Positivity (2011) 15:677–693
DOI 10.1007/s11117-010-0103-7

Positivity

Cone characterization of Grothendieck spaces and Banach spaces containing c_0

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Received: 6 May 2010 / Accepted: 11 November 2010 / Published online: 30 November 2010

Abstract In this article we study the embeddability of cones in a Banach space X. First we prove that c_0 is embeddable in X if and only if its positive cone c_0^+ is embeddable in X and we study some properties of Banach spaces containing c_0 in the light of this result. So, unlike with the positive cone of ℓ_1 which is embeddable in any non-reflexive space, c_0^+ has the same behavior as the whole space c_0. In the second part of this article we give a characterization of Grothendieck spaces X according to the geometry of cones of X^*. By these results we give a partial positive answer to a problem of J.H. Qiu concerning the geometry of cones.

Keywords Cones · Bases for cones · Conic isomorphisms · Grothendieck spaces · c_0^+ · ℓ_1^+

Mathematics Subject Classification (2000) 46B03 · 46B40

1 Introduction

The study of Banach spaces in terms of the geometry of their cones seems to have some interesting and surprising applications in the theory of Banach spaces. The first important result of this kind is based on the articles of Singer [20] and Pelczynski [13], from where we have that X is non-reflexive if and only if X contains a basic sequence of ℓ_+-type. Note that the positive cone of a basic sequence of ℓ_+-type is isomorphic

This article is dedicated to the memory of I.A. Polyrakis's Friend and Collaborator C.D. Aliprantis.

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to the positive cone ℓ_1^+ of ℓ_1 [21, Chap. II, Theorem 10.2]. In 1964, D. Milman and V. Milman stated, see in [12, Theorem 2.9], that a Banach space X is non-reflexive if and only if ℓ_1^+ is embeddable in X. Polyrakis proved in [16], the following dichotomy theorem for cones: If $\langle E, F\rangle$ is a dual system where E, F are normed spaces and P is a $\sigma(E, F)$-closed cone of E and the positive part $U_E^+ = P \cap U_E$ of the unit ball of E is $\sigma(E, F)$-compact, then any base for P defined by an element of F is bounded or any such a base is unbounded. So if X is reflexive, all the bases of a closed cone P are of the same type with respect to the boundness and an analogous result holds for the bases of the weak-star closed cones in dual spaces whenever the bases are defined by elements of the predual.

In [5, Lemma 3.4], Casini and Miglierina proved that if a closed cone P of a Banach space X has a base defined by a vector of X^* and any such a base of P is bounded, then any base defined by a vector of X^* is weakly compact, or equivalently the positive part U_X^+ of the unit ball of X is weakly compact. This result is essential for the proof of Theorem 6.

As we have noted in the abstract, we prove that c_0^+ is embeddable in X if and only if c_0 is embeddable in X. This catholic property of c_0^+ convert the problem of the embeddability of c_0 in Banach spaces to a problem of geometry of cones of X. In Theorem 6 we show that if a closed cone P of a Banach space X is isomorphic to ℓ_1^+ and a standard closed cone defined by P is normal, then c_0 is embeddable in X. Proposition 5, is a cone analogue of the Theorem of James [8], for the embeddability of c_0 in Banach spaces with Schauder bases.

Continuing the study of cones we define a seminorm topology on X induced by a cone P of X^*, weaker than the norm topology of X. We use this topology in order to define a generalized type of quasi-interior points of X. In the sequel we prove that a Banach space X is non-Grothendieck if and only if there exists a well-based cone (see in the notations) P of X^* such that $int(P_0) = \emptyset$ and P_0 has quasi-interior points with respect to the seminorm topology of X defined by P. Qiu proved in [17], that *a Banach space X is non-reflexive if and only if there exists a well-based cone K of X^* such that $int(K_0) = \emptyset$* where K_0 is the dual cone of K in X and in the same paper Qiu posed the problem: *Does any non-reflexive Banach space X contain a closed cone P so that $int(P) = \emptyset$ and P^0 is well based*? In this article we give a positive answer to this problem in the case where X is non-reflexive and separable or more general if X has a non-reflexive, separable complemented subspace.

For cone characterizations of reflexivity see also in [5,16].

2 Notations

Let X be a normed space. Denote by X^* the topological dual and by X' the algebraic dual of X. Also denote by U_X the closed unit ball and by S_X the unit sphere of X. For any $A \subseteq X$ denote by $\overline{A}$, $\overline{A}^w$ the closure of A in the norm, weak, topology of X and if $A \subseteq X^*$ denote by $\overline{A}^{w^*}$, the closure of A in the weak-star topology of X^*.

Suppose that P is a cone of X i.e. P is a non-empty convex subset of X and $\lambda P \subseteq P$, for each $\lambda \in \mathbb{R}_+$. If moreover $P \cap (-P) = \{0\}$, the cone P is **proper** or **pointed**. The cone $P \subseteq X$ induces the partial ordering $\geq$ in X so that $x \geq y$ if and only if

$x - y \in P$, for any $x, y \in X$. If $P - P = X$ the cone P is **generating**. The cone P gives an **open decomposition of** X if there exists $\rho > 0$ so that $\rho U_X \subseteq U_X^+ - U_X^+$, where $U_X^+ = U_X \cap P$. This is equivalent with the property: there exits a real constant α so that for any $x \in X$ there exist $x_1, x_2 \in P$ so that $x = x_1 - x_2$ and $||x_1||, ||x_2|| \leq \alpha ||x||$(in fact $\alpha = \frac{1}{\rho}$). The cone P is **normal** if there exists $c \in \mathbb{R}$ so that for any $x,\ y \in X,\ 0 \leq x \leq y$ implies $\|x\| \leq\ c\ \|y\|$. Then c is **a constant of the normal cone** P.

