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Lattice Banach spaces, order-isomorphic to l_1

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Introduction. It is known, (see [7], theorem 4 or [8], corollary II, 10·1), that a positive generated, ordered Banach space X is order-isomorphic to l_1 iff X has a Schauder basis which generates its positive cone X_+ and X_+ has a bounded base.

In this paper we prove that the existence of a bounded base of the positive cone and the validity of the Krein–Milman property in an infinite-dimensional, separable, locally solid lattice Banach space X, ensures the existence of a Schauder basis of X which generates the positive cone X_+, hence X is order-isomorphic to l_1.

Notation. We denote by $\mathbb{R}_+$ the set of non-negative real numbers. Let X be a (partially) ordered linear topological space. We denote by X_+ its positive cone. X is positive generated if $X = X_+ - X_+$.

We say that X_+ gives an open decomposition of X if for each neighbourhood U of 0, the set $X_+ \cap U - X_+ \cap U$ is also a neighbourhood of 0.

A convex subset of X_+ is called a base for the cone X_+, if $0 \notin B$ and for each $x \in X_+ \backslash \{0\}$, there exists a unique real number $\lambda > 0$, such that $\lambda x \in B$.

We say that a Schauder basis $(e_i)_{i\in\mathbb{N}}$ of X, generates the positive cone X_+, if $X_+ = \{\Sigma_{i=1}^{\infty} \lambda_i e_i \in X \mid \lambda_i \geqslant 0\}$. If X is a lattice normed space then X is locally solid iff there exists $\alpha \in \mathbb{R}_+$, such that for each $x, y \in X$, $|x| \leqslant |y|$ implies $\alpha\|x\| \leqslant \|y\|$. X is order-isomorphic to an ordered linear topological space Y, if there exists an isomorphism T of X on to Y and T, T^{-1} are positive.

A Banach space X has the Krein–Milman property, if each closed, convex and bounded subset A of X is the closed convex hull of its extreme points ($A = \overline{\text{co}}\,\text{ep}\,A$). It is known [6] that in Banach spaces the Radon–Nikodým property implies the Krein–Milman property. Moreover we know [1] that in locally solid lattice Banach spaces the Radon–Nikodým and the Krein–Milman property are equivalent.

It is also known that l_1 has the Krein–Milman property. If X and Y are isomorphic Banach spaces then X has the Krein–Milman property iff Y has it.

THEOREM 1. *Let X be an infinite-dimensional, separable, locally solid lattice Banach space. Then the following statements are equivalent*

(i) *X is order-isomorphic to l_1.*

(ii) *X has the Krein–Milman property and X_+ has a bounded base.*

Proof (i) ⇒ (ii). Let T be an order isomorphism of l_1 on to X. Then X has the Krein–Milman property. The set $B = \{x \in l_1^+ \mid \|x\| = 1\}$, is a bounded base of l_1^+, hence $T(B)$ is a bounded base of X_+.

Proof (ii) ⇒ (i). Let B be a bounded base for the cone X_+. Since X is locally solid we have that the cone X_+ is closed. By ([3], 3·8·1), it follows that the base B is closed. So B is a closed, convex and bounded subset of X, hence

$$B = \overline{\text{co}}\,\text{ep}\,(B). \tag{1}$$

Since B is closed and bounded and $0 \notin B$, it follows that there exist $M_1, M_2 \in \mathbb{R}_+ \backslash \{0\}$ such that

$$M_1 \leqslant \|x\| \leqslant M_2, \quad \text{for each} \quad x \in B. \tag{2}$$

The extreme points of B are extremal points of X_+, hence they are pairwise disjoint. So for each $x, y \in \operatorname{ep}(B)$ we have $x \leqslant x+y = |x-y|$. Since X is locally solid we have $\alpha \|x\| \leqslant \|x-y\|$, hence $\|x-y\| \geqslant \alpha M_1$.

