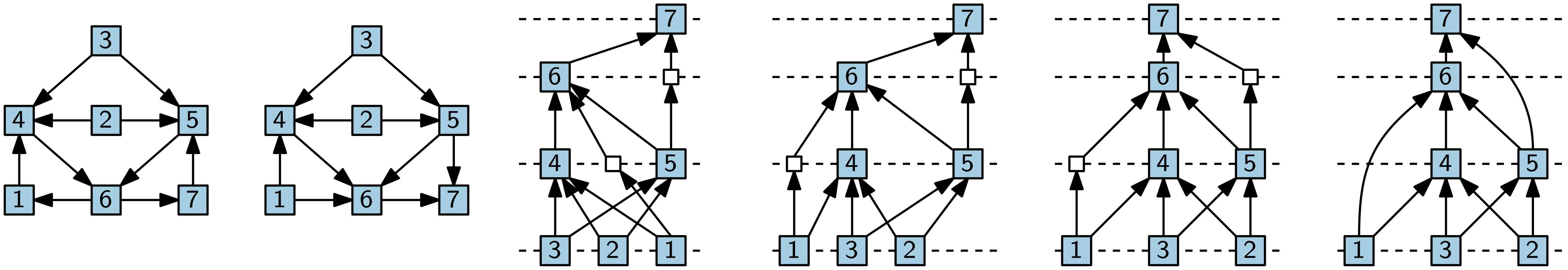
Visualisation of graphs

Hierarchical layouts

Sugiyama framework

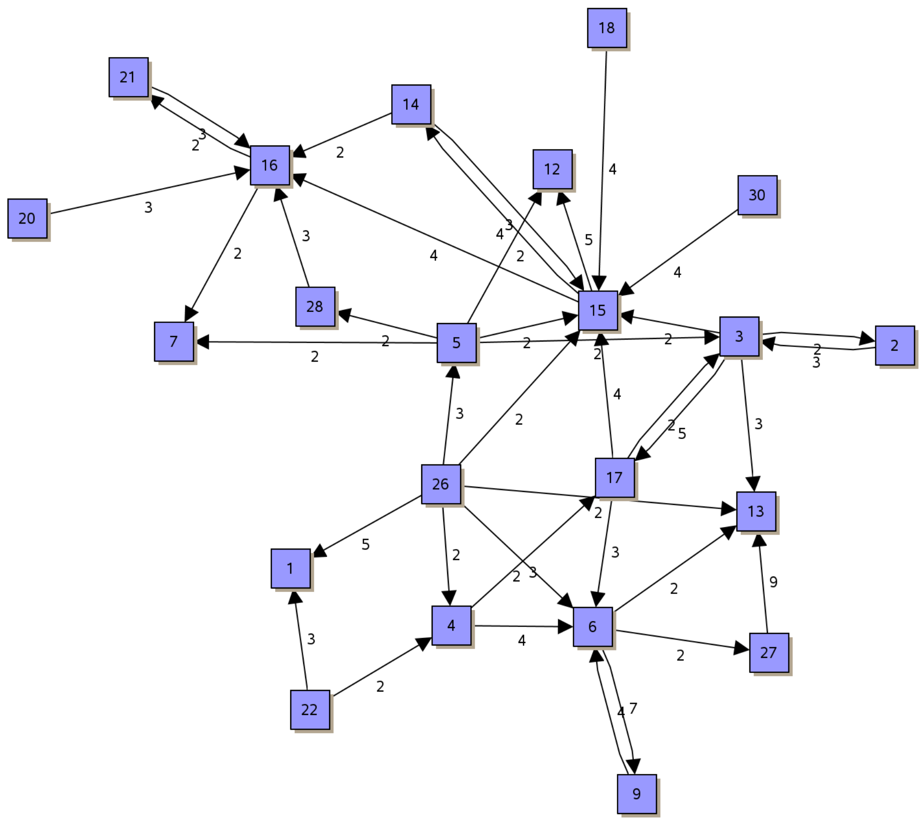
Antonios Symvonis · Chrysanthi Raftopoulou

Fall semester 2022

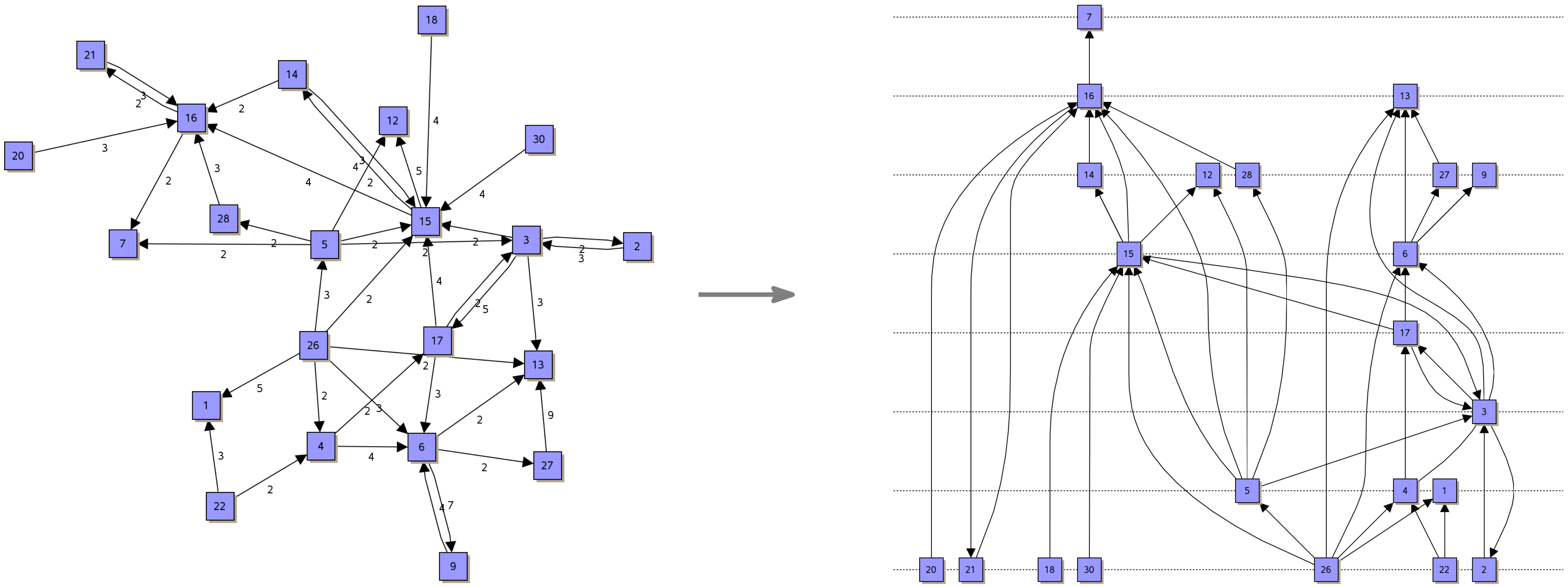


The original slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, ...
The original presentation was modified/updated by A. Symvonis and C. Raftopoulou

Hierarchical drawings – motivation



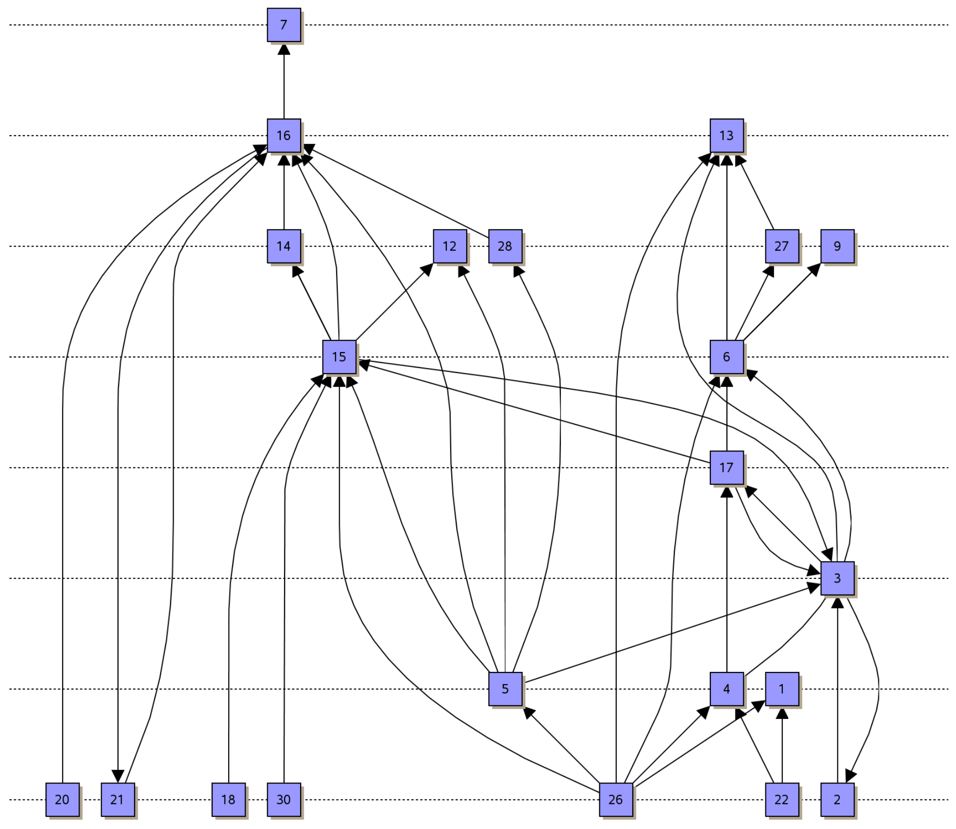
Hierarchical drawings – motivation



Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G



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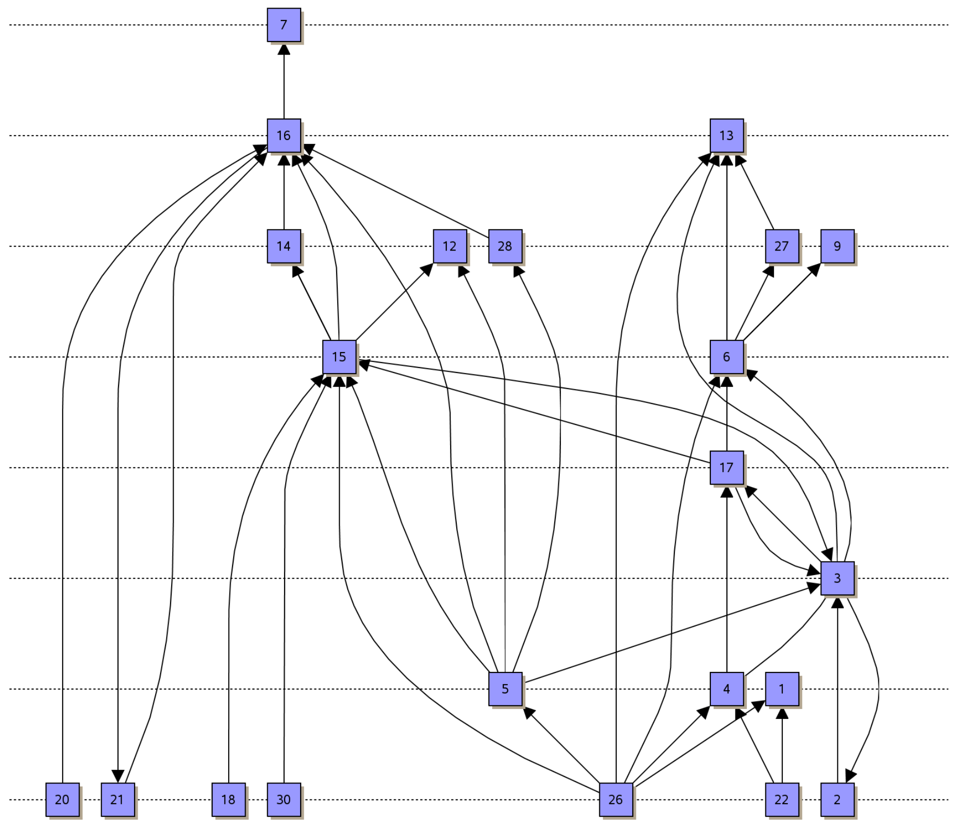
Desireable properties.



