, \cdot)$ on X, then $P'_{x_0}(x) = \rho'(x_0, x)x_0$ is a continuous, positive projection of x_0 defined on X. So we have that X has the CPP iff $Y = P - P$ has the CPP.

Let the cone P be closed, and Y have the Riesz decomposition property (i.e., $\forall x, y, z \in P$ with $x \leqslant y + z$, there exist $x_1, x_2 \in P$ such that $0 \leqslant x_1 \leqslant y$, $0 \leqslant x_2 \leqslant z$, and $x = x_1 + x_2$). Then Y is Archimedean, and by [5, Theorem 1.2], for each $x_0 \in \mathrm{EP}(P)$ there exists a linear functional $f(x_0, \cdot)$, defined on $Y = P - P$ by the formulas

$$\begin{aligned} f(x_0, x) &= \sup\{t \in \mathbb{R}_+ \mid tx_0 \leqslant x\} \qquad \forall x \in P, \\ f(x_0, x - y) &= f(x_0, x) - f(x_0, y) \qquad \forall x, y \in P. \end{aligned} \tag{1}$$

(In [5, Theorem 1.2], the existence of $f(x_0, \cdot)$ is deduced from the fact that Y is Archimedean and Y has the Riesz decomposition property.)

Let $T_{x_0}(x) = f(x_0, x)x_0$ $\forall x \in Y$. Then T_{x_0} is a linear, positive projection of Y onto $[x_0]$. Since the cone P is closed, we have that $T_{x_0}(x) \leqslant x$ $\forall x \in P$. If $f(x_0, \cdot)$ is continuous $\forall x_0 \in \mathrm{EP}(P)$, then x_0 has continuous projection and the space Y has the CPP. If Y has the CPP, then for each $x_0 \in \mathrm{EP}(P)$, there exists a continuous positive projection $P_{x_0}(x) = \rho(x_0, x)x_0$ of Y onto $[x_0]$. This projection is unique, and therefore $f(x_0, \cdot) = \rho(x_0, \cdot)$ is continuous. So, if the cone P is closed and X has the Riesz decomposition property, then X has the CPP iff $Y = P - P$ has the CPP iff for each $x_0 \in \mathrm{EP}(P)$ the linear functional $f(x_0, \cdot)$ defined by the formula (1) is continuous. For example, if $Y = P - P$ is a Banach space, P is closed, and Y has the Riesz decomposition property, then we have that $f(x_0, \cdot)$ is continuous because each positive linear functional of a Banach space ordered by a closed, generating cone is continuous [2, 3.5.6] and therefore that Y has the CPP.

Also, if $Y = P - P$ is a locally solid linear lattice, then we can show that the functional $f(x_0, \cdot)$ is continuous for each $x_0 \in \mathrm{EP}(P)$ and therefore that Y has the CPP [4, Proposition 3.2]. Since the CPP in a normed space X, ordered by the cone P, depends only on P, in this paper we shall say "P has the CPP" instead of "X has the CPP."

4. CONES LOCALLY ISOMORPHIC TO $l_1^+(\Gamma)$

Let G be a closed and convex subset of a Banach space X. The set G has the Krein-Milman property (KMP) if $K = \overline{\mathrm{co}}\,\mathrm{ep}(K)$ for each closed, convex and bounded subset K of G. It is known [1, 3.5.7] that the set G has the Radon-Nikodym property (RNP) iff $K = \overline{\mathrm{co}}\,\mathrm{sep}(K)$ for each closed, convex, and bounded subset K of G.

