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Lattice-Subspaces and Positive Bases. An Application in Economics (**).

Abstract. – *This is a survey article on lattice-subspaces and positive bases but contains also some new results. Especially the first section is a survey on lattice-subspaces and the second on positive bases in subspaces of function spaces. In the third section finite dimensional lattice-subspaces of normed spaces and, in the forth infinite-dimensional lattice-subspaces of* $\mathbb{R}^{\Omega}$ *with positive bases are studied. In the fifth some applications in the determination of positive bases are given. In the last one we give an application of lattice-subspaces in economics.*

1. – Lattice-subspaces.

Let E be a (partially) ordered vector space with positive cone E_+ and X a subspace of E. The cone $X \cap E_+$ will be called the **induced cone** of X, and the ordering defined in X by this cone the **induced ordering.** We will denote by X_+ the induced cone of X, i.e. $X_+ = X \cap E_+$. An **ordered subspace** of E is a subspace of E ordered by the induced cone. A **lattice-subspace** of E is an ordered subspace of E which is also a vector lattice (Riesz space).

Let X be a lattice-subspace of E. Then, for each $x, y \in X$ we will denote by $\sup_X \{x, y\}$ or by $x \nabla y$ the supremum of $\{x, y\}$ in X (note that $\sup_X \{x, y\} = z$ if and only if $z \in X$, $z \geqslant x, y$ and for each $w \in X$, $w \geqslant x, y$

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implies $w \geq z$). Similarly we will denote the infimum of $\{x, y\}$ in X by $\inf_X \{x, y\}$ or by $x \triangle y$. It is clear that

$$x \vee y \leq x \nabla y \quad \text{and} \quad x \triangle y \leq x \wedge y$$

whenever $x \vee y$, $x \wedge y$ exist (remind that $x \vee y$ is the supremum and $x \wedge y$ the infimum of $\{x, y\}$ in E). If E is a vector lattice and $x \nabla y = x \vee y$ for any $x, y \in X$ then X is a **sublattice** (Riesz subspace) of E.

Let E be moreover a Banach space. A sequence $\{e_n\}$ is a **positive basis** of E if $\{e_n\}$ is a (Schauder) basis of E and $E_+ = \{x = \sum_{i=1}^{\infty} \lambda_i e_i \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\}$. A positive basis $\{e_n\}$ of E is unique in the sense of a positive multiple, i.e. if $\{b_n\}$ is an other positive basis of E, then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$.

THEOREM 1 ([25], Theorem 16.3). *If $\{e_n\}$ is a positive basis of E, the following statements are equivalent:*

1) *The basis $\{e_n\}$ is unconditional.*
2) *The cone E_+ is generating and normal.*

The cone E_+ is **generating** if $E = E_+ - E_+$ and E_+ is **normal** (or self-allied) if there exists some $c \in \mathbb{R}_+$ such that $0 \leq x \leq y$ implies $\|x\| \leq c\|y\|$.

THEOREM 2. *If $\{e_n\}$ is a positive, unconditional basis of E then E is a locally solid vector lattice (i.e. E is a Banach lattice with respect to an equivalent norm).*

The proof of the above theorem follows by the fact that for each $x = \sum_{i=1}^{\infty} \lambda_i e_i$ and $y = \sum_{i=1}^{\infty} \mu_i e_i$ in E we have

$$x \vee y = \sum_{i=1}^{\infty} (\lambda_i \vee \mu_i) e_i .$$

Also by the previous theorem we have that the cone E_+ is normal and generating, therefore by [11], Theorem 4.1.5, E is locally solid.

The following result (see [1] or [23]) is very important for the study of finite dimensional lattice-subspaces. It can be proved both elementary or as a partial result of the Choquet-Kentall Theorem.

THEOREM 3. *A finite dimensional ordered vector space E is a vector lattice if and only if E has a positive basis.*

In the theory of ordered spaces have been studied sublattices ideals and bands and many basic results have been formulated in terms of these subspaces. The class of lattice-subspaces has not been systematically studied. The most serious difficulty for the study of lattice-subspaces is that $\sup_X \{x, y\}$ depends on the subspace. So, for example we cannot conclude that the intersection of lattice-subspaces is again a lattice-subspace. In lattice-subspaces we have the induced ordering but the lattice structure is not the induced one.

The following result, of **Lattice-Universality** of $C[0, 1]$ shows the importance of lattice-subspaces in the theory of Banach lattices.

THEOREM 4 ([22], Theorem 4.1). *Each separable Banach lattice is order-isomorphic to a closed lattice-subspace of $C[0, 1]$.*

Since the sublattices of $C[0, 1]$ are not enough for this representation, the lattice-subspaces seems to be the right class of subspaces to study Banach lattices. Also recently lattice-subspaces have been used in economics where they have applications in incomplete markets and in the theory of finance.

Lattice-subspaces appeared in the works of many authors in their attempt to study the subspaces of E which are the range of a positive projection. If X is the range of a positive projection $P : E \to X$ and E is a vector lattice, it is easy to show that X is a lattice-subspace with $x \nabla y = P(x \vee y)$ and $x \triangle y = P(x \wedge y)$, for each $x, y \in X$. In this case X has also some extra properties (for example the positive extension property) arising from the existence of the positive projection, but as it is remarked in [1], there are lattice-subspaces which are not the range of a positive projection. Especially in [1] an example of a two-dimensional lattice-subspace of $C[0, 1]$ which is not the range of a positive projection is given. For results on this special class of lattice-subspaces see in K. Donner, [6], N. Ghoussoub, [7], G. Jameson and A. Pinkus, [12] and for a survey on lattice-subspaces and positive projections see in [1].