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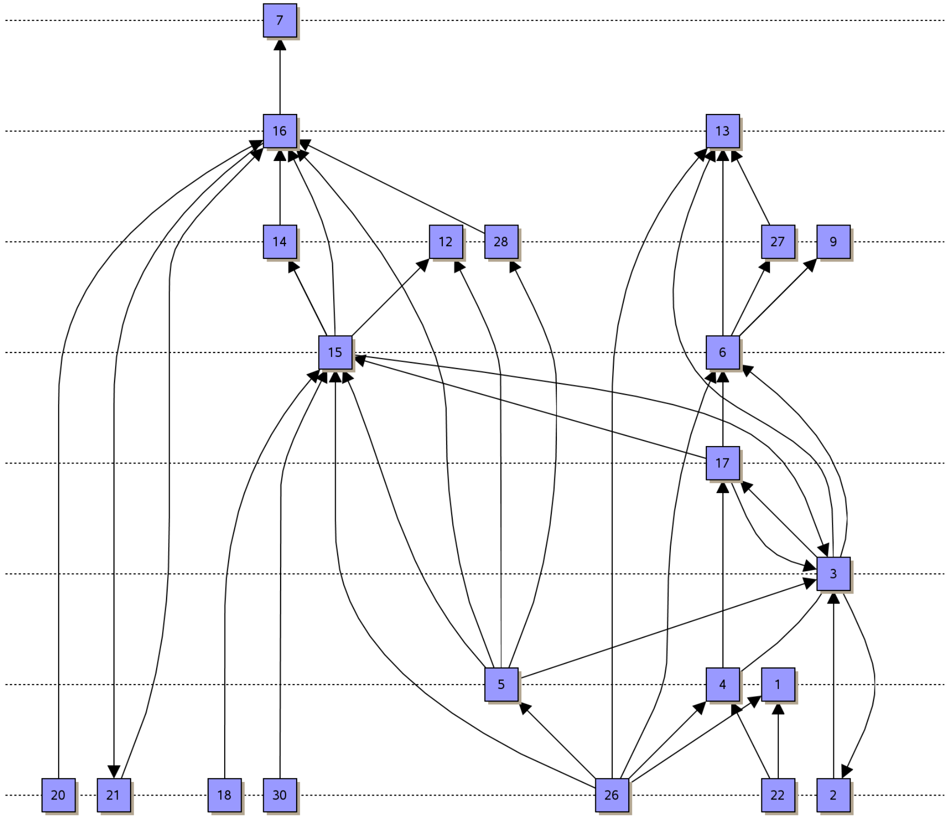

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Problem statement.

- Input: digraph $G = (V, E)$
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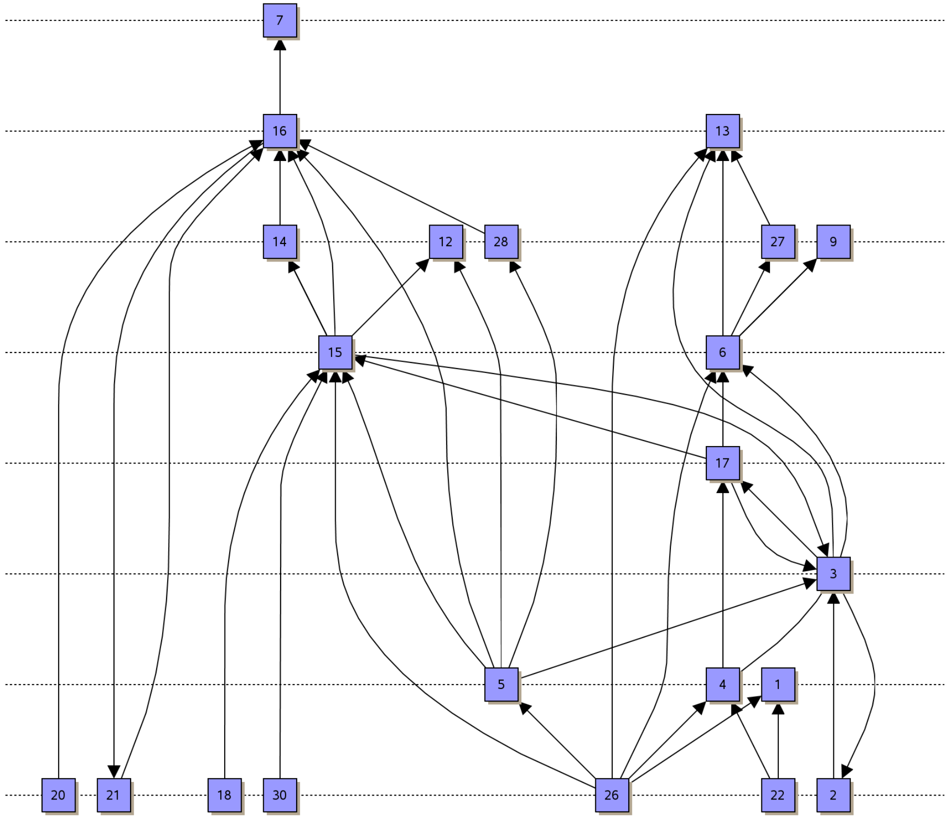
Desireable properties.

- vertices occur on (few) horizontal lines
- edges directed upwards



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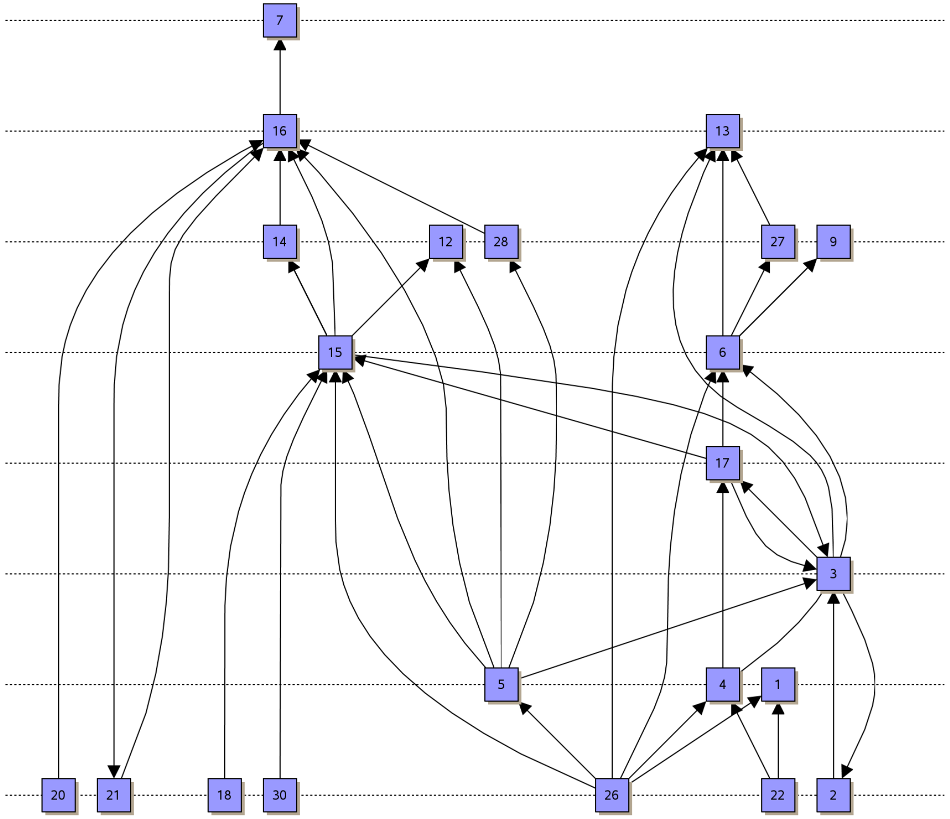
- edge crossings minimized



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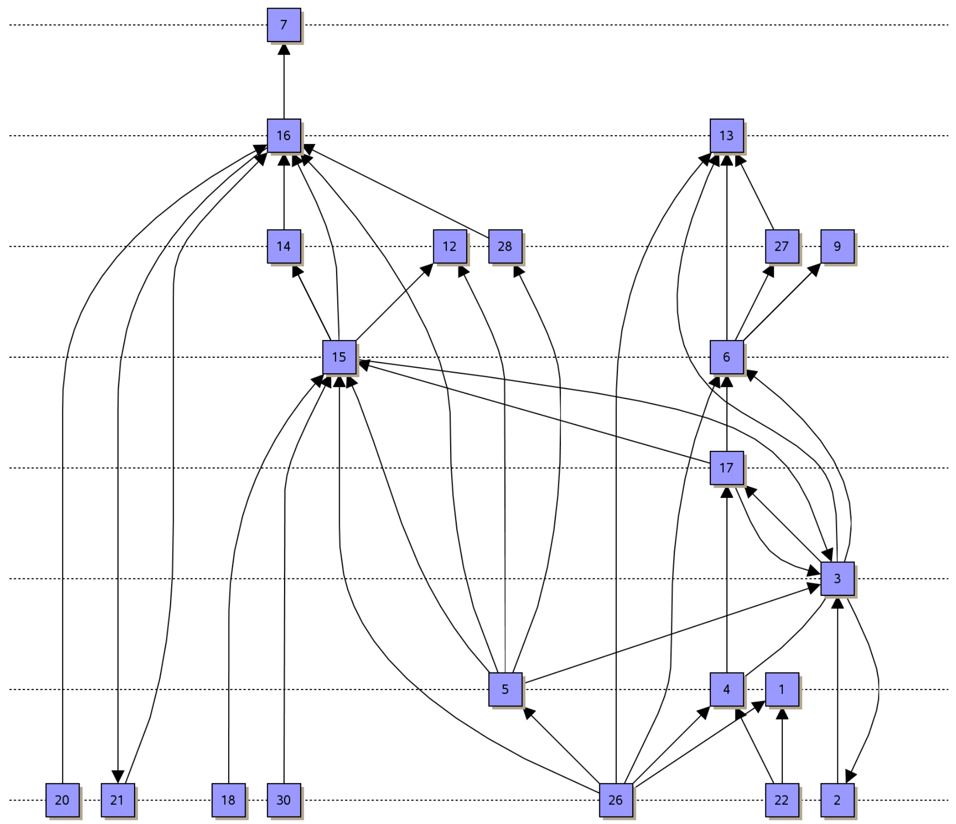
- edges upward, straight, and short as possible



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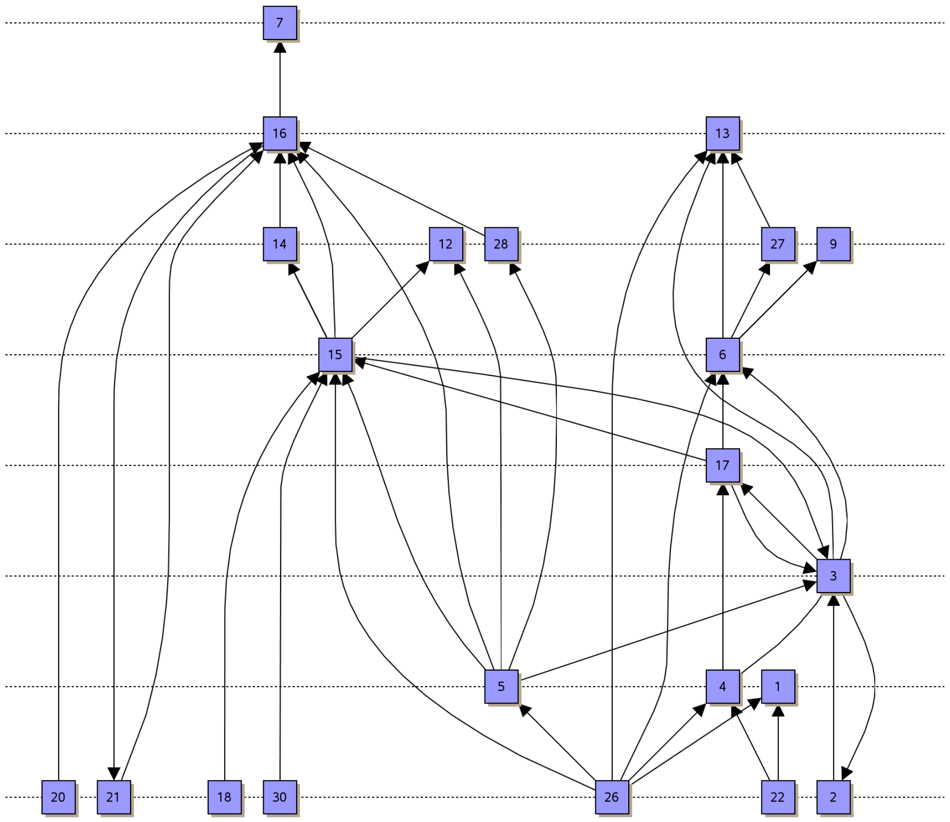

