TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 7, July 1996

FINITE-DIMENSIONAL LATTICE-SUBSPACES OF $C(\Omega)$ AND CURVES OF $\mathbb{R}^n$

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Abstract. Let $x_1, \ldots, x_n$ be linearly independent positive functions in $C(\Omega)$, let X be the vector subspace generated by the x_i and let β denote the curve of $\mathbb{R}^n$ determined by the function $\beta(t) = \frac{1}{z(t)}(x_1(t), x_2(t), \ldots, x_n(t))$, where $z(t) = x_1(t) + x_2(t) + \cdots + x_n(t)$. We establish that X is a vector lattice under the induced ordering from $C(\Omega)$ if and only if there exists a convex polygon of $\mathbb{R}^n$ with n vertices containing the curve β and having its vertices in the closure of the range of β. We also present an algorithm which determines whether or not X is a vector lattice and in case X is a vector lattice it constructs a positive basis of X. The results are also shown to be valid for general normed vector lattices.

1. Introduction

It is well known that each separable Banach space is isometric to a closed subspace of $C[0,1]$. By a minor modification of the existing proof of the universality of $C[0,1]$, we can show (see [12]) that $C[0,1]$ is also a universal Banach lattice. More precisely, we can show that each separable Banach lattice is order isomorphic to a closed lattice-subspace of $C[0,1]$, i.e., it is order isomorphic to a closed subspace of $C[0,1]$ which with the induced ordering is a vector lattice in its own right. Since the class of vector sublattices is not enough for this representation, the class of lattice-subspaces seems to be the proper class of subspaces for studying Banach lattices. The structure of lattice-subspaces has not been yet systematically studied. In [8] it is shown that a subspace X of a vector lattice is a lattice-subspace if and only if there exists a positive projection from the vector sublattice generated by X onto X. In [11] and [12] the existence of positive bases in lattice-subspaces has been studied. A recent survey of lattice-subspaces and positive projections as well as some new results on lattice-subspaces can be found in [1]. Recently, lattice-subspaces have been employed in economics where they appear naturally in incomplete markets and the theory of finance [4, 5].

Now let E be an ordered Banach space with positive cone E_+. A sequence $\{e_n\}$ of E is a *positive basis* if $\{e_n\}$ is a (Shauder) basis and

$$E_+ = \left\{ x = \sum_{i=1}^{\infty} \lambda_i e_i \in E : \lambda_i \in \mathbb{R}_+ \text{ for each } i \right\}.$$

Received by the editors April 24, 1995.
1991 *Mathematics Subject Classification.* Primary 46B42, 52A21, 15A48, 53A04.
This research was supported in part by the NATO Collaborative Research Grant #941059.

A positive basis $\{e_n\}$ of E is unique in the sense that if $\{b_n\}$ is another positive basis of E, then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$. If $\{e_n\}$ is a positive basis of E, then the following statements are equivalent.

(i) The basis $\{e_n\}$ is unconditional.
(ii) The cone E_+ is generating and normal.
(iii) E is a locally solid vector lattice (i.e., E is a Banach lattice with respect to an equivalent norm).

For a proof see [13, Theorem 16.3, p. 473] and [6, Theorems 3.5.2 and 4.1.5]. The cone E_+ is generating if $E = E_+ - E_+$ and E_+ is normal (or self-allied) if there exists some $c \in \mathbb{R}_+$ such that: $0 \leq x \leq y$ implies $\|x\| \leq c\|y\|$. If $\{e_n\}$ is a positive unconditional basis of E, then the lattice operations in E are given by

$$x \vee y = \sum_{i=1}^{\infty}(\lambda_i \vee \mu_i)e_i \quad \text{and} \quad x \wedge y = \sum_{i=1}^{\infty}(\lambda_i \wedge \mu_i)e_i,$$

for each $x = \sum_{i=1}^{\infty} \lambda_i e_i$, $y = \sum_{i=1}^{\infty} \mu_i e_i \in E$.

If E is an n-dimensional space and the cone E_+ is closed and generating, then, by the Choquet-Kendall Theorem [7] (see also [9, Theorem 3.11, p. 30]), E is a vector lattice if and only if a base B for E_+ is an $(n-1)$-dimensional simplex (i.e., B is a convex polygon with n vertices). In such a case, if $b_1, b_2, \ldots, b_n$ are the vertices of B, then the set $\{b_1, b_2, \ldots, b_n\}$ is a positive basis for E. Therefore, for finite-dimensional spaces the Choquet-Kendall Theorem can be stated as follows.

Proposition 1.1 (Choquet-Kendall)**.** *A finite-dimensional ordered vector space E with a closed and generating cone E_+ is a vector lattice if and only if E has a positive basis.*

If E is 2-dimensional and E_+ is closed and generating, then E is a vector lattice. This is true because each base B for E_+ is a closed line segment. Therefore, B is a simplex.

Now let X be a subspace of a partially ordered vector space E. The cone $X \cap E_+$ is called the *induced cone* of X and the ordering defined in X by this cone is called the induced ordering. An *ordered subspace* of E is a subspace of E ordered by the induced cone. A *lattice-subspace* of E is an ordered subspace of E which is also a vector lattice (Riesz space).[1]

If X is a lattice-subspace, then for each $x, y \in X$ we denote by $x \nabla y$ (resp. $x \Delta y$) the supremum (resp. infimum) of $\{x, y\}$ in X. It is clear that $x \Delta y \leq x \wedge y$ and $x \vee y \leq x \nabla y$, whenever $x \wedge y$ and $x \vee y$ exist.

Now let E be a vector lattice. If X is a lattice-subspace and $x \nabla y = x \vee y$, $x \Delta y = x \wedge y$ for all $x, y \in X$, then X is a sublattice (Riesz subspace) of E. If X is the range of a positive projection $P : E \to E$, then X is a lattice-subspace with $x \nabla y = P(x \vee y)$, $x \Delta y = P(x \wedge y)$ for each $x, y \in X$. For notation and terminology not explained here, we refer the reader to [3, 6, 9].

2. Lattice-subspaces of $C(\Omega)$ with positive bases

In this paper, we shall denote by Ω a compact Hausdorff topological space and by $C(\Omega)$ the Banach lattice of continuous real valued functions defined on Ω.

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[1]The term "lattice-subspace" has been introduced in [10]. In [8] a lattice-subspace is called a "quasi-sublattice."

Let Y be a closed subspace of $C(\Omega)$ with a basis $\{b_n\}$. Fix $t \in \Omega$ and $m \in \mathbb{N}$. If $b_m(t) \neq 0$ and $b_n(t) = 0$ for each $n \neq m$, then we shall say that the point t is an m-*node* (or simply a *node*) of the basis $\{b_n\}$. If for each n there exists an n-node t_n of the basis $\{b_n\}$, then we shall say that $\{b_n\}$ is a *basis* of Y *with nodes* and that $\{t_n\}$ is a *sequence of nodes* of $\{b_n\}$. If $\dim Y = n$ and for each $m \in \{1, 2, \dots, n\}$ there exists an m-node t_m of the basis of Y, then we shall say that $\{b_1, b_2, \dots, b_n\}$ is a basis of Y with nodes and that the points $t_1, t_2, \dots, t_n$ are nodes of the basis $\{b_1, b_2, \dots, b_n\}$.

Recall that the *support* of a function $x \in C(\Omega)$, in symbols supp x, is the closure of the set $\{t \in \Omega : x(t) > 0\}$.

