LATTICE-SUBSPACES AND POSITIVE PROJECTIONS

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ABSTRACT

A lattice-subspace of an ordered vector space is a vector subspace which is a vector lattice (Riesz space) with respect to the induced ordering. This work presents a comprehensive study of lattice-subspaces. Among other things, we characterise the collections of positive vectors in $\mathbb{R}^m$ that span lattice-subspaces. This characterisation can be considered as a constructive supplement to the well-known Choquet–Kendall theorem. We also study relationships between lattice-subspaces and positive projections.

1. Introduction

Recall that the vector space $\mathscr{H}$ of all functions which are harmonic in $D = \{(x, y)\colon x^2 + y^2 < 1\}$ and continuous in $\bar{D} = \{(x, y)\colon x^2 + y^2 \leq 1\}$ is a vector lattice under the pointwise ordering (see for instance [11, p. 51]). What is important to emphasise for us here is the fact that $\mathscr{H}$ viewed as a vector subspace of $C(\bar{D})$ fails to be a vector sublattice, i.e. $\mathscr{H}$ is not closed under the pointwise suprema and infima. This property of the space of harmonic functions is possessed by a variety of subspaces of classical vector lattices. That is, there are plenty of subspaces of vector lattices which are not vector sublattices but which are nevertheless vector lattices under the induced ordering. These spaces are referred to as lattice-subspaces in order to distinguish them from the vector sublattices, and they will be the subject of our present study.

Lattice-subspaces have appeared in a fragmented way in the works of several authors. They are closely related to positive projections. Schaefer [18] has shown that the range of a positive projection is a lattice-subspace, but the converse is not true, i.e. a lattice-subspace need not be the range of a positive projection. In spite of this, Miyajima [12] has proven that each lattice-subspace X is the range of a positive projection defined on the vector sublattice generated by X. T. Andô and (independently) Ghoussoub [6] have established that the range of a positive projection on a Banach lattice with order-continuous norm is order-isomorphic to a vector sublattice. Polyrakis ([16]; [17]) has studied lattice-subspaces extensively in connection with Schauder bases, and Tsekrekos [19] and Nassopoulos and Tsekrekos

[13] have studied the existence of positive projections with a given range. Jameson and Pinkus [10] also studied positive projections, and obtained many interesting results with emphasis on the metric structure. Among other things, they, together with Donner [5], considered some Korovkin-type conditions. Christianson [4] has shown that the subspace generated by the positive fixed points of a left-amenable semigroup of positive operators on a Banach lattice is a lattice-subspace—see also [2] for a related result about fixed points and vector sublattices.

In this paper, we put together in a common setting all the results we know concerning lattice-subspaces, obtain some new ones, and present several examples. The main result (Theorem 2.6) presents an algorithm that can be 'followed' by a computer to determine whether a finite number of linearly independent positive vectors in a finite-dimensional space generate a lattice-subspace. Our study was motivated by questions raised in economics [8], where lattice-subspaces appear naturally in incomplete markets and in the theory of finance.

For notation and terminology regarding vector lattices and Banach lattices we refer the reader to [3], [11], and [18].

2. Lattice-subspaces of $\mathbb{R}^m$

We start with the basic notion of 'lattice-subspace'.

Definition 2.1. A vector subspace X of a partially ordered vector space Y is said to be a *lattice-subspace* if X under the induced ordering is a vector lattice (Riesz space).[1]

If Y_+ denotes the cone of Y and X is a lattice-subspace of Y, then we shall also say that $X_+ = X \cap Y_+$ is a *lattice-cone*. Clearly, every vector sublattice is a lattice-subspace but the converse is not true in general.

In this section we will be concerned with the lattice-subspaces of the finite-dimensional vector lattice $Y = \mathbb{R}^m$, ordered by the standard cone $\mathbb{R}^m_+$.[2] Fix n linearly independent vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of $\mathbb{R}^m_+$ (where $1 \leq n < m$) and denote by $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ the n-dimensional vector subspace they generate. Clearly, the cone $X_+ = X \cap \mathbb{R}^m_+$ induced by $\mathbb{R}^m_+$ on X is closed and generating. An internal characterisation of lattice subspaces is given by the following classical Choquet–Kendall Theorem.

Theorem 2.2 (Choquet–Kendall). *The vector space X is a lattice-subspace if and only if X_+ has a base which is a simplex.*

For an extensive discussion of this theorem we refer the reader to [14, chapter 1, sect. 3] and [15]; see also [7]. The Choquet–Kendall Theorem, though very important and elegant, does not tell us which collections of linearly independent

[1] Miyajima [12] refers to a lattice-subspace as a quasi-sublattice. We prefer the term 'lattice-subspace' introduced in [16].

[2] We remind the reader that by the classical Yudin theorem any m-dimensional Archimedean vector lattice is order-isomorphic to $\mathbb{R}^m$.

positive vectors span lattice-subspaces. The question *When does a collection of linearly independent positive vectors generate a lattice-subspace?* is of great interest and it will be solved in the present section.

In order to solve this problem, we need to introduce some notation and terminology. If $\mathbf{x} \in \mathbb{R}^m$, then $\mathbf{x}(i)$ will denote the ith component of $\mathbf{x}$. So the matrix whose rows are the components of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ can be written as

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1(1) & \mathbf{x}_1(2) & \ldots & \mathbf{x}_1(m) \\ \mathbf{x}_2(1) & \mathbf{x}_2(2) & \ldots & \mathbf{x}_2(m) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{x}_n(1) & \mathbf{x}_n(2) & \ldots & \mathbf{x}_n(m) \end{bmatrix}.$$

Next, we introduce the following m vectors of $\mathbb{R}^n$:

$$\begin{aligned} \mathbf{y}_1 &= (\mathbf{x}_1(1), \mathbf{x}_2(1), \ldots, \mathbf{x}_n(1)) \\ \mathbf{y}_2 &= (\mathbf{x}_1(2), \mathbf{x}_2(2), \ldots, \mathbf{x}_n(2)) \\ &\vdots \\ \mathbf{y}_m &= (\mathbf{x}_1(m), \mathbf{x}_2(m), \ldots, \mathbf{x}_n(m)), \end{aligned}$$

which are the column vectors (written as rows) of the matrix $[\mathbf{x}_i(j)]$.[3] The important things to keep in mind are the following.