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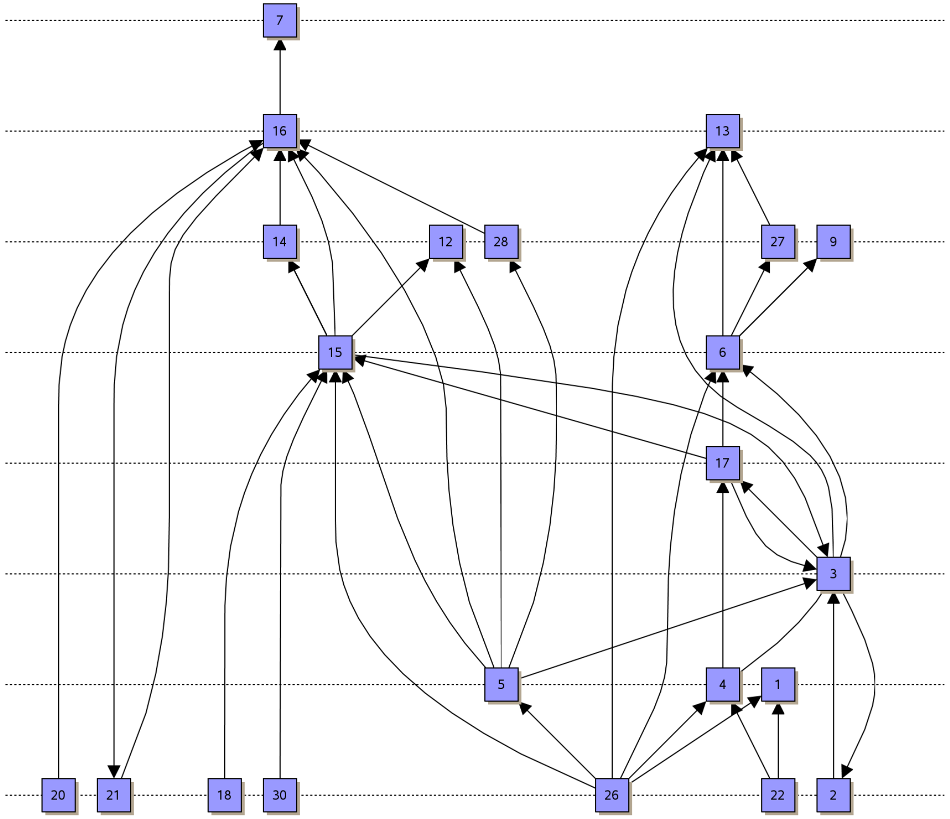
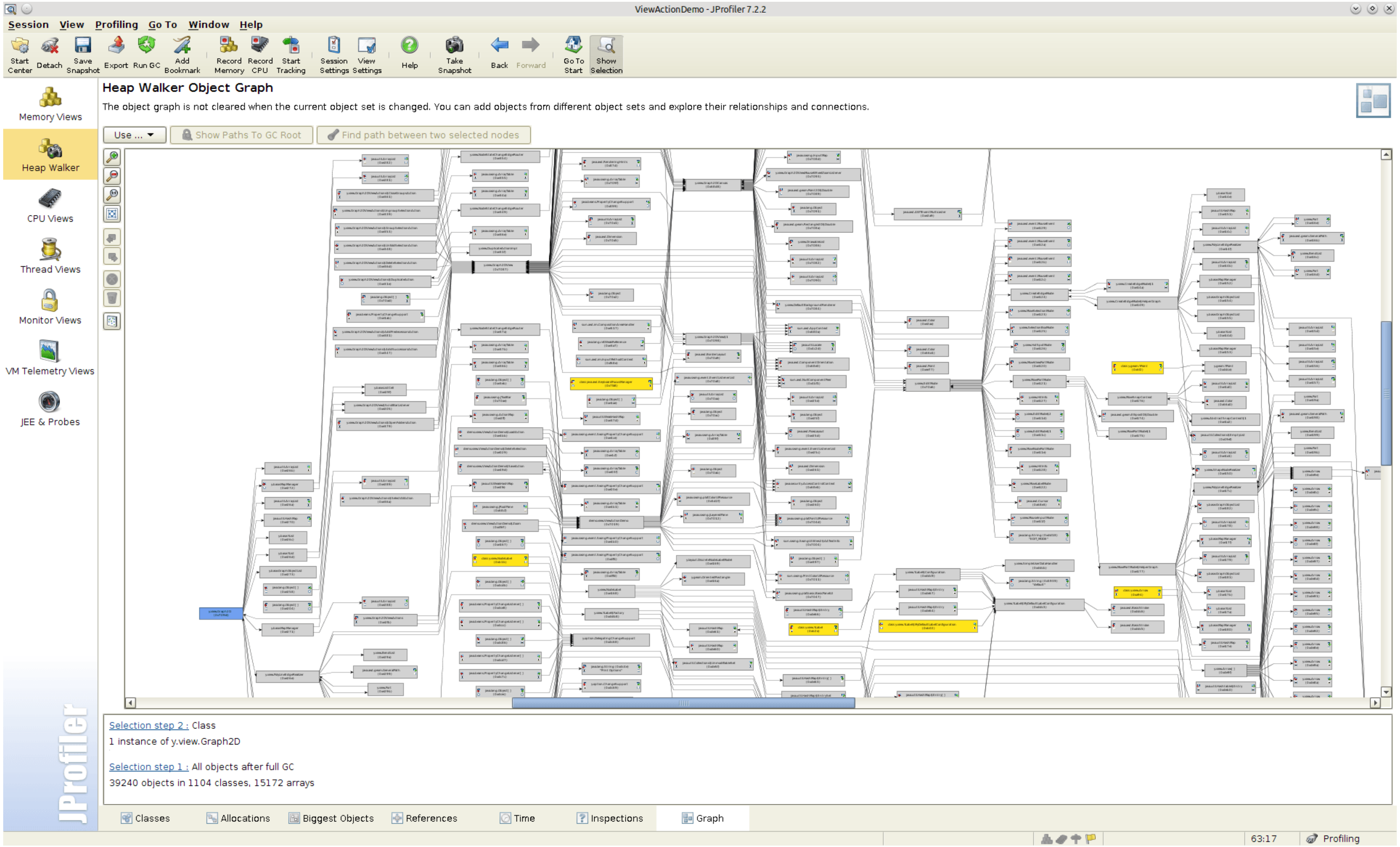
Criteria can be contradictory!



Hierarchical drawing – applications

yEd Gallery: Java profiler JProfiler using yFiles



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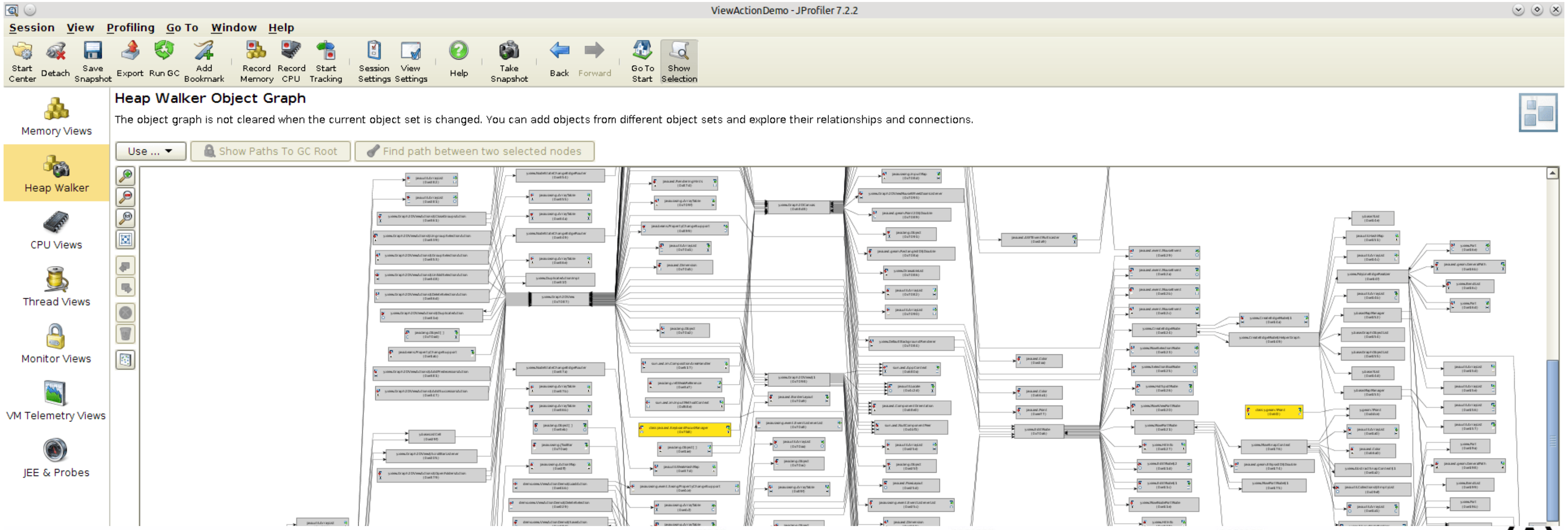