Theorem 2.1. *For a closed ordered subspace Y of $C(\Omega)$ having a basis $\{b_n\}$ consisting of positive functions we have the following.*

(i) *If $\{b_n\}$ is a positive basis of Y, then*
  - (a) *for each m there exists a sequence $\{\omega_\nu\}$ of Ω such that $\lim_{\nu\to\infty} \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} = 0$ for each $i \neq m$, and*
  - (b) *there exists a sequence $\{t_n\}$ of Ω with $t_n \in \operatorname{supp} b_n$ and $b_m(t_n) = 0$ for $m \neq n$.*

(ii) *If $\{t_n\}$ is a sequence of nodes of $\{b_n\}$, then $\{b_n\}$ is a positive basis of Y and for each $x = \sum_{i=1}^{\infty} \lambda_i b_i \in Y$ we have $\lambda_i = \frac{x(t_i)}{b_i(t_i)}$ for each i.*

Proof. (i) For each k let $z_k = -\frac{1}{k} b_m + \sum_{i=1, i\neq m}^{k} b_i$. Since $\{b_n\}$ is a positive basis, $z_k \notin Y_+$ and so there exists some $\omega_k \in \Omega$ (depending on m) such that $z_k(\omega_k) < 0$ or

$$0 \leq \sum_{\substack{i=1 \\ i\neq m}}^{k} \frac{b_i(\omega_k)}{b_m(\omega_k)} < \frac{1}{k}$$

for each k. Also, let t_m be a limit point of the sequence $\{\omega_\nu\}$. Now (a) and (b) follow by letting $k \to \infty$.

(ii) Let $\{t_n\}$ be a sequence of nodes of $\{b_n\}$. It is easy to see that if $x = \sum_{i=1}^{\infty} \lambda_i b_i \in X$, then $\lambda_n = \frac{x(t_n)}{b_n(t_n)}$. In particular, $x \in X_+$ implies $\lambda_i \in \mathbb{R}_+$ for each i which shows that $\{b_n\}$ is also a positive basis. □

Proposition 2.2. *For a closed lattice-subspace Y of $C(\Omega)$ with a positive basis $\{b_n\}$ the following statements are equivalent.*

1. *Y is a sublattice of $C(\Omega)$.*
2. *If $b_m(t) > 0$ for some m and t, then t is an m-node of the basis $\{b_n\}$.*

Proof. (1) ⇒ (2) Suppose that Y is a sublattice of $C(\Omega)$ and that $b_m(t) > 0$ and let $n \neq m$. Then $e_m \wedge e_n = e_m \Delta e_n = 0$. This implies $b_n(t) = 0$ and hence t is an m-node of the basis $\{b_n\}$.

(2) ⇒ (1) Let $x = \sum_{i=1}^{\infty} \lambda_i b_i$, $y = \sum_{i=1}^{\infty} \mu_i b_i$ and let $b_m(t) > 0$. Then t is an m-node of $\{b_n\}$ which implies $b_n(t) = 0$ for each $n \neq m$. Therefore,

$$\begin{aligned}(x \nabla y)(t) &= \sum_{i=1}^{\infty} (\lambda_i \vee \mu_i) b_i(t) = (\lambda_m \vee \mu_m) b_m(t) \\ &= \left[\frac{x(t)}{b_m(t)} \vee \frac{y(t)}{b_m(t)}\right] b_m(t) = x(t) \vee y(t) \\ &= (x \vee y)(t).\end{aligned}$$

If $b_i(t) = 0$ for all i, then $(x \vee y)(t) = (x \nabla y)(t) = 0$ is also true. Thus, $x \nabla y = x \vee y$, and so Y is a sublattice of $C(\Omega)$. $\square$

Proposition 2.3. *Let Y be an n-dimensional subspace of $C(\Omega)$ and let $b_1, b_2, \ldots, b_n$ in Y_+. Then $\{b_1, \ldots, b_n\}$ is a positive basis of Y if and only if for each $1 \leq m \leq n$ there exists a sequence $\{\omega_\nu\}$ of Ω satisfying $\lim_{\nu\to\infty} \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} = 0$ for each $i \neq m$.*

Proof. We show the "only if" part. So, assume that the vectors $b_1, b_2, \ldots, b_n$ satisfy the stated property. We must show that $\{b_1, \ldots, b_n\}$ is a positive basis. To this end, assume $x = \sum_{i=1}^n \lambda_i b_i \in Y_+$. Then from

$$0 \leq \frac{x(\omega_\nu)}{b_m(\omega_\nu)} = \sum_{i=1}^n \lambda_i \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} \xrightarrow[\nu\to\infty]{} \lambda_m,$$

we have $\lambda_m \geq 0$ for each m. Also, if $x = 0$, then as before we see that $\lambda_m = 0$ for each m. Hence $\{b_1, b_2, \ldots, b_n\}$ is a positive basis of Y. $\square$

Proposition 2.4. *Let $\{b_1, \ldots, b_n\}$ be a positive basis of an n-dimensional lattice-subspace Y of $C(\Omega)$. For a function $x = \sum_{i=1}^n \lambda_i b_i \in Y$, we have the following.*

(i) *If a point t_i is an i-node of the basis, then $\lambda_i = \frac{x(t_i)}{b_i(t_i)}$.*

(ii) *If $\{\omega_\nu\}$ is a sequence of Ω such that $\lim_{\nu\to\infty} \frac{b_j(\omega_\nu)}{b_i(\omega_\nu)} = 0$ for each $j \neq i$, then we have $\lambda_i = \lim_{\nu\to\infty} \frac{x(\omega_\nu)}{b_i(\omega_\nu)}$.*

Proof. If the point t_i is an i-node, then $x(t_i) = \lambda_i b_i(t_i)$ and the validity of (i) follows. For (ii) notice that $\lim_{\nu\to\infty} \frac{x(\omega_\nu)}{b_i(\omega_\nu)} = \lim_{\nu\to\infty} \sum_{j=1}^n \lambda_j \frac{b_j(\omega_\nu)}{b_i(\omega_\nu)} = \lambda_i$. $\square$

Theorem 2.5 ([12])**.** *Let E be a Banach lattice with a positive basis. Then there exists a closed lattice-subspace Z of $C[0,1]$ with positive basis $\{b_n\}$ having node-intervals (i.e., besides $\{b_n\}$ being a positive basis of Z there exists a sequence $\{J_n\}$ of intervals of $[0,1]$ satisfying $b_n(t) > 0$ for $t \in J_n$ and $b_n(t) = 0$ for all $t \in J_m$ with $m \neq n$) and an onto order-isomorphism $T : E \to Z$ such that*

$$\tfrac{1}{8}\|x\| \leq \|Tx\| \leq \|x\|$$

for all $x \in E$.

Now let Y be a closed lattice-subspace of $C(\Omega)$ with a positive basis. As we shall see in Example 3.1(iii) below, in general Y need not have a positive basis with nodes. However, according to Theorem 2.5, the space Y is order-isomorphic to a closed lattice-subspace Z of $C[0,1]$ which has a positive basis with node-intervals (and therefore Z also has a positive basis with nodes).

3. Finite-dimensional lattice-subspaces of $C(\Omega)$

For our discussion here, we shall fix n linearly independent positive functions $x_1, x_2, \ldots, x_n$ of $C(\Omega)$. The ordered subspace of $C(\Omega)$ generated by these functions will be denoted by X or by $[x_1, x_2, \ldots, x_n]$, i.e.,

$$X = [x_1, x_2, \ldots, x_n].$$

We now state the main problem of our study.

- *When is X a lattice-subspace of $C(\Omega)$? Or equivalently, when does X have a positive basis?*

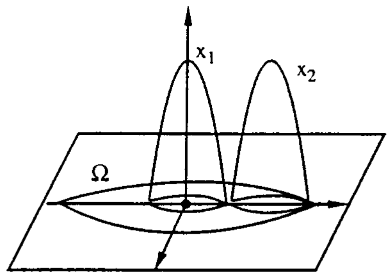

FIGURE 1

We shall answer this question by completely characterizing the lattice-subspaces. As a matter of fact, we shall not only characterize the lattice-subspaces X but we shall also present an algorithm of determining the positive basis of X. This is important since once the positive basis has been found, we can determine the coordinates of the elements of X and the lattice operations in X; see Examples 4.3 and 4.5 below.