The first works on the general class of lattice-subspaces appeared in 1983. The notion of the lattice-subspace was introduced by Polyrakis in [20] when the next result was prooved:

THEOREM 5. *Each closed lattice-subspace of l_1 is order-isomorphic to l_1.*

At the same time the notion of the lattice-subspace was introduced independently by Miyajima in [15], who instant of the term «lattice-subspace» used the term «quasi sublattice». He proved the following:

THEOREM 6. *If E is a vector lattice then X is a lattice-subspace if and only if X is the range of a positive projection from the sublattice $S(X)$ generated by X onto X.*

The existence of positive bases in the lattice-subspaces of E is studied in [21] and [22]. In these articles it is supposed that E is a Banach lattice the positive cone of which is defined by a countable family $\mathbb{F} = \{f_i \mid i \in \mathbb{N}\}$ of positive linear and continuous functionals of E, i.e.

$$E_+ = \{x \in E \mid f_i(x) \geq 0, \text{ for each } i\}$$

and it is supposed that X is a closed lattice-subspace of E. For each $x \in E$ the support of x with respect to the family $\mathbb{F}$ is the set

$$\mathrm{supp}_{\mathbb{F}}(x) = \{i \in \mathbb{N} \mid f_i(x) \neq 0\}$$

and for each ordered subspace Y of E the support of Y_+ with respect to $\mathbb{F}$ is

$$\mathrm{supp}_{\mathbb{F}}(Y_+) = \bigcup_{x \in Y_+} \mathrm{supp}(x).$$

In [21] the notion of the maximal support property for the lattice-subspaces of E is introduced. This property concerns the microstructure of the subspaces and is defined as follows:

DEFINITION 7. *A lattice-subspace X of E has the* **maximal support property** *with respect to the family $\mathbb{F}$, if for each closed ideal I of X and for each $x \in I_+$ it holds: x is a quasi-interior point of I_+ if and only if* $\mathrm{supp}_{\mathbb{F}}(x) = \mathrm{supp}_{\mathbb{F}}(I_+)$.

THEOREM 8. *If X is a closed, countably order complete, separable lattice-subspace of E and X has the maximal support property with respect to $\mathbb{F}$, then X has a positive basis.*

In 1992 Aliprantis and Brown understood the meaning of lattice-subspaces in economics. One of the most important questions at this time was the study of finite-dimensional lattice-subspaces. This problem is interesting, even in $\mathbb{R}^n$, because many economic models, as the famous Arrow-Debreu model, are finite. In [1] the lattice-subspaces of $\mathbb{R}^n$ are studied and in [23] and [24], the finite dimensional lattice-subspaces of $C(\Omega)$ are also studied. For applications of lattice-subspaces in economics see in [10], [3] and [4]. Also for applications of ordered spaces in economics we refer the reader to the book of Aliprantis, Brown and Burkinshaw, [2]. For ordered spaces we refer the reader to the books [18], [11], [5], and [13].

2. – Subspaces of $\mathbb{R}^\Omega$ with positive bases.

This section is a generalization of [23] where the same problems are studied in the space of real valued continuous functions $C(\Omega)$, defined on a compact and Hausdorff topological space Ω. In [23] the reader can also find the original ideas, some simple examples on finite-dimensional lattice-subspaces and many other information on this kind of subspaces. The results of this section in this generalized form can be found by the reader in [19].

In this section, we shall denote by Ω a nonempty set, by $\mathbb{R}^\Omega$ the space of real valued functions defined in Ω and we will suppose that $\mathbb{R}^\Omega$ is ordered by the pointwise ordering.

Suppose that $\{b_r\}$ is a (finite or infinite) sequence of $\mathbb{R}^\Omega$. If t is a point of Ω and there exists $m \in \mathbb{N}$ such that $b_m(t) \neq 0$ and $b_r(t) = 0$ for each $r \neq m$, then we shall say that t is an m-**node** (or simply a **node**) of the sequence $\{b_r\}$. If for each r there exists an r-node t_r of $\{b_r\}$, we shall say that $\{b_r\}$ is a **sequence** of $\mathbb{R}^\Omega$ **with nodes** and also that $\{t_r\}$ is a **sequence of nodes** of $\{b_r\}$. In the following result suppose that E is a subspace of $\mathbb{R}^\Omega$, ordered by the induced ordering. Suppose also that E has a positive basis therefore we suppose that E is a Banach space with respect to a norm which is in general independent on the space $\mathbb{R}^\Omega$.

THEOREM 9. *Suppose that E is a subspace of $\mathbb{R}^\Omega$ and that $\{b_r\}$ is a sequence of E consisting of positive functions.*

(i) If $\{b_r\}$ is a positive basis of E, then for each m there exists a sequence $\{\omega_\nu\}$ of Ω (depending on m) such that for each $k \in \mathbb{N}$ we have

$$0 \leq \sum_{i=1, i \neq m}^{k} \frac{b_i(\omega_k)}{b_m(\omega_k)} < \frac{1}{k}.$$

Therefore $\lim_{\nu \to \infty} \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} = 0$ *for each* $i \neq m$.

(ii) If E is n-dimensional subspace of $\mathbb{R}^\Omega$ and the sequence $\{b_r\}$ is consisting of n vectors $b_1, b_2, \ldots, b_n$, the converse of (i) is also true, i.e. if for each $1 \leq m \leq n$ there exists a sequence $\{\omega_\nu\}$ of Ω (depending on m) satisfying

$$\lim_{\nu \to \infty} \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} = 0 \quad \text{for each } i \neq m,$$

then $\{b_1, \ldots, b_n\}$ is a positive basis of E.

PROPOSITION 10. *Let E be a subspace of $\mathbb{R}^\Omega$ and let $\{b_1, \ldots, b_n\}$ be a positive basis of E. Then for each function $x = \sum_{i=1}^{n} \lambda_i b_i \in E$, we have the following.*

(*i*) *If a point t_i is an i-node of the basis, then $\lambda_i = \frac{x(t_i)}{b_i(t_i)}$.*

(*ii*) *If $\{\omega_\nu\}$ is a sequence of Ω such that $\lim_{\nu\to\infty} \frac{b_j(\omega_\nu)}{b_i(\omega_\nu)} = 0$ for each $j \neq i$, then $\lambda_i = \lim_{\nu\to\infty} \frac{x(\omega_\nu)}{b_i(\omega_\nu)}$.*

2.1. *Finite-dimensional subspaces of $\mathbb{R}^\Omega$.*

Suppose that $x_1, x_2, \ldots, x_n$ are fixed, linearly independent positive elements of $\mathbb{R}^\Omega$, $z = x_1 + x_2 + \ldots + x_n$ is the sum of the functions x_i and we suppose that X is the subspace of $\mathbb{R}^\Omega$ generated by these functions i.e.