Let Γ be any set. We denote by $l_1(\Gamma)$ the Banach space of all real functions $\xi: \Gamma \to \mathbb{R}$, $\xi = (\xi(i))_{i \in \Gamma}$, such that $\sum_{i \in \Gamma} |\xi(i)| < +\infty$, with norm

$\|\xi\| = \sum_{i \in \Gamma} |\xi(i)|$. The space $l_1(\Gamma)$ has the RNP [1, 4.1.9], and the space $l_1(\Gamma)$, ordered by the cone $l_1^+(\Gamma) = \{\xi \in l_1(\Gamma) \mid \xi(i) \geqslant 0 \ \forall i \in \Gamma\}$, is a Banach lattice. The set $B = \{\xi \in l_1^+(\Gamma) \mid \|\xi\| = 1\}$ is a closed, bounded base for the cone $l_1^+(\Gamma)$. We denote by l_1 the space $l_1(\mathbb{N})$.

A closed cone P of a Banach space X is locally isomorphic to a closed cone Q of a Banach space Y if there exists a map T of P onto Q such that $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ $\forall x, y \in P$, $\lambda, \mu \in \mathbb{R}_+$ and if for each sequence x_ν of P we have $\lim_{\nu \to \infty} x_\nu = x$ iff $\lim_{\nu \to \infty} T(x_\nu) = T(x)$. In this case we say that T is a local isomorphism of P onto Q. An ordered Banach space X is order-isomorphic to an ordered Banach space Y iff there exists an isomorphism T of X onto Y and T, T^{-1} are positive.

THEOREM 4.1. *Let X be a Banach space ordered by the infinite-dimensional, closed cone P. If P has the CPP, then*:

(i) *P is locally isomorphic to $l_1^+(\Gamma)$ iff P has a closed, bounded base with the KMP.*

(ii) *P is locally isomorphic to l_1 iff P has a separable, closed, bounded base with the KMP.*

Proof. Let T be a local isomorphism of P onto $l_1^+(\Gamma)$. Since $B = \{\xi \in l_1^+(\Gamma) \mid \|\xi\| = 1\}$ is a closed, bounded base for $l_1^+(\Gamma)$ with the KMP, we have that $T^{-1}(B)$ is a closed, bounded base for P with the KMP.

Let B be a closed, bounded base for P with the KMP. By [2, 3.8.12] there exists a uniformly monotonic continuous linear functional f of X.

Let C be the closed and bounded base for P defined by f. Then $\text{ep}(C) \neq \varnothing$ because $\text{ep}(B) \neq \varnothing$ and the extreme points of C coincides, up to a scalar multiple, with the extreme points of B. Let

$$\text{ep}(C) = \{b_i \mid i \in \Gamma\}.$$

We shall prove that

$$x = \sum_{i \in \Gamma} \rho(b_i, x) b_i \quad \text{and} \quad \sum_{i \in \Gamma} \rho(b_i, x) < +\infty \qquad \forall x \in P. \tag{1}$$

First we shall show that

$$L = \left\{x \in P \middle| P_{b_i}(x) = 0 \ \forall i \in \Gamma\right\} = \{0\}.$$

The set L is a cone, and it is closed, because P_{b_i} is continuous $\forall i \in \Gamma$. If

$L \neq \{0\}$, the set $B' = B \cap L$ is a nonempty, closed, and bounded base for L. So $\mathrm{EP}(L) \neq \varnothing$ because $\mathrm{ep}(B') \neq \varnothing$. Also $\mathrm{EP}(L) \subseteq \mathrm{EP}(P)$, because for each $x \in L$ and $y \in P$, $0 \leqslant y \leqslant x$ implies that $y \in L$. Hence $b_j \in \mathrm{EP}(L)$ for at least one $j \in \Gamma$. This contradicts the definition of L, because $P_{b_j}(b_j) = b_j$. Hence $L = \{0\}$.