In the sequel, we shall denote by z the sum of $x_1, \ldots, x_n$ (i.e., $z = \sum_{i=1}^n x_i$) and by β the function $\beta : \Omega \to \mathbb{R}^n$ such that

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \ldots, \frac{x_n(t)}{z(t)} \right)$$

for each $t \in \Omega$ with $z(t) > 0$. So, β defines a curve on the base $B = \{y \in \mathbb{R}_+^n : \sum_{i=1}^n y_i = 1\}$ for the cone $\mathbb{R}_+^n$. We shall refer to β as the *basic curve* of the vectors $x_1, x_2, \ldots, x_n$. The reader should keep in mind that the cone X_+ of X is always generating and normal.

We start our study with several examples of ordered subspaces of $C(\Omega)$.

Example 3.1. (i) Let $\Omega = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 9\}$ and consider the functions $x_1, x_2 \in C(\Omega)$ defined by

$$x_1(u,v) = \begin{cases} 4(1 - u^2 - v^2) & \text{if } u^2 + v^2 \leq 1, \\ 0 & \text{if } u^2 + v^2 > 1, \end{cases}$$

$$x_2(u,v) = \begin{cases} 4[1 - u^2 - (v-2)^2] & \text{if } u^2 + (v-2)^2 \leq 1, \\ 0 & \text{if } u^2 + (v-2)^2 > 1. \end{cases}$$

See Figure 1. Let X be the subspace of $C(\Omega)$ generated by x_1, x_2. Then X as a 2-dimensional subspace, is a lattice-subspace of $C(\Omega)$. The points $(0,0)$ and $(0,2)$ are nodes of the basis $\{x_1, x_2\}$, and so $\{x_1, x_2\}$ is a positive basis of X. The space X is a sublattice of $C(\Omega)$ because each $(u,v) \in \Omega$ with $(x_1 + x_2)(u,v) > 0$ is a node of the basis $\{x_1, x_2\}$.

(ii) Let $\Omega = [0,1]$, $x_1(t) = 1$, $x_2(t) = t$ and $X = [x_1, x_2]$. Then X is a lattice-subspace and the set $\{b_1(t) = 1 - t, b_2(t) = t\}$ is a basis of X with nodes the points $t_1 = 0$ and $t_2 = 1$. Therefore, $\{b_1, b_2\}$ is a positive basis. The space X is not a sublattice of $C(\Omega)$ because the point $t = \frac{1}{2}$ is not a node of the basis $\{b_1, b_2\}$. Also, it should be clear that $b_1 \nabla b_2 = \mathbf{1}$ and $b_1 \Delta b_2 = \mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ are the constant functions one and zero; see Figure 2.

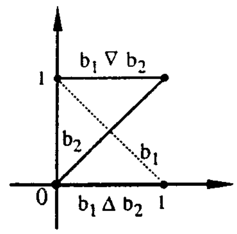

FIGURE 2

(iii) Let $\Omega = [-1, 1]$, $x_1(t) = |t|$ and

$$x_2(t) = \begin{cases} \sqrt{|t|} & \text{if } -1 \leq t < 0, \\ t^2 & \text{if } 0 \leq t \leq 1. \end{cases}$$

Then X is a lattice-subspace of $C(\Omega)$ and $\{x_1, x_2\}$ is a basis of X since

$$\lim_{t \to 0^-} \frac{x_1(t)}{x_2(t)} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{x_2(t)}{x_1(t)} = 0.$$

The space X does not have a positive basis with nodes; see Figure 3.

(iv) Let $\Omega = [0, 1]$, $x_1(t) = 1$, $x_2(t) = t$, $x_3(t) = t^2$ and $X = [x_1, x_2, x_3]$. We claim that X is not a lattice-subspace. To see this, assume by way of contradiction that X is a lattice-subspace of $C[0, 1]$. Let $\{b_1, b_2, b_3\}$ be a positive basis of X. Then for each $a \in [0, 1]$ the function $x_a(t) = (t - a)^2$, as a positive element of X, is a positive linear combination of b_1, b_2, b_3. This implies $b_i(a) = 0$ for at least one i and so $b_i = \mathbf{0}$ for at least one i, a contradiction. Therefore, X is not a lattice-subspace of $C[0, 1]$.

(v) Let X be the subspace of $C[0, 1]$ generated by the functions $x_1(t) = |1 - t|(2 - t)$, $x_2(t) = t(2 - t)$ and $x_3(t) = t|1 - t|$. The points $t_1 = 0$, $t_2 = 1$ and $t_3 = 2$ are nodes of the basis $\{x_1, x_2, x_3\}$. Hence, X is a lattice-subspace with a positive basis with nodes; Figure 4.

(vi) Let $\{t_\nu\}$, $\{\omega_\nu\}$ and $\{r_\nu\}$ be strictly increasing sequences of $[0, 1]$ all convergent to $\frac{1}{2}$ satisfying $t_\nu < \omega_\nu < r_\nu < t_{\nu+1}$ for each ν. Also, let x_1, x_2 and x_3 be elements of $C[0, 1]$ with $x_i(t) > 0$ for each $t \neq \frac{1}{2}$ and each $i = 1, 2, 3$ such that:

$$\begin{aligned} &x_1(t_\nu) = \tfrac{1}{2} - t_\nu, && x_1(\omega_\nu) = (\tfrac{1}{2} - \omega_\nu)^2, && x_1(r_\nu) = (\tfrac{1}{2} - r_\nu)^3, \\ &x_2(t_\nu) = (\tfrac{1}{2} - t_\nu)^3, && x_2(\omega_\nu) = (\tfrac{1}{2} - \omega_\nu), && x_3(r_\nu) = (\tfrac{1}{2} - r_\nu)^2, \\ &x_3(t_\nu) = (\tfrac{1}{2} - t_\nu)^2, && x_2(\omega_\nu) = (\tfrac{1}{2} - \omega_\nu)^3, && x_3(r_\nu) = (\tfrac{1}{2} - r_\nu). \end{aligned}$$

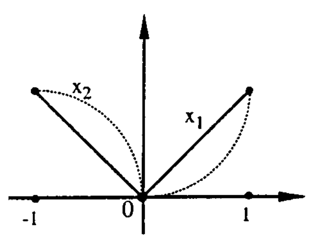

FIGURE 3

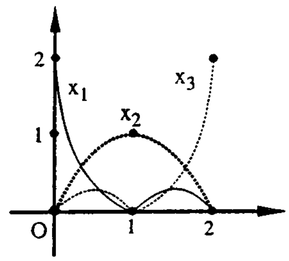

FIGURE 4

Then $x_i(\frac{1}{2}) = 0$ for each i. The graph of x_1, x_2 and x_3 over the interval $[t_\nu, t_{\nu+1}]$ might be as in Figure 5. Now note that

$$\lim_{\nu\to\infty} \frac{x_i(t_\nu)}{x_1(t_\nu)} = 0 \quad \text{for } i = 2, 3; \qquad \lim_{\nu\to\infty} \frac{x_i(\omega_\nu)}{x_2(\omega_\nu)} = 0 \quad \text{for } i = 1, 3;$$

and

$$\lim_{\nu\to\infty} \frac{x_i(r_\nu)}{x_3(r_\nu)} = 0 \quad \text{for } i = 1, 2.$$

By Proposition 2.3, the set $\{x_1, x_2, x_3\}$ is a positive basis of X and so X is a lattice-subspace of $C[0, 1]$. Since $x_i(t) > 0$ for each $t \neq \frac{1}{2}$ and each i, it follows that X does not have a positive basis with nodes. □

Recall that the Wronskian of the functions $\varphi_i \in C^{(m-1)}(a, b)$, $i = 1, 2, \ldots, m$, is the determinant function:

$$W(\varphi_1, \varphi_2, \ldots, \varphi_m)(t) = \det \begin{bmatrix} \varphi_1(t) & \varphi_2(t) & \ldots & \varphi_m(t) \\ \varphi_1'(t) & \varphi_2'(t) & \ldots & \varphi_m'(t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(m-1)}(t) & \varphi_2^{(m-1)}(t) & \ldots & \varphi_m^{(m-1)}(t) \end{bmatrix}.$$

Proposition 3.2. *Assume $[a, b] \subseteq (c, d)$ and $f_0, f_1, \ldots, f_{m-1} \in C(c, d)$. If $m > 2$, then the vector space of all solutions of the linear differential equation*

$$(*) \qquad x^{(m)} + f_{m-1}(t)x^{(m-1)} + \cdots + f_0(t)x = 0$$

is not a lattice-subspace of $C[a, b]$.