$$X = [x_1, x_2, \ldots, x_n].$$

We study the problem: **Does X have a positive basis?**

DEFINITION 11. *We will refer the function*

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \ldots, \frac{x_n(t)}{z(t)}\right), \quad t \in \Omega \ \text{with}\ z(t) > 0\,,$$

as **the basic function (curve)** (1) *of $x_1, x_2, \ldots, x_n$. We will denote by $D(\beta)$ the domain, by $R(\beta)$ the range of β and by K the convex hull of the closure of the range of β, i.e. $K = co\overline{R(\beta)}$.*

The range of β is a subset of the simplex (base)

$$\Delta_n = \left\{\xi \in \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^{n} \xi_i = 1\right\} \quad \text{of } \mathbb{R}^n_+.$$

A subset C of $\mathbb{R}^m$ is an **r-simplex** if it is the convex hull of $r+1$ affinely independent vectors of $\mathbb{R}^m$. These vectors are called **vertices** of the simplex.

(1) If Ω is a real interval, then β defines a curve in $\mathbb{R}^n$. For this reason we refer also β as «basic curve».

THEOREM 12. *The following statements are equivalent.*

(*i*) *X is a lattice-subspace*

(*ii*) *K is an $(n-1)$-simplex.*

Suppose that statement (ii) is true and that $P_1, P_2, \ldots, P_n$ are the vertices of K. Then for each $i = 1, 2, \ldots, n$ there exists a sequence $\{\omega_{i\nu}\}$ of Ω such that

$$P_i = \lim_{\nu\to\infty} \beta(\omega_{i\nu})\,.$$

Suppose also that A is the $n \times n$ matrix whose, for each $i=1, 2, \ldots, n$, the i^{th} column is the vector P_i and $b_1, b_2, \ldots, b_n$ are the functions defined by the formula

(1) $$(b_1(t), b_2(t), \ldots, b_n(t))^T = A^{-1}(x_1(t), x_2(t), \ldots, x_n(t))^T.$$

Then the set $\{b_1, b_2, \ldots, b_n\}$ is a positive basis of X and

$$\lim_{\nu\to\infty}\left(\frac{b_j}{b_i}\right)(\omega_{i\nu}) = 0 \quad \text{for each} \quad j \neq i\,.$$

If $P_k = \beta(t_k)$, then t_k is a k-node of the basis $\{b_1, \ldots, b_n\}$.

THEOREM 13. *Let K be a $(n-1)$-simplex and let $\beta(t_0)$ be a vertex of K. Suppose that $\{a_r\}$ is a sequence of real numbers convergent to zero with $a_{2r} > 0$ and $a_{2r+1} < 0$ for each r and suppose also that $\{t_r\}$ is a sequence of Ω. If $\lim_{r\to\infty} \frac{\beta(t_r) - \beta(t_0)}{a_r} = l$, $l \in \mathbb{R}^n$, then $l = 0$.*

COROLLARY 1 (The derivative's criterion). *Let K be a $(n-1)$-simplex and let $\beta(t_0)$ be a vertex of K. Suppose that σ is a function defined on the real interval $(-\varepsilon, \varepsilon)$ with values in Ω with $\sigma(0) = t_0$ and suppose that $\varphi = \beta o \sigma$ is the composition of σ, β. Then*

$$\varphi'(0) = 0\,,$$

whenever the derivative $\varphi'(0)$ of φ at the point 0 exists.

REMARK 14. *In order to study if X has a positive basis (or equivalently if X is a lattice-subspace) we study if the convex hull K of the closure of the range of β is a simlex. In general it is difficult to study whether K is a simplex and if it is it is also difficult to determine the vertices of K. If K is a simplex, $\beta(t_0)$ is a vertex of K and t_0 is an interior point of a curve c of Ω, then the derivative at the point t_0 of the restriction of β on the curve c is equal to zero, whenever the derivative exists. If for example Ω is a subset of $\mathbb{R}^m$ and t_0 is an interior point of Ω, then the* *partial derivatives of β at the point t_0 are equal to zero and if t_0 belongs to the boundary $\vartheta(\Omega)$ of Ω the derivative at t_0 of the restriction of β at any curve of $\vartheta(\Omega)$ having t_0 as an interior point, is equal to zero. Hence, by Corollary 1, the points t_0 of Ω whose the images $\beta(t_0)$ are vertices of the simplex K can be obtained as the solution of a system of equations or among the points of Ω which are not interior points of a differentiable curve of Ω. In many cases, the extreme points of Ω have this property, Example 22.*

3. Linear functions.

In this section we suppose also that $x_1, x_2, \ldots, x_n$ are linearly independent positive elements of $\mathbb{R}^{\Omega}$, z is the sum of x_i and X is the subspace of $\mathbb{R}^{\Omega}$ generated by the functions x_i but we add the assumption that Ω is a convex set and the functions x_i are **linear**, i.e

$$x_i\left(\sum_{k=1}^{m} \lambda_k t_k\right) = \sum_{k=1}^{m} \lambda_k x_i(t_k),$$

for each convex combination $\sum_{k=1}^{m} \lambda_k t_k$ of Ω and also $x_i(\lambda t) = \lambda x_i(t)$, for each positive real number λ with $t, \lambda t \in \Omega$.

The results of this section can be found by the reader in [19].

THEOREM 15. *Suppose that $x_1, x_2, \ldots, x_n$ are linear functions. Then*

(i) the basic function β is homogeneous of degree zero in the sense that $\beta(\lambda t) = \beta(t)$ for each positive real number λ with $t, \lambda t \in D(\beta)$.

(ii) for each positive, linear combination $t = \sum_{k=1}^{m} \lambda_k t_k \in D(\beta)$ of elements of $D(\beta)$ we have

$$\beta(t) = \sum_{k=1}^{m} \frac{\lambda_k z(t_k)}{z(t)} \beta(t_k),$$

therefore $\beta(t)$ is a convex combination of the vectors $\beta(t_i)$, $i = 1, 2, \ldots, m$.