We denote by F the set of finite subsets of Γ, and for each $x \in P$ and $\delta \in F$ we denote by x_δ the sum

$$x_\delta = \sum_{i \in \delta} P_{b_i}(x).$$

Let $x \in C$. Then $(x_\delta)_{\delta \in F}$ is an upward-directed net of P (if $\delta_1, \delta_2 \in F$, we say that $\delta_1 \leqslant \delta_2$ iff $\delta_1 \subseteq \delta_2$). We shall show that $x_\delta = \sup\{P_{b_i}(x) \mid i \in \delta\}$. If $z \geqslant P_{b_i}(x)$ $\forall i \in \delta$, then $w = z - P_{b_i}(x) \geqslant 0$. If $j \in \delta$ and $j \neq i$, then $w \geqslant P_{b_j}(w) = P_{b_j}(z) \geqslant P_{b_j}(x)$ and therefore $z \geqslant P_{b_i}(x) + P_{b_j}(x)$. Similarly we have that $z \geqslant x_\delta$; hence

$$x_\delta = \sup\left\{P_{b_i}(x) \middle| i \in \delta\right\} \leqslant x \qquad \forall \delta \in F.$$

By [2, 3.8.8], we have that

$$y = \lim x_\delta \leqslant \sup_{\delta \in F} (x_\delta) \leqslant x.$$

This implies that $P_{b_i}(x) \leqslant y \leqslant x$ $\forall i \in \Gamma$ and therefore that $P_{b_i}(x - y) = 0$ $\forall i \in \Gamma$, because $P_{b_i}(x) \leqslant P_{b_i}(y) \leqslant P_{b_i}(x)$ $\forall i \in \Gamma$. Hence $x = y$, and therefore

$$x = \sum_{i \in \Gamma} \rho(b_i, x) b_i.$$

Since $f \in X^*$ and f define the base C, we have that

$$f(x) = \sum_{i \in \Gamma} \rho(b_i, x) = 1.$$

So (1) is true because it is true for each $x \in C$. We shall prove that the map

$$T(x) = \left(\rho(b_i, x)\right)_{i \in \Gamma} \in l_1^+(\Gamma),$$

is a local isomorphism of P onto $l_1^+(\Gamma)$.

It is clear that $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ $\forall x, y \in P$ and $\lambda, \mu \in \mathbb{R}_+$. T is one-to-one, because if $T(x) = T(y)$ then $\rho(b_i, x) = \rho(b_i, y)$ $\forall i \in \Gamma$; hence by (1) we have that $x = y$.

Since the set C is bounded, there exists $M \in \mathbb{R}_+$ such that

$$\|x\| \leqslant M \qquad \forall x \in C.$$

We shall show that the map T is onto.

Let $\xi = (\xi(i))_{i \in \Gamma} \in l_1^+(\Gamma)$. For each $\delta \in F$ we put

$$x_\delta = \sum_{i \in \delta} \xi(i) b_i \quad \text{and} \quad \xi_\delta = T(x_\delta).$$

Let $\varepsilon > 0$. Since $\lim \xi_\delta = \xi$, there exists $\delta_0 \in F$ such that

$$\|\xi_{\delta_1} - \xi_{\delta_2}\| < \varepsilon \qquad \forall \delta_1, \delta_2 > \delta_0.$$

If $\delta_1' = \delta_1 \setminus \delta_2$ and $\delta_2' = \delta_2 \setminus \delta_1$, then $\xi_{\delta_1'} - \xi_{\delta_2'} = \xi_{\delta_1} - \xi_{\delta_2}$. Since $\xi_{\delta_1'}, \xi_{\delta_2'}$ are disjoint, we have that

$$|\xi_{\delta_1} - \xi_{\delta_2}| = |\xi_{\delta_1'} - \xi_{\delta_2'}| = \xi_{\delta_1'} + \xi_{\delta_2'}$$

and therefore

$$\|\xi_{\delta_1} - \xi_{\delta_2}\| = \|\xi_{\delta_1'} + \xi_{\delta_2'}\| = \|\xi_{\delta_1'}\| + \|\xi_{\delta_2'}\| \leqslant \varepsilon.$$

So

$$\|x_{\delta_1} - x_{\delta_2}\| = \|x_{\delta_1'} - x_{\delta_2'}\| \leqslant M\big(\|\xi_{\delta_1'}\| + \|\xi_{\delta_2'}\|\big) < M\varepsilon.$$

Hence the net $(x_\delta)_{\delta \in F}$ is Cauchy. If $x = \lim x_\delta$, then $x \in P$ and $\rho(b_i, x) = \xi(i)$ $\forall i \in \Gamma$. So $T(x) = \xi$ and the map T is onto.