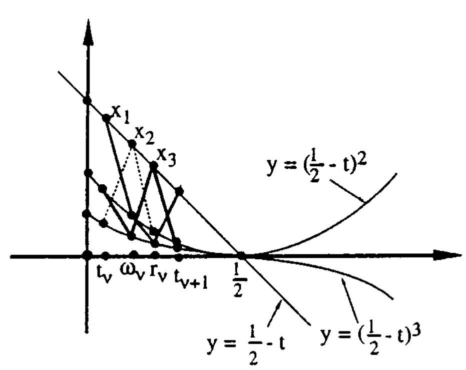

FIGURE 5

Proof. Suppose that the vector space L of all solutions of $(*)$ is a lattice-subspace having a positive basis $\{b_1, b_2, \ldots, b_m\}$. Choose $t_j \in [a, b]$, $j = 1, \ldots, m$, such that $b_i(t_j) = 0$ for each $i \neq j$. Since $W(b_1, b_2, \ldots, b_m)(t_i) \neq 0$, we infer that $b_i(t_i) > 0$. This implies that $t_i \neq t_j$ for $i \neq j$ and so (in view of $m > 2$) there exists some $t_k \in (a, b)$. But then for each $i \neq k$, the function b_i attains a local minimum at t_k, which implies that $b_i'(t_k) = 0$ for each $i \neq k$. In turn, the latter implies $W(b_1, b_2, \ldots, b_m)(t_k) = 0$, a contradiction. □

Corollary 3.3. *Assume* $[a, b] \subseteq (c, d)$ *and* $\varphi_1, \varphi_2, \ldots, \varphi_m \in C^{(m)}(c, d)$, *where* $m > 2$. *If* $W(\varphi_1, \varphi_2, \ldots, \varphi_m)(t) \neq 0$ *for each* $t \in (c, d)$, *then the subspace generated by the functions* $\varphi_1, \ldots, \varphi_m$ *is not a lattice-subspace of* $C[a, b]$.

Theorem 3.4. *The following statements are equivalent.*

1. *The subspace* X *has a positive basis with nodes.*
2. *The subspace* X *is the range of a positive projection* $P : C(\Omega) \to C(\Omega)$.

In particular, if $\{b_1, b_2, \ldots, b_n\}$ *is a positive basis with nodes the points* $t_1, t_2, \ldots, t_n$, *then the operator* $P : C(\Omega) \to C(\Omega)$, *defined by*

$$P(x) = \sum_{i=1}^{n} \frac{x(t_i)}{b_i(t_i)} b_i,$$

is a positive projection with range X.

Proof. (1) ⇒ (2) Clearly, if $\{b_1, b_2, \ldots, b_n\}$ is a positive basis with nodes the points $t_1, t_2, \ldots, t_n$, then the operator $P : C(\Omega) \to C(\Omega)$, defined by

$$P(x) = \sum_{i=1}^{n} \frac{x(t_i)}{b_i(t_i)} b_i,$$

is a positive projection with range X.

(2) ⇒ (1) Since X is the range of a positive projection, we know that X is a lattice-subspace of $C(\Omega)$. Let $\{b_1, b_2, \ldots, b_n\}$ be a positive basis of X. Then there are $t_1, t_2, \ldots, t_n$ in Ω such that $b_i(t_j) = 0$ for each $i \neq j$. Let $\psi_i = \varphi_i \circ P$, where φ_i denotes the ith coefficient functional of the basis $\{b_1, \ldots, b_n\}$. By the Riesz Representation Theorem, for each i there exists a unique Borel regular measure μ_i such that

$$\psi_i(x) = \int_\Omega x \, d\mu_i$$

for each $x \in C(\Omega)$. Let $b = \sum_{i=1}^n b_i$, $u_i = b - b_i$, $A = b^{-1}((0, \infty))$, $A_i = b_i^{-1}((0, \infty))$ and $B_i = u_i^{-1}((0, \infty))$. Since

$$\varphi_i(b_i) = \int_A b_i \, d\mu_i = 1 \quad \text{and} \quad \varphi_i(u_i) = \int_A u_i \, d\mu_i = 0,$$

it follows that the measure μ_i restricted to A is supported by a subset S_i of $A \backslash B_i$ such that $S_i \cap A_i \neq \varnothing$. Now notice that if $t_i \in S_i \cap A_i$, then the points $t_1, \ldots, t_n$ are nodes of the basis $\{b_1, b_2, \ldots, b_n\}$. □

In Banach spaces each finite-dimensional subspace is complemented. The above result shows that an order-analogue result is not valid in Banach lattices. For instance, the lattice-subspace of Example 3.1(iii) is not positively complemented because it does not have a positive basis with nodes. On the other hand, the lattice-subspace X of Example 3.1(v) has a positive basis with nodes and therefore is positively complemented. As a matter of fact, a positive projection $P : C[0,2] \to C[0,2]$ with range X is given by $P(x) = \frac{x(0)}{2}x_1 + x(1)x_2 + \frac{x(2)}{2}x_3$.

Proposition 3.5. *Let X be a lattice-subspace of $C(\Omega)$. If $z(t) > 0$ for each $t \in \Omega$, then X has a positive basis with nodes* (*and therefore X is the range of a positive projection*).

Proof. Let $\{b_1, b_2, \dots, b_n\}$ be a positive basis of X. Then there exist $t_1, t_2, \dots, t_n \in \Omega$ such that $b_j(t_i) = 0$ for each $j \neq i$. If $b_i(t_i) = 0$, then $z(t_i) = 0$, a contradiction. Hence $b_i(t_i) > 0$ for each i and so the points $t_1, t_2, \dots, t_n$ are nodes of $\{b_1, b_2, \dots, b_n\}$. □

We shall denote by $D(\beta)$ the domain and by $R(\beta)$ the range of the basic curve β of $x_1, x_2, \dots, x_n$. As usual, if K is a subset of a topological space F, we shall denote by $\mathrm{Int}(K)$ the interior of K, by $\overline{K}$ the closure of K and by ∂K the boundary of K. Also (whenever F is a linear topological space) we shall denote by $\mathrm{co}\,K$ the convex hull of K, by $\overline{\mathrm{co}}K$ the closed convex hull of K (i.e., the closure of $\mathrm{co}\,K$). If A is a matrix, then we shall denote by A^T the transpose of A.

Theorem 3.6. *The following statements are equivalent.*

(i) *X is a lattice-subspace of $C(\Omega)$.*

(ii) *There exist n linearly independent vectors $P_1, P_2, \dots, P_n$ of $\mathbb{R}^n$, belonging to the closure of the range of β such that for each $t \in D(\beta)$ the vector $\beta(t)$ is a convex combination of the vectors of $P_1, \dots, P_n$, i.e., $R(\beta) \subseteq \mathrm{co}\{P_1, \dots, P_n\}$.*

If (ii) *is true, $P_i = \lim_{\nu\to\infty} \beta(\omega_{i\nu})$ for each i, A is the $n \times n$ matrix whose ith column is the vector P_i and $b_1, b_2, \dots, b_n$ are the functions defined by the formula*

$$(b_1, b_2, \dots, b_n)^T = A^{-1}(x_1, x_2, \dots, x_n)^T, \tag{1}$$

then X has the following properties:

(a) *The set $\{b_1, b_2, \dots, b_n\}$ is a positive basis of X. In addition, if t_i is a limit point of the sequence $\{\omega_{i\nu} : \nu = 1, 2, \dots\}$, then $t_i \in \mathrm{supp}\, b_i$ and $b_k(t_i) = 0$ for each $k \neq i$.*

(b) *The closed convex hull of $R(\beta)$ and the convex polygon with vertices the points $P_1, P_2, \dots, P_n$ coincide.*

(c) *If $P_k = \beta(t_k)$, then t_k is a k-node of the basis $\{b_1, \dots, b_n\}$.*

(d) *If $\Omega \subseteq \mathbb{R}^m$, $P_k = \beta(t_k)$ for some interior point t_k of Ω and the functions x_i are C^2-functions in a neighborhood of t_k, then*

$$D_j\beta(t_k) = \mathbf{0}, \qquad j = 1, 2, \dots, m,$$

where D_j denotes the operator of the jth partial derivative.