The cone $P^0 = \{x^* \in X^* : x^*(x) \geq 0\ \text{ for each } x \in P\}$ is the **dual cone** of P in X^*. If P is a cone of X^*, the set $P_0 = \{x \in X\ : x^*(x) \geq 0\ \text{ for each } x^* \in P\}$ is the dual cone of P in X or the **predual cone** of P. A linear functional f of X is **positive on** P if $f(x) \geq 0$ for each $x \in P$ and **strictly positive on** P if $f(x) > 0$ for each $x \in P, x \neq 0$. A convex subset B of P is a **base for the cone** P if for each $x \in P, x \neq 0$ a unique real number $f(x) > 0$ exists such that $\frac{x}{f(x)} \in B$. Then the functional f is additive and positively homogeneous on P and f can be extended to a linear functional on $P - P$ by the formula $f(x_1 - x_2) = f(x_1) - f(x_2), x_1, x_2 \in P$, and in the sequel this linear functional can be extended to a linear functional on X. So we have: *B is a base for the cone P if and only if a strictly positive (not necessarily continuous) linear functional f of X exists so that,* $B = \{x \in P \mid f(x) = 1\}$. Then we say that **the base B is defined by the functional** f and we write $B = B_f$. It is known, [9, Theorem 3.8.4], that *a cone P of a normed space X has a bounded base B with $0 \notin \overline{B}$ if and only if the dual cone P^0 of P in X^* has interior points.*

We have: *the base B defined by the functional f is norm bounded if and only if f is uniformly monotonic on P*, i.e. a real constant $a > 0$ exists so that $f(x) \geq a||x||$ for each $x \in P$ [16, Proposition 2]. We say that P is **well-based** if P has a bounded base B, defined (the base B) by a continuous linear functional of X. Suppose that X is a Banach space and $\{x_n\}$ is a sequence of X. Denote by $[x_n]$ the closed linear span of the sequence $\{x_n\}$. If any $x \in X$ is of the form $x = \sum_{i=1}^{\infty} \lambda_i x_i$ and this expansion is unique, we say that $\{x_n\}$ is a Schauder basis or simply a basis of X. $\{x_n\}$ is a **basic sequence** if it is a basis of $[x_n]$. If $\{x_n\}$ is a basic sequence, $P = \{\sum_{i=1}^{\infty} a_i x_i \in X\ |a_i \geq 0 \text{ for each } i\}$ is the **positive cone of** $\{x_n\}$. The sequence $y_n = \sum_{i=k_n}^{k_{n+1}-1} \lambda_i x_i$, where $\{k_n\}$ is a strictly increasing sequence of $\mathbb{N}$ with $k_1 = 1$ and $\lambda_i \in \mathbb{R}$ for any i is a **block sequence** of $\{x_n\}$.

3 Conic isomorphisms

Suppose that $X,\ Y$ are normed spaces and $P,\ Q$, are closed cones of $X,\ Y$ respectively. We say that **the cone P is isomorphic to the cone Q** if there exits an one-to-one, map T of P onto Q so that $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for any $x,\ y \in P$ and $\lambda, \mu \in \mathbb{R}_+$ and also the maps T, T^{-1} are continuous with respect to the induced topologies of $P,\ Q$. By the continuity of T and T^{-1} at zero, two real constants $A, B > 0$ exist, so that

$$A||x|| \leq ||T(x)|| \leq B||x||, \tag{1}$$

for any $x \in P$.

Proposition 1 *Suppose that X, Y are normed spaces ordered by the cones P, Q, respectively. If the cones P, Q are isomorphic, we have: P is normal if and only if Q is normal.*

Proof Suppose that $T : P \to Q$ is an isomorphism of P onto Q and that P is normal with constant c. T can be extended to a linear operator from $P - P$ onto $Q - Q$ via the formula

$$T(x - y) = T(x) - T(y),$$

for any $x, y \in P$. Suppose that $T(x),\ T(y) \in Q$ such that

$$0 \leq T(x) \leq T(y).$$

Then $0 \leq x \leq y$ therefore $||x|| \leq c||y||$. So by (1) we have

$$||T(x)|| \leq B\|x\| \leq B\ c\|y\| \leq \frac{B\ c}{A}||T(y)||,$$

therefore Q is normal. □

Proposition 2 *If a cone P of a normed space X is isomorphic to the positive cone ℓ_1^+ of ℓ_1 and $Y = P - P$, we have: P gives an open decomposition of Y if and only if Y, ordered by the cone P, is order-isomorphic to ℓ_1.*

Proof Suppose $T : \ell_1^+ \to P$ is an isomorphism of ℓ_1^+ onto P. Then there exist $A, B > 0$ such that

$$A||\xi|| \leq ||T(\xi)|| \leq B||\xi||,$$

for each $\xi \in \ell_1^+$. T can be extended to a linear and one to one operator from ℓ_1 onto $P - P$ by the formula $T(\xi) = T(\xi^+) - T(\xi^-)$. Then for each $\xi \in \ell_1$ we have

$$||T(\xi)|| = ||T(\xi^+ - \xi^-)|| \leq ||T(\xi^+)|| + ||T(\xi^-)|| \leq B(||\xi^+|| + ||\xi^-||) = B||\xi||.$$

P is normal, with constant c, because it is isomorphic to ℓ_1^+. By our assumptions P gives an open decomposition of Y, therefore there exists a real constant $a > 0$ such that for each $x \in Y$ there exist $x_1, x_2 \in P$ so that $x = x_1 - x_2$ and $\|x_1\|,\ \|x_2\| \leq a\|x\|$. So for any $\xi \in \ell_1$, there exist $\eta_1, \eta_2 \in \ell_1^+$ so that $T(\xi) = T(\eta_1) - T(\eta_2)$ with

$$||T(\eta_1)||,\ ||T(\eta_2)|| \leq a||T(\xi)||.$$

By the lattice structure of ℓ_1 we have $\xi^+ \leq \eta_1$ and $\xi^- \leq \eta_2$, therefore

$$||T(\xi^+)|| \leq c||T(\eta_1)|| \quad \text{and} \quad ||T(\xi^-)|| \leq c||T(\eta_2)||.$$

Therefore for any $\xi \in \ell_1$ we have

$$||\xi|| = || \ |\xi| \ || \leq \frac{1}{A}(||T(|\xi|)||) = \frac{1}{A}(||T(\xi^+) + T(\xi^-)||)$$
$$\leq \frac{1}{A}(||T(\xi^+)|| + ||T(\xi^-)||) \leq \frac{c}{A}(||T(\eta_1)|| + ||T(\eta_2)||) \leq \frac{2ca}{A}||T(\xi)||,$$

hence T is an order-isomorphic of ℓ_1 onto Y. The converse is obvious. □

Theorem 3 *In any Banach space X the following statements are equivalent:*

(i) *the positive cone c_0^+ of c_0 is embeddable in X,*
(ii) *the space c_0 is embeddable in X.*

Proof Only the direct proof is needed. So we suppose that X is ordered by the cone P and also that $T : c_0^+ \to P$ is an isomorphism of c_0^+ onto P with $b_i = T(e_i)$ for each i. Then there exist $A, B > 0$ so that

$$A\|\xi\| \leq \|T(\xi)\| \leq B\|\xi\|,$$

for any $\xi \in c_0^+$. Therefore

$$\left\|\sum_{i=1}^{n} b_i\right\| = \left\|T\left(\sum_{i=1}^{n} e_i\right)\right\| \leq B\left\|\sum_{i=1}^{n} e_i\right\| = B.$$

