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Cones and Geometry of Banach Spaces

Abstract. *This is a survey article on cones and geometry of Banach spaces. It is shown that the geometry of cones and especially the geometry of their bases (strongly exposed points, dentability and also the existence of bounded bases) are related with important properties of the whole space. Recall that a base for a cone P is an intersection of P with an affine hyperplane defined by a strictly positive linear functional. Especially in this article the above properties of bases for cones are studied and some important properties of cones in reflexive spaces and also properties of cones of dual spaces are proved. In the sequel characterizations of the positive cone of ℓ_1 and also characterizations of reflexive Banach spaces based on the above properties of cones are given. Finally it is shown that if T is an one-to-one, continuous, linear operator of an $L_1(\mu)$ space into the topological dual E' of a Banach space E, the geometry of the images of the positive cone of $L_1(\mu)$ and its subcones in E' is connected with the geometry of the space $L_1(\mu)$ and also with the geometry of the space E. In the last section an application of the geometry of cones in vector optimization is given.*

Key Words: *Bases for cones, Reflexivity, Characterizations of l_1, L_1-preduals, Vector optimization.*

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1. Introduction

One of the first results connecting cones and geometry of Banach spaces is that of D. and V. Milman, 1964, that *a Banach space is non-reflexive if and only if it does not contain the positive cone of ℓ_1*. This re-

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sult shows also the importance of characterizations of the positive cone ℓ_1^+ of ℓ_1 for the theory of Banach spaces. In this article it is shown that properties of cones as the existence of bounded bases, the existence of dentable bases and also the existence of strongly exposed points in the bases for cones, are related with the geometry of the whole space. We start with the study of the strongly exposed points. An important property for this study is the *continuous projection property*. This property of cones was defined in [29], and for any extreme point of a base for a cone, it assumes the existence of a positive, continuous, order contractive projection onto the one dimensional subspace generated by this point. It is shown that in many cases, the continuous projection property is weaker than the lattice property and also than the Riesz decomposition property. For example *if P is a closed, generating cone of a Banach space E and E ordered by the cone P has the Riesz decomposition property, then E has the continuous projection property*. Also note that although the lattice property and the Riesz decomposition property are global properties of ordered spaces, the continuous projection property can be defined only for an extreme point of a base for a cone. It is shown that in cones with the continuous projection property, *an extreme point of a base for a cone is a strongly exposed point if and only if the cone has a bounded base*. In the sequel the existence of dentable bases is studied. It is shown that *a reflexive space cannot have a closed cone with an unbounded, closed, dentable base*. This is an important property of reflexive spaces. As it is shown in Section 5, the converse is also true and the following characterization of reflexive Banach spaces in terms of the geometry of the bases of cones is given: *a Banach space E is reflexive if and only if E does not have a closed cone with an unbounded, closed, dentable base*. Also the following property of the bases for cones of dual spaces is proved: *a dual Banach space cannot have a weak-star closed cone with an unbounded, weak-closed and weak-star dentable base*. In Section 4 some characterizations of the positive cone of of ℓ_1 based on the geometry of the bases of cones are given. Especially it is shown that *if a closed cone P of a Banach space E has the Radon-Nikodým property and E, ordered by the cone P, has the the continuous projection property, the following statements are equivalent: (i) P is isomorphic to $\ell_1^+(\Gamma)$, (ii) P has a closed, bounded base, (iii) a base for P defined (the base) by a continuous linear functional, has at least one strongly exposed point, (iv) the zero is a strongly exposed point of P, (v) the cone P has a dentable base, defined by a continuous linear functional, (vi) the cone P is dentable, (vii) each closed and convex subset of P has at least one strongly exposed point.*

The next result of Section 5, shows also the importance of cones in the geometry of Banach spaces: *a Banach lattice E, is a non-reflexive KB-space if and only if the positive cone of ℓ_1 is embeddable in E_+ and the positive cone of c_0 is not embeddable in E_+*. In Section 6 it is supposed that (Ω, Σ, μ) is a measure space and T an one-to-one, linear, continuous

operator of $L_1(\mu)$ into the norm dual E' of a Banach space E. It is shown that the geometry of the images of the positive cone of $L_1(\mu)$ and its subcones in E' is closely connected with the geometry of the space $L_1(\mu)$ as well as with the geometry of E. Especially necessary conditions based on the geometry of these cones are given so that the measure μ to be purely atomic and $L_1(\mu)$ to be lattice isometric to $\ell_1(\mathcal{A})$, where $\mathcal{A}$ is the set of atoms of μ. Also a new characterization of $c_0(\Gamma)$ based on the above properties of cones is given.

In this article we will denote by E a normed space and by E' the topological dual of E. Suppose that P is a **wedge** of E, i.e. P is a convex subset of E with $\lambda x \in P$ for each $x \in P$ and each $\lambda \in \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of real numbers $\lambda \geq 0$. If $E = P - P$ the wedge P is **generating**. P defines the following, not necessarily antisymmetric, linear ordering in E: $x \leq y$ if and only if $y - x \in P$. Then we say that E is ordered by the wedge P and we denote P by E_+. A linear functional f of E is **order bounded** if it maps order intervals of E into order intervals of $\mathbb{R}$, **positive** if $f(x) \geq 0$ for each $x \in P$ and f is **strictly positive** if $f(x) > 0$ for each $x \in P, x \neq 0$. The set $P^0 = \{f \in E' | f(x) \geq 0 \text{ for each } x \in P\}$ is the **dual wedge** of P in E'. The wedge P is a **cone** if $P \cap (-P) = \{0\}$. The linear ordering defined by a cone is antisymmetric. A nonzero element x_0 of P is an **extremal point** of P if for any $x \in E$, $x_0 \geq x \geq 0$ implies that $x = \lambda x_0$ for some real number $\lambda \in \mathbb{R}_+$.