Let (x_ν) be a sequence of P, and $x \in P$. We shall show that

$$\lim_{\nu \to \infty} x_\nu = x \quad \Leftrightarrow \quad \lim_{\nu \to \infty} T(x_\nu) = T(x).$$

Let $\lim_{\nu\to\infty} T(x_\nu) = T(x)$. Then

$$\lim_{\nu\to\infty} \|T(x_\nu) - T(x)\| = \lim_{\nu\to\infty}\left(\sum_{i\in\Gamma} |\rho(b_i, x_\nu - x)|\right) = 0.$$

Then for each $\nu \in \mathbb{N}$ we have that

$$\|x_\nu - x\| = \left\| \sum_{i\in\Gamma} \rho(b_i, x_\nu - x) b_i \right\| \leqslant M \sum_{i\in\Gamma} |\rho(b_i, x_\nu - x)|$$

and therefore $\lim_{\nu\to\infty} x_\nu = x$.

Let $\lim_{\nu\to\infty} x_\nu = x$. We shall show that $T(x_\nu) \to T(x)$. To show this it is enough to show that for each $\varepsilon > 0$ there exists $\nu_0 \in \mathbb{N}$ such that

$$\|T(x_\nu) - T(x)\| = \sum_{i\in\Gamma} |\rho(b_i, x_\nu - x)| < \varepsilon \qquad \forall \nu > \nu_0.$$

Let $\varepsilon > 0$. Since $T(x) \in l_1^+(\Gamma)$, there exists $\delta_0 \in F$ such that

$$\sum_{i\in\delta'\setminus\delta_0} \rho(b_i, x) < \varepsilon \qquad \forall \delta' \geqslant \delta_0. \tag{2}$$

Now $x_\nu \to x$, so there exists $\nu_1 \in \mathbb{N}$ such that

$$\|x_\nu - x\| < \varepsilon \qquad \forall \nu > \nu_1. \tag{3}$$

Since δ_0 is finite and $\lim_{\nu\to\infty} \rho(b_i, x_\nu) = \rho(b_i, x)$ $\forall i \in \Gamma$, there exists $\nu_2 \in \mathbb{N}$ such that

$$\sum_{i\in\delta_0} |\rho(b_i, x_\nu - x)| < \varepsilon \qquad \forall \nu > \nu_2. \tag{4}$$

Let a constant $\nu \in \mathbb{N}$, and $\nu \geqslant \nu_0 = \max\{\nu_1, \nu_2\}$. By (3), there exists $\delta_1 \in F$ such that

$$\left\| \sum_{i\in\delta} \rho(b_i, x_\nu - x) b_i \right\| < 2\varepsilon \qquad \forall \delta \geqslant \delta_1. \tag{5}$$

Let $\delta_2 > \delta_0, \delta_1$ and $\delta > \delta_2$. Then

$$\sum_{i\in\delta} |\rho(b_i, x_\nu - x)| \leqslant \sum_{i\in\delta_0} |\rho(b_i, x_\nu - x)| + \sum_{i\in\delta\setminus\delta_0} \rho(b_i, x) + \sum_{i\in\delta\setminus\delta_0} \rho(b_i, x_\nu) \qquad \forall\delta > \delta_2, \quad (6)$$