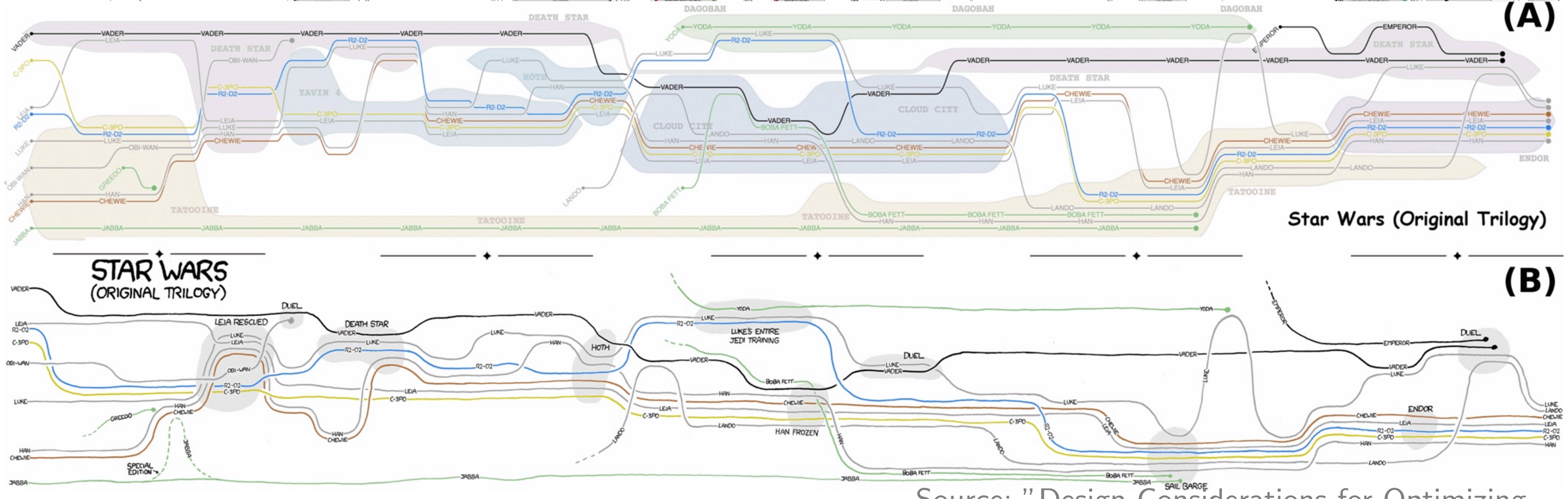

Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

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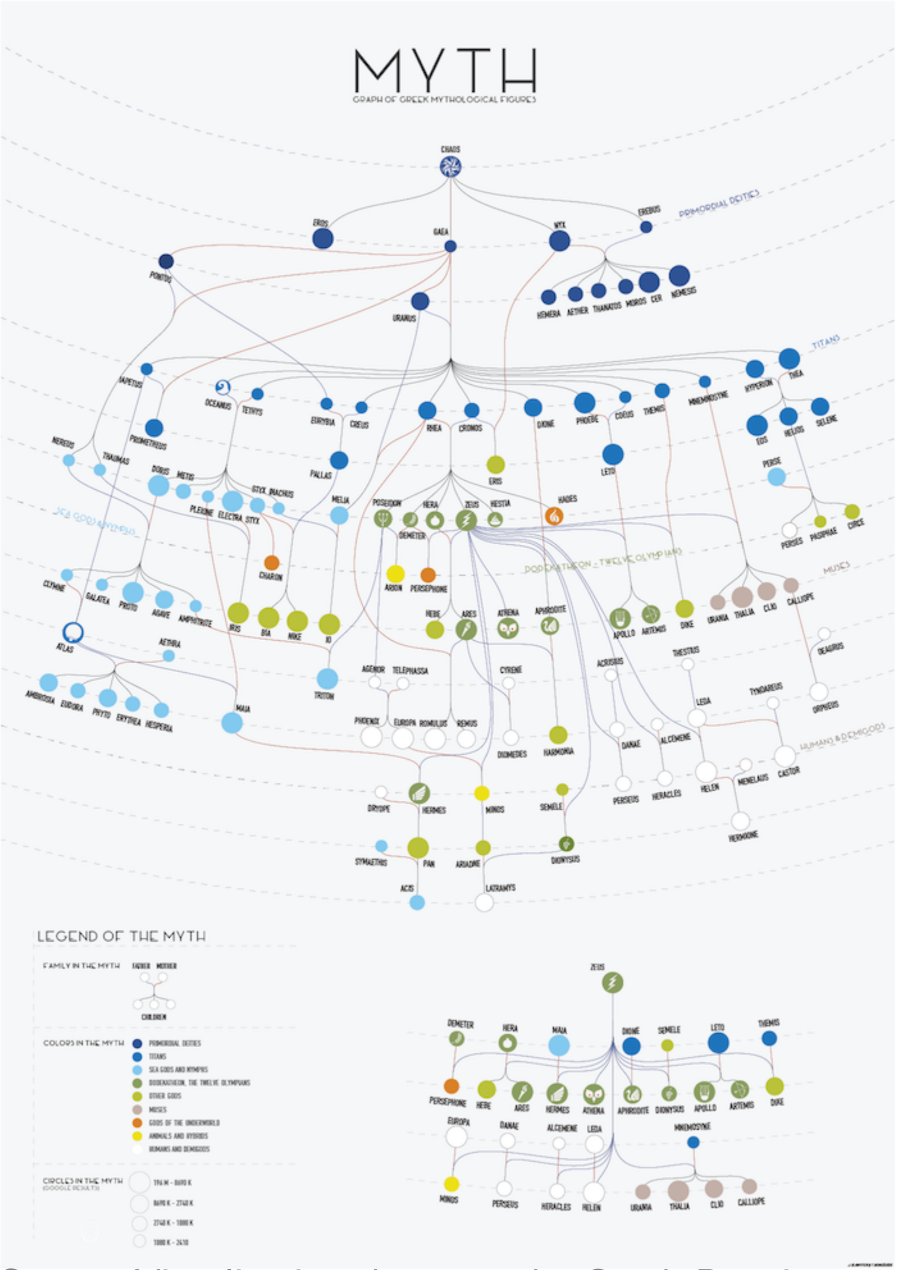
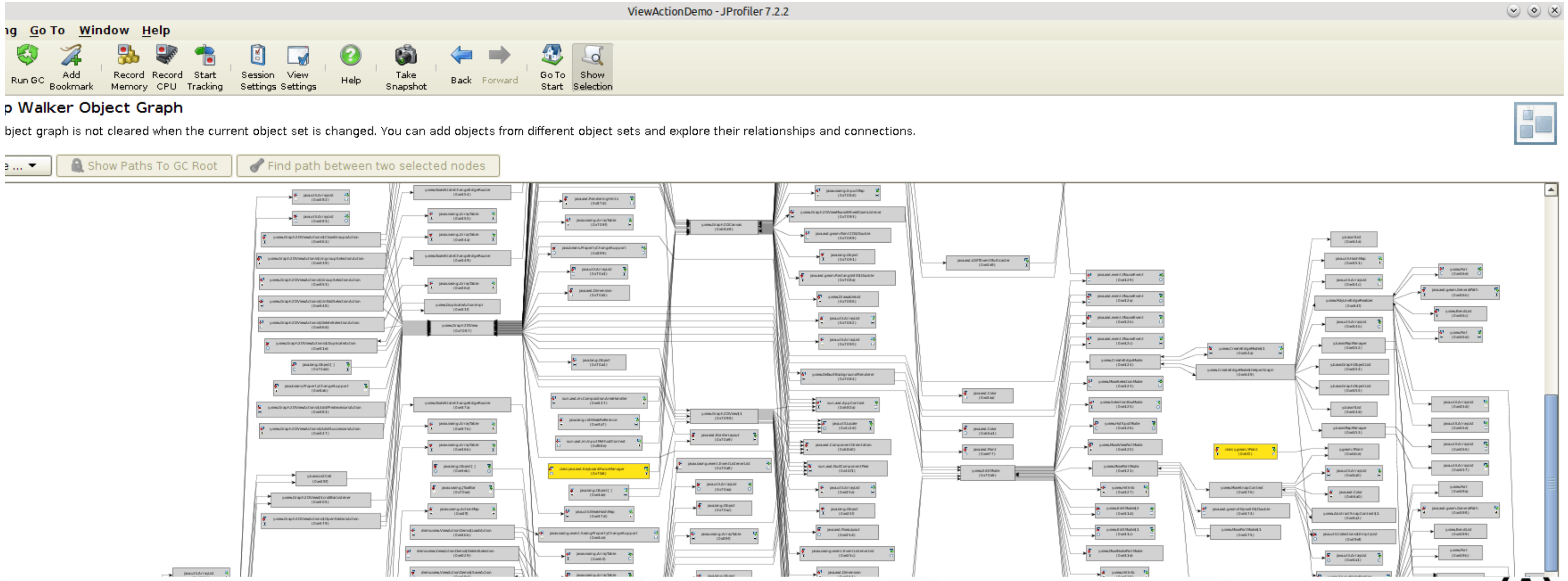
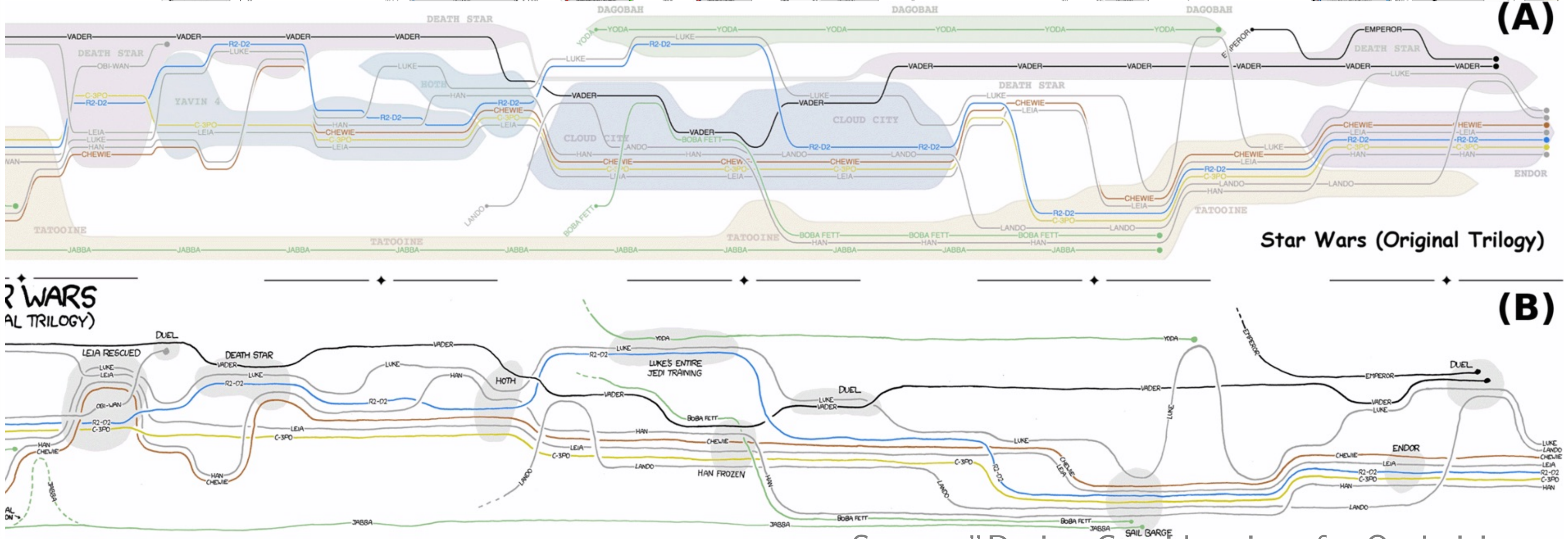






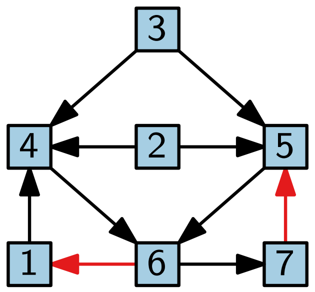
Source: Visualization that won the Graph Drawing contest 2016. Klawitter & Mchedlidze

Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda '81]

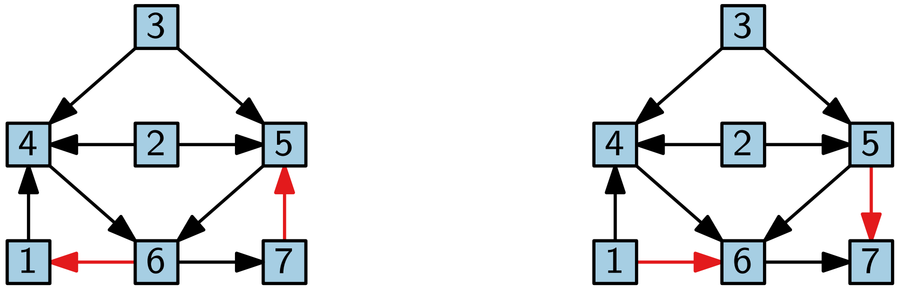
Input



Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda '81]

Input ⟶ Cycle breaking



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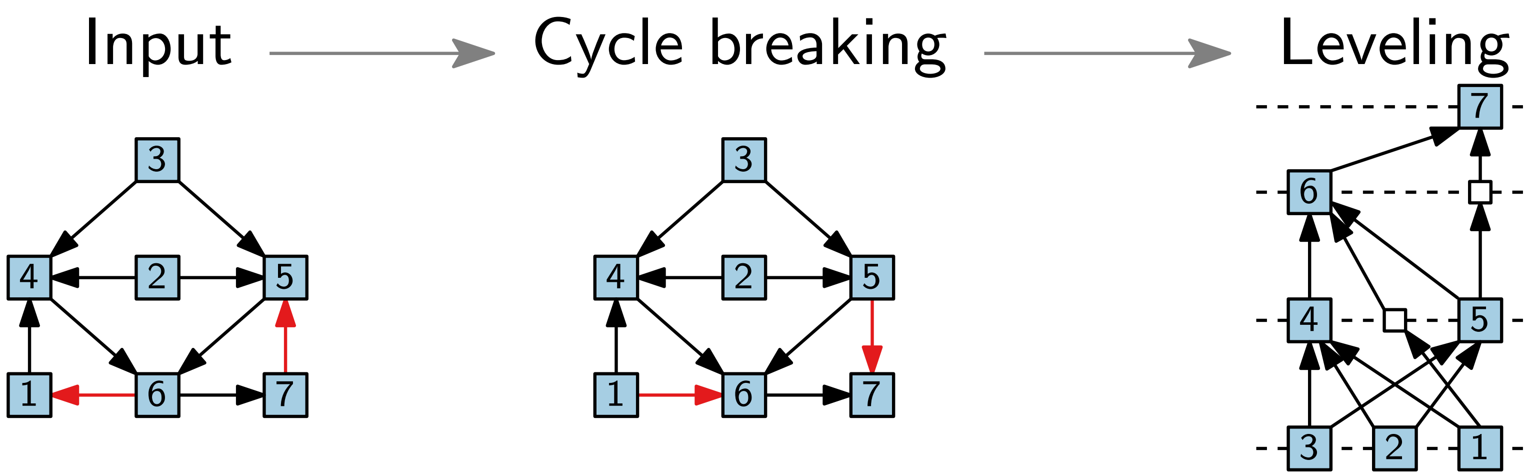