Suppose now that E is ordered by the cone P. We say that E is a **vector lattice** if for any $x, y \in E$ the supremum and infimum of $\{x, y\}$ in E exist. Then we denote by $x \vee y$ and $x \wedge y$ the supremum and infimum of $\{x, y\}$ respectively and also we denote by $|x|$ the supremum of $\{x, -x\}$. The space E has the **Riesz Decomposition Property** if for any $x, y, z \in E_+$, $x \leq y + z$ implies that $x = x_1 + x_2$ where $x_1, x_2 \in E_+$ with $x_1 \leq y$, $x_2 \leq z$. Remind that every vector lattice has the Riesz Decomposition Property but the converse is not always true. The cone P is **normal** if there exists a real number a such that: for any $x, y \in E$, $0 \leq x \leq y$ implies that $\|x\| \leq a\|y\|$. Also we say that the cone P gives an **open decomposition** in E if $U_+ - U_+$ is a neighborhood of 0, where $U_+ = U \cap P$ is the positive part of the closed unit ball U of E. Finally note that the cone P is a lattice cone if the subspace $X = P - P$, ordered by the cone P is a vector lattice.

The following are important results of the theory of ordered normed spaces. For their proof see in [15], Proposition 3.5.2, 3.5.6, 3.5.11 and Theorem 4.1.5.

THEOREM 1. *If E is complete and the cone P is closed and generating, then P gives an open decomposition of E.*

THEOREM 2. *If E is complete and the cone P is closed and generating, then each order bounded linear functional of E is continuous.*

THEOREM 3. *If the cone P gives an open decomposition of E and each increasing Cauchy sequence of P has a limit (in E), then E is complete.*

THEOREM 4. *If E is a vector lattice, the following statements are equivalent:*

(i) E is locally solid
(ii) the cone P is normal and gives an open decomposition of E.

For notions not defined here we refer to the books [26], [15], [2], [1] and [22].

2. Bases for cones

In this article we will denote by E a linear space ordered by the cone P. A subset B of P is a **base for the cone** P if a strictly positive linear functional f of E exists such that B is the intersection of the cone P with the affine hyperplane $\{x \in E \mid f(x) = 1\}$, i.e.

$$B = \{x \in P \mid f(x) = 1\}.$$

Then we say that **the base B is defined by the functional** f and it is easy to show that the base B is a convex subset of P. The following is an example of an ordered space E without strictly positive linear functionals. Therefore does not exists a base for the cone E_+.

EXAMPLE 5. Suppose that $E = c_0(\Gamma)$ where Γ is an uncountable set, ordered by the pointwise ordering. Then the set of positive linear functional of E is the positive cone $\ell_1^+(\Gamma)$ of $\ell_1(\Gamma)$. Since the support of any element of $\ell_1(\Gamma)$ is at most countable we have that the space E does not have strictly positive linear functionals. This holds because if we suppose that f is a strictly positive linear functional of E then $f_i = f(e_i) > 0$ for each i, therefore the support of f is uncountable.

In the above example the space E is not separable. In the case where the space E is separable the following result is true.

PROPOSITION 6. *If E is a separable normed space ordered by the cone P and $P^0 - P^0$ is weak star dense in E' then E has strictly positive, continuous, linear functionals.*

The proof is the following: P^0 is weak-star closed and the positive part $U_+^0 = U^0 \cap P^0$ of the closed unit ball U^0 of E' is weak-star metrizable. Since each compact metric space is separable we have that U_+^0 is separable in the weak-star topology and therefore $P^0 = \bigcup_{n=1}^{\infty} nU_+^0$ is weak-star separable. Suppose that $\{x_n'\}$ is a weak-star dense sequence of P^0. Then we can show that

$$x_0' = \sum_{i=1}^{\infty} \frac{x_n'}{2^n \|x_n'\|},$$

is a strictly positive linear functional of E as follows: since $P^0 - P^0$ is weak-star dense in E', the sequence $\{\pm x_n'\}$ is weak-star dense in E'. Also

$$0 \le x_n' = 2^n \|x_n'\| \frac{x_n'}{2^n \|x_n'\|} \le 2^n \|x_n'\| x_0',$$

therefore if we suppose that $x_0'(x) = 0$ for some $x \in P, x \neq 0$, we have that $x_n'(x) = 0$ for any n and therefore $x = 0$, contradiction. Therefore x_0' is strictly positive. We give also the following result:

PROPOSITION 7. *A point x_0 of a base B for P is an extreme point of B if and only if x_0 is an extremal point of P.*

PROOF. Suppose that x_0 is an extremal point of P and $x_0 = \lambda x + {}+(1-\lambda)y$ with $x, y \in B$. Then $0 \le \lambda x, (1-\lambda)y \le x_0$, therefore x, y are positive multiples of x_0 and by the fact that x, y are elements of B we have that $x = y = x_0$. For the converse, suppose that the base B is defined by the linear functional f, x_0 is an extreme point of B and that $0 < x \le x_0$. Then $x_0 = f(x)\frac{x}{f(x)} + f(x_0 - x)\frac{x_0 - x}{f(x_0 - x)}$ and by the fact that x_0 is an extreme point of B we have that x is a positive multiple of x_0, therefore x_0 is an extremal point of P. ∎

The next theorem characterizes a lattice cone in terms of the geometry of it's bases. For the proof see in [18] or [26].