Now, from the separability of X, it follows that the set of extreme points of B is countable. (See also [**9**], proposition 3.) We assume that

$$\operatorname{ep}(B) = \{u_i | i \in \mathbb{N}\}.$$

We shall prove that $(u_i)_{i \in \mathbb{N}}$, is a Schauder basis of X which generates X_+, where by ([**8**], II corollary 10·1), it follows that X is order-isomorphic to l_1.

Let $x \in B$. Then $0 \leqslant x \wedge u_i \leqslant u_i$, so for each $i \in \mathbb{N}$ there exists $x(i) \in \mathbb{R}_+$ such that $x \wedge u_i = x(i)\, u_i$. We shall prove that

$$x = \sum_{i=1}^{\infty} x(i)\, u_i.$$

For each $n \in \mathbb{N}$ we have

$$0 \leqslant \sum_{i=1}^{n} x(i)\, u_i = x(1)\, u_1 \vee x(2)\, u_2 \vee \ldots \vee x(n)\, u_n \leqslant x. \tag{3}$$

Since u_i, x belongs to the base B of X_+, we have

$$\sum_{i=1}^{n} x(i) \leqslant 1, \quad \text{for each } n \in \mathbb{N}.$$

But $\|u_i\| \leqslant M_2$, for each $i \in \mathbb{N}$, so the series $\sum_{i=1}^{\infty} x(i)\, u_i$ converges to a point y of X_+ and from (3) it follows that

$$0 \leqslant y = \sum_{i=1}^{\infty} x(i)\, u_i \leqslant x.$$

On the other hand, from (1) we have that there exists a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ which converges to x and $x_\nu \in \operatorname{co}\{u_i | i \in \mathbb{N}\}$ for each $\nu \in \mathbb{N}$. So each element of this sequence has the form

$$x_\nu = \sum_{i \in \Phi(\nu)} \lambda_i(\nu)\, u_i,$$

where $\Phi(\nu)$ is a finite subset of $\mathbb{N}$ and $\lambda_i(\nu) \in \mathbb{R}_+$ such that

$$\sum_{i \in \Phi(\nu)} \lambda_i(\nu) = 1.$$

So we have

$$x \wedge x_\nu = x \wedge \sum_{i \in \Phi(\nu)} \lambda_i(\nu)\, u_i \leqslant \sum_{i \in \Phi(\nu)} x \wedge \lambda_i(\nu)\, u_i \leqslant \sum_{i \in \Phi(\nu)} x \wedge u_i = \sum_{i \in \Phi(\nu)} x(i)\, u_i \leqslant y.$$

Hence

$$x \wedge x_\nu \leqslant y. \tag{4}$$

Because X is locally solid, we have that the operator $(z, w) \to z \wedge w$, from $X \times X$ onto X, is uniformly continuous, hence from (4) we have that $x = \lim (x \wedge x_\nu) \leqslant y$. So $x = y$.

So we have proved that each element x of B has the form

$$x = \sum_{i=1}^{\infty} x(i)\, u_i.$$

From the definition of $x(i)$, $i \in \mathbb{N}$, we have that they are uniquely determinate. Since X is a linear lattice and the cone X_+ is closed, it follows that $(u_i)_{i \in \mathbb{N}}$ is a Schauder basis of X which generates X_+.

Corollary 1. *Let X be an infinite-dimensional, separable, locally solid lattice Banach space. If X_+ has a bounded base, then the following statements are equivalent*

(i) *X has the Krein–Milman property.*

(ii) *X has a Schauder basis which generates X_+.*

Proposition 1. *Let E be a Banach space with the Krein–Milman property and $S \subseteq E$ bean infinite-dimensional, closed, bounded andlinearly compact simplex. If $C = \{\lambda x | x \in S$ and $\lambda \in \mathbb{R}_+\}$ and $X = C - C$ is separable, then the following statements are equivalent:*

(i) *X is closed, the set of extreme points of S is countable* ($\operatorname{ep}(S) = \{u_i | i \in \mathbb{N}\}$) *and each element x of S is a unique (infinite) convex combination of its extreme points* ($x = \sum_{i=1}^{\infty} \lambda_i u_i$, *where* $\lambda_i \in \mathbb{R}_+$ *and* $\sum_{i=1}^{\infty} \lambda_i = 1$).