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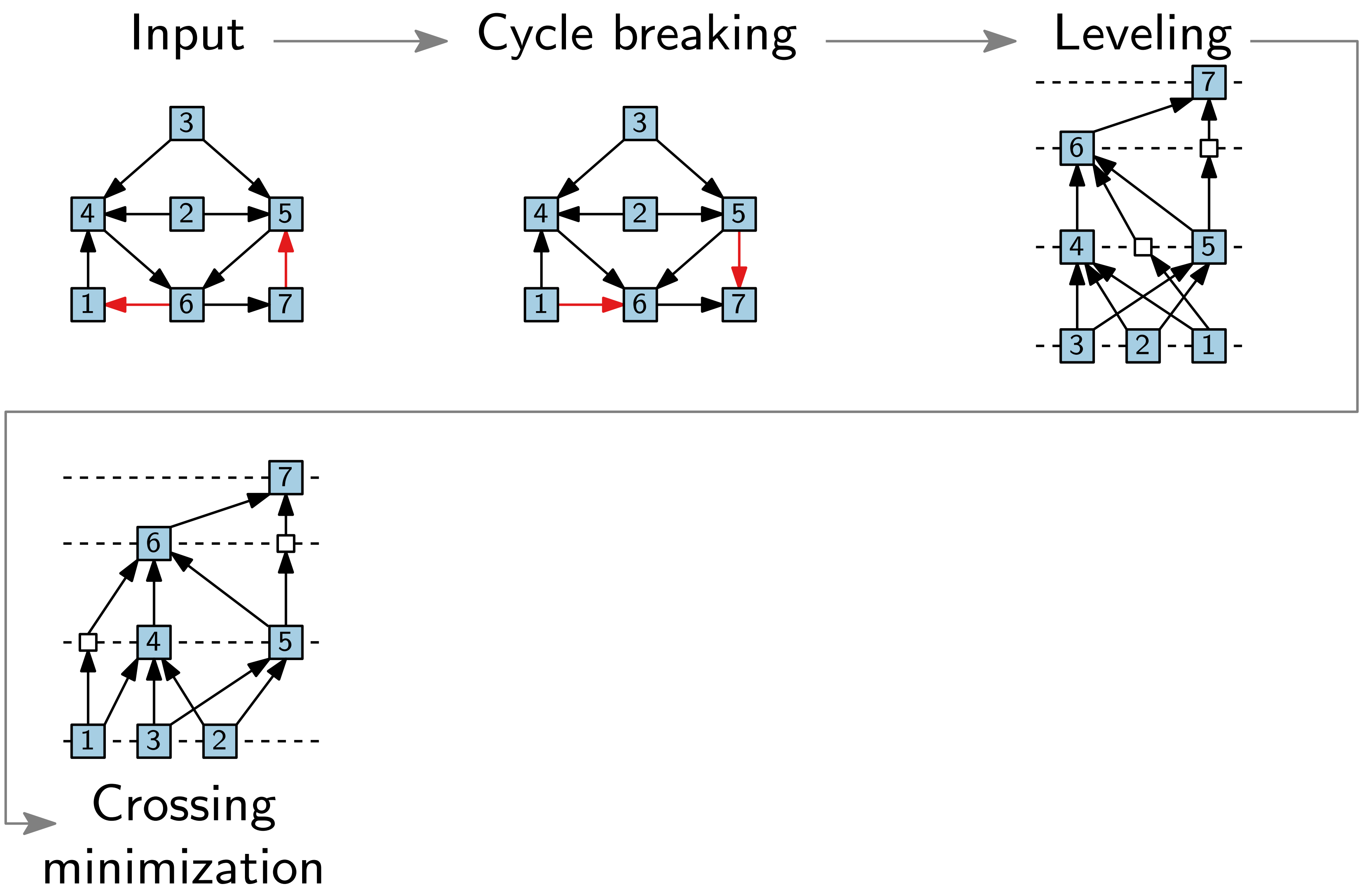

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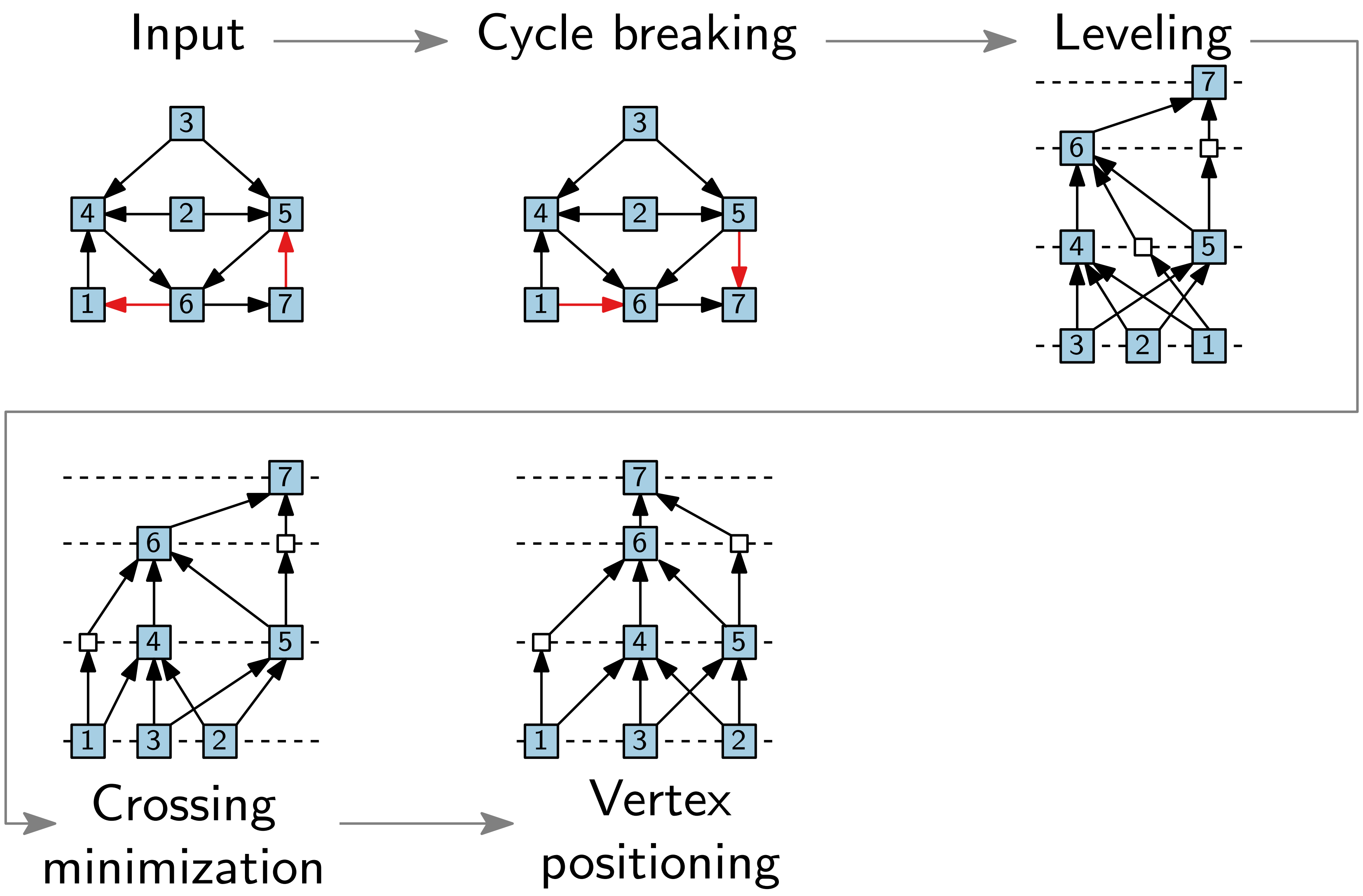

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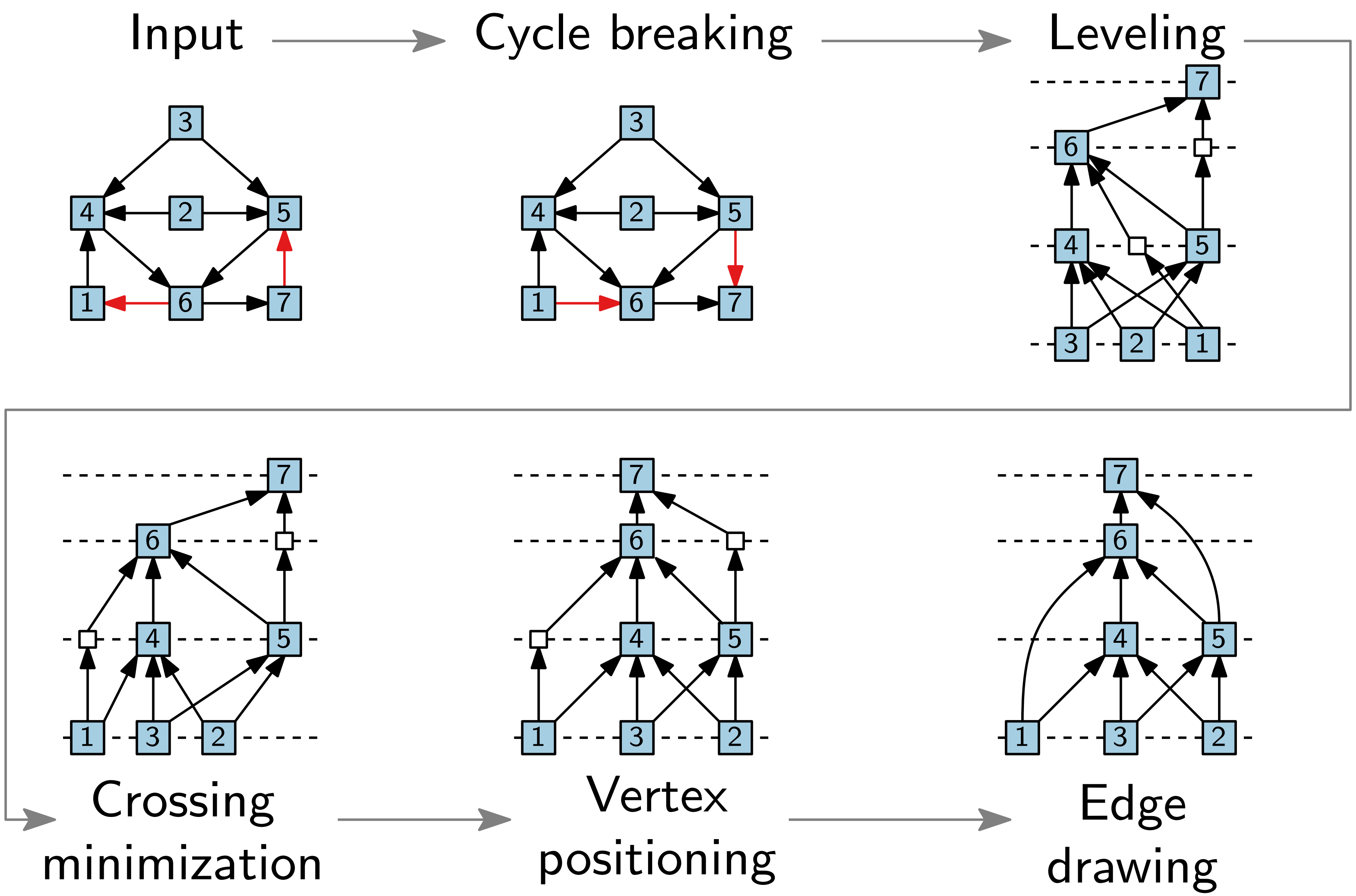
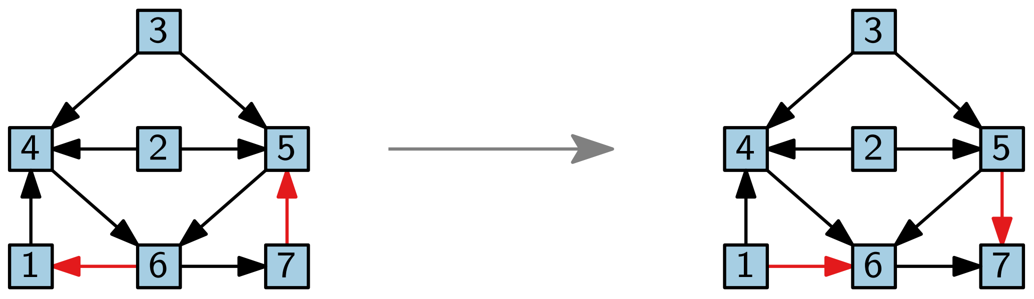


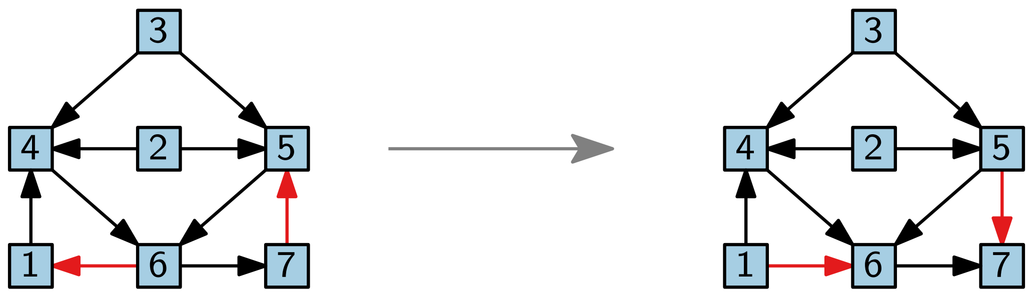
Step 1: Cycle breaking



Approach.

- Find minimum set $E^\star$ of edges which are not upward.
- Remove $E^\star$ and insert reversed edges.