(1) The arbitrary vector $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{x}_i \in X$ satisfies

$$\mathbf{x} = (\mathbf{c} \cdot \mathbf{y}_1, \mathbf{c} \cdot \mathbf{y}_2, \ldots, \mathbf{c} \cdot \mathbf{y}_m),$$

where $\mathbf{c} \cdot \mathbf{y}_i$ denotes the standard dot product in $\mathbb{R}^n$ of the vectors $\mathbf{y}_i$ and $\mathbf{c} = (c_1, c_2, \ldots, c_n)$.

(2) Since the matrix with rows $\mathbf{x}_1, \ldots, \mathbf{x}_n$ has rank n, it follows that there are n linearly independent vectors amongst the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_m$.

Recall that a sequence $\{\mathbf{e}_n\}$ of positive vectors of a partially ordered Banach space Z is a *positive (Schauder) basis* whenever it is a Schauder basis of Z and

$$Z_+ = \left\{ z = \sum_{n=1}^{\infty} c_n \mathbf{e}_n \in Z : c_n \geq 0 \quad \text{for all } n \right\},$$

i.e. $z = \sum_{n=1}^{\infty} c_n \mathbf{e}_n \geq 0$ if and only if $c_n \geq 0$ for each n.

In general, a sequence $\{\mathbf{e}_n\}$ of positive vectors of Z is said to be a *positive basic sequence* whenever it is basic and $z = \sum_{n=1}^{\infty} c_n \mathbf{e}_n \geq 0$ holds if and only if $c_n \geq 0$ for

[3] Warning: The boldface letters $\mathbf{y}_1, \ldots, \mathbf{y}_m$ will be exclusively reserved to designate these column vectors.

each n. It is easy to see that existence of a positive basis in a finite-dimensional Archimedean vector space Z implies that Z is a vector lattice.

Lemma 2.3. *For n vectors $\mathbf{e}_1,\dots,\mathbf{e}_n \in X_+$ the following statements are equivalent.*
(1) *The set of vectors $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is a positive basis.*
(2) *There exists a subset $\{m_1,\dots,m_n\}$ of $\{1, 2,\dots, m\}$ consisting of n indices such that*

$$\mathbf{e}_i(m_i) > 0 \quad \textit{and} \quad \mathbf{e}_i(m_j) = 0 \quad \textit{for} \quad j \neq i.$$

PROOF. (1) $\Rightarrow$ (2) Assume that $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is a positive basis for X and let $1 \leq l \leq n$ be fixed. Then for each k the vector

$$\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{l-1} - \frac{1}{k}\mathbf{e}_l + \mathbf{e}_{l+1} + \cdots + \mathbf{e}_n \in X$$

does not belong to X_+. So, for some $r_k \in \{1,\dots,m\}$ we have $\mathbf{x}(r_k) < 0$, or

$$0 \leq \sum_{i \neq l} \mathbf{e}_i(r_k) < \frac{1}{k}\mathbf{e}_l(r_k). \qquad (*)$$

Since $1 \leq r_k \leq m$ holds for each k, there exists some $m_l \in \{1,\dots,m\}$ such that $r_k = m_l$ holds for infinitely many k. From $(*)$, we see that

$$\mathbf{e}_l(m_l) > 0 \quad \text{and} \quad \mathbf{e}_i(m_l) = 0 \quad \text{for } i \neq l.$$

Now note that the set of indices $\{m_1, m_2, \dots, m_n\}$ satisfies the desired properties.

(2) $\Rightarrow$ (1) In this case $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is clearly a basis for X. Moreover, notice that if $\mathbf{x} = \sum_{l=1}^n c_l\mathbf{e}_l \in X$, then $\mathbf{x}(m_l) = c_l\mathbf{e}_l(m_l)$. So, if $\mathbf{x} \geq 0$, then $c_l\mathbf{e}_l(m_l) \geq 0$ for each l, and since $\mathbf{e}_l(m_l) > 0$, we see that $c_l \geq 0$ for each l. That is, $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is a positive basis for X. ■

Lemma 2.4. *If $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is a positive basis for X, and a set of n indices $\{m_1,\dots,m_n\}$ satisfies statement* (2) *in Lemma* 2.3, *then the corresponding n vectors $\mathbf{y}_{m_1},\dots,\mathbf{y}_{m_n}$ are linearly independent.*

PROOF. We can assume without loss of generality that $\mathbf{e}_i(m_j) = \delta_{ij}$. Put $\mathbf{c} = (c_1,\dots,c_n)$ and note that the n (n-dimensional) vectors $\mathbf{y}_{m_1},\dots,\mathbf{y}_{m_n}$ are linearly independent if and only if $\mathbf{c} = 0$ is the only solution of the homogeneous system of linear equations

$$\mathbf{c}\cdot\mathbf{y}_{m_r} = 0, \quad r = 1,\dots,n.$$

To establish this, suppose that $\mathbf{c}\cdot\mathbf{y}_{m_r}=0$ for each $r=1,\dots,n$ and consider the vector

$$\mathbf{x}=\sum_{i=1}^{n} c_i\mathbf{x}_i=(\mathbf{c}\cdot\mathbf{y}_1,\mathbf{c}\cdot\mathbf{y}_2,\dots,\mathbf{c}\cdot\mathbf{y}_m)\in X.$$

Clearly, $\mathbf{x}(m_r)=\mathbf{c}\cdot\mathbf{y}_{m_r}=0$ for each $r=1,\dots,n$. Now if $\mathbf{x}=\sum_{r=1}^{n}\xi_r\mathbf{e}_r$ is the representation of $\mathbf{x}$ in terms of the basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, then certainly $\xi_r=\mathbf{x}(m_r)=0$, implying that $\mathbf{x}=0$. Since $\mathbf{x}=\sum_{i=1}^{n}c_i\mathbf{x}_i$, if follows that $c_i=0$ for each $i=1,\dots,n$, and the proof is finished. ■