THEOREM 8 (Choquet-Kendall). *Suppose that E is a linear space ordered by the generating cone P and suppose that B is a base for the cone P. Then E is a vector lattice if and only if B is a linearly compact simplex.*

Let E be a normed space. A linear functional f of E is **uniformly monotonic** if a real number $a > 0$ exists such that $f(x) \ge a\|x\|$ for each $x \in P$.

PROPOSITION 9. *Let E be a normed space ordered by the cone P and suppose that B is a base for P defined by the functional f. The base B is bounded if and only if the functional f is uniformly monotonic.*

PROOF. If we suppose that $\|x\| \leq M$ for each $x \in B$, then for each $x \in P, x \neq 0$ we have $\|\frac{x}{f(x)}\| \leq M$, therefore $\|x\| \leq Mf(x)$, for each $x \in P$, hence f is uniformly monotonic. For the converse suppose that $f(x) \geq \geq a\|x\|$ for each $x \in P$. Then for each $x \in B$ we have $1 = f(x) \geq a\|x\|$, therefore the base B is bounded. ■

Note also that if P is a finite dimensional closed cone then each base B for P is bounded. This holds because if we suppose that the base B is defined by the linear functional f and $x_n \in B$ with $\|x_n\| \longrightarrow \infty$, then $f(\frac{x_n}{\|x_n\|}) \longrightarrow 0$. Since the set $P \cap U$, where U is the closed unit ball of E is compact, a subsequence of $\{\frac{x_n}{\|x_n\|}\}$ exists which converges to an element x_0 of P. Then we have that $\|x_0\| = 1$ and $f(x_0) = 0$, contradiction because f is strictly positive on P.

EXAMPLE 10. Let $E = L_1[0,1]$. Then each element $y \in L_\infty^+[0,1]$ with $y(t) \geq a > 0$ for any $t \in [0,1]$ is a strictly positive linear functional of E and defines the base

$$B = \left\{ x \in E_+ \mid \int_{[0,1]} x(t)y(t)dt = 1 \right\},$$

for the cone E_+. This base is bounded because $y(x) \geq a\|x\|$ for any $x \in E_+$. Suppose that t_0 is a point of $[0,1]$. Then any $y \in C[0,1]$ with $y(t_0) = 0$ and $y(t) > 0$, for each $t \neq t_0$ is a strictly positive linear functional of E and it is easy to show that y defines an unbounded base for the cone E_+.

For the proof of the following result see in [15], Theorem 3.8.4.

PROPOSITION 11. *If E is a normed space ordered by the cone P, the following statements are equivalent:*

(*i*) *the cone P has a bounded base B with $0 \notin \overline{B}$,*
(*ii*) *the dual wedge P^0 of P in E' has interior points.*

3. Geometry of the bases for cones

In [28], [30] and [33] the geometry (dentability, extreme points) of the bases for cones is studied. From these articles we refer below some basic notions and results. Note also that some results of [28] for normal cones, have been generalized in [33] without the assumption that the cone is normal. We start with the notion of the continuous positive projection which has been defined in [30], as follows: let x_0 be an **extremal point** of P. If there exists a continuous, order contractive projection Π of E

onto the one-dimensional subspace generated by x_0 i.e. $\Pi : E \longrightarrow [x_0]$ is a continuous projection such that

$$0 \leq \Pi(x) \leq x \text{ for each } x \in P,$$

then we say that the point x_0 **has (admits) a continuous, positive projection**. Then it is easy to show that a positive continuous linear functional π of E exists such that

$$\Pi(x) = \pi(x)x_0 \text{ for each } x \in E, \text{ with } \pi(x_0) = 1.$$

In other words, if we suppose that x_0 admits a continuous, positive projection Π, then

$$E = [x_0] \oplus Y,$$

where Y is the kernel of Π and for any $x \in E$ we have:

$$x \in E_+ \text{ if and only if } \Pi(x) \in E_+ \text{ and } x - \Pi(x) \in E_+.$$

If each extremal point of P (whenever such points exist) admits a continuous positive projection then we say that E has the **continuous projection property**. The following result gives necessary conditions in order an ordered normed space to have the continuous projection property. Recall that E is a **locally solid lattice** if it is a lattice and for each $x, y \in E, |x| \leq |y|$ implies that $\|x\| \leq a\|y\|$, for some constant real number $a > 0$.

PROPOSITION 12 ([30], Proposition 3.2). *Let E be a normed space ordered by the cone P. If*

(*i*) *E is a locally solid lattice, or*
(*ii*) *E is a Banach space, the cone P is closed and generating and E has the Riesz decomposition property,*

then E has the continuous projection property.