(iii) If $t = t_1 + t_2$ with $z(t) > 0$ and $z(t_2) = 0$, then $\beta(t) = \beta(t_1)$.

(iv) Let τ be a topology on Ω and suppose that $T = \{t_i, i \in I\}$ and A are subsets of the domain $D(\beta)$ of β.

(a) If A is contained in the positive linear span of(2) *T, then the image $\beta(A)$ of A is contained in the convex hull of the family $\{\beta(t_i) \mid i \in I\}$.*

(b) If A is contained in the τ-closure Ψ of the positive linear span of T and the basic function β is τ-continuous on Ψ, then the image $\beta(A)$ of A is contained in the closed convex hull of the family $\beta(t_i)$, $i \in I$.

DEFINITION 16. *Suppose that Y, G are ordered spaces. We will say that the dual system $\langle Y, G\rangle$ is* (3) an **ordered dual system** *if*

$$Y_+ = \{y \in Y \mid \langle y, g\rangle \geq 0 \text{ for each } g \in G_+\}$$

and

$$G_+ = \{g \in G \mid \langle y, g\rangle \geq 0 \text{ for each } y \in Y_+\}.$$

Suppose $\langle Y, G\rangle$ is an ordered dual system and Y, G are ordered norned spaces. The sets

$$U_Y^+ = \{y \in Y_+ \mid \|y\| \leq 1\} \text{ and } U_G^+ = \{g \in G_+ \mid \|g\| \leq 1\}$$

are the positive part of the closed unit balls of Y and G respectively. Any element of Y can be considered as an element of $\mathbb{R}^{U_G^+}$ and each element of G as an element of $\mathbb{R}^{U_Y^+}$ but the equality is not the same. If for example we suppose that $y_1, y_2 \in Y$, then $y_1 = y_2$ in Y, if $\langle y_1, g\rangle = \langle y_2, g\rangle$ for each $g \in G$ and $y_1 = y_2$ in $\mathbb{R}^{U_G^+}$, if $y_1(t) = y_2(t)$ for each $t \in U_G^+$. It is clear that the equality in Y implies equality in $\mathbb{R}^{U_G^+}$ but the converse is not true in general. If the cone G_+ is generating (i.e. $G = G_+ - G_+$) it is easy to show that the two equalities are equivalent, therefore we can identify algebraically Y with a subspace of $\mathbb{R}^{U_G^+}$. If for example G is a Banach lattice, the cone G_+ is generating therefore we may suppose that Y is a subspace of $\mathbb{R}^{U_G^+}$. For the space G hold also similar results.

Suppose that $x_1, x_2, \ldots, x_n$ are linearly independent positive elements of G and that X is the subspace of $\mathbb{R}^{U_Y^+}$ generated by these functions. As in the previous section denote by z the sum of the functions x_i.

(2) The positive linear span of T is the set of positive linear combinations of elements of T.

(3) In any dual system $\langle Y, G\rangle$ it is supposed that G separates the points of Y and conversely.

The basic function is

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \ldots, \frac{x_n(t)}{z(t)} \right), \quad t \in U_Y^+ \text{ with } z(t) > 0 .$$

As before, denote by $D(\beta)$ the domain and by $R(\beta)$ the range of β.

The following result is an easy consequence of statements (*ii*) and (*iv*) of the previous theorem.

THEOREM 17. *Let the ordered dual system $\langle Y, G \rangle$ and let Y be a Banach lattice with a positive basis $\{d_i\}$. If $x_1, x_2, \ldots, x_n$ are linearly independent positive elements of G and β is the basic function of the vectors x_i, then*

$$R(\beta) \subseteq \overline{co}\{\beta(d_i) \mid i \in \Phi\},$$

where $\Phi = \{i \in \mathbb{N} \mid d_i \in D(\beta)\}$.

If Y is an m-dimensional space with a positive basis $\{d_1, d_2, \ldots, d_m\}$, then

$$co\overline{R(\beta)} = co\{\beta(d_i) \mid i \in \Phi\},$$

where $\Phi = \{i = 1, 2, \ldots, m \mid d_i \in D(\beta)\}$ and the following statements are equivalent:

(*i*) *The space X generated by $x_1, x_2, \ldots, x_n$ has a positive basis.*

(*ii*) *There exist $i_1, i_2, \ldots, i_n \in \Phi$ such that $\beta(d_{i_k})$, $k = 1, 2, \ldots, n$ are linearly independent and*

$$co\overline{R(\beta)} = co\{\beta(d_{i_1}), \beta(d_{i_2}), \ldots, \beta(d_{i_n})\}.$$

Then a positive basis of X is given by the formula

$$(b_1(t), b_2(t), \ldots, b_n(t))^T = A^{-1}(x_1(t), x_2(t), \ldots, x_n(t))^T, \quad \textit{for each } t \in U_Y^+,$$

where A is the matrix with columns the vectors $\beta(d_{i_1})$, $\beta(d_{i_2}), \ldots, \beta(d_{i_n})$.

THEOREM 18. *Let $\langle Y, G \rangle$ be an ordered dual system. Suppose that Y is a normed space and that $\{y_{ij} \mid i \in \mathbb{N}, 0 \leq j \leq 2^i\}$ where $y_{ij} \in Y_+$ and $y_{ij} = y_{i+1, 2j-1} + y_{i+1, 2j}$, for each i, j, is a decomposition tree of Y_+.*

If $x_1, x_2, \ldots, x_n$ are linearly independent, positive elements of E and β is the basic function of the vectors $x_1, x_2, \ldots, x_n$, then

$$\beta(y_{ij}) = \frac{z(y_{i+1, 2j-1})}{z(y_{ij})} \beta(y_{i+1, 2j-1}) + \frac{z(y_{i+1, 2j})}{z(y_{ij})} \beta(y_{i+1, 2j}),$$

for each i, j, whenever $y_{i+1,2j-1}, y_{i+1,2j}$ belong to the domain $D(\beta)$ of β.