Also $\left\|\sum_{i=n}^{n+m} b_i\right\| = \left\|T\left(\sum_{i=n}^{n+m} e_i\right)\right\| \geq A\left\|\sum_{i=n}^{n+m} e_i\right\| = A$. Therefore the sequence $\sum_{i=1}^{n} b_i$ is not convergent. For any $x^* \in X^*$, positive on P we have

$$0 \leq x^*\left(\sum_{i=1}^{n} b_i\right) \leq ||x^*|| \ \left\|\sum_{i=1}^{n} b_i\right\| \leq B \ ||x^*||.$$

Since the sequence $\sum_{i=1}^{n} x^*(b_i)$ is increasing we have $\sum_{i=1}^{\infty} x^*(b_i) \in \mathbb{R}_+$. Also the cone P is isomorphic to c_0^+ therefore it is normal. By Krein's Theorem, see for example [3] Theorem 2.26, the dual cone P^0 of the normal cone P is generating. So for any $x^* \in X^*$ we have $x^* = x_1^* - x_2^*$ with $x_1^*, \ x_2^* \in P^0$. Therefore $\sum_{i=1}^{\infty} |x^*(b_i)| \leq \sum_{i=1}^{\infty} x_1^*(b_i) + \sum_{i=1}^{\infty} x_2^*(b_i)$, hence $\sum_{i=1}^{\infty} |x^*(b_i)| \in \mathbb{R}$. Therefore c_0 is embeddable in X, see in [2], Theorem 4.49. □

To complete our study on the embeddability of c_0 in Banach spaces in connection with the embeddability of c_0^+, we give an equivalent formulation of the classical result of Bessaga–Pelczynski [4], for positive a_i.

Theorem 4 (Bessaga–Pelczynski) *The Schauder basis $\{x_n\}$ of a Banach space X is equivalent to the standard unit vector basis $\{e_n\}$ of c_0 if and only if $inf_{n\in\mathbb{N}} \|x_n\| > 0$ and there exists a real constant $c > 0$ such that*

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leq c \ max\{a_1, \ldots, a_n\} \quad \textit{for each} \ \ a_1, \ldots, a_n \in \mathbb{R}_+ \ \ \textit{and each} \ \ n \in \mathbb{N}. \qquad (2)$$

In [4], the above result is formulated with the condition

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leq c\ max\{|a_1|, \dots, |a_n|\}, \quad \text{for any real sequence } \{a_i\}. \tag{3}$$

Conditions (2) and (3) are equivalent. Indeed if we suppose that (2) is true and $\{a_i\}$ is a real sequence, then for any n we put $F_+ = \{i = 1, \dots, n \mid a_i \geq 0\}$, $F_- = \{i = 1, \dots, n \mid a_i < 0\}$ and we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i x_i\right\| &\leq \left\|\sum_{i\in F_+} a_i x_i\right\| + \left\|\sum_{i\in F_-} a_i x_i\right\| = \left\|\sum_{i\in F_+} a_i x_i\right\| + \left\|\sum_{i\in F_-} -a_i x_i\right\| \\ &\leq c(\max\{a_i : i \in F_+\} + \max\{-a_i : i \in F_-\}) \\ &\leq 2c \max\{|a_i| : i = 1, \dots, n\},\end{aligned}$$

therefore (2) implies (3).

Let $\{x_n\}$ be basic sequence of a Banach space X and let P be the positive cone of $\{x_n\}$. If for any real sequence $a = (a_i)$ we have

$$sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{n} a_i x_i\right\| < +\infty \Longrightarrow \sum_{i=1}^{\infty} a_i x_i \in X,$$

$\{x_n\}$ is **boundedly complete** and if the above condition is true for any positive real sequence $a = (a_i)$, then $\{x_n\}$ is **boundedly complete on the cone** P.

James proved in [8] that *if X is a Banach space with an unconditional and non-boundedly complete Schauder basis $\{x_n\}$ then c_0 is embeddable in X*. It is known that a Schauder basis $\{x_n\}$ of a Banach space X is unconditional if and only if the positive cone P of the basis is generating and normal [21, Chap. II, Theorem 16.3]. In the next result we weaken the first assumption of James that the basis $\{x_n\}$ is unconditional by assuming that the positive cone P of $\{x_n\}$ is normal, but our second assumption that $\{x_n\}$ is non-boundedly complete on P is stronger than the one of James (the summing basis $\{b_n\}$ of c_0 is not boundedly complete but it is boundedly complete on it's positive cone). In the next result, our proof is the cone analogue of the one of James.

Proposition 5 *Suppose that X is a Banach space with a Schauder basis $\{x_n\}$. If the positive cone P of $\{x_n\}$ is normal and the basis $\{x_n\}$ is non-boundedly complete on P, there exists a block basic sequence $\{y_n\}$ of $\{x_n\}$ consisting of positive linear combinations of the vectors of $\{x_n\}$, equivalent to the standard basis of c_0.*

Proof Since $\{x_n\}$ is non-boundedly complete on P, there exists a sequence $\{a_i\}$ of positive real numbers such that

$$\left\|\sum_{i=1}^{n} a_i x_i\right\| \leq M, \ \text{for each } n \in \mathbb{N},$$

and the series $\sum_{i=1}^{\infty} a_i x_i$ is not convergent. So there exists a strictly increasing sequence $\{q_n\}$ of $\mathbb{N}$ such that $\inf \|\sum_{i=q_n}^{q_{n+1}-1} a_i x_i\| = d > 0$. If c is a constant of the normal cone P we have

$$\left\|\sum_{i=q_1}^{q_2-1} a_i x_i\right\| \leq c \left\|\sum_{i=1}^{q_2-1} a_i x_i\right\|,$$

therefore we may assume that $q_1 = 1$. Then the sequence $y_n = \sum_{i=q_n}^{q_{n+1}-1} a_i x_i$, as a block sequence of $\{x_n\}$, is basic with

$$\left\|\sum_{i=1}^{n} y_i\right\| = \left\|\sum_{i=1}^{q_{n+1}-1} a_i x_i\right\| \leq M,$$

for each $n \in \mathbb{N}$. So for any $n \in \mathbb{N}$ and $\beta_1, \ldots, \beta_n \in \mathbb{R}_+$ we have

$$0 \leq \sum_{i=1}^{n} \beta_i y_i \leq \max\{\beta_1, \ldots, \beta_n\} \sum_{i=1}^{n} y_i,$$

therefore

$$\left\|\sum_{i=1}^{n} \beta_i y_i\right\| \leq c \max\{\beta_1, \ldots, \beta_n\} \left\|\sum_{i=1}^{n} y_i\right\| \leq c\, M \max\{\beta_1, \ldots, \beta_n\},$$

and by Theorem 4, $\{y_n\}$ is equivalent to the standard basis of c_0. □

Let $\{x_n\}$ be a sequence of a Banach space X. The set

$$cone\{x_n\} = \left\{\sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N}, \lambda_i \in \mathbb{R}_+\right\},$$