Definition 2.5. A set of n indices $\{m_1,\dots,m_n\}$ is said to be *fundamental* for the collection of vectors $\mathbf{x}_1,\dots,\mathbf{x}_n\in\mathbb{R}^m_+$ whenever

(1) the n vectors $\mathbf{y}_{m_1},\dots,\mathbf{y}_{m_n}$ are linearly independent; and

(2) for each $j\notin\{m_1,\dots,m_n\}$ all the coefficients ξ_{jr} in the expansion

$$\mathbf{y}_j=\sum_{r=1}^{n}\xi_{jr}\mathbf{y}_{m_r}$$

are non-negative, i.e. for each $j\notin\{m_1,\dots,m_n\}$ the vector $\mathbf{y}_j$ belongs to the cone generated by the vectors $\mathbf{y}_{m_1},\dots,\mathbf{y}_{m_n}$.

And now we come to the main result of this section describing the collections $\{\mathbf{x}_1,\dots,\mathbf{x}_n\}$ of linearly independent positive vectors in $\mathbb{R}^m$ whose span $X=[\mathbf{x}_1,\dots,\mathbf{x}_n]$ is a lattice-subspace of $\mathbb{R}^m$.

Theorem 2.6. *The vector space X is a lattice-subspace of $\mathbb{R}^m$ if and only if the vectors $\mathbf{x}_1,\dots,\mathbf{x}_n$ admit a fundamental set of indices $\{m_1,\dots,m_n\}$. In this case, the vectors $\mathbf{e}_1,\dots,\mathbf{e}_n$ defined by*

$$\mathbf{e}_r(j)=\begin{cases}1, & \textit{if } j=m_r\\ 0, & \textit{if } j\in\{m_1,\dots,m_n\}\setminus\{m_r\}\\ \xi_{jr}, & \textit{if } j\notin\{m_1,\dots,m_n\},\end{cases}$$

where the ξ_{jr} are the coefficients of the expansion $\mathbf{y}_j=\sum_{r=1}^{n}\xi_{jr}\mathbf{y}_{m_r}$, form a positive basis for X.

PROOF. Assume first that $\{m_1,\dots,m_n\}$ is a fundamental set of indices for the vectors $\mathbf{x}_1,\dots,\mathbf{x}_n$ as defined in Definition 2.5. For simplicity of notation we may assume that $m_j=j$ for each $j\le n$, i.e. that $\{m_1,\dots,m_n\}=\{1,\dots,n\}$. We want to prove that X is a lattice-subspace. The claim will follow if we show that the vectors $\mathbf{e}_r$ introduced above form a positive basis in X. Notice that under our agreement we have

$$\mathbf{e}_r(j) = \begin{cases} \delta_{rj}, & \text{if } j \le n \\ \xi_{jr}, & \text{if } j > n. \end{cases}$$

Clearly these vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are linearly independent, so we need to verify that they belong to X and form a positive basis there.

Since $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ are linearly independent vectors, for each $1 \le s \le n$ the system of linear equations

$$\mathbf{c} \cdot \mathbf{y}_r = \delta_{sr}, \quad r = 1, \dots, n,$$

has a unique solution, say $\mathbf{c}_s = (c_{s1}, \dots, c_{sn})$. Moreover, for each $j > n$ we have

$$\mathbf{c}_s \cdot \mathbf{y}_j = \mathbf{c}_s \cdot \left(\sum_{r=1}^{n} \xi_{jr} \mathbf{y}_r \right) = \sum_{r=1}^{n} \xi_{jr} \mathbf{c}_s \cdot \mathbf{y}_r = \xi_{js} \ge 0.$$

So $\mathbf{e}_s = (\mathbf{c}_s \cdot \mathbf{y}_1, \mathbf{c}_s \cdot \mathbf{y}_2, \dots, \mathbf{c}_s \cdot \mathbf{y}_n) = \sum_{r=1}^{n} c_{sr} \mathbf{x}_r$, from which it follows that $\mathbf{e}_s \ge 0$ and $\mathbf{e}_s \in X$ for each $s = 1, \dots, n$. In view of Lemma 2.3, $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a positive basis of X.

For the converse, suppose that X is a lattice-subspace of $\mathbb{R}^m$. Hence X has a positive basis, say $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. By Lemma 2.3, there exist n indices $\{m_1, \dots, m_n\} \subset \{1, 2, \dots, m\}$ satisfying

$$\mathbf{e}_i(m_r) = \delta_{ir}, \quad i, r = 1, \dots, n.$$

Again we will assume for simplicity that $m_j = j$ for each $j \le n$. We claim that $\{1, \dots, n\}$ is a fundamental set of indices for $\mathbf{x}_1, \dots, \mathbf{x}_n$. Invoking Lemma 2.4, we see that the vectors $\mathbf{y}_1, \dots, \mathbf{y}_n$ are linearly independent, and hence for each $j > n$ there exists a unique representation

$$\mathbf{y}_j = \sum_{r=1}^{n} \xi_{jr} \mathbf{y}_r.$$

It remains to be shown that $\xi_{jr} \ge 0$ for each j and r.