Therefore in many cases the continuous projection property is weaker than the lattice property and also than the Riesz decomposition property. Indeed in a Banach space ordered by a closed generating cone, the Riesz decomposition property implies the continuous projection property. Also note that although the Riesz decomposition property is a global property of ordered spaces, the continuous projection property can be defined only for one extremal point of the cone. Therefore the continuous projection property is more useful for the study of the extreme points of a base for a cone. Recall also that a continuous linear functional f of E **strongly exposes** the point x in D if $x \in D \subseteq E$, $f(x) > f(y)$ for each $y \in D$

and for any sequence $\{x_n\}$ of D we have: $f(x_n) \longrightarrow f(x)$, implies that $\|x_n - x\| \longrightarrow 0$.

THEOREM 13 ([30], Proposition 3.4). *Suppose that B is a base for a cone P of a normed space E, defined (the base) by a continuous linear functional $f \in E'$ and suppose that x_0 is an extreme point of B which admits a continuous positive projection. Then we have:*

(i) *x_0 is a strongly exposed point of B if and only if there exists a uniformly monotonic, continuous linear functional of E.*
(ii) *If $\Pi(x) = \pi(x)x_0$, $x \in E$ is a continuous positive projection of E onto $[x_0]$ and h is a uniformly monotonic, continuous linear functional of E, then the functional $g = h(x_0)\pi - h$, strongly exposes the point x_0 in B with $g(x_0) = 0$.*

From the above result we have:

COROLLARY 1 ([30], Corollary 3.1). *Let E be a Banach space ordered by the closed, generating cone P and let B be a base for P and suppose that x_0 is an extreme point of B. If x_0 admits a continuous positive projection (especially if E has the Riesz decomposition property) the following statements are equivalent:*

(i) *x_0 is a strongly exposed point of B,*
(ii) *the cone P has a closed, bounded base.*

The following examples are applications of the above results in the geometry of cones.

EXAMPLE 14. (i) Suppose that $E = \ell_1$ and that ℓ_1^+ is the positive cone of ℓ_1. It is easy to see that each element e_n of the usual Schauder basis $\{e_n\}$ of ℓ_1 is an extremal point of ℓ_1^+ which admits the continuous, positive projection

$$\Pi_n(x) = \pi_n(x)e_n, \text{ where } \pi_n(x) = x_n \text{ for each } x = (x_1, x_2, \dots) \in \ell_1.$$

Suppose that $y \in \ell_\infty$ with $y_n = \frac{1}{n}$, for each n. Then y defines the base

$$B = \{x \in \ell_1^+ \mid y(x) = 1\},$$

for the cone ℓ_1^+ and we remark that B is unbounded because $ne_n \in B$ for each $n \in \mathbb{N}$. Also the element $h \in \ell_\infty$ with $h_i = 1$ for each i is a uniformly monotonic, continuous linear functional of ℓ_1, therefore by the previous theorem each extreme point ne_n of B is a strongly exposed point of B and also the functional $g_n = h(ne_n)\pi_n - h$ strongly exposes the point ne_n in B with $g(ne_n) = 0$. Especially for the point e_1 we have that the functional $g_1 = (0, -1, -1, -1, \dots)$ strongly exposes e_1 with $g(e_1) = 0$. (ii) Suppose that $E = \ell_p$ with $1 < p < +\infty$, ℓ_p^+ is the positive cone of ℓ_p and $\{e_n\}$ is the usual Schauder basis of E. Then $B = \{x \in \ell_p^+ \mid f(x) = 1\}$, is the base for ℓ_p^+ which is defined by the linear functional $f = (f_1, f_2, \dots) \in \ell_q$. Since f is strictly positive we have that $f_i > 0$ for each i. Also $\frac{e_n}{f_n}$ is an extreme point of B which admits a continuous positive projection and by the above theorem $\frac{e_n}{f_n}$ is a strongly exposed point of B if and only if there exists a uniformly monotonic and continuous linear functional of E. If we suppose that h is a uniformly monotonic and continuous linear functional of E with $h(x) \geq a\|x\|$ for each $x \in E_+$, then $h_n = h(e_n) \geq a\|e_n\| = a$ for each n, therefore $a = 0$, contradiction. So we have that E does not have a uniformly monotonic and continuous linear functional, therefore the cone ℓ_p^+ does not have a bounded base and $\frac{e_n}{f_n}$ is not a strongly exposed point of B. Hence the base B does not have strongly exposed points.

We continue now with the results of [31] and [33]. We start with a general definition of dentable sets.

Suppose that $\langle E, F\rangle$ is a dual system where E is a normed space and suppose also that $K \subseteq E$. Denote by $\overline{K}$, $\overline{K}^{\sigma(E,F)}$ the closure of K in the norm, $\sigma(E,F)$-topology of E, respectively. Also denote by co$K$ the convex hull of K, and by $\overline{\text{co}}K$, $\overline{\text{co}}^{\sigma(E,F)}K$ the closed convex hull of K in the norm, $\sigma(E,F)$-topology of E, respectively. The set K is **F-dentable** if for each real number $\epsilon > 0$ there exists $y \in K$ (depending on ϵ) such that the $\sigma(E,F)$-closed convex hull of the set $\{x \in K \mid \|y - x\| \geq \epsilon\}$ does not contain y, i.e. $y \notin \overline{\text{co}}^{\sigma(E,F)}\{x \in K \mid \|y - x\| \geq \epsilon\}$. A point x_0 of K is an **F-strongly exposed point** of K if there exists $f \in F$ such that:

(i) $\langle x_0, f\rangle > \langle x, f\rangle$ for each $x \in K$, $x \neq x_0$, and
(ii) for each sequence $\{x_\nu\}$ of K, $\lim_{\nu\to\infty}\langle x_\nu, f\rangle = \langle x_0, f\rangle$ implies that $\lim_{\nu\to\infty}\|x_\nu - x_0\| = 0$.