Also D_m is a subset of the convex hull of D_{m+1}, for each $m \in \mathbb{N}$, where

$$D_m = co\{\beta(y_{mj}) \mid 0 \leq j \leq 2^m, y_{mj} \in D(\beta)\}.$$

PROOF. For each $k = 1, 2, \ldots, n$ we have:

$$\frac{x_k(y_{ij})}{z(y_{ij})} = \frac{z(y_{i+1, 2j-1})}{z(y_{ij})} \frac{x_k(y_{i+1, 2j-1})}{z(y_{i+1, 2j-1})} + \frac{z(y_{i+1, 2j})}{z(y_{ij})} \frac{x_k(y_{i+1, 2j})}{z(y_{i+1, 2j})},$$

therefore

$$\beta(y_{ij}) = \frac{z(y_{i+1, 2j-1})}{z(y_{ij})} \beta(y_{i+1, 2j-1}) + \frac{z(y_{i+1, 2j})}{z(y_{ij})} \beta(y_{i+1, 2j}).$$

Hence the first assertion of the Theorem is true. Since $\beta(y_{ij})$ is a convex combination of $\beta(y_{i+1, 2j-1})$, $\beta(y_{i+1, 2j})$, the second assertion is also true. ■

4. – The infinite dimensional case.

Suppose E is a subspace of $\mathbb{R}^\Omega$ generated by the linearly independent positive elements x_i, $i \in \mathbb{N}$, of E, i.e. E is the closure of the linear span of the vectors x_i. Then we may also suppose that

$$z = \sum_{i=1}^{\infty} x_i$$

exists because E is also generated by the elements $\frac{x_i}{2^i |||x_i|||}$, $i \in \mathbb{N}$ (4).

Therefore as in the finite dimensional case, we can define the *basic function (curve)* of the countable set of linearly independent positive elements $\{x_i \mid i \in \mathbb{N}\}$ of $\mathbb{R}^\Omega$ with sum z as follows:

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \ldots \right) \quad \text{for each } t \in \Omega \text{ with } z(t) > 0 .$$

The function β takes values in the simplex

$$S = \{\xi \in l_1^+ \mid \|\xi\|_1 = 1\} \quad \text{of } l_1^+ .$$

THEOREM 19. *Let E be a sublattice of $\mathbb{R}^\Omega$. If $\{b_n\}$ is a positive basis of E, then the range $R(\beta)$ of the basic function of the elements x_i is countable.*

PROOF Since E is a sublattice of $\mathbb{R}^\Omega$, we have:

$$b_i \triangle b_j = b_i \wedge b_j = 0 ,$$

therefore the sets $I_i = b_i^{-1}(0, \infty)$, $i \in \mathbb{N}$, are disjoint subsets of Ω.

Let $x_n = \sum_{i=1}^{\infty} \lambda_{ni} b_i$ be the expansion of x_n in the basis b_n. Then

$$z = \sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} \sigma_i b_i ,$$

where $\sigma_i = \sum_{j=1}^{\infty} \lambda_{ji}$. Let $\Phi = \{i \in \mathbb{N} \mid \sigma_i > 0\}$. Then $D = \bigcup_{i \in \Phi} I_i$ is the domain of the basic function

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \ldots \right).$$

The set Φ is infinite. This holds because if we suppose that Φ is finite, then $\lambda_{ni} = 0$ for each $i \notin \Phi$ therefore $x_n = \sum_{i \in \Phi} \lambda_{ni} b_i$, for each n. Hence

(4) We suppose that E is a Banach space with respect to some norm $|||\cdot|||$.

each x_n belongs to the space generated by the set $\{b_i \mid i \in \Phi\}$, therefore E is finite dimensional, contradiction.

For each $i \in \Phi$ and $t \in I_i$ we have that $b_i(t) > 0$ and $b_j(t) = 0$, for each $i \neq j$, therefore

$$\beta(t) = \left(\frac{\lambda_{1i}}{\sigma_i}, \frac{\lambda_{2i}}{\sigma_i}, \frac{\lambda_{3i}}{\sigma_i}, \ldots \right) = P_i .$$

Also for each $i, j \in \Phi$ with $i \neq j$ we have that $P_i \neq P_j$, because in the contrary case we would have $\frac{\lambda_{ki}}{\sigma_i} = \frac{\lambda_{kj}}{\sigma_j}$ for each k, therefore $\frac{x_i}{\sigma_i} = \frac{x_j}{\sigma_j}$, contradiction because the vectors x_i are linearly independent. Therefore the range $R(\beta) = \{P_i \mid i \in \Phi\}$ of the basic function is countable. ■

We close this section by the question if an analogous result of Theorem 12 holds also in the infinite dimensional case. So we suppose that E is a infinite dimensional oredered subspace of $\mathbb{R}^\Omega$ generated by the linearly independent positive elements x_i, $i \in \mathbb{N}$, of E. As we have remarked before we may also suppose that

$$\sum_{i=1}^{\infty} x_i$$

exists therefore the basic function β of the vectors x_i takes values in the simplex

$$S = \{\xi \in l_1^+ \mid \|\xi\|_1 = 1\} \quad \text{of } l_1^+ .$$

Suppose that K is the closure of the convex hull of the range of β.

PROBLEM 20. *Is E a lattice-subspace if and only if K is a simplex?*

5. – Examples of computation of positive bases.

EXAMPLE 21. *Let $\Omega = [0, 2]$, $x_1(t) = t^2 - 2t + 2$, $x_2(t) = -t^3 + 2t^2 - t + 2$, $x_3(t) = t^3 - 3t^2 + 3t$ and X be the subspace of $C[0, 2]$ generated by x_1, x_2, x_3.*

Then $z(t) = x_1(t) + x_2(t) + x_3(t) = 4$ and $\beta = \frac{1}{4}(x_1(t), x_2(t), x_3(t))$ is the basic function of x_1, x_2, x_3. We remark that $D(\beta) = [0, 2]$ and that the range $R(\beta)$ of β is closed. To study whether X has a positive ba-

sis we study if the convex hull K of $R(\beta)$ is a simplex. If we suppose that K is a simplex with vertices P_1, P_2, P_3 then we have that these vertices belong to $R(\beta)$ therefore we may suppose that $\beta(t_i) = P_i$ for each i. The points t_i are pairwise different, therefore at least one of them is an interior point of $[0, 2]$. In Corollary 1, we have proved that if $\beta(t_i)$ is an extreme point of K and t_i an interior point of $[0, 2]$, then the derivative of β at the point t_i is equal to zero.