is **the cone generated by** $\{x_n\}$ and the closed hull $\overline{cone}\{x_n\}$, of $cone\{x_n\}$ is **the closed cone generated by** $\{x_n\}$. It is easy to show that $cone\{x_n\}$ is the smallest cone and $\overline{cone}\{x_n\}$ is the smallest closed cone of X which contains the sequence $\{x_n\}$. The sequence $y_1 = x_1$ and $y_{n+1} = x_{n+1} - x_n$ for any $n \geq 1$ is the **difference sequence** of the sequence $\{x_n\}$. Suppose that $\{x_n\}$ is a basic sequence. $\{x_n\}$ is of **type** p if $inf_{n\in\mathbb{N}}\|x_n\| > 0$ and $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^{n} x_i\| < +\infty$, and $\{x_n\}$ is of **type** p^* if the sequence of the coefficient functionals $\{x_n^*\}$ of $\{x_n\}$ is a basis of $[x_n^*]$, of type p, see in [21, p. 308]. If $sup_{n\in\mathbb{N}}\|x_n\| < \infty$, *then* $\{x_n\}$ *is of type* p^* *if and only if the difference sequence* $\{y_n\}$ *of* $\{x_n\}$ *is a basic sequence of type* p [21, Chap. II, Theorem 9.2], (1) ⇔ (7). $\{x_n\}$ is **strongly summing** if it is weakly Cauchy and for any sequence of scalars $\{c_i\}$ we have:

$$\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} c_i x_i \right\| < +\infty \Longrightarrow \sum_{i=1}^{\infty} c_i \text{ exists.}$$

Note that *any strongly summing basic sequence* $\{x_n\}$ *is of type* p^*, because the difference sequence $\{y_n\}$ of $\{x_n\}$ is of type p, see [19, Proposition 2.1, Definitions 2.1 and 2.2].

In the proof of the next theorem we use the classical ℓ_1 and c_0 dichotomy theorems of Rosenthal [18,19] and also the following recent result of Casini and Miglierina [5, Theorem 4.5]: *If a closed cone* P *of a Banach space is isomorphic to* ℓ_1^+, *then* P *is a mixed base cone, i.e.* P *has both bounded and unbounded bases defined* (*the bases*) *by continuous linear functionals on* X.

Theorem 6 *Suppose that* T *is an isomorphism of* ℓ_1^+ *onto a closed cone* P *of a Banach space* X *and* $x_n = T(e_n)$ *for each* n. *If the closed cone generated by the difference sequence* $\{y_n\}$ *of the sequence* $\{x_n\}$ *is normal,* c_0 *is embeddable in* X.

Proof Suppose that $T : \ell_1^+ \to P$ is an isomorphism of ℓ_1^+ onto P with $x_n = T(e_n)$ for each n. Then there exist real numbers $A, B > 0$ so that

$$A||\xi|| \leq ||T(\xi)|| \leq B||\xi||, \quad \text{for any } \xi \in \ell_1^+,$$

therefore $A \leq ||x_n|| \leq B$ for each n. We will show first that the sequence $\{x_n\}$ is not weakly convergent. So we suppose that $x_n \xrightarrow{w} x_0$. Then $x_0 \in \overline{co}^w\{x_n\} = \overline{co}\{x_n\}$, so $T^{-1}(x_0) \in \overline{co}\{e_n\}$ and $\overline{co}\{e_n\}$ is the positive part of the unit sphere of ℓ_1, hence $x_0 \neq 0$. By [5, Theorem 4.5], there exists a functional $x^* \in X^*$ which defines an unbounded base for P. Since $x^*(x_n) \to x^*(x_0) > 0$, we have $x^*(x_n) \geq a > 0$ for each n because $x_0 \in P$, $x_0 \neq 0$ and x^* is strictly positive on P. For each $x = T(\xi) \in P$ where $\xi = (\xi_i) \in \ell_1^+$ we have $x = \sum_{i=1}^{\infty} \xi_i x_i$, therefore

$$x^*(x) = x^*\left(\sum_{i=1}^{\infty} \xi_i x_i\right) \geq a \sum_{i=1}^{\infty} \xi_i = a||\xi|| \geq \frac{a}{B}||x||.$$

Hence x^* is uniformly monotonic on P, therefore by [16, Proposition 2], x^* defines a bounded base for P, a contradiction. Therefore x_n is not weakly convergent. By Rosenthal ℓ_1 and c_0 Theorem, one of the following is true: (i) $\{x_n\}$ has a subsequence equivalent to the standard basis of ℓ_1, or (ii) $\{x_n\}$ has a strongly summing subsequence, or (iii) $\{x_n\}$ has a convex block basis equivalent to the summing basis. In case (iii) the theorem is trivially true. If (i) or (ii) is true, there exists a subsequence x_{k_n} of x_n which is of type p^*. If z_n is the difference sequence of x_{k_n} then z_n is a basic sequence of type p. Also the positive cone K of $\{z_n\}$ is contained in $\overline{cone}\{y_n\}$ because

$$z_n = x_{k_n} - x_{k_{n-1}} = \sum_{i=k_{n-1}+1}^{k_n} y_i \in cone\{y_n\},$$

for each n, therefore K as a subcone of the normal cone, $\overline{cone}\{y_n\}$ is normal. $\{z_n\}$ is equivalent to the standard basis of c_0. Indeed, $inf_{n\in\mathbb{N}}\|z_n\| > 0$ and $\left|\left|\sum_{i=1}^{n} z_i\right|\right| \leq M$, for each n because $\{z_n\}$ is a basic sequence of type p. So for each $n \in \mathbb{N}$ and $a_1, \ldots, a_n \geq 0$ we have

$$0 \leq \sum_{i=1}^{n} a_i z_i \leq \max\{a_1, \ldots a_n\} \sum_{i=1}^{n} z_i,$$

where $\leq$ is the ordering defined by K, therefore

$$\left|\left|\sum_{i=1}^{n} a_i z_i\right|\right| \leq cM \max\{a_1, \ldots, a_n\},$$

because K is normal. By Theorem 4, $\{z_n\}$ is equivalent to the standard basis of c_0. □

4 Cone-seminorms

Let X be a normed space, P be a cone of X^*, $U_{X^*}^+ = U_{X^*} \cap P$ be the positive part of the closed unit ball U_{X^*} of X^* and let $V = co(U_{X^*}^+ \cup (-U_{X^*}^+))$ be the convex hull of $U_{X^*}^+ \cup (-U_{X^*}^+)$. For any $x \in X$ we put

$$d_P(x) = \sup_{x \in V} x^*(x).$$

Since V is a subset of the closed unit ball U_{X^*} of X^*, $d_P(x)$ exists and we have

$$d_P(x) \leq \|x\|, \quad \text{for any } x \in X.$$

It is easy to show that

$$d_P(x) = \sup_{x^* \in U_{X^*}^+} |x^*(x)| = \sup_{x^* \in S_{X^*}^+} |x^*(x)|,$$

where $S_{X^*}^+ = S_{X^*} \cap P$ is the positive part of the unit sphere S_{X^*} of X^*. The function d_P is a seminorm on X. It is clear that the seminorm depends on the cone P. We will say that d_P is **the seminorm defined by the cone** P. Also d_P can be referred as a **cone-seminorm**. If d_P is a norm of X we will denote this norm by $|| \cdot ||_P$.