Then we say that f **F-strongly exposes** x_0 in K. If $F = E'$ and $\langle x, f\rangle = f(x)$ for each $x \in E$ and each $f \in E'$ then, instead to say that the set K is E'-dentable or that x_0 is an E'-strongly exposed point of K, we say that K is **dentable** and that x_0 is a **strongly exposed point** of K, respectively. Note that if a subset K of E is not F-dentable then K does not have F-strongly exposed points. Indeed if we suppose that f, F-strongly exposes the point x_0 of K then for any real number $\varepsilon > 0$, f separates x_0 and the set $\{x \in K \mid \|x - x_0\| \geq \varepsilon\}$ which is impossible because the set K is not F-dentable. Consider the dual system $\langle E', E\rangle$ with $\langle f, x\rangle = f(x)$, for each $f \in E'$ and each $x \in E$. Then instead to say that a subset K of E' is E-dentable we say that K is **weak-star dentable** and instead to say that a point x_0 of K is an E-stronly exposed point we say that x_0 is a **weak-star stronly exposed point**. Therefore an element $x_0' \in K$ is a **weak-star strongly exposed point** of K if there exists $x \in E$ which, as a linear functional on E, strongly exposes the point x_0' in K.

For a further study of the geometry of convex sets (dentability, extreme points) we refer to the book of J. Diestel and J. J. Uhl, [8]. In [28], the geometry of the bases of cones is studied where for any subset K of E and for any real number $\rho > 0$ they are denoted by K_ρ and $K_{S,\rho}$ the following subsets of K:

$$K_\rho = \{x \in K \mid \|x\| \le \rho\}, \quad K_{S,\rho} = \{x \in K \mid \|x\| = \rho\},$$

whenever these sets are nonempty.

THEOREM 15 ([28], Theorem 1). *Let E be a Banach space ordered by the closed, normal cone P and let B be a base for P. If K is an unbounded, convex subset of B and for each ρ the set K_ρ is weakly compact, then*

(*i*) $K_\rho = \overline{\mathrm{co}} K_{S,\rho} = \overline{K_{S,\rho}}^{\sigma(E,E')}$, *for each ρ.*
(*ii*) *The set K is non-dentable (therefore K does not have strongly exposed points).*

From this result we obtain the following important property of reflexive spaces:

COROLLARY 2. *A reflexive Banach space E does not have a closed, normal cone with an unbounded, closed, dentable base.*

In 2001, the above result was generalized for non normal cones as follows:

THEOREM 16 ([33], Theorem 1). *Suppose that $\langle E, F\rangle$ is a dual system, where E is a normed space, and suppose that P is a $\sigma(E,F)$-closed cone of E.*
If we suppose that B is a base for the cone P and K a convex, unbounded subset of B, then the following statements hold:

(*i*) *If the set $\overline{K}_{S,\rho}^{\sigma(E,F)}$ is a $\sigma(E,F)$-compact subset of B, then*

$$K_\rho \subseteq \overline{K}_{S,\rho}^{\sigma(E,F)};$$

(*ii*) *If $K_\rho \subseteq \overline{\mathrm{co}}^{\sigma(E,F)} K_{S,\rho}$ for each ρ, the set K is not F-dentable.*

From the above result we have the following corollaries:

COROLLARY 3 ([33], Corollary 1). *Let P be a closed cone of a reflexive Banach space E and suppose that B is an unbounded base for the cone P. Then each unbounded, closed and convex subset K of B is not dentable.*

COROLLARY 4 ([33], Corollary 2). *Let P be a weak-star closed cone of the dual E' of a normed space E and let B be a base for the cone P. Then each unbounded, weak-star closed and convex subset K of B is not weak-star dentable.*

EXAMPLE 17. Suppose that $E = c_0$ and suppose that $B = \{x \in \ell_1^+ \mid y(x) = 1\}$ is the base for the positive cone ℓ_1^+ of E' defined by the linear functional $y = (\frac{1}{n}) \in \ell_\infty$. The cone ℓ_1^+ is weak-star closed and the base B is unbounded and weak-star closed, therefore by the previous theorem, B is not weak-star dentable, therefore B does not have weak-star strongly exposed points. As we have remarked in Example 14, ne_n is a strongly exposed point of B for each n, therefore ne_n is strongly exposed by an element of $\ell_\infty \setminus c_0$. In fact as we have shown in Example 14, the point $e_1 \in B$ is strongly exposed by the linear functional $g = (0, -1, -1, -1, \dots) \in \ell_\infty \setminus c_0$.