Since the derivative of β is equal to zero only in the point 1 of $(0, 2)$ we have that the set

$$\{\beta(0), \beta(1), \beta(2)\}$$

is the only set of possible extreme points of β and suppose that $P_1 = \beta(0)$, $P_2 = \beta(1)$, $P_3 = \beta(2)$.

We check now if each $\beta(t)$ is a convex combination of the vectors P_i and it is easy to show that this is true, therefore K is a simplex with vertices P_1, P_2, P_3.

By Theorem 12, X is a lattice-subspace and a positive basis $\{b_1, b_2, b_3\}$ of X is given by the formula:

$$(b_1(t), b_2(t), \ldots, b_n(t))^T = A^{-1}(x_1(t), x_2(t), \ldots, x_n(t))^T$$

where A is the matrix with columns the vectors P_1, P_2, P_3.

After the computations we have that $b_1(t) = 2(t-1)^2(2-t)$, $b_2(t) = = 4t(2-t)$ and $b_3(t) = 2t(t-1)^2$. The points $t_1 = 0$, $t_2 = 1$ and $t_3 = 2$ are nodes for the basis $\{b_1, b_2, b_3\}$, therefore the expansion of the element x of X in this basis is

$$x = \frac{x(0)}{b_1(0)} b_1 + \frac{x(1)}{b_2(1)} b_2 + \frac{x(2)}{b_3(2)} b_3 .$$

EXAMPLE 22. *Suppose that $x_1(u, v) = 1$, $x_2(u, v) = u$, $x_3(u, v) = v$ and X is the space of $F(\Omega)$ generated by x_1, x_2, x_3. We distinguish the following two cases:*

(i) $\Omega = \{(u, v) \in \mathbb{R}^2 \mid (u-1)^2 + (v-1)^2 \leq 1\}$.

The basic function is

$$\beta(u, v) = \left(\frac{1}{1+u+v}, \frac{u}{1+u+v}, \frac{v}{1+u+v}\right)$$

and its partial derivatives

$$\beta_u(u, v) = \left(\frac{-1}{(1+u+v)^2}, \frac{1+v}{(1+u+v)^2}, \frac{1+u}{(1+u+v)^2}\right) \neq 0$$

and

$$\beta_v(u, v) = \left(\frac{-1}{(1+u+v)^2}, \frac{1+u}{(1+u+v)^2}, \frac{1+v}{(1+u+v)^2}\right) \neq 0 .$$

Suppose that $K = \mathrm{co}\overline{R(\beta)}$ is a simplex with vertices P_1, P_2, P_3. Since the function β is continuous on the compact set Ω, the range of β is closed, therefore $K = \mathrm{co}R(\beta)$. So there exist three different points $(u_1, v_1), (u_2, v_2), (u_3, v_3)$ of Ω with $\beta(u_i, v_i) = P_i$, for each i. Since the partial derivatives of β are nonzero at any point (u, v) we have that (u_i, v_i) cannot be an interior point of Ω. The restriction of β on the boundary of Ω is

$$\sigma(\vartheta) = \beta(1+\cos\vartheta, 1+\sin\vartheta) = \frac{1}{3+\cos\vartheta+\sin\vartheta}(1, 1+\cos\vartheta, 1+\sin\vartheta)$$

and it is easy to see that the derivative $\sigma'(\vartheta)$ of σ is nonzero for each ϑ. Therefore (u_i, v_i) cannot be also a point of the boundary of Ω. From these remarks we have that K cannot be a simplex, therefore X does not have a positive basis or equivalently X is not a lattice-subspace.

(ii) The set Ω is the triangle $\Omega = \{(u, v) \in \mathbb{R}^2 \mid u \geq 0,\ v \geq 0$ and $u + v \leq 1\}$.

Suppose that K is a simplex and that $\beta(u_0, v_0)$ is an extreme point of K. Since the partial derivatives of β at any point (u, v) are nonzero we have that (u_0, v_0) is the point $(0, 0)$ or it is a point of the hypotenuse of the triangle. The restriction of β on the line $u + v = 1$ is

$$\sigma(v) = \left(\frac{1}{2}, \frac{1-v}{2}, \frac{v}{2}\right)$$

and the derivative of σ is

$$\sigma'(v) = \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

which is different from zero, therefore the point (u_0, v_0) cannot be an *interior point of the hypotenuse. Hence the only possible set of extreme points of K is the set*

$$\left\{\beta(0,0)=(1,0,0),\ \beta(1,0)=\left(\frac{1}{2},\frac{1}{2},0\right),\ \beta(0,1)=\left(\frac{1}{2},0,\frac{1}{2}\right)\right\}.$$

It is easy to see that any point $\beta(u,v)$ *is of the form (convex combination)*

$$\beta(u,v)=\frac{1-u-v}{1+u+v}(1,0,0)+\frac{2u}{1+u+v}\left(\frac{1}{2},\frac{1}{2},0\right)+\frac{2v}{1+u+v}\left(\frac{1}{2},0,\frac{1}{2}\right),$$

therefore X has a positive basis $\{b_1, b_2, b_3\}$ *which is given by the formula:*

$$(b_1,b_2,b_3)^T=\begin{bmatrix}1&-1&-1\\0&2&0\\0&0&2\end{bmatrix}(x_1,x_2,x_3)^T,$$

therefore

$$b_1=1-u-v,\ b_2=2u,\ b_3=2v,$$

is a positive basis of X.

We show below the way we can work in $\mathbb{R}^n$ and also in the space of matrices

EXAMPLE 23. *We will denote by* M_n *the space of the* $n\times n$ *real matrices. Note that* M_n *is order-isomorphic to the space* $\mathbb{R}^{n\times n}$.