Proposition 7 *If d_P is the seminorm of X defined by the cone $P \subseteq X^*$, the following are equivalent:*

(i) *d_P is a norm of X,*
(ii) *the subspace $Y = P - P$ of X^* separates the points of X.*

Proof For any $x \in X$, we have:

$$\begin{aligned} d_P(x) &= 0 \Leftrightarrow x^*(x) = 0 \quad \text{for any } x^* \in U_{X^*}^+ \Leftrightarrow \\ &\Leftrightarrow x^*(x) = 0 \quad \text{for any } x^* \in P \Leftrightarrow x^*(x) = 0 \quad \text{for any } \ x^* \in Y, \end{aligned}$$

therefore statements (i) and (ii) are equivalent. □

Recall that if $\langle E, F\rangle$ is a dual system, where E, F are normed spaces, we say that F is **norming** to E, if $||x|| = \sup\{|y(x)| \mid y \in U_F\}$ for each $x \in E$.

Proposition 8 *Suppose that P is a cone of X^*. If $Y = P - P$ is norming to X and P gives an open decomposition of Y, then the seminorm of X defined by P is a norm which is equivalent to the norm of X.*

Proof Y separates the points of X because Y is norming to X, therefore d_P is a norm. If $\rho U_Y \subseteq U_Y^+ - U_Y^+$, then for any $x \in X$ we have

$$\rho \sup_{x^* \in U_Y} x^*(x) \leq \sup_{x^* \in U_Y^+} x^*(x) + \sup_{x^* \in (-U_Y^+)} x^*(x) \leq 2 \sup_{x^* \in U_Y^+} |x^*(x)|,$$

where $U_Y^+ = P \cap U_Y$ is the positive part of U_Y. Therefore $\rho||x|| \leq 2||x||_P$. Also $||x||_P \leq ||x||$, therefore the norms are equivalent. □

In [20], Singer introduces the notion of the basic sequence of ℓ_+-type. A basic sequence $\{x_n\}$ of a Banach space X is of **ℓ_+-type** if it is bounded and there exists a real constant $C > 0$ such that for any n and every finite real numbers $a_1, \ldots, a_n \geq 0$, we have $|| \sum_{i=1}^n a_i x_i || \geq C \sum_{i=1}^n a_i$. Let $\{x_n\}$ be a basic sequence of a Banach space X. It is known that if $sup\{||x_n||\} < +\infty$ then $\{x_n\}$ is of ℓ_+-type if and only if there exists $x^* \in X^*$ so that $x^*(x_n) \geq 1$ for each $n \in \mathbb{N}$ [21, Chap. II, Theorem 10.1].

If $\{x_n\}$ is of ℓ_+-type, there exists an isomorphism T of ℓ_1^+ onto the positive cone P of $\{x_n\}$ with $T(e_n) = x_n$ for each n, where $\{e_n\}$ is the standard unit basis of ℓ_1 [21, Chap. II, Theorem 10.2].

In the next example the norms $\|\cdot\|$ and $\|\cdot\|_P$ are not equivalent.

Example 9 Suppose that $X = \ell_1$. Then $X^* = \ell_\infty$, the sequence $b_n = \sum_{i=1}^n e_i$ is a basic sequence of ℓ_+-type in ℓ_∞ and suppose that P is the positive cone of $\{b_n\}$. The set

$$B = \left\{ x^* = \sum_{i=1}^{\infty} \xi_i b_i \mid \xi \in \ell_1^+, ||\xi|| = 1 \right\},$$

is a bounded base of P and we remark that $B = S_+$ is the positive part of the closed unit sphere in X^*. Also we remark that $Y = P - P$ separates the points of X, therefore d_P is a norm of X, which we denote by $||\cdot||_P$. Then $||x|| \geq ||x||_P$, for any $x \in \ell_1$. Suppose that these norms are equivalent. Then there exists $A > 0$ so that

$$||x||_P > A||x||, \text{ for any } x \in \ell_1.$$

For any $x^* = \sum_{i=1}^\infty \xi_i b_i \in B$ we have $x^* = \left(\sum_{i=n}^\infty \xi_i\right)_{n \in \mathbb{N}} \in c_0^+$. Also for any $x = (x_i) \in \ell_1$ we have

$$x^*(x) = \sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} \xi_i \right) x_n = \left(\sum_{i=1}^{\infty} \xi_i \right) x_1 + \left(\sum_{i=2}^{\infty} \xi_i \right) x_2 + \cdots$$

Since $\sum_{i=1}^{\infty} \xi_i = 1$ we have

$$\begin{aligned} x^*(x) &= x_1 + (1-\xi_1)x_2 + (1-(\xi_1+\xi_2))x_3 + \cdots \\ &= \sum_{i=1}^{\infty} x_i - \left(\sum_{i=2}^{\infty} x_i\right)\xi_1 - \left(\sum_{i=3}^{\infty} x_i\right)\xi_2 - \cdots \\ &= \left(\sum_{i=1}^{\infty} x_i, \sum_{i=2}^{\infty} x_i, \sum_{i=3}^{\infty} x_i, \ldots\right) \cdot (1, -\xi_1, -\xi_2, \ldots) = z(\eta) \end{aligned}$$

where $z \in c_0$ and $\eta \in \ell_1$. Therefore $|x^*(x)| = |z(\eta)| \leq ||z|| ||\eta|| = 2||z||$, therefore

$$|x^*(x)| \leq 2 \sup_{n\in\mathbb{N}} \left\{ \left| \sum_{i=n}^{\infty} x_i \right| \right\}.$$

By the definition of $|| \cdot ||_P$, for any $x \in \ell_1$, there exits $x^* = \sum_{i=1}^{\infty} \xi_i b_i$, where $\xi \in \ell_1^+$, $||\xi|| = 1$, so that

$$2 \sup_{n\in\mathbb{N}} \left\{ \left| \sum_{i=n}^{\infty} x_i \right| \right\} \geq |x^*(x)| \geq A||x||.$$