$$\begin{aligned}\sum_{i\in\delta\setminus\delta_0} \rho(b_i, x_\nu) &= f\Bigl(\sum_{i\in\delta\setminus\delta_0} \rho(b_i, x_\nu) b_i\Bigr)\\
&\leqslant \|f\|\Bigl(\Bigl\|\sum_{i\in\delta\setminus\delta_0} \rho(b_i, x_\nu - x) b_i\Bigr\| + \Bigl\|\sum_{i\in\delta\setminus\delta_0} \rho(b_i, x) b_i\Bigr\|\Bigr)\\
&\leqslant \|f\|\Bigl(\Bigl\|\sum_{i\in\delta_0} \rho(b_i, x_\nu - x) b_i\Bigr\| + \Bigl\|\sum_{i\in\delta} \rho(b_i, x_\nu - x) b_i\Bigr\|\\
&\qquad + \Bigl\|\sum_{i\in\delta\setminus\delta_0} \rho(b_i, x) b_i\Bigr\|\Bigr).\end{aligned}$$

By (4), (5), and (2) we have that

$$\sum_{i\in\delta\setminus\delta_0} \rho(b_i, x_\nu) \leqslant 2\varepsilon\|f\|(M+1). \qquad (7)$$

By (6), (4), (2), and (7) we have

$$\sum_{i\in\delta} |\rho(b_i, x_\nu - x)| \leqslant 2\varepsilon + 2\varepsilon\|f\|(M+1) \qquad \forall\delta > \delta_2. \qquad (8)$$

Since (8) is true for each $\nu > \nu_0$, we have that

$$\|T(x_\nu) - T(x)\| = \sum_{i\in\Gamma} |\rho(b_i, x_\nu - x)| \leqslant 2\varepsilon + 2\varepsilon\|f\|(M+1) \qquad \forall\nu > \nu_0,$$

and therefore $T(x_\nu) \to T(x)$.

Hence P is locally isomorphic to $l_1^+(\Gamma)$ and statement (i) is true.

Let P be locally isomorphic to $l_1(\Gamma)$. Then P has a separable closed and bounded base iff $l_1^+(\Gamma)$ has a separable closed and bounded base. If D is such a base for $l_1^+(\Gamma)$, then $\mathrm{ep}(D) = \{\xi_i \mid i \in \Gamma\} \neq \varnothing$ and $0 < \rho_1 < \|\xi\| < \rho_2$ $\forall \xi \in D$.

Since $|\xi_i - \xi_j| = \xi_i + \xi_j$ $\forall i \neq j$, we have that

$$\|\xi_i - \xi_j\| = \|\xi_i + \xi_j\| = \|\xi_i\| + \|\xi_j\| \geqslant 2\rho_1 \qquad \forall i \neq j.$$

So the set Γ is countable, because D is separable and $\|\xi_i - \xi_j\| \geqslant 2\rho_1$ $\forall i \neq j$. Hence P is locally isomorphic to l_1^+, and the statement (ii) is true. ∎

COROLLARY 4.1. *Let X be a Banach space ordered by the closed, generating cone P, and P have the CPP. Then*:

(i) *X is order-isomorphic to $l_1(\Gamma)$ iff P has a closed, bounded base with the KMP.*

(ii) *X is order-isomorphic to l_1 iff P has a separable, closed, bounded base with the KMP.*

Proof. The "only if" part of (i) is obvious. Let P have a closed, bounded base with the KMP. Then there exists a local isomorphism T of $l_1^+(\Gamma)$ onto P. If

$$G(\xi) = T(\xi^+) - T(\xi^-) \qquad \forall \xi \in l_1(\Gamma),$$

then G is linear, and G is onto because P is generating. G is one-to-one, because if $G(\xi_1) = G(\xi_2)$, then $T(\xi_1^+ + \xi_2^-) = T(\xi_1^- + \xi_2^+)$ and therefore $\xi_1 = \xi_2$. By the definition of G we have that G and G^{-1} are positive. The local isomorphism T is defined by the formula