Let X be the subspace of M_3 *generated by*

$$x_1=\begin{bmatrix}1&1&1\\2&0&1\\0&1&2\end{bmatrix},\quad x_2=\begin{bmatrix}0&2&1\\5&1&0\\1&2&2\end{bmatrix},\quad x_3=\begin{bmatrix}1&0&2\\1&1&1\\1&0&1\end{bmatrix}.$$

In order to study whether X has a positive basis we use Theorem 17 *as follows: Consider the dual system* $\langle M_3, M_3\rangle$ *and consider X as a subspace of the second space. The usual basis* $\{e_1, e_2, \ldots, e_9\}$ *of* M_3 *is a positive basis of* M_3. *To study whether X has a positive basis we study*

if the convex hull K of the vectors $\beta(e_i)$ *is a 2-simplex. We have:*

$$\beta(e_1)=\frac{1}{2}(1,0,1)=\beta(e_6),\quad \beta(e_5)=\frac{1}{2}(0,1,1)=\beta(e_7),$$

$$\beta(e_2)=\frac{1}{3}(1,2,0)=\beta(e_8),$$

$$\beta(e_3)=\frac{1}{4}(1,1,2),\ \beta(e_4)=\frac{1}{8}(2,5,1),\ \beta(e_9)=\frac{1}{5}(2,2,1).$$

Suppose that K is a simplex. Since the second coordinate of the vector $\beta(e_1)$ *is zero and the corresponding coordinates of the other vectors* $\beta(e_i)$ *are non-zero we have that* $\beta(e_1)$ *cannot be a convex combination of the other vectors, therefore* $\beta(e_1)$ *must be a vertex of K. Similarly we have that* $\beta(e_5)$, $\beta(e_2)$ *are also vertices of K, therefore* $\{\beta(e_1), \beta(e_5), \beta(e_2)\}$ *is a possible set of extreme points of K. It is easy to show that each* $\beta(e_i)$ *is a convex combination of* $\beta(e_1)$, $\beta(e_5)$, $\beta(e_2)$, *therefore X has a positive basis* $\{b_1, b_2, b_3\}$ *which is given by the formula:*

$$(b_1,b_2,b_3)^T=A^{-1}(x_1,x_2,x_3)^T,$$

where A is the matrix with columns the vectors $\beta(e_1)$, $\beta(e_5)$, $\beta(e_2)$. *After the computation of* A^{-1} *we have*

$$(b_1,b_2,b_3)^T=\begin{bmatrix}\frac{4}{3}&-\frac{2}{3}&\frac{2}{3}\\-\frac{4}{3}&\frac{2}{3}&\frac{4}{3}\\1&1&1\end{bmatrix}(x_1,x_2,x_3)^T,$$

hence $b_1=\frac{2}{3}(2x_1-x_2+x_3)$, $b_2=\frac{2}{3}(-2x_1+x_2+2x_3)$, $b_3=x_1+x_2-x_3$.

Therefore the positive basis of X is

$$b_1=2\begin{bmatrix}1&0&1\\0&0&1\\0&0&1\end{bmatrix},\quad b_2=2\begin{bmatrix}0&0&1\\1&1&0\\1&0&0\end{bmatrix},\ b_3=3\begin{bmatrix}0&1&0\\2&0&0\\0&1&1\end{bmatrix}.$$

The points e_1, e_5, e_2 *of* $\Omega=U^+_{M_3}$ *are nodes of the basis of* $\{b_1, b_2, b_3\}$. *By statement (i) of Proposition* 10, *the coordinates of* x_1

(and of any other vector of X) in the basis are given by the formula

$$\left(\frac{x_1(e_1)}{b_1(e_1)} = \frac{1}{2}, \frac{x_1(e_5)}{b_2(e_5)} = 0, \frac{x_1(e_2)}{b_3(e_2)} = \frac{1}{3}\right),$$

therefore $x_1 = \frac{1}{2}b_1 + \frac{1}{3}b_3$.

EXAMPLE 24. *Let X be the subspace of the the space M_2 generated by*

$$x_1 = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

We study whether the convex hull K of the vectors $\beta(e_i)$, $i=1, 2, 3, 4$, where $\{e_i \mid i=1, 2, 3, 4\}$ is the usual basis of M_2, is a simplex.

$$\beta(e_1) = \frac{1}{2}(1, 1, 0), \beta(e_2) = \frac{1}{6}(1, 4, 1),$$

$$\beta(e_3) = \frac{1}{2}(0, 1, 1), \beta(e_4) = \frac{1}{6}(3, 2, 1).$$

Suppose that K is a simplex. Since the third coordinate of $\beta(e_1)$ is zero, $\beta(e_1)$ cannot be a convex combination of the other vectors $\beta(e_i)$, therefore $\beta(e_1)$ is a vertex of K. Similarly we have that $\beta(e_3)$ is also a vertex of K, therefore the possible sets of extreme points of K are the following:

$$G_1 = \{\beta(e_1), \beta(e_3), \beta(e_2)\}$$

and

$$G_2 = \{\beta(e_1), \beta(e_3), \beta(e_4)\}.$$

It is easy to see that $\beta(e_4)$ is not a convex combination of $\beta(e_1)$, $\beta(e_3)$, $\beta(e_2)$, therefore G_1 is not a set of extreme points of K. Similarly we have that G_2 is not a set of extreme points of K, therefore X does not have a positive basis.

6. – An application of lattice-subspaces in economics.

Competitive security markets with finitely many securities have been studied by Hart, [9] and also by Hammond [8], Nielsen [16], Page [17] and Werner [26]. Formally the Hart's model is very similar to the standard model of competitive commodity markets. The space of portfolios plays a role similar to the role of the commodity space. Equilibrium theory of security markets exploits this similarity and relies on the methods of the Arrow-Debreu equilibrium theory. For an extensive discussion on the relationship between Hart's models and the Arrow-Debreu model see in Milne [14].