Step 1: Cycle breaking



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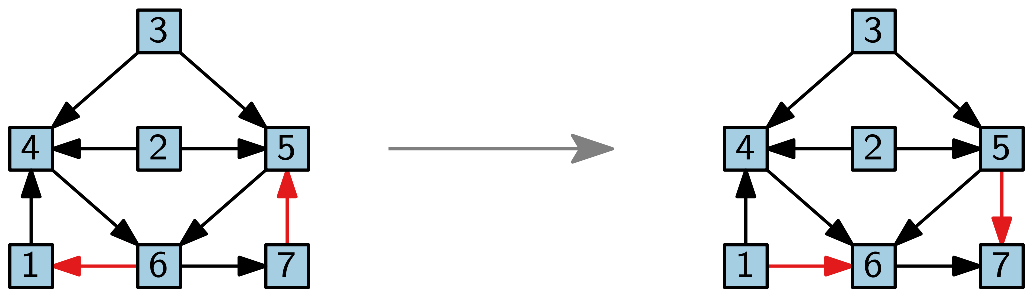
Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

Step 1: Cycle breaking



Approach.

- Find minimum set $E^\star$ of edges which are not upward.
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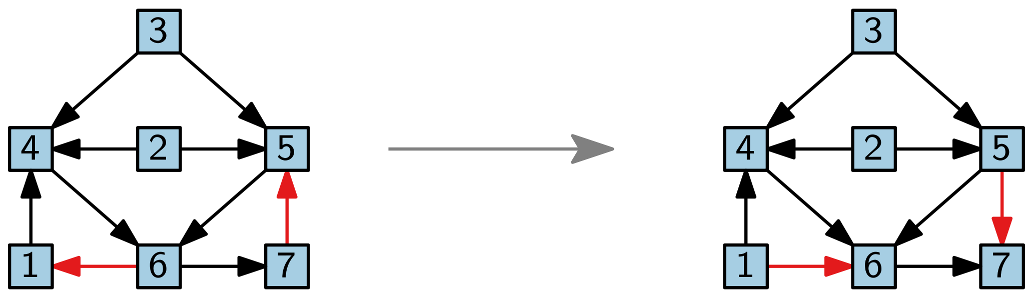
Problem MINIMUM FEEDBACK ~~ARC~~ SET **(F~~A~~S).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that ~~$G - E^\star$~~ acyclic

$G - E^\star + E_r^\star$

Step 1: Cycle breaking



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...NP-hard :-(

Step 1: Cycle breaking

Problem Minimum Feedback Arc Set **(FAS).**

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Step 1: Cycle breaking

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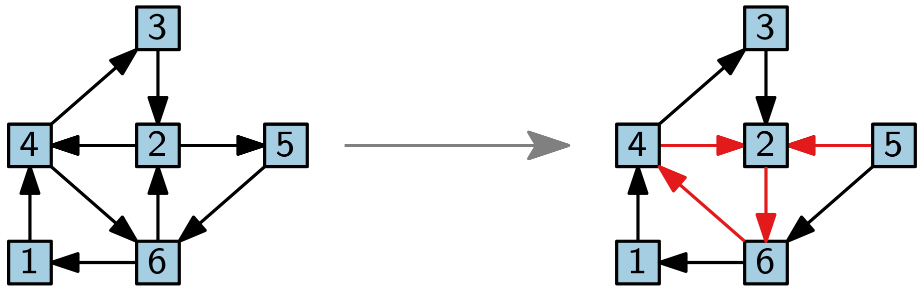

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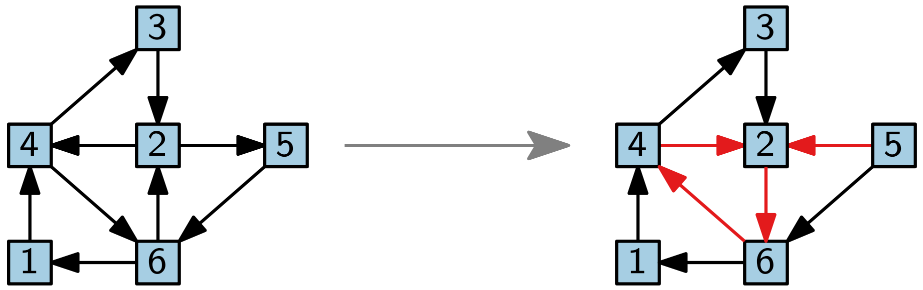
$G - E^\star + E_r^\star$ not acyclic



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- Input: directed graph $G = (V, E)$
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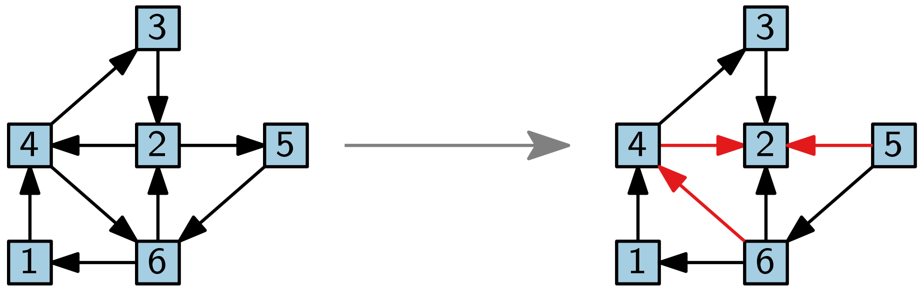
If $E^\star$ is minimum: $G - E^\star + E_r^\star$ **is** acyclic



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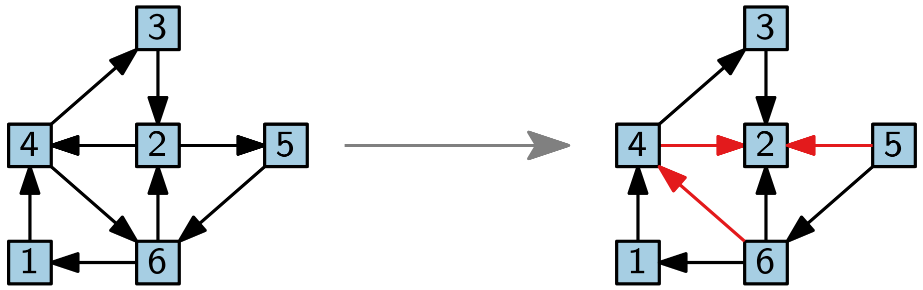

Problem MINIMUM FEEDBACK SET **(FS).**

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Step 1: Cycle breaking

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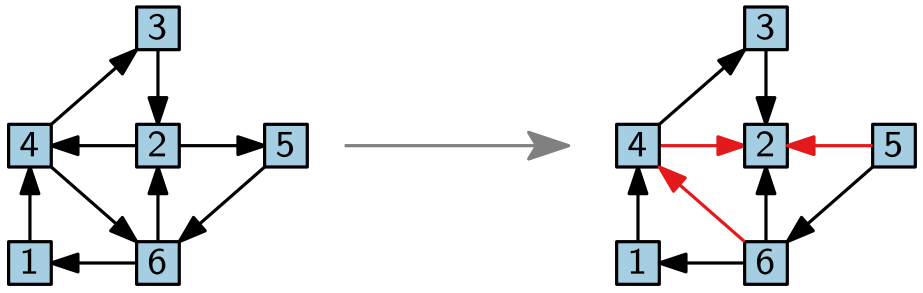

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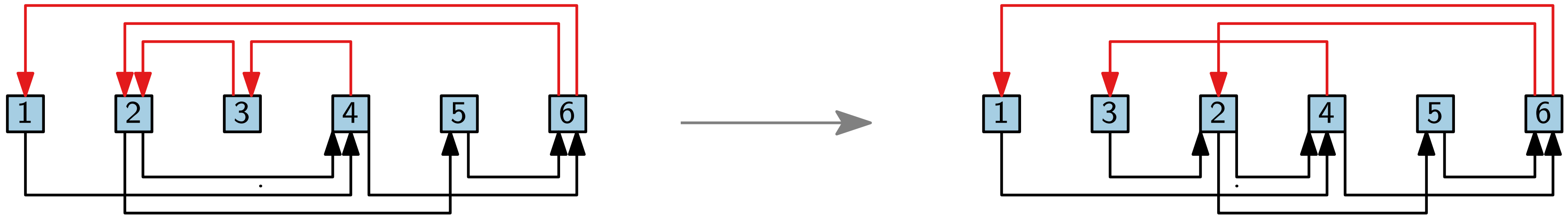

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

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  remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$

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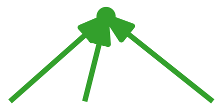

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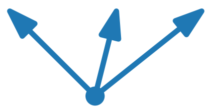

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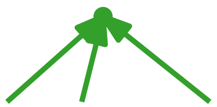

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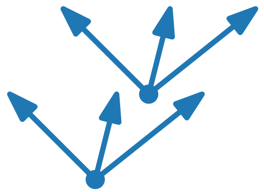

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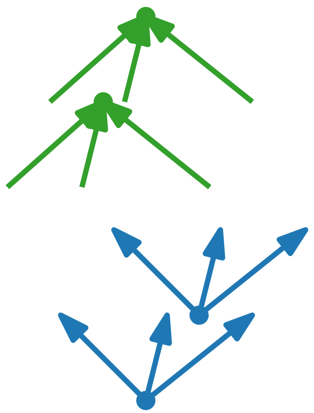

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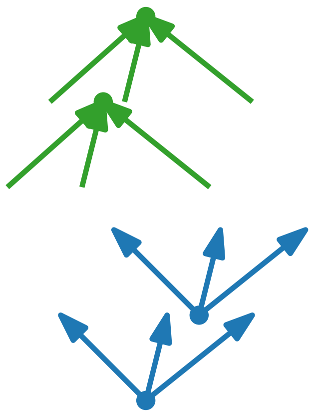

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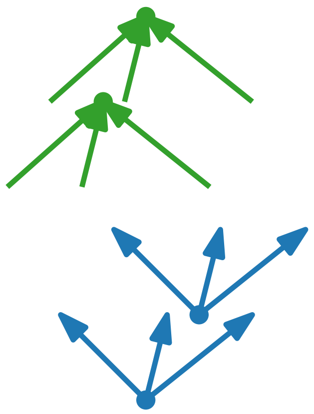
- $G' = (V, E')$ is a DAG
  - we create an order on V
- $E \setminus E'$ is a feedback arc set



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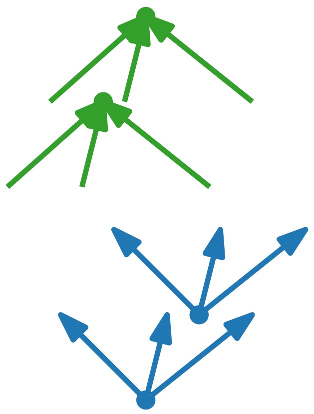

- $G' = (V, E')$ is a DAG
  - we create an order on V
- $E \setminus E'$ is a feedback arc set

- Time:

Heuristic 1

[Berger, Shor '90]

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foreach $v \in V$ **do**
  if $|N^{\to}(v)| \geq |N^{\leftarrow}(v)|$ **then**
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    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  remove v and $N(v)$ from G.
return (V, E')

$$N^{\to}(v) := \{(v,u) | (v,u) \in E\}$$
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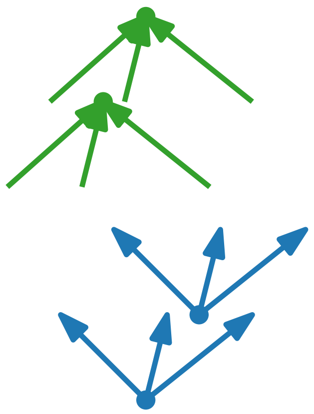

- $G' = (V, E')$ is a DAG
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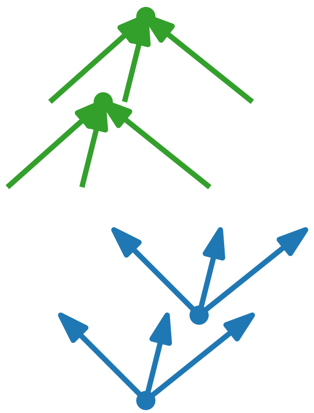

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  - we create an order on V
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- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq$

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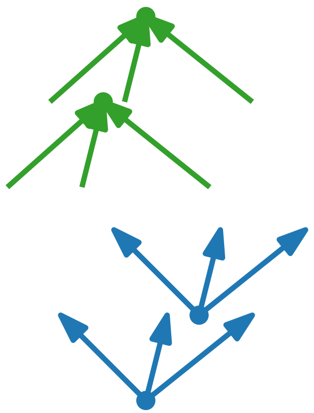