Proof. We assume that (ii) and the other assumptions are true. We shall establish that (a), (b), (c) and (d) (and therefore (i)) are true.

Since $\{x_1, x_2, \dots, x_n\}$ is a basis of X, it follows that $\{b_1, b_2, \dots, b_n\}$ is likewise a basis of X. Let

$$P_i = (a_{i1}, a_{i2}, \dots, a_{in}), \qquad i = 1, 2, \dots, n.$$

Since each P_i is a vector of the base $B = \{y \in \mathbb{R}^n_+ : \sum_{r=1}^n y_r = 1\}$, it follows that $\sigma_i = \sum_{j=1}^n a_{ij} = 1$ for each i. Therefore

$$z = \sum_{i=1}^n x_i = \sum_{i=1}^n \sigma_i b_i = \sum_{i=1}^n b_i.$$

Let $\beta(t) = \sum_{i=1}^n \xi_i(t) P_i$ be the expansion of $\beta(t)$ relative to the basis $\{P_1, P_2, \dots, P_n\}$. Then

$$\frac{1}{z(t)}(x_1(t), x_2(t), \dots, x_n(t))^T = A(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T,$$

and in view of (1) we get

$$(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T = \frac{1}{z(t)}(b_1(t), b_2(t), \dots, b_n(t))^T.$$

Since $\beta(t)$ is a convex combination of the vectors P_i, we get $\xi_i(t) \in \mathbb{R}_+$ and so $b_i(t) \in \mathbb{R}_+$ for each i. Thus, $b_i \in X_+$ for each i. From (1), we have

$$(\beta(t))^T = A\left(\frac{b_1(t)}{z(t)}, \frac{b_2(t)}{z(t)}, \dots, \frac{b_n(t)}{z(t)}\right)^T.$$

Replacing t by $\omega_{i\nu}$ and taking limits, we get

$$(a_{i1}, a_{i2}, \dots, a_{in})^T = A(\eta_{i1}, \eta_{i2}, \dots, \eta_{in})^T, \tag{2}$$

where $\eta_{ij} = \lim_{\nu\to\infty} \frac{b_j(\omega_{i\nu})}{z(\omega_{i\nu})}$.

Since the solution of the system (2) is unique, we have $\eta_{ii} = 1$ and $\eta_{ij} = 0$ for $j \neq i$. Since $z(\omega_{i\nu}) > 0$ for each ν and $\eta_{ii} = 1$, we have $b_i(\omega_{i\nu}) > 0$ for each ν. Therefore,

$$\lim_{\nu\to\infty}\left(\frac{b_i}{z}\right)(\omega_{i\nu}) = \lim_{\nu\to\infty} \frac{1}{1 + \left(\sum_{\substack{j=1\\ j\neq i}}^n \frac{b_j}{b_i}\right)(\omega_{i\nu})} = 1.$$

Hence

$$\lim_{\nu\to\infty}\left(\frac{b_j}{b_i}\right)(\omega_{i\nu}) = 0 \quad \text{for each } j \neq i, \tag{3}$$

and so, from Proposition 2.3, the set $\{b_1, \dots, b_n\}$ is a positive basis of X. In other words, X is a lattice-subspace.

Now let t_i be an accumulation point of the sequence $\{\omega_{i\nu} : \nu = 1, 2, \dots\}$. We have shown that $b_i(\omega_{i\nu}) > 0$ for each ν and therefore $t_i \in \operatorname{supp} b_i$. Also, from (3), we see that $b_j(t_i) = 0$ for each $j \neq i$. Thus, the validity of (a) has been established.

By our assumptions, $R(\beta) \subseteq \operatorname{co}\{P_1, P_2, \dots, P_n\}$ and so

$$\overline{\operatorname{co}}\, R(\beta) \subseteq \overline{\operatorname{co}}\{P_1, \dots, P_n\} = \operatorname{co}\{P_1, \dots, P_n\}.$$

Since $P_i \in \overline{R(\beta)} \subseteq \overline{\operatorname{co}}\, R(\beta)$, we get $\operatorname{co}\{P_1, \dots, P_n\} \subseteq \overline{\operatorname{co}}\, R(\beta)$, and the validity of (b) follows.

To establish (c) let $P_k = \beta(t_k)$. Without loss of generality, we can assume that $\omega_{k\nu} = t_k$ for each ν. Then $b_k(t_k) > 0$. Also, by (a), we have $b_j(t_k) = 0$ for each $j \neq k$. Hence t_k is a k-node of the basis $\{b_1, \dots, b_n\}$, and therefore statement (c) is true.

Finally, assume that the hypotheses of claim (d) are valid. Then t_k is a k-node and so $b_\mu(t_k) = 0$ for each $\mu \neq k$. Since t_k is an interior point of Ω, for each $\mu \neq k$

the function b_μ attains a local minimum at the point t_k. This implies $D_j b_\mu(t_k) = 0$ for each j and all $\mu \neq k$. Now let $x_\mu = \sum_{i=1}^n c_{\mu i} b_i$. Then $x_\mu(t_k) = c_{\mu k} b_k(t_k)$ and $D_j x_\mu(t_k) = c_{\mu k} D_j b_k(t_k)$. Hence

$$\begin{aligned} &z(t_k) D_j x_\mu(t_k) - x_\mu(t_k) D_j z(t_k) \\ &\quad = \left(\sum_{r=1}^n c_{rk} b_k(t_k)\right) c_{\mu k} D_j b_k(t_k) - c_{\mu k} b_k(t_k) \left(\sum_{r=1}^n c_{rk} D_j b_k(t_k)\right) = 0, \end{aligned}$$

and the validity of (d) also follows.

(i) $\Rightarrow$ (ii) Suppose that X is a lattice-subspace of $C(\Omega)$ and that $\{b_1, b_2, \ldots, b_n\}$ is a positive basis of X. Then, by Proposition 2.3, for each i there exists a sequence $\{\omega_{i\nu}\}$ such that $\lim_{\nu\to\infty} \frac{b_j(\omega_{i\nu})}{b_i(\omega_{i\nu})} = 0$ for each $j \neq i$. Let

$$x_i = \sum_{j=1}^n \lambda_{ij} b_j. \tag{4}$$

Since $\{b_1, \ldots, b_n\}$ is a positive basis, we see that $\lambda_{ij} \in \mathbb{R}_+$ for all i and j. Moreover, we have $z = \sum_{i=1}^n x_i = \sum_{i=1}^n \sigma_i b_i$, where $\sigma_i = \sum_{j=1}^n \lambda_{ji}$. Hence,

$$\lim_{\nu\to\infty} \left(\frac{x_j}{z}\right)(\omega_{i\nu}) = \lim_{\nu\to\infty} \left(\frac{\sum_{k=1}^n \lambda_{jk} \frac{b_k}{b_i}}{\sum_{k=1}^n \sigma_k \frac{b_k}{b_i}}\right)(\omega_{i\nu}) = \frac{\lambda_{ji}}{\sigma_i},$$

from which it follows that

$$\lim_{\nu\to\infty} \beta(\omega_{i\nu}) = \left(\frac{\lambda_{1i}}{\sigma_i}, \frac{\lambda_{2i}}{\sigma_i}, \ldots, \frac{\lambda_{ni}}{\sigma_i}\right) = P_i.$$

Now let A be the $n \times n$ matrix with columns the vectors $P_1, P_2, \ldots, P_n$. Then, from (4), we see that

$$(x_1, x_2, \ldots, x_n)^T = A(\sigma_1 b_1, \sigma_2 b_2, \ldots, \sigma_n b_n)^T. \tag{5}$$