Suppose that $x = \left(\frac{(-1)^{i+1}}{\omega^i}\right)_{i\in\mathbb{N}} = (x_i)_{i\in\mathbb{N}} \in \ell_1$, where $\omega \in (1, 2)$. Then $||x|| = \frac{1}{\omega-1}$ and for each $n \in \mathbb{N}$ we have

$$\left| \sum_{i=n}^{\infty} x_i \right| = \left| \sum_{i=n}^{\infty} x_i^+ - \sum_{i=n}^{\infty} x_i^- \right| = \frac{1}{\omega^{n-1}(\omega+1)}.$$

So we have that $\sup_{n\in\mathbb{N}} \left\{ \left| \sum_{i=n}^{\infty} x_i \right| \right\} = \sum_{i=1}^{\infty} x_i = \frac{1}{\omega+1}$. So we have

$$2\frac{1}{\omega+1} \geq A\frac{1}{\omega-1} \Rightarrow \frac{2}{A} \geq \frac{\omega+1}{\omega-1} \quad \text{for any } \omega \in (1, 2).$$

Therefore $\frac{2}{A} = +\infty$, a contradiction. Therefore the norms $|| \cdot ||_P$ and $|| \cdot ||$ are not equivalent.

Suppose that X is ordered by the cone K. $x_0 \in K$ is a **quasi interior point** of K with respect to the seminorm d_P, if the set $\bigcup_{n=1}^{\infty} [-nx_0, \ nx_0]$ is d_P-dense in X. We denote by $Q_P(K)$ the set of quasi interior points of K with respect to d_P.

Proposition 10 *Suppose that X is a normed space, P is a cone of X^* and that X is ordered by the cone P_0. If $x_0 \in Q_P(P_0)$, then x_0 is strictly positive on P.*

Proof Suppose that $x^*(x_0) = 0$ for at least one $x^* \in U_{X^*}^+$, $x^* \neq 0$. Then for any $x \in X$ and $\epsilon > 0$, there exists $y \in [-nx_0, \ nx_0]$ so that $d_P(x-y) < \epsilon$, therefore $|x^*(x-y)| < \epsilon$, by the definition of d_P. So we have that $|x^*(x)| \leq |x^*(x-y)| + |x^*(y)|$. x^* as a point of P is positive on P_0. Since $-nx_0 \leq y \leq nx_0$ and $x^*(x_0) = 0$ we have that $x^*(y) = 0$. Therefore $|x^*(x)| \leq \epsilon$ for any $\epsilon > 0$, hence $x^*(x) = 0$. So $x^*(x) = 0$ for any $x \in X$, a contradiction therefore x_0 is strictly positive on P. □

5 Cones in Grothendieck spaces

A Banach space X is **Grothendieck** if every weak-star convergent sequence in X^* is weakly convergent. Trivial examples of Grothendieck spaces are the reflexive spaces and of non-Grothendieck the non-reflexive, separable spaces. $\ell_\infty(\Gamma)$ was the first example of a non reflexive, Grothendieck space [14]. Also for any Stonian space Ω, $C(\Omega)$ is a Grothendieck space [7]. In Banach-lattices we have: any σ-Dedekind complete AM-space with a unit is a Grothendieck space, see in [2, Theorem 4.44]. For a study of Grothendieck spaces and different equivalent definitions we refer to [6] and for some resent results on Grothendieck spaces we refer to [1,10].

The following is a simple generalization of a well known result, see in [9, Theorem 3.8.4] with a similar proof.

Theorem 11 *Suppose that $\langle E, F\rangle$ is a dual system where E, F are normed spaces. If P is a cone of E, $P^0 = \{g \in F \mid g(x) \geq 0$ for each $x \in P\}$, is the dual cone of P in F and $f \in P^0$, we have:*

(i) *If E is norming to F and B_f is a bounded base of P then $f \in int(P^0)$.*
(ii) *If F is norming to E and $f \in int(P^0)$ then B_f is a bounded base of P.*

In this article we use the next result:

Corollary 12 *Suppose that X is a normed space, P is a cone of X^* and P_0 is the dual cone of P in X. If $x \in P_0$ we have: $x \in int(P_0)$ if and only if x defines a bounded base B_x for P.*

Proposition 13 *Suppose that P is the positive cone of a basic sequence $\{x_n^*\}$ of X^*. If $x_n^* \xrightarrow{w^*} 0$ and $\inf_{n\in\mathbb{N}} ||x_n^*|| > 0$, the dual cone P_0 of P in X has empty interior.*

Proof Suppose that $x \in int(P_0)$. By Corollary 12, B_x is a bounded base of P and x, as a linear functional of X^*, is uniformly monotonic on P. So there exists a real number $a > 0$ such that $x_n^*(x) \geq a||x_n^*|| \geq a \inf_{n\in\mathbb{N}} ||x_n^*|| > 0$ for each $n \in \mathbb{N}$. This contradicts the fact that $x_n^* \xrightarrow{w^*} 0$, therefore $int(P_0) = \emptyset$. □

We recall now two fundamental theorems of the theory of Schauder bases due to A. Pelczynski and W.B. Johnson–H.P. Rosenthal which we will use below.

Suppose that X is a Banach space. If the subspace G of X^* is norming to X, and $\{x_n\}$ is a sequence of S_X such that $x_n \xrightarrow{\sigma(X,G)} 0$, then there exists a subsequence $\{y_n\}$ of $\{x_n\}$, which is a basic sequence [13].

Let X be a separable Banach space. If $\{x_n^*\}$ is a sequence of X^*, so that $x_n^* \xrightarrow{w^*} 0$ and $\inf_{n\in\mathbb{N}} ||x_n^*|| > 0$, then there exists a subsequence $\{b_n^*\}$ of $\{x_n^*\}$ which is a w^*-basic sequence [11].

Suppose that X is a Banach space. The sequence $\{(x_n, y_n)\}$ where $x_n \in X$ and $y_n \in X^*$ is a **biorthogonal system** if $y_n(x_m) = \delta_{nm}$ for each $n, m \in \mathbb{N}$. A sequence $\{b_n^*\}$ of X^* is a w^***-basic sequence** if a sequence $\{c_n\}$ of X exists so that $\{(c_n, b_n^*)\}$ is a biorthogonal system and for each $x^* \in \overline{[b_n^*]}^{w^*}$ we have $\sum_{i=1}^n x^*(c_i)b_i^* \xrightarrow{w^*} x^*$. The notion of the w^*-basic sequence has been defined in [11].

It is known that *if* $\{b_n^*\}$ *is a* w^**- basic sequence then* $\{b_n^*\}$ *is a basic sequence in* X^*, [11, Proposition II.1].