- $G' = (V, E')$ is a DAG
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- Time: $\mathcal{O}(|V| + |E|)$
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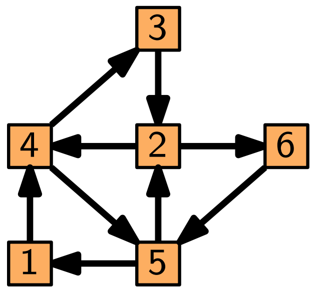

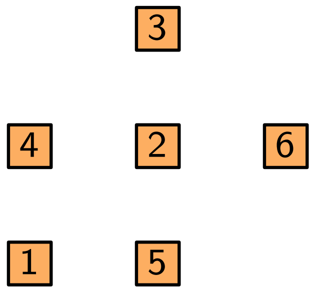

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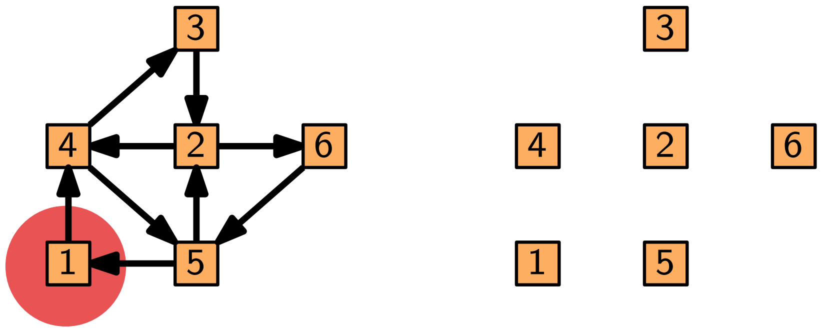

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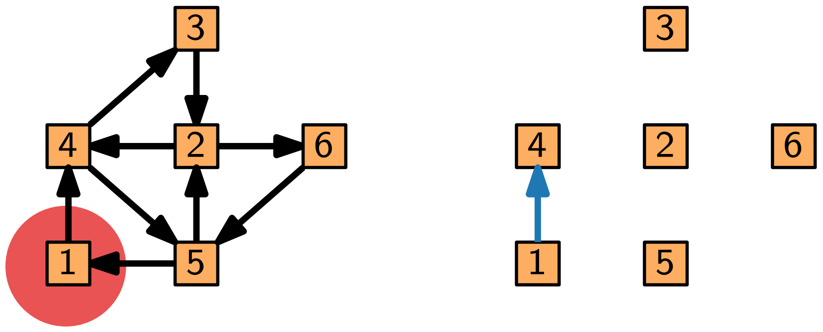

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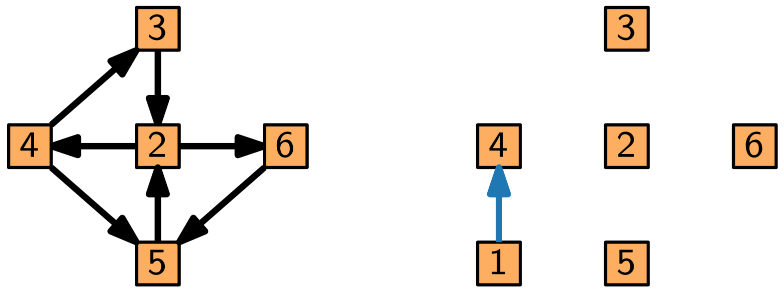

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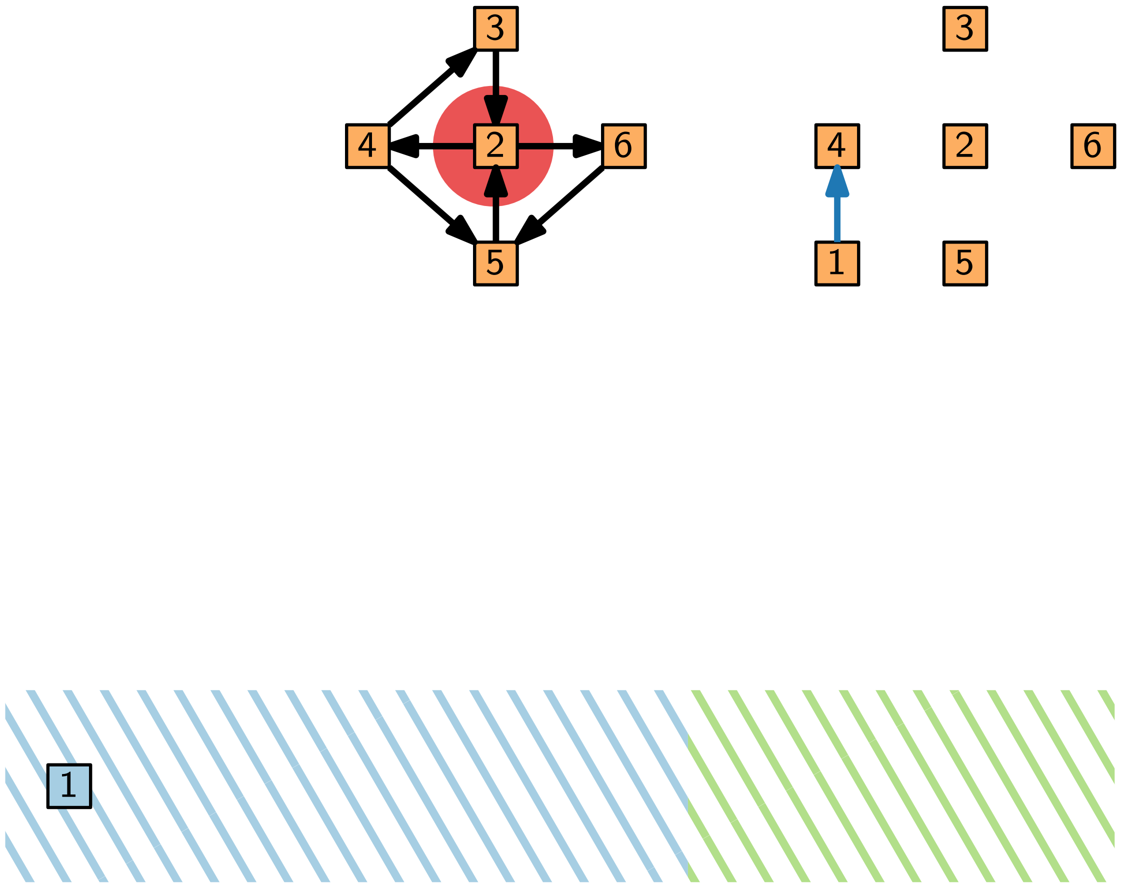

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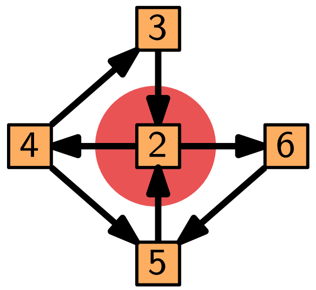

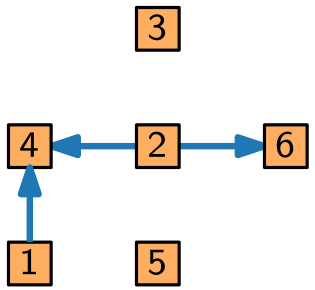

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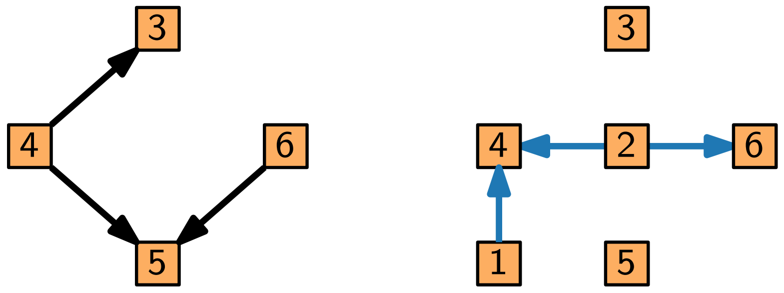

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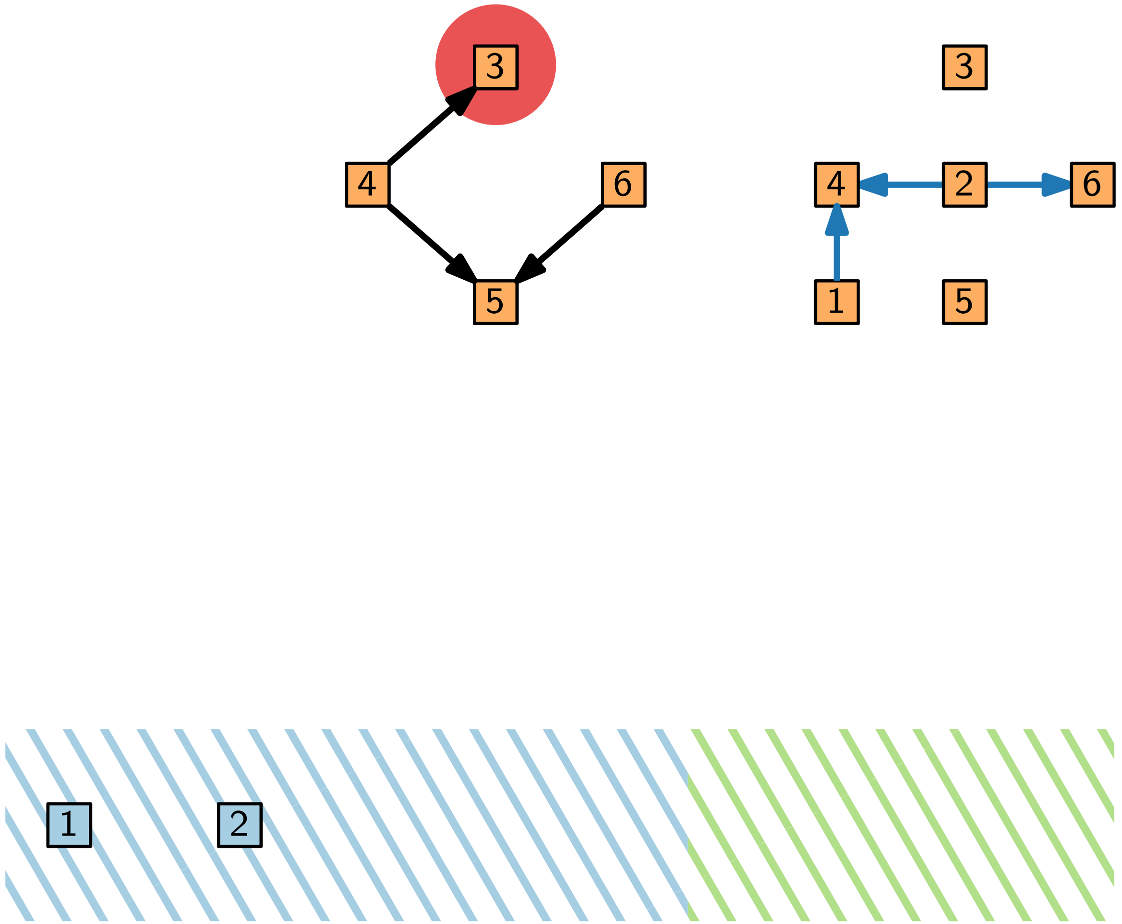

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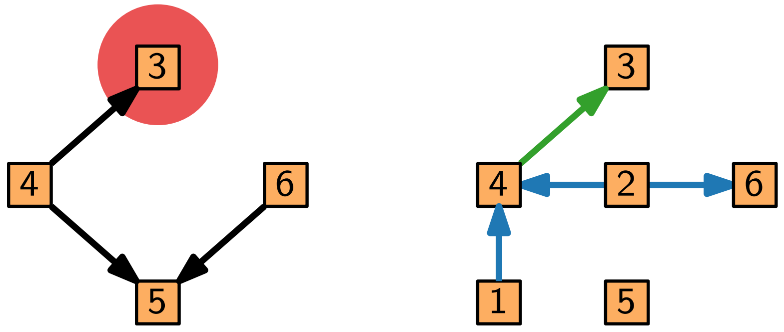

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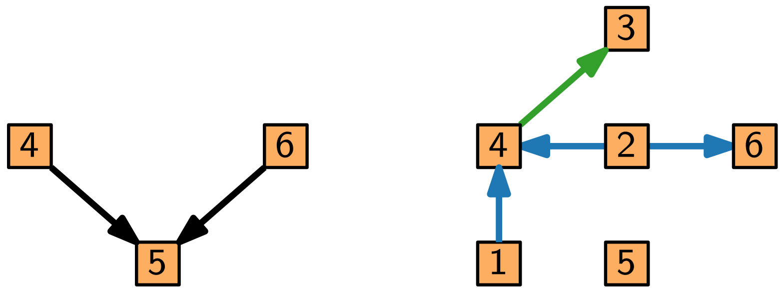