So, the vectors $P_1, \ldots, P_n$ are linearly independent since $\{x_1, \ldots, x_n\}$ and $\{b_1, \ldots, b_n\}$ are both bases of X. Let $\beta(t) = \sum_{i=1}^n \xi_i(t) P_i$ be the expansion of $\beta(t)$ relative to the basis $\{P_1, \ldots, P_n\}$. Then

$$(\beta(t))^T = A(\xi_1(t), \xi_2(t), \ldots, \xi_n(t))^T$$

and from (5) we get

$$(\xi_1(t), \xi_2(t), \ldots, \xi_n(t))^T = \frac{1}{z(t)} (\sigma_1 b_1(t), \sigma_2 b_2(t), \ldots, \sigma_n b_n(t))^T.$$

Hence $\xi_i(t) \in \mathbb{R}_+$ for each i and $\sum_{i=1}^n \xi_i(t) = 1$, and the proof is finished. $\square$

Now we define the subset $E(\beta)$ of $\overline{R(\beta)}$ as follows: If there exists a subset G of $\overline{R(\beta)}$ consisting of n linearly independent vectors such that $R(\beta) \subseteq \operatorname{co} G$, then we put $E(\beta) = G$, otherwise we put $E(\beta) = \varnothing$. We shall refer to the set $E(\beta)$ as the *extreme subset of* β.

From the preceding definition the following result should be immediate.

Proposition 3.7. *The subspace X satisfies the properties*

1. *X is a lattice-subspace if and only if $E(\beta) \neq \varnothing$.*
2. *If $\beta(t) \in E(\beta)$, then t is a node of the positive basis of X.*
3. *X has a positive basis with nodes if and only if $E(\beta)$ is a nonempty subset of $R(\beta)$.*

Now let $P \in E(\beta)$. If $P \notin R(\beta)$, then by Theorem 3.6, we have $P = \lim_{\nu\to\infty} \beta(t_\nu)$, where $\{t_\nu\}$ is a sequence of $D(\beta)$ having all its limit points in the boundary $\partial D(\beta)$ of $D(\beta)$. Accordingly, we define the *limit set* $L(\beta)$ of the curve β as follows:

$$L(\beta) = \{P \in \mathbb{R}^n : \exists \{t_\nu\} \subseteq D(\beta) \text{ with its limit points in } \partial D(\beta), P = \lim_{\nu\to\infty} \beta(t_\nu)\}.$$

It should be clear that $\overline{R(\beta)} \setminus R(\beta) \subseteq L(\beta)$.

Example 3.8. (i) Let x_1, x_2 be the functions defined in Example 3.1(iii), β be the basic curve of x_1, x_2 and X be the lattice-subspace generated by x_1, x_2. It is easy to show that the range $R(\beta)$ of β is the open line segment joining the points $P_1 = (1,0)$ and $P_2 = (0,1)$. The extreme subset of β is the set $E(\beta) = \{P_1, P_2\}$ because $P_1, P_2 \in \overline{R(\beta)}$ and each $\beta(t)$ is a convex combination of P_1 and P_2. Also, the limit set of β is $L(\beta) = \{P_1, P_2\}$ because $\partial D(\beta) = \{0\}$ and $\lim_{t\to 0^-} \beta(t) = P_2$ and $\lim_{t\to 0^+} \beta(t) = P_1$.

According to Proposition 3.7, $E(\beta) \neq \varnothing$ implies that X is a lattice-subspace and $E(\beta) \nsubseteq R(\beta)$ guarantees that X does not have a positive basis with nodes. See Figure 6.

(ii) Let x_1, x_2 and x_3 be the functions introduced in Example 3.1(vi), β be the basic curve of x_1, x_2, x_3 and X the subspace generated by x_1, x_2, x_3. Then $D(\beta) = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and so $\partial D(\beta) = \{\frac{1}{2}\}$.

Now let the sequences $\{t_\nu\}$, $\{\omega_\nu\}$ and $\{r_\nu\}$ be as in Example 3.1(vi). Then

$$\lim_{\nu\to\infty} \beta(t_\nu) = P_1(1,0,0), \quad \lim_{\nu\to\infty} \beta(\omega_\nu) = P_2(0,1,0),$$

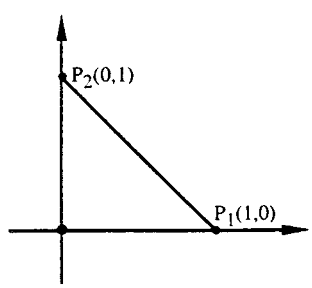

FIGURE 6

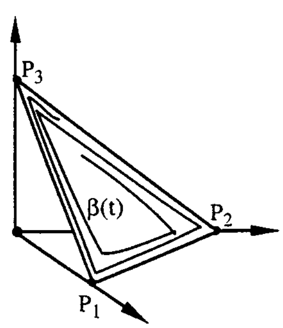

FIGURE 7

and

$$\lim_{\nu\to\infty} \beta(r_\nu) = P_3(0,0,1).$$

Therefore $P_1, P_2, P_3 \in L(\beta)$. Also $E(\beta) = \{P_1, P_2, P_3\}$ because $P_i \in \overline{R(\beta)}$ for each i and $\beta(t)$ is a convex combination of the P_i. See Figure 7.

4. The case $\Omega \subseteq \mathbb{R}^m$

Suppose now that Ω is a compact subset of $\mathbb{R}^m$. Again, we shall study in this special case when the space X generated by the linearly independent positive elements $x_1, \dots, x_n$ of $C(\Omega)$ is a lattice-subspace. If t is an interior point of Ω and $\beta(t) \in E(\beta)$, then the point t is a root of the equations

$$D_j\beta(t) = 0, \qquad j = 1, \dots, m. \tag{6}$$

We shall denote by $I(\beta)$ the images of the roots of (6), i.e.,

$$I(\beta) = \{\beta(t) : t \in \operatorname{Int}(\Omega) \cap D(\beta) \text{ and } t \text{ is a root of equations (6)}\}.$$

Also, we shall denote by $\beta(\partial\Omega)$ the set $\beta(\partial\Omega) = \{\beta(t) : t \in \partial\Omega\}$.

Proposition 4.1. *If the functions $x_1, \dots, x_n$ are C^2-functions in the set* $\operatorname{Int}(\Omega) \cap D(\beta)$*, then $E(\beta) \subseteq L(\beta) \cup I(\beta) \cup \beta(\partial\Omega)$.*

Proof. Let $P \in E(\beta)$. If $P \in R(\beta)$, then there exists some t such that $P = \beta(t)$. If $t \in \operatorname{Int}(\Omega)$, then by Theorem 3.6, we get $P \in I(\beta)$. If $t \notin \operatorname{Int}(\Omega)$, then $t \in \partial\Omega$ and so $P \in \beta(\partial\Omega)$. If $P \notin R(\beta)$, then $P \in \overline{R(\beta)} \setminus R(\beta) \subseteq L(\beta)$ and the proof of the proposition is finished. □

We continue with one more definition. Any subset of $L(\beta) \cup I(\beta) \cup \beta(\partial\Omega)$ consisting of n linearly independent vectors will be called a *possible extreme subset* of β. In order to study when X is a lattice-subspace, we shall determine the set $L(\beta) \cup I(\beta) \cup \beta(\partial\Omega)$ and shall investigate when one of the possible extreme subsets of β is indeed an extreme subset of β.

In case $\Omega = [a,b] \subset \mathbb{R}$, we make the following remark. The set $\beta(\partial\Omega)$ is known because $\partial\Omega = \{a,b\}$. If $D(\beta) = [a,b]$, then $\partial D(\beta) = \varnothing$ and so $L(\beta) = \varnothing$. If we assume that

$$D(\beta) = [a, \omega_1) \cup (t_1, \omega_2) \cup \cdots \cup (t_{n-1}, \omega_n) \cup (t_n, b]$$

and the limits

$$R_i = \lim_{t\to t_i^+} \beta(t), \quad Q_i = \lim_{t\to\omega_i^-} \beta(t)$$

exist for each i, then $L(\beta) = \{R_1, R_2, \dots, R_n\} \cup \{Q_1, \dots, Q_n\}$.