(ii) *The cone C gives an open decomposition of X.*

(iii) *The set* $\operatorname{co}(S \cup (-S))$, *contains a neighbourhood of zero.*

Proof (i) ⇒ (ii). The set of extreme points of S, $(u_i)_{i \in \mathbb{N}}$, is a Schauder basis of X which generates its positive cone and the set S is a bounded base for the cone C. Hence X is order-isomorphic to l_1. So the cone C gives an open decomposition of X.

Proof (ii) ⇒ (i). By [**4**], X is a linear lattice. The cone C gives an open decomposition of X and S is a bounded base for the cone C, hence by ([**3**], 3·8·2 and 4·1·5), we have that X is locally solid. Because S is a closed and bounded base of C we have that the cone C is closed. X is closed in E because C is complete and gives an open decomposition of X, hence X has the Krein–Milman property. So, by Theorem 1, we have that X is order-isomorphic to l_1 and the statement (i) is true.

Proof (ii) ⇒ (iii). Let $\rho \in \mathbb{R}_+ \backslash \{0\}$ such that for each $x \in S$ $\|x\| > 2\rho$. If

$$U_\rho^+ = \{x \in X_+ \,|\, \|x\| \leqslant \rho\},$$

then $U_\rho^+ - U_\rho^+$ is a neigbourhood of zero. Let $x \in U_\rho^+ - U_\rho^+$. Then there exist $y, z \in U_\rho^+$ such that $x = y - z$. We put $2y = \lambda_1 b_1$, $2z = \lambda_2 b_2$, where $b_1, b_2 \in S$. Since $\|b_1\|, \|b_2\| > 2\rho$ we have that $0 < \lambda_1, \lambda_2 < 1$. So we have:

$$x = \frac{2y}{2} - \frac{2z}{2} = \frac{\lambda_1}{2} b_1 + \frac{\lambda_2}{2}(-b_2) = \left(\frac{1}{2} + \frac{\lambda_1}{4} - \frac{\lambda_2}{4}\right) b_1 + \frac{\lambda_2}{2}(-b_2)$$
$$+ \left(\frac{1}{2} - \frac{\lambda_1}{4} - \frac{\lambda_2}{4}\right)(-b_1) \in \operatorname{co}(S \cup (-S)).$$

Hence $U_\rho^+ - U_\rho^+ \subseteq \operatorname{co}(S \cup (-S))$.

Proof (iii) ⇒ (ii). Let $M \in \mathbb{R}_+$ such that for each $x \in S$ implies $\|x\| < M$. Then we have $\operatorname{co}(S \cup (-S)) \subseteq \operatorname{co}(S \cup \{0\}) - \operatorname{co}(S) \cup \{0\} \subseteq U_M^+ - U_M^+$. Hence $U_M^+ - U_M^+$ is a neighbourhood of zero, hence C gives an open decomposition of X.

Definition. Let E be a linear lattice. A closed subspace X of E is called *lattice-subspace* of E if X, ordered by the cone $X \cap E_+$, is a linear lattice.

Proposition 2. *Each infinite-dmensional lattice-subspace X of l_1 is order-isomorphic to l_1.*

Proof. It is clear that X has the Krein–Milman property and that the set

$$B = \{x \in X_+ \mid \|x\| = 1\}$$

is a bounded base for the cone X_+. The cone X_+ gives an open decomposition of X because X is complete and X_+ generating ([3], 3·5·2). So X is a locally solid linear lattice; hence it is order isomorphic to l_1.

PROPOSITION 3. *Let X be an infinite-dimensonal lattice-subspace of $L_1[0, 1]$. Then the following statements are equivalent:*

(i) *X is order-isomorphic to l_1.*

(ii) *X has the Krein–Milman property.*

Proof. It is clear that the positive cone X_+ gives an open decomposition of X. So X is locally solid. The set $B = \{x \in X_+ \mid \|x\| = 1\}$ is a bounded base for the cone X_+. Since $L_1[0, 1]$ is separable we have that X is separable. Hence the statements (i) and (ii) are equivalent.

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