Since $\mathbf{x}_1, \dots, \mathbf{x}_n$ form a basis in X, for each $i = 1, \dots, n$ there exists a (unique) vector $\mathbf{c}_i = (c_{i1}, \dots, c_{in})$ such that

$$\mathbf{e}_i = \sum_{r=1}^{n} c_{ir} \mathbf{x}_r = (\mathbf{c}_i \cdot \mathbf{y}_1, \mathbf{c}_i \cdot \mathbf{y}_2, \dots, \mathbf{c}_i \cdot \mathbf{y}_m).$$

Consequently,

$$\mathbf{c}_i \cdot \mathbf{y}_j = \mathbf{e}_i(j) = \delta_{ij} \text{ for } 1 \le j \le n \text{ and } \mathbf{c}_i \cdot \mathbf{y}_j \ge 0 \text{ for } j > n.$$

Now fix $j > n$ and let $1 \leq r \leq n$. Then

$$\xi_{jr} = \sum_{l=1}^{n} \xi_{jl}\delta_{rl} = \sum_{l=1}^{n} \xi_{jl}(\mathbf{c}_r \cdot \mathbf{y}_l) = \mathbf{c}_r \cdot \left(\sum_{l=1}^{n} \xi_{jl}\mathbf{y}_l \right) = \mathbf{c}_r \cdot \mathbf{y}_j \geq 0,$$

and the proof is finished. (Notice that the above proof can be formulated also in matrix language but at the expense of introducing some additional notation.) ■

Corollary 2.7. *If $\{m_1, \dots, m_n\}$ is a fundamental set of indices for the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$, then*

$$\mathbf{x}_1(m_r) + \mathbf{x}_2(m_r) + \cdots + \mathbf{x}_n(m_r) > 0$$

for each $r = 1, \dots, n$.

PROOF. Assume by way of contradiction that $\mathbf{x}_i(m_r) = 0$ for each $i = 1, \dots, n$. Now if $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the positive basis described in Theorem 2.6, then $1 = \mathbf{e}_r(m_r) = 0$, which is impossible, and our conclusion follows. ■

Next, we shall illustrate Theorem 2.6 with two examples.

Example 2.8. Consider the three positive vectors in $\mathbb{R}^4$ defined by

$$\mathbf{x}_1 = (1,1,1,1), \quad \mathbf{x}_2 = (1,1,2,2), \quad \text{and} \quad \mathbf{x}_3 = (0,1,1,2).$$

Clearly, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent. Also, notice that

$$\mathbf{y}_1 = (1,1,0), \quad \mathbf{y}_2 = (1,1,1), \quad \mathbf{y}_3 = (1,2,1), \quad \text{and} \quad \mathbf{y}_4 = (1,2,2).$$

There are four possible sets of fundamental indices. They are:

$$I_1 = \{1,2,3\}, \quad I_2 = \{1,2,4\}, \quad I_3 = \{1,3,4\}, \quad \text{and} \quad I_4 = \{2,3,4\}.$$

From the identities

$$\begin{aligned} \mathbf{y}_4 &= -\mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3, \\ \mathbf{y}_3 &= \mathbf{y}_1 - \mathbf{y}_2 + \mathbf{y}_4, \\ \mathbf{y}_2 &= \mathbf{y}_1 - \mathbf{y}_3 + \mathbf{y}_4, \\ \mathbf{y}_1 &= \mathbf{y}_2 + \mathbf{y}_3 - \mathbf{y}_4, \end{aligned}$$

and Definition 2.5, we see that none of the sets I_1, I_2, I_3, or I_4 is a fundamental set of indices for $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$. Consequently, the vector space generated by $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ is not a lattice-subspace of $\mathbb{R}^4$.

In connection with Theorem 2.2, it is interesting to notice that the preceding example shows that if we pick n arbitrary linearly independent vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$

from the standard S_{m-1} simplex of $\mathbb{R}^m_+$, then the set $S_{m-1} \cap [\mathbf{x}_1, \dots, \mathbf{x}_n]$ can fail to be a simplex, provided $n \geq 3$. For $n = 2$ this set is a simplex in view of the fact that in $\mathbb{R}^2$ each closed and generating cone is a lattice-cone.

Example 2.9. Consider the four vectors in $\mathbb{R}^5$ defined by

$$\mathbf{x}_1 = (4,1,2,1,2),\ \mathbf{x}_2 = (3,0,2,1,2),\ \mathbf{x}_3 = (3,1,2,1,0), \text{ and } \mathbf{x}_4 = (3,1,0,1,2).$$

Here $\mathbf{y}_1 = (4,3,3,3)$, $\mathbf{y}_2 = (1,0,1,1)$, $\mathbf{y}_3 = (2,2,2,0)$, $\mathbf{y}_4 = (1,1,1,1)$, and $\mathbf{y}_5 = (2,2,0,2)$.

We claim that $\{2,3,4,5\}$ is a fundamental set of indices for the vectors $\mathbf{x}_i$. To see this, note first that $\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ and $\mathbf{y}_5$ are linearly independent. Moreover,

$$\mathbf{y}_1 = \mathbf{y}_2 + \tfrac{1}{2}\mathbf{y}_3 + \mathbf{y}_4 + \tfrac{1}{2}\mathbf{y}_5,$$

which establishes our claim. Hence (by Theorem 2.6) $X = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4]$ is a lattice-subspace of $\mathbb{R}^5$. It should be noted that the positive basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ described in Theorem 2.6 is given by

$$\mathbf{e}_1 = (1,1,0,0,0), \quad \mathbf{e}_2 = (\tfrac{1}{2},0,1,0,0), \quad \mathbf{e}_3 = (1,0,0,1,0), \quad \mathbf{e}_4 = (\tfrac{1}{2},0,0,0,1).$$

We shall close our discussion here by mentioning that in applications (see [8]) it is important to decide whether or not a given collection of positive vectors generates a lattice-subspace. And in connection with this, notice that our results present in actuality an algorithm that can be used by a computer to determine whether or not a given set $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ of n linearly independent positive vectors of $\mathbb{R}^m$ $(n < m)$ generates a lattice-subspace. The computer can check whether a subset $\{m_1, \dots, m_n\}$ of $\{1, \dots, m\}$ is a fundamental set of indices by following the three steps below.

(1) If for some $r \in \{1, \dots, n\}$ we have $\mathbf{x}_i(m_r) = 0$ for each $i = 1, \dots, n$, then (by Corollary 2.7) $\{m_1, \dots, m_n\}$ cannot be a set of fundamental indices and we stop here. If this is not the case, then we go to the next step.