4. Cones isomorphic to the positive cone of ℓ_1

Suppose that E, X are normed spaces ordered by the cones P, Q respectively. The linear operator $T : E \longrightarrow X$ is an **order-isomorphism** of E into X if T is an isomorphism of E into X and for any $x \in E$ we have: $x \in P$ if and only if $T(x) \in Q$. We say that the cone P is **isomorphic (or locally-isomorphic)** to the cone Q of X if there exists an additive, positively homogeneous, one-to-one, map T of P onto Q such that T and T^{-1} are continuous in the induced topologies. Then we say also that the cone P is **embeddable** in X and that T is an **isomorphism** of P onto Q. Suppose that T is an isomorphism of P onto Q. If we suppose that E, X are Banach spaces, the cone P gives an open decomposition of E and that $P - P$ is a closed subspace of E and also that $Q - Q$ is a closed subspace of F, then T is an order-isomorphism of $P - P$ onto $Q - Q$. So in order to show that a Banach lattice E is order-isomorphic to ℓ_1 it is enough to show that the positive cone E_+ of E is order-isomorphic to the positive cone of ℓ_1. The next theorem is a result of this type. For the proof see in [37], II, Corollary 10.1.

THEOREM 18. *A bounded Schauder basis $\{e_n\}$ of a Banach space E is equivalent to the unit vector basis of l_1 if and only if the positive cone $P = \{\sum_{i=1}^{\infty} \lambda_i e_i \in E \mid \lambda_i \ge 0, \text{for each } i\}$ of the basis $\{e_n\}$ is generating and the cone P has a bounded base.*

The following is the main result of [27]. The existence of a bounded base for the positive cone of E is the crucial hypothesis in this theorem. Note also that the existence of a bounded base is a crucial property for the characterizations of the positive cone of ℓ_1.

THEOREM 19 ([27], Theorem 1). *An infinite-dimensional separable Banach lattice E is order-isomorphic to ℓ_1 if and only if E has the Krein-Milman property and E_+ has a bounded base.*

The following characterizations of the positive cone of ℓ_1 are based on existence of a bounded base for the cone and also on the results of [30], where the geometry of the bases for cones is studied. These results can be considered also as characterizations of the Banach lattice ℓ_1 because as we have remarked above a Banach space E ordered by a generating cone P is order-isomorphic to ℓ_1 if and only if P is isomorphic to ℓ_1^+.

THEOREM 20 ([29], Proposition 4.1). *Let E be a Banach space ordered by the infinite dimensional, closed cone P and let E have the continuous projection property. If the cone P has the Krein-Milman property, statements (i), (ii), (iii) and (iv) are equivalent. If P has the Radon-Nikodým property, all the following statements are equivalent:*

(i) the cone P is isomorphic to $\ell_1^+(\Gamma)$,
(ii) the cone P has a closed, bounded base,
(iii) a base for P defined (the base) by a continuous linear functional, has at least one strongly exposed point,
(iv) the zero is a strongly exposed point of P,
(v) the cone P has a dentable base, defined by a continuous linear functional,
(vi) the cone P is dentable,
(vii) each closed and convex subset of P has at least one strongly exposed point.

It is easy to show that if the cone P is isomorphic to ℓ_1^+, then P is a lattice cone and that P has the Radon-Nikoýme property.

The following is an important property of the theory of cones.

PROPOSITION 21. *The positive cone of ℓ_1 is embeddable in the positive cone of c_0.*

The proof is the following: suppose that $\{e_n\}$ is the usual Shauder basis of c_0 and consider the summing basis $\{b_n = \sum_{i=1}^n e_i\}$ of c_0. Recall that for each $x = (x_i) \in c_0$ and each n we have

$$\sum_{i=1}^n (x_i - x_{i+1})b_i = \sum_{i=1}^n x_i e_i - x_{n+1}b_{n+1}.$$

The positive cone

$$C = \left\{ \sum_{i=1}^{\infty} \lambda_i b_i \in c_0 \mid \lambda_i \in \mathbb{R}_+, \text{ for each } i \right\},$$

of the basis $\{b_n\}$, is isomorphic to the positive cone of ℓ_1, see in [31].

5. Bases for cones and reflexivity

The next result was proved in 1964 and it is one of the first applications of cones in the geometry of Banach spaces. For it's proof see in [23], Theorem 2.9.

THEOREM 22 (D. and V. Milman). *A Banach space E is non-reflexive if and only if the positive cone of ℓ_1, is embeddable in E.*

In 1984 it was proved in [28], that each closed, unbounded base for a closed, normal cone of a reflexive Banach space is non-dentable, therefore it was proved that reflexivity is related with the geometry (dentability) of the unbounded, closed bases for cones. Later, in 1987, V. M. Kadets proved that the converse of the previous result is also true. Especially he proved that the positive cone of ℓ_1 has a closed, unbounded base which is not dentable, therefore by the theorem of Milman it follows that each non-reflexive Banach space, has a closed, normal cone with an unbounded, closed, dentable base. (This follows also immediately by Theorem 20 of the present article.) So Kadets stated in [16] the following characterization of non-reflexive Banach spaces:

THEOREM 23. *A Banach space E is non-reflexive if and only if E has a closed, normal cone with an unbounded, closed, dentable base.*

Recently it was also proved in [33], that each closed, unbounded base for a closed cone of a reflexive Banach space is non-dentable, and the following characterization of reflexivity is obtained:

THEOREM 24 ([33], Theorem 7). *A Banach space E is non-reflexive if and only if E has a closed cone with an unbounded, closed, dentable base.*

In 2001 Jing Hui Qiu, [36], proved the following characterization of reflexive spaces:

THEOREM 25. *A Banach space E is reflexive if and only if for each cone Q of the norm dual E' of E with a closed, bounded base, the dual wedge $Q_0 = \{x \in E \mid x'(x) \geq 0$ for each $x' \in Q\}$ of Q in E has interior points.*

In 1968 G. Lozanovskii proved the following characterization of reflexive Banach lattices in terms of the lattice-embeddability of c_0 and ℓ_1 in the space.