$$T(\xi) = \sum_{i \in \Gamma} \xi(i) b_i \qquad \forall \xi = (\xi(i))_{i \in \Gamma} \in l_1^+(\Gamma),$$

where $\{b_i \mid i \in \Gamma\}$ is the set of extreme points of a bounded base C for P. If $\|x\| \leqslant M$ $\forall x \in C$, then for each $\xi \in l_1(\Gamma)$ we have

$$\|G(\xi)\| \leqslant \|T(\xi^+)\| + \|T(\xi^-)\| \leqslant M(\|\xi^+\| + \|\xi^-\|) = M\|\xi\|.$$

So the map G is continuous. By the closed-graph theorem we have that G^{-1} is continuous and therefore statement (i) is true.

Statement (ii) follows from (i) and from statement (ii) of Theorem 4.1. ∎

Let X be a Banach space ordered by the cone P, P have the CPP, and B be a base for P defined by the functional $f \in X^*$. In [4] it is proved that:

(i) If $\text{sep}(B) \neq \varnothing$ then P is well based [4, Theorem 3.1].

(ii) If $\text{ep}(B) \neq \varnothing$ and P is well based, then $\text{sep}(B) \neq \varnothing$ [4, Proposition 3.3].

(iii) If the cone P is well based and the cone P has the RNP, then $\text{sep}(K) \neq \varnothing$ for each closed and convex subset K of P [4, Proposition 4.1].

Let K be a closed and convex subset of a Banach space X. For each $\rho \in \mathbb{R}_+$ we denote by K_ρ, $K_{s,\rho}$ the sets $\{x \in K \mid \|x\| \leqslant \rho\}$, $\{x \in K \mid \|x\| = \rho\}$, whenever these sets are nonempty. In [3] the following result is proved: If the space X has the RNP and K is a closed, convex and unbounded subset of X, then $K_\rho = \overline{\text{co}}K_{s,\rho}$ for at least one $\rho \in \mathbb{R}_+$ iff $\text{sep}(K) \neq \varnothing$ iff K is dentable [3, Proposition 3]. The proof of this statement is simple for the case where K is a closed, convex, unbounded subset of a closed, convex subset A of X and the set A has the RNP.

By means of these results, in the following proposition we give necessary and sufficient conditions in order for a cone P to be locally isomorphic to $l_1^+(\Gamma)$.

PROPOSITION 4.1. *Let X be a Banach space ordered by the infinite-dimensional closed cone P, and P have the CPP. If P has the KMP, statements* (i), (ii), (iii) *and* (iv) *below are equivalent. If P has the RNP, all the following statements are equivalent*:

(i) *P is locally isomorphic to $l_1^+(\Gamma)$.*

(ii) *P has a closed, bounded base.*

(iii) $\text{sep}(B) \neq \varnothing$ *for at least one base B for P defined by $f \in X^*$.*

(iv) $0 \in \text{sep}(P)$.

(v) *P has a dentable base defined by $f \in X^*$.*

(vi) *P is dentable.*

(vii) $\text{sep}(K) \neq \varnothing$ *for each closed and convex subset K of P.*

Proof. Let P have the KMP. Then each closed base for P has the KMP, and by Theorem 4.1 we have that (i) $\Leftrightarrow$ (ii). By [4, Theorem 3.1 and Proposition 3.3] we have that (ii) $\Leftrightarrow$ (iii). If P has a closed bounded base, there exists a uniformly monotonic, continuous linear functional g of X [2, 3.8.12]. The functional $-g$ strongly exposes 0 in P. So (ii) $\Rightarrow$ (iv). Also if g strongly exposes the 0 in P then it is easy to show that $-g$ is uniformly monotonic; hence the base B for P defined by $-g$ is closed and bounded. So (iv) $\Leftrightarrow$ (ii). Let P have the RNP. Then for each closed and convex subset A of