(2) At this step we check (by solving, using the computer, an appropriate linear system of n equations in n unknowns) whether or not the n vectors $\mathbf{y}_{m_1}, \dots, \mathbf{y}_{m_n}$ are linearly independent. If they are not, then the set of indices $\{m_1, \dots, m_n\}$ cannot be fundamental and we stop here. If they are independent, then we go to the next step.

(3) At this step we must solve (using the computer again) at most $m - n$ systems

$$\mathbf{y}_j = \sum_{r=1}^{n} \xi_{jr} \mathbf{y}_{m_r}, \quad j \notin \{m_1, \dots, m_n\}.$$

If all the ξ_{jr} are non-negative, then $\{m_1, \dots, m_n\}$ is a fundamental set of indices and X is a lattice-subspace.

The above algorithm can be checked by the computer with at most $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ steps. Notice that when we find a set of fundamental indices, then we can automatically construct a positive basis for X.

3. Lattice-subspaces and positive projections

When X is a lattice-subspace of a vector lattice E and $x, y \in X$, then (following [12]) we shall denote the supremum and the infimum of the set $\{x, y\}$ in X by $x \curlyvee y$ and $x \curlywedge y$ respectively. As usual, $x \vee y$ and $x \wedge y$ will denote the supremum and infimum respectively of the set $\{x, y\}$ in E. The absolute value of an element x in X will be denoted $|x|_X$.

There is a close relationship between lattice-subspaces and positive projections. Schaefer [18, proposition 11.5, p. 214] has shown that the range of a positive projection on a vector lattice is always a lattice-subspace.

Theorem 3.1 (Schaefer). *Let $P: E \to E$ be a positive projection on a vector lattice, i.e. $P \geq 0$ and $P^2 = P$. Then the range $F = P(E)$ of P satisfies the following properties.*

(1) *The vector space F is a lattice-subspace of E. Its lattice operations are given by*

$$x \curlyvee y = P(x \wedge y), \quad x \curlywedge y = P(x \wedge y), \quad \textit{and} \quad |x|_F = P(|x|).$$

(2) *If E has a strong unit, then F has a strong unit.*

(3) *If E is Dedekind (resp. σ-Dedekind) complete, then F is Dedekind (resp. σ-Dedekind) complete.*

(4) *If P is strictly positive, then F is a vector sublattice.*

(5) *If E is a Banach lattice, then the norm*

$$|||x||| = \| |x|_F \| = \|P|x|\|, \quad x \in F,$$

is a lattice norm on F. Moreover, $|||\cdot|||$ is equivalent to $\|\cdot\|$ and $(F, |||\cdot|||)$ is a Banach lattice.

Two partially ordered Banach spaces X and Y are *order-isomorphic* whenever there exists a one-to-one and onto continuous linear operator $T: X \to Y$ such that

$$x \geq 0 \quad \text{if and only if} \quad Tx \geq 0.$$

A partially ordered Banach space X is said to be *order-embeddable* into another partially ordered Banach space Y whenever there exists a (closed) vector subspace Z of Y such that X is order-isomorphic to Z when the latter is equipped with the induced ordering. In case X and Y are Banach lattices, note that X is order-embeddable in Y if and only if X is order-isomorphic to a (closed) lattice-subspace of Y. Also, let us say that a closed subspace X of a Banach lattice E is *positively complemented* whenever there exists a positive projection on E whose range is X.

The following result is essentially due to T. Andô (pers. comm.) and independently to Ghoussoub [6, lemma III.3, p. 463]. It shows that the range of a positive projection is often 'very close' to a vector sublattice.

Theorem 3.2 (Andô–Ghoussoub). *If E is a Dedekind complete vector lattice, then the range of every positive order-continuous projection on E is order-isomorphic to a positively complemented vector sublattice of E*[4].

PROOF. Let $P:E \to E$ be a positive order-continuous projection and let $X = P(E)$. Also, put $N = \{x \in E\colon P|x| = 0\}$ and note that since P is order-continuous, N is a band in E. So $E = N \oplus N^d$ holds. We shall denote by Q the band projection of E onto N^d. Notice that if $x = x_1 + x_2 \in N \oplus N^d$, then $Px = Px_2 = PQx$, and so $PQ = P$.

Now let $R = QP$. Clearly, $R \geq 0$ and

$$R^2 = (QP)(QP) = Q(PQ)P = QPP = QP = R,$$

i.e. R is a positive projection on E. Obviously, R leaves N^d invariant and is strictly positive on N^d. Indeed, if $0 < x \in N^d$ and $Rx = QPx = 0$, then $Px \in N$, and consequently $Px = P(Px) = 0$, contrary to $x \in N^d$. That is, $0 < x \in N^d$ implies $Rx > 0$. Clearly, $R(E) = R(N^d)$. So, letting $Y = R(E)$ and using Theorem 3.1(4), we see that Y is a positively complemented vector sublattice of N^d and hence of E.

Next, we claim that $R:X \to Y$ is an onto order isomorphism. To see this, note first that $x \in X$ and $Rx = 0$ imply $Qx = QPx = Rx = 0$ or $x \in N$, from which it follows that $x = Px = 0$. That is, R is one-to-one. Also, it should be obvious that R is positive and onto. To finish the proof, it remains to be shown that $R^{-1}:Y \to X$ is also positive. To this end, let $x \in X$ satisfy $0 \leq Rx = QPx = Qx$. If $x = x_1 + x_2 \in N \oplus N^d$, then $x_2 = Qx \geq 0$. So, $x = Px = Px_1 + Px_2 = Px_2 \geq 0$, and the proof is finished. ■

It is well known that not every vector sublattice of a Banach lattice is complemented. Therefore the converse of Schaefer's theorem is not true. Nevertheless, it was proven by Miyajima [12, proposition 1, p. 85] that an important partial converse holds.