THEOREM 26 ([21], G. Ya. Lozanovskii). *A Banach lattice E is reflexive if and only if neither ℓ_1 nor c_0 are lattice embeddable in E.*

The next result was stated in [31] with the extra assumption in statement (iii) that the cone is normal and in its present form in [33]. Its proof is based on the fact that each closed, unbounded base for a closed cone of a reflexive Banach space is non-dentable and on Lozanovskii's theorem.

THEOREM 27 ([33], Theorem 9). *For a Banach lattice E, the following statements are equivalent:*

(*i*) *E is non-reflexive,*
(*ii*) *ℓ_1^+ is embeddable in E_+,*
(*iii*) *E_+ contains a closed cone with an unbounded, closed, dentable base.*

Also in [31] the following characterization of non-reflexive KB-spaces in terms of the embeddability of the positive cones of ℓ_1 and c_0 is proved. Recall that a Banach lattice E is a KB-spaces if each increasing, norm bounded sequence of E is norm-convergent.

THEOREM 28 ([31], Theorem 3). *A Banach lattice E, is a non-reflexive KB-space if and only if the positive cone ℓ_1^+ of ℓ_1 is embeddable in E_+ and the positive cone c_0^+ of c_0 is not embeddable in E_+.*

As the first result of characterization of reflexive Banach lattices in terms of the embeddability of ℓ_1 and c_0 can be considered that of James, [14], in which it is proved that a Banach space E with an unconditional basis is reflexive if and only if neither ℓ_1 nor c_0 are embeddable in E. Note that any Banach space E with an unconditional basis, ordered by the positive cone of the basis, is a Banach lattice with respect to an equivalent norm. The proof of this result follows by [37], Theorem 16.3 and by [15], Theorem 3.5.2 and Theorem 4.1.5. Kalton, [17], proved the following significant generalization of the result of James: a complete barrelled space E with an unconditional basis is reflexive if and only if neither ℓ_1 nor c_0 are contained in E as complemented subspaces.

S. Dias and A. Fernandez proved in [10] that in Banach lattices, Lozanowskii's theorem is true with the extra assumption that one of the spaces ℓ_1 and c_0 is not embeddable in E as a sublattice and the other is not contained in E as a complemented subspace.

6. Geometry of cones and embeddability of $L_1(\mu)$ in dual spaces

Suppose that (Ω, Σ, μ) is a measure space and that T is an one-to-one, linear, continuous operator of $L_1(\mu)$ into the dual E' of a Banach space E. In this section it is shown that the geometry of the images of the positive cone of $L_1(\mu)$ and its subcones in E' affect drastically the geometry of the space $L_1(\mu)$ as well as the geometry of E. Especially for each measurable subset A of Ω which is not the union of a finite number

of atoms we study the geometry of the cone $T(L_1^+(\mu_A))$ of E', where μ_A is the restriction of μ on A and it is shown that the geometry of these cones is closely connected with the geometry of the space $L_1(\mu)$ and also with the geometry of E. The results of this section can be found in [34]. Note also that the basic idea and also the basic proof of [34] are based on the following property (Corollary 4 of the present article) of cones in dual spaces: *any weak-star closed cone of a dual space cannot have a norm-unbounded, weak-star closed and weak-star dentable base.* For each infinitely decomposable subset A of Ω (i.e. for any measurable subset A of Ω which is not the union of a finite number of atoms) we denote by $Q(A)$ the weak-star closure of the cone $T(L_1^+(\mu_A))$, where μ_A is the restriction of μ on A. The basic hypothesis of [34] is the following: for each infinitely decomposable subset A of Ω, we suppose that a measurable subset D of A and an element $y \in E$ exist such that the element y, as a linear functional on E', defines an unbounded base for the cone $Q(D)$. In other words, for each infinitely decomposable subset A of Ω, a measurable subset D of A and an element $y \in E$ exist such that $y(x') > 0$ for each $x' \in Q(D)$ with $x' \neq 0$ and the set $\{x' \in Q(D) : y(x') = 1\}$ is unbounded.