In general, the set $D(\beta) = z^{-1}((0,\infty))$, as an open subset of $[a,b]$, is the union of at most countably many disjoint open intervals of $[a,b]$ and therefore $\partial D(\beta)$ is at most countable. Finally, we note that a necessary condition for X to be a lattice-subspace of $C[a,b]$ is that the Wronskian of the functions $x_1, x_2, \dots, x_n$ have at least one root in the interval $[a,b]$ (Corollary 3.3).

So, in order to determine whether X is a lattice-subspace we must follow the steps of the algorithm below:

1. Does the Wronskian of the functions $x_1, \dots, x_n$ have at least one root in the interval $[a,b]$?
2. Determine the sets $\beta(\partial\Omega), L(\beta), I(\beta)$ and the possible extreme subsets of β.

3. Is one of the possible extreme subsets an extreme subset of β?
4. Determine a positive basis of X.

Example 4.2. Let $\Omega = [0,1]$ and $x_i(t) = t^i, i = 0,1,\dots,k$. Then the space generated by the functions x_i is not a lattice-subspace of $C[0,1]$ since the Wronskian does not have any root in $[0,1]$. □

Example 4.3. Let $\Omega = [0,2]$, $x_1(t) = t^2 - 2t + 2$, $x_2(t) = -t^3 + 2t^2 - t + 2$, $x_3(t) = t^3 - 3t^2 + 3t$ and X be the subspace of $C[0,2]$ generated by x_1, x_2, x_3.

The Wronskian of these functions $W(t) = -24(t-1)^2$ has a root in the interval $[0,2]$. Therefore, according to Step 2, we must determine the possible extreme subsets of β. We have $z(t) = x_1(t) + x_2(t) + x_3(t) = 4$ and $\beta(t) = \frac{1}{4}(x_1(t), x_2(t), x_3(t))$ is the basic curve of the triplet. Since $D(\beta) = [0,2]$, we see that $L(\beta) = \varnothing$. Also,

$$\beta(\partial\Omega) = \{\beta(0) = (\tfrac{1}{2}, \tfrac{1}{2}, 0), \beta(2) = (\tfrac{1}{2}, 0, \tfrac{1}{2})\}.$$

Now the equation $\beta'(t) = \mathbf{0}$ has one solution in the interval $(0,2)$ (namely $r = 1$) and so

$$I(\beta) = \{\beta(1) = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4})\}.$$

Hence

$$\beta(\partial\Omega) \cup L(\beta) \cup I(\beta) = \{P_1(\tfrac{1}{2}, \tfrac{1}{2}, 0), P_2(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}), P_3(\tfrac{1}{2}, 0, \tfrac{1}{2})\},$$

and so the set $\{P_1, P_2, P_3\}$ is the only possible extreme subset of β.

We claim that $\{P_1, P_2, P_3\}$ is an extreme subset of β. To this end, suppose that

$$\beta(t) = \xi_1(t)P_1 + \xi_2(t)P_2 + \xi_3(t)P_3 \tag{7}$$

is the expansion of $\beta(t)$ relative to the basis $\{P_1, P_2, P_3\}$. We must show that $\xi_i(t) \geq 0$ for each i. From (7), it follows that

$$\xi_1(t) = \tfrac{1}{2}(t-1)^2(2-t), \quad \xi_2(t) = t(2-t) \quad \text{and} \quad \xi_3(t) = \tfrac{1}{2}t(t-1)^2,$$

from which it follows that $\xi_i(t) \geq 0$ for each i and each $t \in [0,2]$. Since $\beta(t)$ belongs to the base $B = \{y \in \mathbb{R}^3_+ : y_1 + y_2 + y_3 = 1\}$ of $\mathbb{R}^3_+$ we observe that $\xi_1(t) + \xi_2(t) + \xi_3(t) = 1$, which shows that $\{P_1, P_2, P_3\}$ is an extreme subset of β.

A positive basis $\{b_1, b_2, b_3\}$ of X is the following:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ -2 & 2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2(x_1 - x_3) \\ 2(-x_1 + x_2 + x_3) \\ x_1 - x_2 + x_3 \end{bmatrix};$$

hence $b_1(t) = 2(t-1)^2(2-t)$, $b_2(t) = 4t(2-t)$ and $b_3(t) = 2t(t-1)^2$. The points $t_1 = 0$, $t_2 = 1$ and $t_3 = 2$ are nodes for the basis $\{b_1, b_2, b_3\}$. See Figure 8.

Now, we can determine the coefficients of the points of X and the lattice operations of X. For example,

$$x_1 = \frac{x_1(0)}{b_1(0)}b_1 + \frac{x_1(1)}{b_2(1)}b_2 + \frac{x_1(2)}{b_3(2)}b_3 = \frac{1}{2}b_1 + \frac{1}{4}b_2 + \frac{1}{2}b_3,$$

and similarly $x_2 = \frac{1}{2}b_1 + \frac{1}{2}b_2$. Therefore,

$$x_1 \nabla x_2 = \tfrac{1}{2}b_1 + \tfrac{1}{2}b_2 + \tfrac{1}{2}b_3 \quad \text{and} \quad x_1 \Delta x_2 = \tfrac{1}{2}b_1 + \tfrac{1}{4}b_2.$$

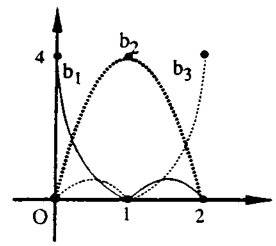

FIGURE 8

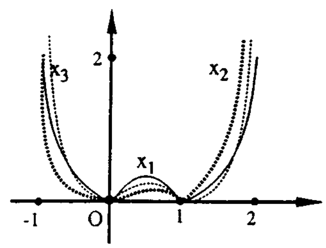

FIGURE 9

A positive projection P with range X is given by

$$P(x) = \frac{x(t_1)}{b_1(t_1)} b_1 + \frac{x(t_2)}{b_2(t_2)} b_2 + \frac{x(t_3)}{b_3(t_3)} b_3 = \frac{x(0)}{4} b_1 + \frac{x(1)}{4} b_2 + \frac{x(2)}{4} b_3,$$

for each $x \in C[0,2]$. □

Example 4.4. Let $\Omega = [-1,2]$, $x_1(t) = t^2(t-1)^2$, $x_2(t) = t^4(t-1)^2$ and $x_3(t) = t^4(t-1)^4$; see Figure 9. Also let X be the subspace of $C[-1,2]$ generated by x_1, x_2, x_3. A computation shows that the Wronskian of these functions is $W(t) = 4t^8(t-1)^5(4t^2-7t+3)$ which has roots in the interval $[-1,2]$. Notice that $z(t) = x_1(t) + x_2(t) + x_3(t) = x_1(t)a(t)$, where $a(t) = 1 + t^2 + t^2(t-1)^2$ and

$$\beta(t) = \left(\frac{1}{a(t)}, \frac{t^2}{a(t)}, \frac{t^2(t-1)^2}{a(t)} \right), \qquad t \in [-1,0) \cup (0,1) \cup (1,2],$$

is the basic curve of x_1, x_2, x_3. Also,

$$\beta(\partial\Omega) = \left\{ \beta(-1) = P_1\left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{4}{6}\right), \beta(2) = P_2\left(\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{4}{9}\right) \right\}$$

and $\lim_{t\to 0} \beta(t) = P_3(1,0,0)$ and $\lim_{t\to 1} \beta(t) = P_4(\frac{1}{2}, \frac{1}{2}, 0)$. Therefore, $L(\beta) = \{P_3, P_4\}$. Now it is easy to show that the equation $\beta'(t) = \mathbf{0}$ does not have any root in the set $(-1,2) \cap D(\beta)$. Therefore, $I(\beta) = \varnothing$ and hence

$$I(\beta) \cup L(\beta) \cup \beta(\partial\Omega) = \{P_1, P_2, P_3, P_4\}.$$

The possible extreme subsets of β are the following:

$$S_1 = \{P_1, P_2, P_3\}, \quad S_2 = \{P_1, P_2, P_4\},$$
$$S_3 = \{P_1, P_3, P_4\}, \quad \text{and} \quad S_4 = \{P_2, P_3, P_4\}.$$

We can show that no set from the above candidates is an extreme subset of β. For instance if the set S_1 is an extreme subset of β, then let $\beta(t) = \xi_1(t)P_1 + \xi_2(t)P_2 + \xi_3(t)P_3$ and note that $\xi_1(t) = \frac{2t^3(t-2)}{a(t)} < 0$ for each $t \in (0,2)$, a contradiction.