Theorem 14 *Suppose that* X *is a non-Grothendieck space. Then there exists a biorthogonal system* $\{(c_n, b_n^*)\}$ *where* $c_n \in X$, $b_n^* \in X^*$, *so that* $\{b_n^*\}$ *is a basic sequence in* X^* *of* ℓ_+*-type with* $\|b_n^*\| = 1$ *for each* n *and* $b_n^* \xrightarrow{w^*} 0$. *If we suppose moreover that* X *is separable, the sequence* $\{b_n^*\}$ *can be chosen so that* $\{b_n^*\}$ *is in addition a* w^**-basic sequence.*

Proof Let X be a non-Grothendieck space. Then there exists a sequence $\{x_n^*\}$ of X^*, such that $x_n^* \xrightarrow{w^*} 0$ and $x_n^* \not\xrightarrow{w} 0$. Therefore there exists a subsequence of $\{x_n^*\}$ which we denote again by $\{x_n^*\}$ and a functional $f \in X^{**}$, such that $f(x_n^*) \geq 1$ for each n, so $||x_n^*|| \geq \frac{1}{||f||} > 0$ for each n. The sequence $\{x_n^*\}$ is w^*- convergent, therefore $\sup_{n\in\mathbb{N}} ||x_n^*|| = d < +\infty$.

Consider the sequence $y_n^* = \frac{x_n^*}{||x_n^*||}$. It is clear that $y_n^* \xrightarrow{w^*} 0$ and $y_n^* \not\xrightarrow{w} 0$. By [13], there exists a basic subsequence $\{b_n^*\}$ of $\{y_n^*\}$. Since $f(x_n^*) \geq 1$ we have that $f(b_n^*) \geq \frac{1}{d}$ for each n, therefore the sequence $\{b_n^*\}$ is of ℓ_+-type. Also $b_n^* \xrightarrow{w^*} 0$.

Let Y be the closed subspace of X^* generated by $\{b_n^*\}$ and let Z be a closed, separable subspace of X with the property $\|y^*\| = sup\{y^*(x) \mid x \in U_Z\}$. (Such a subspace Z exists). Then by [11], there exists a subsequence of $\{b_n^*\}$ which we denote again by $\{b_n^*\}$ which is $\sigma(Z, Z^*)$-basic sequence in Z^*. Hence there exists a sequence $\{c_n\}$ of Z so that $\{(c_n, b_n^*)\}$ is a biorthogonal system and also for any x^* in the $\sigma(Z, Z^*)$-closure of $[b_n^*]$ in Z^* we have $x^* = \sum_{i=1}^{\infty} x^*(c_i)b_i^*$ in the $\sigma(Z, Z^*)$-topology of Z^*. If we assume that X is separable, then we may assume that $Z = X$ and we conclude that $\{b_n^*\}$ is a w^*-basic sequence of ℓ_+-type. □

In the next result we give a cone characterization of Grothendieck spaces. For the direct we use the method developed in [15, Theorem 3.1], and for the converse we use the previous theorem.

Theorem 15 *A Banach space* X *is non-Grothendieck if and only if there exists a well-based cone* P *of* X^* *such that*

$$int(P_0) = \emptyset \quad and \quad Q_P(P_0) \neq \emptyset.$$

Proof Suppose that there exists a well-based cone P of X^* such that $int(P_0) = \emptyset$ and $Q_P(P_0) \neq \emptyset$. Since P is well-based, there exists $f \in int(P^0)$. Then the base B_f of P defined by f is bounded. We shall show that $0 \in \overline{B_f}^{w^*}$. If we suppose that $0 \notin \overline{B_f}^{w^*}$, there exists $x \in X$ which, as a linear functional of X^*, separates 0 and B_f.

It is easy to show that x is strictly positive on P and also that the base of P defined by x is bounded, therefore by Corollary 12, $x \in int(P_0)$, a contradiction and our assertion is true. We will show now that there exists a sequence $\{x_n^*\}$ in B_f such that $x_n^* \xrightarrow{w^*} 0$. Let $x_0 \in Q_P(P_0)$. Since $0 \in \overline{B_f}^{w^*}$ we have that $0 \in \overline{nB_f}^{w^*}$ for each $n \in \mathbb{N}$, therefore there exists $y_n^* \in nB_f$ with $y_n^*(x_0) < 1$.

Then $x_n^* = \frac{y_n^*}{n} \in B_f$. We will show that $x_n^* \xrightarrow{w^*} 0$. Let $x \in X$ and $\epsilon > 0$. Since $x_0 \in Q_P(P_0)$ there exists $n_0 \in \mathbb{N}$ and $z \in [-n_0x_0,\ n_0x_0]$ such that $d_P(z - x) < \epsilon$. Then we remark that for each $n \in \mathbb{N}$ we have that $|x_n^*(z)| \leq n_0 x_n^*(x_0) \leq \frac{n_0}{n}$. So, we conclude that

$$\begin{aligned} |x_n^*(x)| &\leq |x_n^*(x - z)| + |x_n^*(z)| \leq d_P(x - z) + \frac{n_0}{n} \\ &< \epsilon + \frac{n_0}{n} \quad \text{for each } n \in \mathbb{N}. \end{aligned}$$

Hence $x_n^* \xrightarrow{w^*} 0$. Since $f(x_n^*) = 1$ we have that X is non-Grothendieck.

Suppose now that X is non-Grothendieck. Then by Theorem 14, there exists a biorthogonal system $\{(c_n,\ b_n^*)\}$ with $c_n \in X$, $b_n^* \in X^*$, so that $\{b_n^*\}$ is a basic sequence in X^* of ℓ_+-type with $\|b_n^*\| = 1$ for each n and $b_n^* \xrightarrow{w^*} 0$. Let P be the closed cone of X^* generated by b_n^*. By Proposition 13, $int(P_0) = \emptyset$. Since $\{b_n^*\}$ is of ℓ_+-type, there exists an isomorphism T of ℓ_1^+ onto P with $T(e_i) = b_i^*$ and suppose that $A\|\xi\| \leq \|T(\xi)\| \leq M\|\xi\|$ for any $\xi \in \ell_1^+$. Then $B = T(S_+)$, where S_+ is the positive part of the unit sphere of ℓ_1, is a closed, bounded base for the cone P with the property $A \leq \|x^*\| \leq M$ for any $x^* \in B$. Let

$$x_0 = \sum_{i=1}^{\infty} \frac{c_i}{2^i ||c_i||}.$$

We will show that $x_0 \in Q_P(P_0)$. So suppose that $x \in X$ and $\epsilon > 0$. For each $y^* = \sum_{i=1}^{\infty} a_i b_i^* \in B$ we have that $y^* = T(\sum_{i=1}^{\infty} a_i e_i)$, therefore $a = (a_i) \in \ell_1^+$ with $\sum_{i=1}^{\infty} a_i = 1$. Since $b_n^* \xrightarrow{w^*} 0$, there exists $n_0 \in \mathbb{N}$ such that $|b_n^*(x)| < \epsilon$ for each $n \geq n_0$. We put

$$y = \sum_{i=1}^{n_0} b_i^*(x) c_i.$$

Then it is easy to see that $y \in \cup_{n=1}^{\infty}[-nx_0, nx_0]$ and also that $|b_i^*(x - y)| < \epsilon$ for each $i \in \mathbb{N}$. For any $y^* = \sum_{i=1}^{\infty} a_i b_i^* \in B$ we have

$$|y^*(x - y)| = \left|\sum_{i=1}^{\infty} a_i b_i^*(x - y)\right| \leq \sum_{i=1}^{\infty} a_i |b_i^*(x - y)| < \epsilon \sum_{i=1}^{\infty} a_i = \epsilon,$$

therefore $sup_{y^* \in B} |y^*(x - y)| \leq \epsilon$.