We consider here a model of security markets extending over two days. This model has been studied in [3]. We suppose that there are countably many securities traded at day 0 labeled by the natural numbers 1, 2, ... Securities are described by their payoffs at day 1. The payoff of security n is an element x_n of a partially ordered vector space X which is called **payoff space**. Typically X is the space of continuous functions $C(\Omega)$ on a compact, Hausdorff space Ω, or X is the space of real-valued random variables on some underling probability space (Ω, Σ, μ), such as a $L_p(\Omega, \Sigma, \mu)$-space for $1 \leq p \leq \infty$. Securities can be combined in portfolios. A **portfolio** is a sequence of share holdings $\theta = (\theta_1, \theta_2, \ldots)$, where θ_n is the number of shares of security n. In the case of a short position in security n, the holding θ_n is negative. In this model we suppose that each portfolio θ has a finite number of non-zero holdings θ_n, therefore each portfolio is formed from a finite subset of securities. The space of portfolios is the vector space ϕ of eventually zero real sequences and it is called **the portfolio space**. The **payoff of portfolio** $\theta \in \phi$ is

$$R(\theta) = \sum_{n=1}^{\infty} \theta_n x_n \in X.$$

R is a linear operator of ϕ into X which we call the **payoff operator**.

Suppose also that the vectors $x_1, x_2, \ldots$ are linearly independent (non-redundant securities). This implies that the payoff operator is one-to-one.

The partial order $\geq$ of X induces the partial order $\geq_R$ in the payoff space ϕ as follows

$$\theta \geq_R \varphi, \quad \text{if and only if } R(\theta) \geq R(\varphi), \text{ for each } \theta, \varphi \in \phi.$$

The order $\geq_R$ is called **portfolio dominance order** and its positive cone in the portfolio space is the set

$$\phi_R^+ = \{\theta \in \phi \mid R(\theta) \geq 0\}.$$

Therefore ϕ_R^+ is the set of portfolios with positive payoff. We assume that ϕ ordered by $\geqslant_R$ is a vector lattice. Then for each $\theta, \varphi \in \phi$ denote by

$$\theta \vee_R \varphi, \ \vartheta \wedge_R \varphi$$

the supremum and infimum of $\{\theta, \varphi\}$ respectively.

The range of the operator R, i.e. the subspace

$$M = R(\phi)$$

of X, is the set of payoffs of portfolios and is called the **asset span** of securities. M is also known as the **space of marketed securities**. It is easy one to show that

PROPOSITION 25. *The asset span M is a lattice-subspace of X if and only if the portfolio space ϕ in the portfolio dominance order $\geqslant_R$ is a vector lattice.*

Any vector $q = (q_1, q_2, \ldots) \in \mathbb{R}^\infty$, where q_n is the price of the security n, is called **security price system** or **vector of security price**. The value of security θ at price q is the real number

$$q.\theta = \sum_{n=1}^{\infty} q_n \theta_n .$$

The portfolio space ϕ and the space of security prices $\mathbb{R}^\infty$ form a dual system, $\langle \phi, \mathbb{R}^\infty \rangle$, the **portfolio-price duality**. The dual cone $(\phi_R^+)'$ of ϕ_R^+ is defined by

$$(\phi_R^+)' = \{q \in \mathbb{R}^\infty \mid q.\theta \geqslant 0, \quad \text{for each } \theta \in \phi_R^+\}.$$

6.1. *Equilibrium in security markets.*

Suppose that ϕ is equipped with the inductive limit topology ξ. Suppose also that in our model there are m investors indexed by $i = 1, 2, \ldots m$ and that each investor has,

(i) the cone ϕ_R^+ as his **feasible portfolio set,**

(ii) an **initial portfolio** $\omega^i \in \phi_R^+$,

(iii) a **utility function** u_i: $\phi_R^+ \to \mathbb{R}$ such that u_i is quasi-concave, *ξ-continuous* and monotone in the order $\geqslant_R$ of ϕ, and we suppose also that

(iv) **the marked portfolio** ω (i.e. the sum $\sum_{i=1}^{m} \omega_i$ of the initial portfolios) is desirable i.e. $u_i(\theta + a\omega) > u_i(\theta)$, for each $\theta \in \phi_R^+$ and each real number $a > 0$.

Suppose that $q \in \mathbb{R}^\infty$ is a security price system. The set

$$B_i(q) = \{\theta \in \phi_R^+ \mid q.\theta \leqslant q.\omega_i\}$$

is the **budget set** of the investor i in the price q.

A portfolio $\theta \in B_i(q)$ in which the utility function takes maximum in $B_i(q)$ is called **optimal portfolio** for the investor i in the price q.

Any m-tuple $(\theta^1, \theta^2, \ldots\theta^m)$ with $\theta^i \in \phi_R^+$ and $\sum_{i=1}^{m} \theta^i = \omega$ is a **portfolio allocation.**

A portfolio allocation $(\theta^1, \theta^2, \ldots\theta^m)$ is called **portfolio equilibrium** if there exists a non-zero security price system q such that each θ^i is optimal for the investor i in price q.

A portfolio allocation $(\theta^1, \theta^2, \ldots\theta^m)$ is called **portfolio quasiequilibrium** if there exists a non zero security price system q such that for each $\theta^i \in \phi_R^+$

$$u_i(\theta) \geqslant u_i(\theta^i) \ \text{ imply } \ q.\theta \geqslant q.\omega^i .$$

It is clear that each equilibrium is a quasiequilibrium.

A portfolio allocation $(\theta^1, \theta^2, \ldots\theta^m)$ is called **optimal,** if there exists no other allocation $\varphi = (\varphi^1, \varphi^2, \ldots\varphi^m)$ such that $u_i(\theta^i) \geqslant u_i(\varphi^i)$ for each i and $u_i(\theta^i) > u_i(\varphi^i)$, for at least one i.

Also a portfolio utility function u_i is said **φ-uniformly ξ-proper** on ϕ_R^+, if there exists a neighborhood (in the inductive limit topology) V of zero such that $u_i(\theta - a\varphi + \gamma) \geqslant u_i(\theta)$ implies that $\gamma \notin aV$ for each real number $a > 0$ and $\varphi \in \phi_R^+$ with $\theta - a\varphi + \gamma \in \phi_R^+$.

THEOREM 26 ([3], Theorem 6.1]). *If the payoff space ϕ has a Yudin basis* ([5]) and each portfolio utility function u_i is also ω-uniformly ξ-proper on ϕ_R^+, then there exists a portfolio quasi-equilibrium.

([5]) i.e. ϕ has a positive basis $\{b_i\}$ with the property: each element of ϕ is a finite linear combination of elements of $\{b_i\}$.

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