THEOREM 29 ([34], Theorem 12 and Corollary 13). *Let T be an one-to-one, continuous, linear operator of $L_1(\mu)$ into the norm dual E' of a Banach space E. If the weak-star closure of the set $T(L_1^+(\mu))$ is a cone and for each measurable subset A of Ω which is not the union of a finite number of atoms, there are a measurable subset D of A and an element $y \in E$ such that the element y defines an unbounded base for the cone $Q(D)$, then the following statements are true:*

(*i*) *$L_1(\mu)$ is lattice isometric to $\ell_1(\mathcal{A})$, where $\mathcal{A}$ is the set of equivalence classes of atoms of μ and the image $T'(E)$ of E via the adjoint $T' :$ $: E'' \longrightarrow L_\infty(\mu)$ of T is a subset of $c_0(\mathcal{A})$.*
(*ii*) *If moreover T is an isomorphism of $L_1(\mu)$ into E' and the range $T(L_1(\mu))$ of T is weak-star dense in E', then the adjoint T' of T is an isomorphism of E onto $c_0(\mathcal{A})$.*

Recall that the empeddability of $L_1(\mu)$ in dual spaces is an old problem of Functional Analysis. In 1938, Gelfand [12], proved that $L_1[0,1]$ is not isomorphic to a conjugate Banach space and in 1959 Dieudonné [9], raised the problem: *characterize the $L_1(\mu)$ spaces which are isomorphic to a dual Banach space.* In 1961 Pelczynski, [24], proved that $L_1[0,1]$ is not isomorphic to a subspace of a separable, dual Banach space and in the sequel the problem was studied by many authors. We refer to the papers of Pelczynski [25], Lewis and Stegall [20], Stegall [38], Fonf [11], Bourgain and Delbaen [7], but we can refer also to many other significant works. For an extensively study of L_1-predual spaces we refer to the book of Lacey [19]. In 1981, Bourgain and Delbaen [7], gave an example of a

Banach space E whose dual E' is isomorphic to ℓ_1 but E does not contain any copy of c_0. Moreover E is a separable, $\mathcal{L}_\infty$ space with the Radon-Nikodým property and E is somewhat reflexive. So the Banach spaces whose dual is isomorphic to ℓ_1 seems to be a big class of spaces and an interesting problem is the characterization of c_0 among the elements of this class of spaces.

As a corollary of Theorem 29 we obtain the following characterization of $c_0(\Gamma)$:

THEOREM 30 ([34], Corollary 14). *An infinite-dimensional Banach space E is isomorphic to $c_0(\Gamma)$ if and only if there exists an isomorphism T of $\ell_1(\Gamma)$ into the dual space E' of E with weak-star dense range in E' such that: the weak-star closure of $T(\ell_1^+(\Gamma))$ is a cone and for each infinite subset A of Γ, there are a countable subset D of A and an element $y \in E$ such that the element y defines an unbounded base for the weak-star closure of the cone $\{T(\xi\chi_D) \mid \xi \in \ell_1^+(\Gamma)\}$, where χ_D is the characteristic function of D.*

The following theorem is proved in [32]. The basic steep for it's proof is the existence of a positive Schauder basis in the dual E' of E. Also note that the methods and the results of [32] are quite different from the methodology and the results of [34]. Therefore the next theorem and the previous one are quite different.

THEOREM 31 ([32], Theorem 1). *An ordered Banach space E is order isomorphic to c_0 if and only if E is a σ-Dedekind complete vector lattice and its dual E' is order isomorphic to ℓ_1.*

7. Bases for cones and Vector optimization

The geometry of cones and mainly the geometry of their bases has significant applications in Vector optimization. For a further study of this theory we refer to the articles [4], [5], [6], [13], [35], [3] and also to the references of these papers. Suppose that C is a subset of a normed space E and suppose that E is ordered by the cone P and that $x_0 \in C$. We say that x_0 is a **Pareto efficient point** of C with respect to the cone P if $(C - x_0) \cap (-P)) = \{0\}$, x_0 is a **positive proper efficient** point of C (with respect to the cone P) if a strictly positive, continuous linear functional f of E exists such that $f(x) \geq f(x_0)$ for each $x \in C$ and we say also that x_0 is a **proper Pareto efficient point** of C with respect to the cone P if $[\overline{\text{cone}}(C - x_0)] \cap (-P)) = \{0\}$, where $\overline{\text{cone}}(C - x_0)$ is the closure of the set $\{\lambda(x - x_0) | \lambda \in \mathbb{R}_+, x \in C\}$. We say also that x_0 is a **super efficient point** of C with respect to the cone P if and only if there exists a real number $a > 0$ such that $[\overline{\text{cone}}(C - x_0)] \cap (U - P) \subseteq aU$, where U is the closed unit ball of E. Denote by $E(C, P)$, $posE(C, P)$, $PE(C, P)$, and by $SE(C, P)$, the set of Pareto efficient, positive proper efficient, proper efficient and super efficient points of C respectively. The proof of existence and density results for these points are the main problems of this subject.

The geometry of the cone P and especially the existence of a bounded base for the cone P is very important for this theory as it is shown in the next results.

THEOREM 32 ([6], Theorem 2.4). *If E is a normed space ordered by the cone P and the cone P has a bounded base, then each weakly compact subset of E has strongly efficient points.*

THEOREM 33 ([6], Theorem 2.7 and Theorem 4.2). *Let E be a Banach space ordered by the* cone P. *Suppose that the* cone P *has a closed, bounded base B and that C is a subset of E.*

(*i*) *If C is closed and bounded, then E has super efficient points.*
(*i*) *If the set C is weakly compact, the set of super efficient points of C is norm-dense in the set of efficient points of C.*

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