Recall that if X is a vector subspace of a vector lattice E, then the vector sublattice R_X generated by X in E is given by

$$R_X = \{x \in E\colon \exists\ x_{ij} \in X,\quad i = 1, \ldots, n;\ j = 1, \ldots, m, \text{ with } x = \textstyle\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} x_{ij}\}.$$

For details see [9, p. 47]. If $P:R_X \to R_X$ is a positive projection with range X, then it follows from Schaefer's theorem that X is a lattice-subspace of E and necessarily

$$P(\textstyle\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} x_{ij}) = \mathbb{V}_{i=1}^{n} \mathbb{\Lambda}_{j=1}^{m} x_{ij}. \qquad (*)$$

In particular, this implies the uniqueness of the positive projection from R_X onto X. Conversely, if X is a lattice-subspace, then, as shown in [12], the formula in $(*)$

[4]The theorem is also true if E has only the projection property.

defines a positive projection from R_X onto X. Thus we have the following remarkable result of Miyajima.

Theorem 3.3 (Miyajima). *A vector subspace X of a vector lattice E is a lattice-subspace if and only if there is a (unique) positive projection $P: R_X \to R_X$ whose range is X. In such a case, P is defined by*

$$P(\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} x_{ij}) = \mathop{\bigvee\!\!\!\!\bigvee}_{i=1}^{n} \mathop{\bigwedge\!\!\!\!\bigwedge}_{j=1}^{m} x_{ij},$$

and is also an interval preserving lattice homomorphism.

Lemma 3.4. *Let X be a lattice-subspace of a vector lattice E such that X is a Dedekind complete vector lattice in its own right (in particular, let X be a finite-dimensional lattice-subspace of E). If A is the ideal generated by X in E, then there exists a positive projection P on A with range X.*

PROOF. Since the cone $X_+ = X \cap E_+$ is generating, we see that

$$A = \{y \in E : \exists x \in X_+ \text{ such that } -x \leq y \leq x\}.$$

Now consider the identity operator $I: X \to X$. Since X as a vector subspace of A majorises the vector lattice A and X is Dedekind complete, there exists by Kantorovich's Theorem [3, theorem 2.8, p. 26] a positive linear extension P of I to all of A. Any such positive extension has the desired properties. ∎

The converse of Schaefer's theorem is true for finite-dimensional spaces.

Theorem 3.5. *Let X be a vector subspace of some $\mathbb{R}^m$ such that the induced cone $X_+ = X \cap \mathbb{R}^m_+$ is generating. Then X is a lattice-subspace of $\mathbb{R}^m$ if and only if X is the range of a positive projection on $\mathbb{R}^m$.*

PROOF. If X is the range of a positive projection, then by Theorem 3.1 the induced cone X_+ is a lattice-cone.

For the converse, assume that X is a lattice-subspace of $\mathbb{R}^m$. By Lemma 3.4, there exists a positive projection P on A with range X, where A is the ideal generated by X in $\mathbb{R}^m$. Since, in this case, A is necessarily a band, the composition PP_A of the band projection P_A onto A and P produces the desired positive projection whose range is X. ∎

An alternate proof of Theorem 3.5 can be obtained by using Theorem 3.3 instead of Lemma 3.4. The next theorem tells us how to construct a positive projection onto a lattice-subspace of $\mathbb{R}^n$.

Theorem 3.6. *Let X be a lattice-subspace of $\mathbb{R}^m$. If a fundamental set of indices $\{m_1, \ldots, m_n\}$ for a positive basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ satisfies $\mathbf{e}_i(m_j) = \delta_{ij}$, then the operator*

$P:\mathbb{R}^m \to \mathbb{R}^m$ defined by

$$P(\mathbf{x}) = \sum_{r=1}^{n} \mathbf{x}(m_r)\mathbf{e}_r, \quad \mathbf{x} \in \mathbb{R}^m,$$

is a positive projection whose range is X.

PROOF. If $\mathbf{x} = \sum_{r=1}^{n} \alpha_r \mathbf{e}_r \in X$, then $\mathbf{x}(m_r) = \alpha_r$ for each r. The latter easily implies that P is a positive projection having range X. ■

We mention once more that a lattice-subspace F of a vector lattice E need not be a vector sublattice, and consequently two disjoint elements of F are not necessarily disjoint in E. However, as our next result shows, there is a convenient way of passing from disjointness in F to disjointness in E.

Theorem 3.7. *If a lattice-subspace F of a vector lattice E is the range of a positive projection $P:E \to E$ and $x_1, \ldots, x_n \in F_+$ are pairwise disjoint in F, then there exist pairwise disjoint elements $y_1, \ldots, y_n$ in E satisfying*

$$0 \leq y_i \leq x_i \quad and \quad Py_i = x_i$$

for each i.

PROOF. The proof is by induction on n. For $n = 1$, the result is obvious. Next, we establish it for $n = 2$. So, let $x_1, x_2 \in F_+$ satisfy $x_1 \wedge\!\!\!\wedge\, x_2 = 0$. Put $y_i = x_i - x_1 \wedge x_2$ and note that $0 \leq y_i \leq x_i$ $(i = 1, 2)$ and $y_1 \wedge y_2 = 0$ in E. Moreover, from Theorem 3.1, we see that

$$P(x_1 \wedge x_2) = x_1 \wedge\!\!\!\wedge\, x_2 = 0,$$

and so $Py_i = Px_i = x_i$ for each i.