The conclusion here is that X is not a lattice-subspace of $C[-1,2]$. □

Example 4.5. Let $\Omega = \{(s,t) \in \mathbb{R}^2 : s^2+t^2 \leq 1\}$, $x_1(s,t) = s^2+t^2$, $x_2(s,t) = 1 - e^{-(s^2+t^2)}$ and let X be the lattice-subspace of $C(\Omega)$ generated by x_1 and x_2. We shall determine a positive basis of X. Notice that $z(s,t) = (x_1+x_2)(s,t) = s^2+t^2+1-e^{-(s^2+t^2)}$ and that the basic curve of x_1 and x_2 is $\beta = (\frac{x_1}{z}, \frac{x_2}{z})$. For each $(s,t) \in \partial\Omega$, we have $\beta(s,t) = P_1(\frac{e}{2e-1}, \frac{e-1}{2e-1})$ and so $\beta(\partial\Omega) = \{P_1\}$. Also, $\partial D(\beta) = \{(0,0)\}$. Since $\lim_{(s,t)\to(0,0)} \beta(s,t) = P_2(\frac{1}{2}, \frac{1}{2})$, we see that $L(\beta) = \{P_2\}$. Now the system of equations

$$\frac{\partial\beta(s,t)}{\partial s} = \mathbf{0} \quad \text{and} \quad \frac{\partial\beta(s,t)}{\partial t} = \mathbf{0}$$

does not have solutions in the set $\operatorname{Int}(\Omega) \cap D(\beta)$ and so $I(\beta) = \varnothing$. This implies that extreme subset of β is $E(\beta) = \{P_1, P_2\}$. A positive basis $\{b_1, b_2\}$ of X is the following:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (4e-2) \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{e-1}{2e-1} & \frac{e}{2e-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2e-1)(x_1-x_2) \\ 2[(1-e)x_1 + ex_2] \end{bmatrix}.$$

Therefore

$$b_1(s,t) = (2e-1)(s^2+t^2-1+e^{-(s^2+t^2)}) \text{ and}$$
$$b_2(s,t) = 2[(1-e)(s^2+t^2) + e - e^{1-(s^2+t^2)}].$$

The restriction of b_1 and b_2 in the s-axis is as in Figure 10.

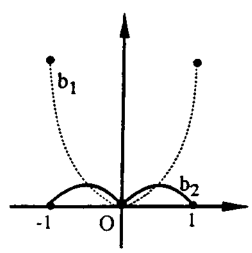

FIGURE 10

Next, we shall determine the coefficients of x_1 relative to the basis $\{b_1, b_2\}$. Let $x_1 = \lambda_1 b_1 + \lambda_2 b_2$. Then $\lambda_1 = \frac{x_1(0,1)}{b_1(0,1)} = \frac{e}{2e-1}$ because the point $(0,1)$ is a 1-node of the basis of X. Also, $\lim_{s\to 0} \frac{b_1(s,0)}{b_2(s,0)} = 0$, which (in view of Proposition 2.4) implies $\lambda_2 = \lim_{s\to 0} \frac{x_1(s,0)}{b_2(s,0)} = \frac{1}{2}$. Therefore, $x_1 = \frac{e}{2e-1} b_1 + \frac{1}{2} b_2$. □

5. THE CASE $\Omega = \{1, 2, \ldots, m\}$

Suppose now that $\Omega = \{1,2,\ldots,m\}$. Then $C(\Omega) = \mathbb{R}^m$ and X is the subspace of $\mathbb{R}^m$ generated by the linearly independent positive vectors

$$x_i = (x_i(1), x_i(2), \ldots, x_i(m)), \qquad i = 1,2,\ldots,n,$$

of $\mathbb{R}^m$. Then $z = x_1 + x_2 + \cdots + x_n$ and the basic curve of $x_1, x_2, \ldots, x_n$ is the function $\beta : \Omega \to \mathbb{R}^m$ defined by

$$\beta(k) = \left(\frac{x_1(k)}{z(k)}, \frac{x_2(k)}{z(k)}, \ldots, \frac{x_n(k)}{z(k)}\right) \quad \text{for each } k \text{ with } z(k) > 0.$$

Then $R(\beta) = \overline{R(\beta)}$. Therefore, if X is a lattice-subspace, then there exists $\{j_1, j_2, \ldots, j_n\} \subseteq D(\beta)$ such that $E(\beta) = \{\beta(j_i) : i = 1, \ldots, n\}$. So, in order to determine whether X is a lattice-subspace of $\mathbb{R}^m$ we find the subsets of $R(\beta)$ consisting of n linearly independent vectors (i.e., we find the possible extreme subsets of β) and examine if one of them is an extreme subset of β. As a consequence of Theorem 3.6 we have the following result—which was also proven in [1, Theorem 2.6].

Theorem 5.1. *The space X is a lattice-subspace of $\mathbb{R}^m$ if and only if there exist indices $j_1, j_2, \ldots, j_n$ of $D(\beta)$ such that for each $k \in D(\beta)$ the vector $\beta(k)$ is a convex combination of the vectors $\beta(j_1), \beta(j_2), \ldots, \beta(j_n)$.*

In this case, a positive basis $\{b_1, b_2, \ldots, b_n\}$ of X, with nodes the points $j_1, j_2, \ldots, j_n$ is given by the formula

$$(b_1, b_2, \ldots, b_n)^T = A^{-1}(x_1, x_2, \ldots, x_n)^T,$$

where A is the $n \times n$ matrix whose columns are the vectors $\beta(j_i)$, $i = 1, \ldots, n$.

6. Lattice-subspaces of normed lattices

Now let E be a normed vector lattice (normed Riesz space) and let U_+° denote the positive part of the closed unit ball of the norm dual E^* of E, i.e.,

$$U_+^\circ = \{f \in E_+^* : \|f\| \le 1\}.$$

Suppose that U_+° is equipped with the weak* topology and let Ω be a weak* closed subset of U_+° defining the positive cone of E, i.e.,

$$E_+ = \{x \in E : f(x) \ge 0 \quad \text{for each } f \in \Omega\}.$$

For each $x \in E$ denote by $\hat{x}$ the function $\hat{x}(f) = f(x)$, $f \in \Omega$, and for any subspace X of E denote by $\widehat{X}$ the subspace $\widehat{X} = \{\hat{x} \in C(\Omega) : x \in X\}$ of $C(\Omega)$. Then, $x \mapsto T(x) = \hat{x}$ is a linear, one-to-one operator from E into $C(\Omega)$ such that T and T^{-1} are both positive.

Theorem 6.1. *A subspace X of E is a lattice-subspace if and only if $\widehat{X}$ is a lattice-subspace of $C(\Omega)$.*

This study of lattice-subspaces is more interesting if there exists a sequence $\{f_n\}$ of U_+° defining the positive cone of E and Ω is the weak* closure of $\{f_n\}$. For instance, if E is a Banach lattice with positive basis, then we can take $\{f_n\}$ to be the sequence of the coefficient functionals of the basis and $\Omega = \{f_n : n \in \mathbb{N}\} \cup \{0\}$. On the other hand, if $E = l_\infty$, then we can take $\{f_n\}$ to be the sequence of functionals of the standard biorthogonal system of l_∞ and Ω to be the weak* closure of the sequence $\{f_n\}$. For details on determining Ω see [2, p. 448].

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