Since $A \leq \|y^*\|$ for any $y^* \in B$ for any $x^* \in AU^+_{X^*}$ there exists $y^* \in B$ so that $x^* = \lambda y^*$ with $0 < \lambda \leq 1$, therefore $d_P(x - y) < \frac{\epsilon}{A}$, and $x_0 \in Q_P(P_0)$. □

Corollary 16 *Let E be an AM-space. If $int(E_+) = \emptyset$ and E_+ has quasi interior points with respect to the norm topology, then E is non-Grothendieck.*

Proof The positive cone $P = E^*_+$ of E^* is well-based because E^* is an AL-space. Then $P_0 = E_+$. By Proposition 8, the norms $\|.\|$ and $\|.\|_P$ of E are equivalent, therefore E_+ has quasi interior points with respect the norm $\|.\|_P$ and by Theorem 15, E is non-Grothendieck. □

Remark 17 In Theorem 15 we have proved that in any non-Grothendieck space X there exists a well-based cone P of X^* such that $int(P_0) = \emptyset$ and $Q_P(P_0) \neq \emptyset$. If $x \in Q_P(P_0)$ then x, as a linear functional of X^*, is strictly positive on P therefore x defines a base for P. This base is unbounded because $int(P_0) = \emptyset$. So P has a bounded and an unbounded base defined (the bases) by elements of X^{**}, therefore P is a mixed base cone.

6 The problem of Qiu

As we have noted in the introduction, in this section we give a positive answer to the problem of Qiu in the case where X is separable. Specifically we show something stronger, i.e. we show that if X is separable there exists a closed cone P of X with empty interior so that P^0 is isomorphic to the positive cone of ℓ^+_1. So in non-reflexive dual spaces with separable predual, we find a cone P isomorphic to ℓ^+_1 with some extra properties, i.e. P is weak-star closed and $int(P_0) = \emptyset$.

Theorem 18 *Any separable, non-reflexive Banach space X, contains a closed cone P such that $int(P) = \emptyset$ and P^0 is isomorphic to the positive cone of ℓ_1.*

Proof Let X be a separable non-reflexive Banach space. Then X is non-Grothendieck, therefore by Theorem 14, there exists a sequence $\{b^*_n\}$ of X^* so that $b^*_n \in S_{X^*}$ for each n, $b^*_n \xrightarrow{w^*} 0$ and $\{b^*_n\}$ is a basic sequence of ℓ_+-type. Also $\{b^*_n\}$ a w^*-basis of X^*. Denote by K the positive cone of $\{b^*_n\}$. Then K is well-based and K_0 is norm closed. Also $int(K_0) = \emptyset$ by Proposition 13. We will show that K is w^*-closed. First we remark that $\overline{K}^{w^*} = (K_0)^0$. Let $x^* \in \overline{K}^{w^*}$. Since $\{b^*_n\}$ is a w^*-basic sequence there exits a biorthogonal system $(c_n, x^*_n) \in X \times X^*$ such that $\sum_{i=1}^n x^*(c_i)b^*_i \xrightarrow{w^*} x^*$. The sequence $\left\{\sum_{i=1}^n x^*(c_i)b^*_i\right\}$ is w^*-convergent, therefore norm-bounded. Let $M > 0$, such that $\left\|\sum_{i=1}^n x^*(c_i)b^*_i\right\| \leq M$ for each $n \in \mathbb{N}$. We remark that $c_i \in K_0$ for each $i \in \mathbb{N}$ and since $x^* \in \overline{K}^{w^*} = (K_0)^0$ we have that $x^*(c_i) \geq 0$ for each $i \in \mathbb{N}$. So $\sum_{i=1}^n x^*(c_i)b^*_i \in K$ for each $n \in \mathbb{N}$. Since $\{b^*_n\}$ is a basic sequence of ℓ_+-type, there exits an isomorphism T of ℓ^+_1 onto K with $T(e_n) = b^*_n$ for each n and suppose that

$$A\|\xi\| \leq \|T(\xi)\| \leq N\|\xi\|$$

for any $\xi \in \ell_1^+$. Therefore

$$\left\|\sum_{i=1}^{n} x^*(c_i)e_i\right\| \leq \frac{1}{A}\left\|\sum_{i=1}^{n} x^*(c_i)b_i^*\right\| \leq \frac{M}{A} \quad \text{for each } n \in \mathbb{N}.$$

Since the basis $\{e_i\}$ of ℓ_1 is boundedly complete, we have that $\sum_{i=1}^{\infty} x^*(c_i)e_i \in \ell_1^+$, therefore $x^* = \sum_{i=1}^{\infty} x^*(c_i)b_i^* \in K$. So the cone $P = K_0$ is closed, $int(P) = \emptyset$, $P^0 = \overline{K}^{w^*} = K$ and K is isomorphic to ℓ_1^+. □

The next result was suggested to us by E. Casini and E. Miglierina in a communication we had during the preparation of this article. This extends the above theorem in the case where X has a non-reflexive, separable complemented subspace.

Proposition 19 *If a Banach space X has a non-reflexive, separable, complemented subspace, then there exists a closed cone P of X so that $int(P) = \emptyset$ and P^0 is isomorphic to ℓ_1^+.*

Proof Suppose that $X = Y \oplus Z$, where Y is non-reflexive and separable. Then there exists a closed cone K of Y so that $int(K) = \emptyset$ and K^0 is isomorphic to ℓ_1^+. If $P = K \oplus Z$, then P is a closed cone with empty interior and $P^0 = K^0 \oplus \{0\}$ is isomorphic to ℓ_1^+. □

By Theorem 18 we have as a corollary, the following characterization of reflexivity for separable Banach spaces.

Theorem 20 *A separable Banach space X is reflexive if and only if $int(P) \neq \emptyset$, for any closed cone P of X whose dual P^0 is well-based.*

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