Now, for the induction step, assume that our claim is true for some $n \geq 2$ and let $x_1, \ldots, x_n, x_{n+1} \in F_+$ be pairwise disjoint in F. Note that

$$\left(\sum_{i=1}^{n} x_i - x\right) \wedge (x_{n+1} - x) = 0, \qquad (*)$$

where $x = x_{n+1} \wedge (\sum_{i=1}^{n} x_i)$. Clearly,

$$Px = P\left(x_{n+1} \wedge \sum_{i=1}^{n} x_i\right) = x_{n+1} \wedge\!\!\!\wedge \left(\sum_{i=1}^{n} x_i\right) = 0.$$

From $0 \leq x \leq \sum_{i=1}^{n} x_i$ and the Riesz Decomposition Property, it follows that there exist elements $v_1, \ldots, v_n \in E_+$ satisfying $0 \leq v_i \leq x_i$ for each i and $x = \sum_{i=1}^{n} v_i$. In particular, from $\Sigma_{i=1}^n Pv_i = Px = 0$, it follows that $Pv_i = 0$ for each i. Now a glance at $(*)$ yields

$$\left[\sum_{i=1}^{n} (x_i - v_i)\right] \wedge (x_{n+1} - x) = 0,$$

and so $(x_i - v_i) \wedge (x_{n+1} - x) = 0$ for $i = 1, \ldots, n$. Since the n positive vectors $x_1, \ldots, x_n$, are pairwise disjoint in F, there exist (by our induction hypothesis) pairwise disjoint elements $w_1, \ldots, w_n$ in E satisfying $0 \leq w_i \leq x_i$ and $Pw_i = x_i$ for each i. Now put

$$y_i = (w_i - v_i)^+, \quad i = 1, \ldots, n, \quad \text{and} \quad y_{n+1} = x_{n+1} - x,$$

where the positive parts are taken in E. Clearly, $y_i \wedge y_j = 0$ holds in E for all $i \neq j$. Also, $w_i - v_i \leq (w_i - v_i)^+ \leq x_i - v_i$ implies

$$x_i = P(w_i - v_i) \leq P(w_i - v_i)^+ = Py_i \leq P(x_i - v_i) = x_i$$

for each i, and the desired conclusion follows. ■

4. Examples of lattice-subspaces

In this section we shall present several examples to illustrate the lattice-subspace notion. First, we shall slightly improve the result mentioned in the introduction, that the vector space $\mathscr{H}$ of all functions in $C(\overline{D})$ which are harmonic in D forms a lattice-subspace of the vector lattice $C(\overline{D})$, by proving that there exists a unique positive projection from $C(\overline{D})$ onto $\mathscr{H}$.

Recall that the classical Dirichlet problem states that if f is a continuous function on the unit circle $\partial D = \{(x, y)\colon x^2 + y^2 = 1\}$, then there exists a unique continuous extension $\hat{f}$ of f to all of $\overline{D}$ which is harmonic on D; we shall call $\hat{f}$ the *harmonic extension* of f. The harmonic extension $\hat{f}$ in polar coordinates is given by Poisson's classical formula

$$\hat{f}(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\varphi)(1 - r^2)}{1 - 2r \sin(\theta - \varphi) + r^2}\, d\varphi, \quad (r, \theta) \in D.$$

From Poisson's formula, it is easy to see that $\hat{f} \geq 0$ if and only if $f \geq 0$.

Theorem 4.1. *The vector space $\mathscr{H}$ is the range of a unique positive projection P on $C(\overline{D})$ (and hence $\mathscr{H}$ is a lattice-subspace of $C(\overline{D})$). The projection $P\colon C(\overline{D}) \to C(\overline{D})$ is given by*

$$Pf = \widehat{f|_{\partial D}},$$

where $f|_{\partial D}$ denotes the restriction of f to ∂D. Moreover, the lattice-subspace is an AM-space with unit, the constant function one, and the mapping $f \mapsto \hat{f}$, from $C(\partial D)$ onto $\mathscr{H}$, is a lattice isometry.

PROOF. The linear operator $P:C(\overline{D}) \to C(\overline{D})$ defined by $Pf = \widehat{f|_{\partial D}}$ is a positive projection whose range is $\mathscr{H}$. Therefore $\mathscr{H}$ is a lattice-subspace of $C(\overline{D})$.

To see that P is uniquely determined, notice first that $\mathbf{1} \in \mathscr{H}$ and that $\mathscr{H}$ separates the points of $\overline{D}$. For instance, let $(x_1, y_1), (x_2, y_2) \in D$ satisfy $(x_1, y_1) \neq (x_2, y_2)$. If $x_1 \neq x_2$, then the function $g(x, y) = x$ satisfies $g \in \mathscr{H}$ and $g(x_1, y_1) \neq g(x_2, y_2)$. From the Stone–Weierstrass theorem, we infer that the vector sublattice $R_{\mathscr{H}}$ generated by $\mathscr{H}$ is norm-dense in $C(\overline{D})$. Since P is (by Theorem 3.3) uniquely determined as a projection on $R_{\mathscr{H}}$ and since every positive projection on $C(\overline{D})$ is continuous, we infer that P is uniquely determined on $C(\overline{D})$.

That the operator $f \mapsto \hat{f}$, from $C(\partial D)$ onto $\mathscr{H}$, is a lattice isometry follows from Theorem 3.1 and the fact that the modulus of every function in $\mathscr{H}$ attains its maximum value on ∂D. ■

Before presenting more examples, let us establish one more case when the converse of Theorem 3.1 is true.

Lemma 4.2. *Let Ω be a compact topological space and let $x_1, \ldots, x_n \in C(\Omega)$ be positive functions such that $x_1(\omega) + \cdots + x_n(\omega) > 0$ holds for each $\omega \in \Omega$. Then the vector subspace $X = [x_1, \ldots, x_n]$ generated by $\{x_1, \ldots, x_n\}$ is a lattice-subspace if and only if X is the range of a positive projection on $C(\Omega)$.*

PROOF. The 'if' part follows immediately from Theorem 3.1. For the 'only if' part, assume that X is a lattice-subspace of $C(\Omega)$. Since the ideal generated by X is $C(\Omega)$, it follows from Lemma 3.4 that the identity operator $I:X \to X$ extends to a positive operator $T:C(\Omega) \to X$. Now note that T as an operator on $C(\Omega)$ is a positive projection whose range is X. ■

As each finite-dimensional subspace of any Banach lattice is complemented, one might be tempted to suggest that each finite-dimensional lattice-subspace is positively complemented, i.e. that the extra assumption in the previous lemma is redundant. Surprisingly, as the next example shows, this is not the case.