P we have that $\text{sep}(A) \neq \varnothing$ iff A is dentable. Hence (v) $\Leftrightarrow$ (iii) and (vi) $\Leftrightarrow$ (v). By [4, Proposition 4.1] we have that (ii) $\Rightarrow$ (vii). Also (vii) $\Rightarrow$ (iii) $\Rightarrow$ (ii), and the proof is complete. ∎

In Theorem 4.1 we proved that if P is an infinite-dimensional, closed cone of a Banach space X, B is a base for P, and

(S1) P has the CPP,
(S2) B is closed and bounded,
(S3) B has the KMP,

then P is locally isomorphic to $l_1^+(\Gamma)$. We shall show that none of the above conditions can be omitted.

The positive cone l_p^+ of l_p with $1 < p < +\infty$ is not locally isomorphic to l_1^+, and l_p^+ has the CPP. The space l_p has strictly positive, continuous linear functionals; hence l_p^+ has a closed base B. This base has the KMP, because l_p, as a reflexive Banach space, has the RNP. Hence condition (S2) cannot be omitted.

The positive cone $L_1^+[0,1]$ of $L_1[0,1]$ is not locally isomorphic to $l_1^+(\Gamma)$. The cone $L_1^+[0,1]$ has the CPP, and the set $B = \{x \in L_1^+[0,1] \mid \|x\| = 1\}$ is a closed and bounded base for P. Hence statement (S3) cannot be omitted. In the following example we show the existence of an infinite-dimensional, closed cone P of a Banach space X that is not locally isomorphic to $l_1^+(\Gamma)$ and that has a closed, bounded base with the KMP. So condition (S1) cannot be omitted.

EXAMPLE 4.1. Let B be a closed, unbounded base for the positive cone l_p^+ of the Banach space l_p with $1 < p < +\infty$. Let $x_0 \in B$, $\rho \in \mathbb{R}_+$ with $\rho > \|x_0\|$, $B_\rho = \{x \in B \mid \|x\| \leqslant \rho\}$, $B_{s,\rho} = \{x \in B \mid \|x\| = \rho\}$, and P be the cone $P = \{\lambda x \mid \lambda \in \mathbb{R}_+ \text{ and } x \in B_\rho\}$. The cone P is closed, because B_ρ is a closed bounded base for P [2, 3.8.3]. The base B_ρ has the KMP.

For each $x \in B \setminus B_\rho$, the line segment $x_0 x$ cuts $B_{s,\rho}$ at a point, and we shall denote this point of $B_{s,\rho}$ by $F(x)$. Then, for each $x \in B \setminus B_\rho$, we have that $x = x_0 + \lambda[F(x) - x_0]$ with $\lambda \in \mathbb{R}_+$ and therefore $B \subseteq P - P$. Hence $l_p = P - P$, and the cone P is infinite-dimensional. (This shows that the cone P has not the CPP—see Corollary 4.1—and that l_p, ordered by the cone P, has not the Riesz decomposition property [4, Theorem 4.1].)

Let T be a local isomorphism of P onto $l_1^+(\Gamma)$. The base B_ρ of P is separable; hence, as in the proof of statement (ii) of Theorem 4.1, we have that the set of extreme points of B_ρ is countable. Since the closed unit ball of l_p is strictly convex, we have that $B_{s,\rho} \subseteq \text{ep}(B_\rho)$ and therefore that the set $B_{s,\rho}$ is countable. Let $y, z \in B$, $y \neq z$, and $\|y\| = \|z\| = 4\rho$. Then for each point x of the line segment yz we have that $\|x\| \geqslant 2\rho$ and therefore that

$F(x)$ exists in $B_{s,\rho}$. Since the map F is a one-to-one map of the line segment yz into $B_{s,\rho}$, we have that the set $B_{s,\rho}$ is uncountable and therefore that the cone P is not locally isomorphic to $l_1^+(\Gamma)$.

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