Example 4.3. Consider the positive functions $x_1, x_2 \in C[0, 1]$ defined by

$$x_1(t) = \begin{cases} (t - \frac{1}{2})^2 & \text{if } 0 \le t \le \frac{1}{2} \\ t - \frac{1}{2} & \text{if } \frac{1}{2} < t \le 1 \end{cases} \quad \text{and} \quad x_2(t) = \begin{cases} \frac{1}{2} - t & \text{if } 0 \le t \le \frac{1}{2} \\ (t - \frac{1}{2})^2 & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

As usual, let $X = [x_1, x_2]$ be the vector space generated in $C[0, 1]$ by the linearly independent vectors x_1 and x_2. We claim that X is a lattice-subspace of $C[0, 1]$. As a matter of fact, we claim that $\{x_1, x_2\}$ is a positive basis of X.

To this end, assume $\lambda_1 x_1 + \lambda_2 x_2 \geq 0$. We must show that $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. To see this, notice that for $0 \leq t < \frac{1}{2}$, we have $\lambda_1(t-\frac{1}{2})^2 + \lambda_2(\frac{1}{2}-t) \geq 0$, or $\lambda_1(\frac{1}{2}-t) + \lambda_2 \geq 0$. Letting $t \to \frac{1}{2}^-$, we obtain $\lambda_2 \geq 0$. Similarly, from $\lambda_1(t-\frac{1}{2}) + \lambda_2(t-\frac{1}{2})^2 \geq 0$ for each $\frac{1}{2} < t \leq 1$, we infer that $\lambda_1 \geq 0$.

Next, we claim that there is no positive projection on $C[0,1]$ whose range is X. To see this, assume by way of contradiction that there exists a positive projection $P: C[0,1] \to C[0,1]$ such that $P(C[0,1]) = X$. If $f: X \to \mathbb{R}$ is the positive linear functional defined by

$$f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1,$$

then $f \circ P$ is a positive linear functional on $C[0,1]$. So, by the Riesz Representation Theorem, there exists a (unique) regular Borel measure μ such that

$$f \circ P(x) = \int_{[0,1]} x \, d\mu,$$

for each $x \in C[0,1]$. In particular, from $0 = f \circ P(x_2) = \int_{[0,1]} x_2 \, d\mu$, we see that Supp $\mu = \{\frac{1}{2}\}$. Consequently,

$$1 = f \circ P(x_1) = \int_{[0,1]} x_1 \, d\mu = x_1(\tfrac{1}{2})\mu(\{\tfrac{1}{2}\}) = 0,$$

which is impossible. Hence, X cannot be the range of a positive projection.

Example 4.4. We consider the Banach lattice $C[0,1]$ and the three functions y_1, y_2, $y_3 \in C[0,1]$ defined by

$$y_1(x) = 1, \quad y_2(x) = x, \quad \text{and} \quad y_3(x) = x^2$$

for each $x \in [0,1]$. We let $Y = [y_1, y_2]$, the vector subspace generated by y_1 and y_2, and $Z = [y_1, y_2, y_3]$, the vector subspace generated by the vectors y_1, y_2, and y_3. We omit a straightforward verification of the following two claims.

(1) The vector space Y is a lattice-subspace of $C[0,1]$ but not a vector sublattice. Moreover, there exists a unique positive projection on $C[0,1]$ whose range is Y.

(2) The vector space Z is not a lattice-subspace.

In general, if $m \geq 2$, then the vector subspace X generated by the functions $\mathbf{1}$, x, $x^2, \ldots, x^m$ in $C[0,1]$ cannot be a lattice-subspace. To see this, assume that $\mathbf{e}_1, \ldots, \mathbf{e}_m$, $\mathbf{e}_{m+1}$ is a positive basis in X. Since for each $s \in [0,1]$ the function $u(t) = (t-s)^2$ satisfies $0 < u \in X_+$, it follows that $\mathbf{e}_i(s) = 0$ must hold for some i. Therefore, for some i the (at most mth degree) polynomial $\mathbf{e}_i$ satisfies $\mathbf{e}_i(s) = 0$ for infinitely many s, i.e. $\mathbf{e}_i = 0$, which is impossible. Consequently X cannot be a lattice-subspace in any function space that contains the functions $\mathbf{1}, x, x^2, \ldots, x^m$.

We close the paper with several remarks.

(1) It is well known that every separable Banach space is isometrically embeddable into $C[0,1]$ (and because of this property $C[0,1]$ is called a universal Banach space). It is shown in [17] that $C[0,1]$ is also lattice-universal in the sense that every separable Banach lattice E is order-embeddable in $C[0,1]$, and there is an order-embedding $T:E \to C[0,1]$ (which is not necessarily an isometry) satisfying

$$\|Tx\|_\infty \leq \|x\| \leq 2\|Tx\|_\infty$$

for each $x \in E$.

(2) While the Banach lattices c_0, l_p, and $L_p\,[0,1]$ $(1 \leq p < \infty)$ can be considered as closed lattice-subspaces of $C[0,1]$, none of them is positively complemented—they all lack strong units.

(3) It is well known that every positive linear functional defined on a vector sublattice extends to a positive linear functional on the whole space. However, as the proof of Example 4.3 shows, this is not the case if the positive linear functional is defined on a lattice-subspace. That is, a positive linear functional defined on a lattice-subspace need not extend to a positive linear functional on the whole space. The same conclusion can be deduced also from theorem 2 in [10], which gives a necessary and sufficient condition for a finite-dimensional subspace of $C(K)$ space to be positively complemented. This theorem implies as well our Lemma 4.2.

(4) It was shown in [1, theorem 1] that if $T:E \to F$ is a positive linear isometry from a normed vector lattice E into another normed vector lattice F, then $T^{-1}:T(F) \to E$ is also a positive operator. This readily implies that $T(E)$ is automatically a lattice-subspace of F. This fact gives us a rich source for obtaining lattice-subspaces.

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