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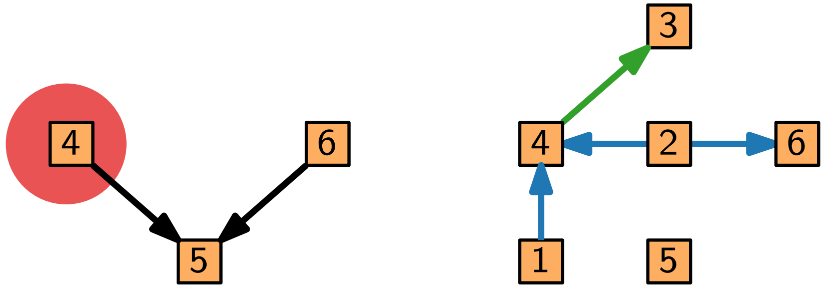

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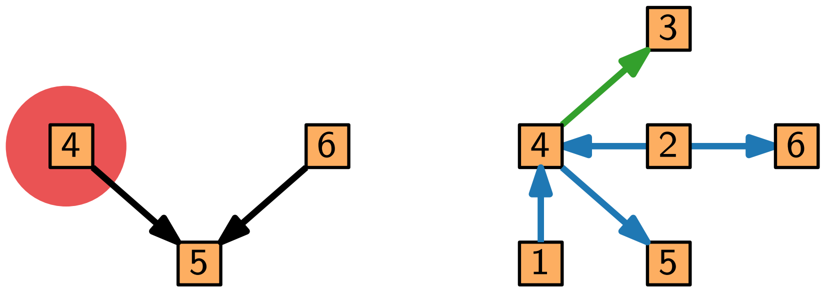

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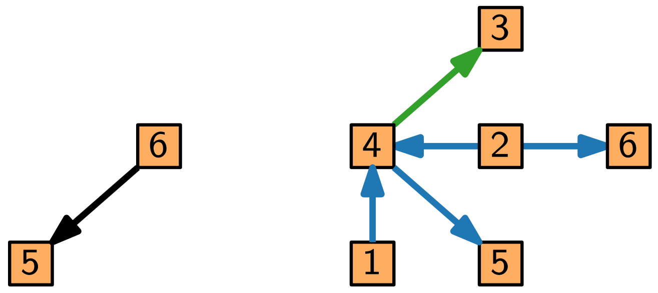

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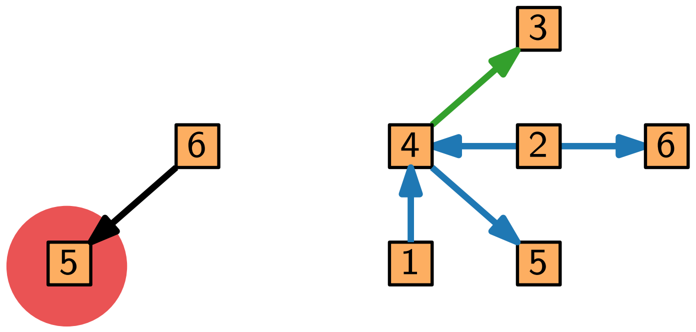

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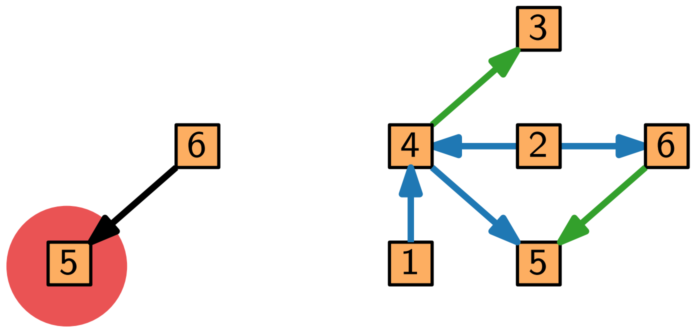

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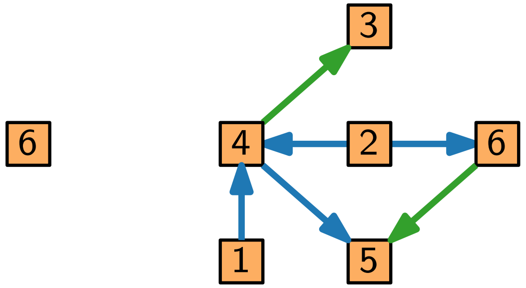

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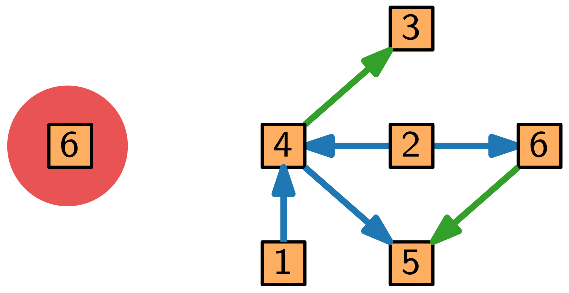

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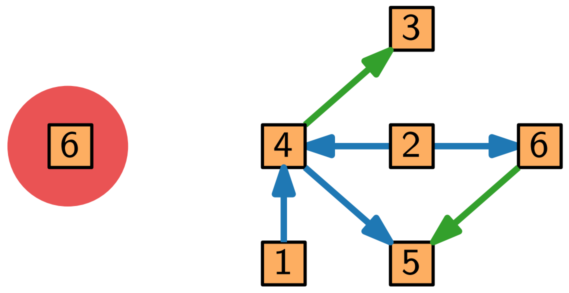

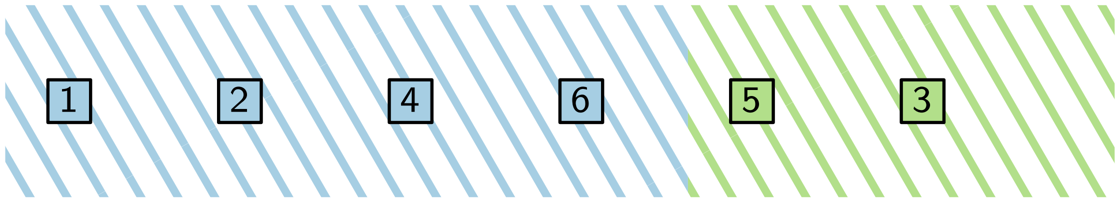

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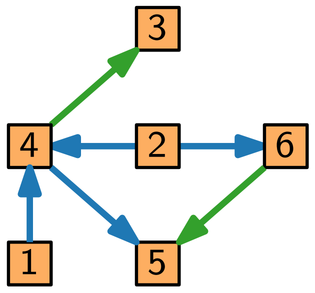

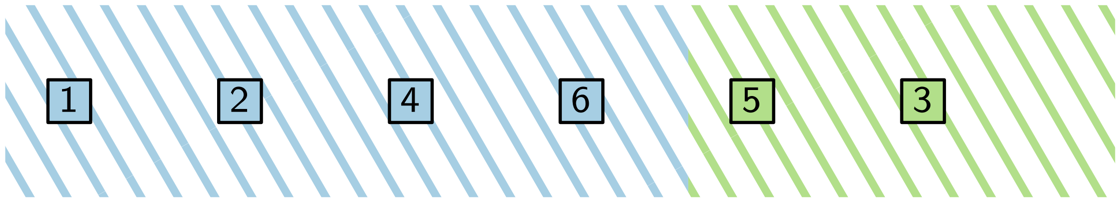

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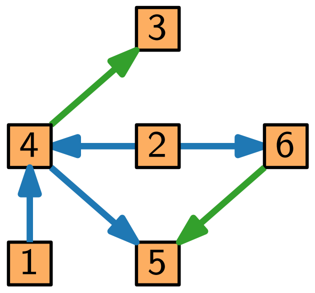

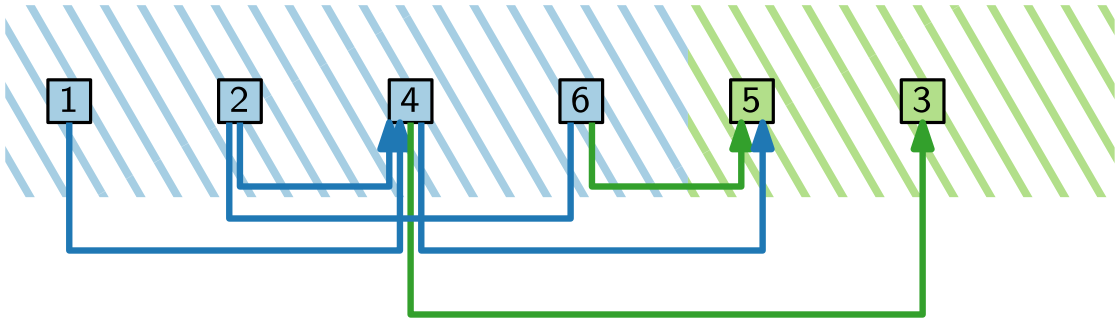

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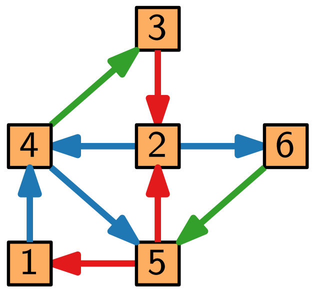

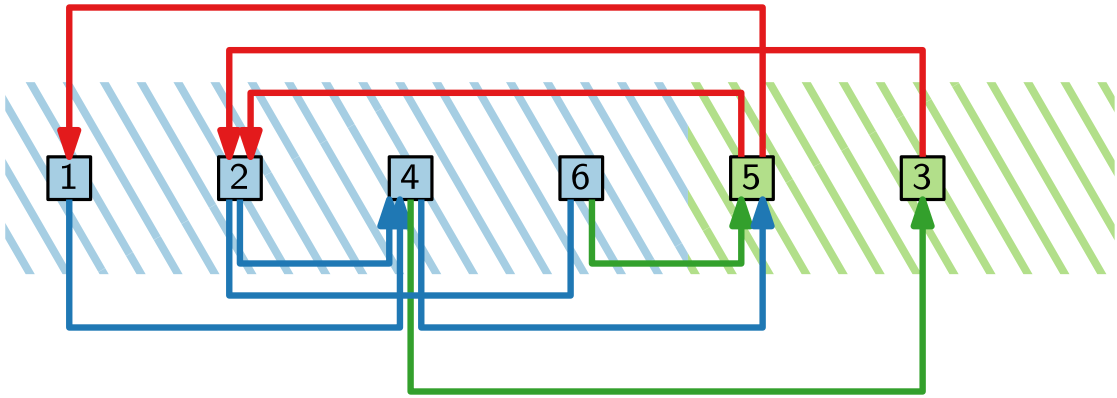

Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

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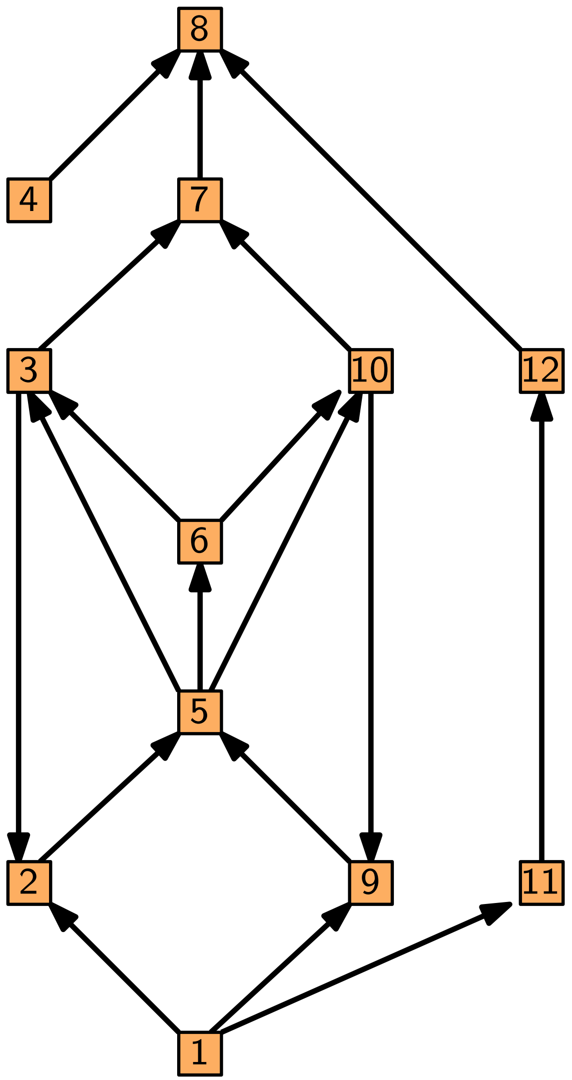

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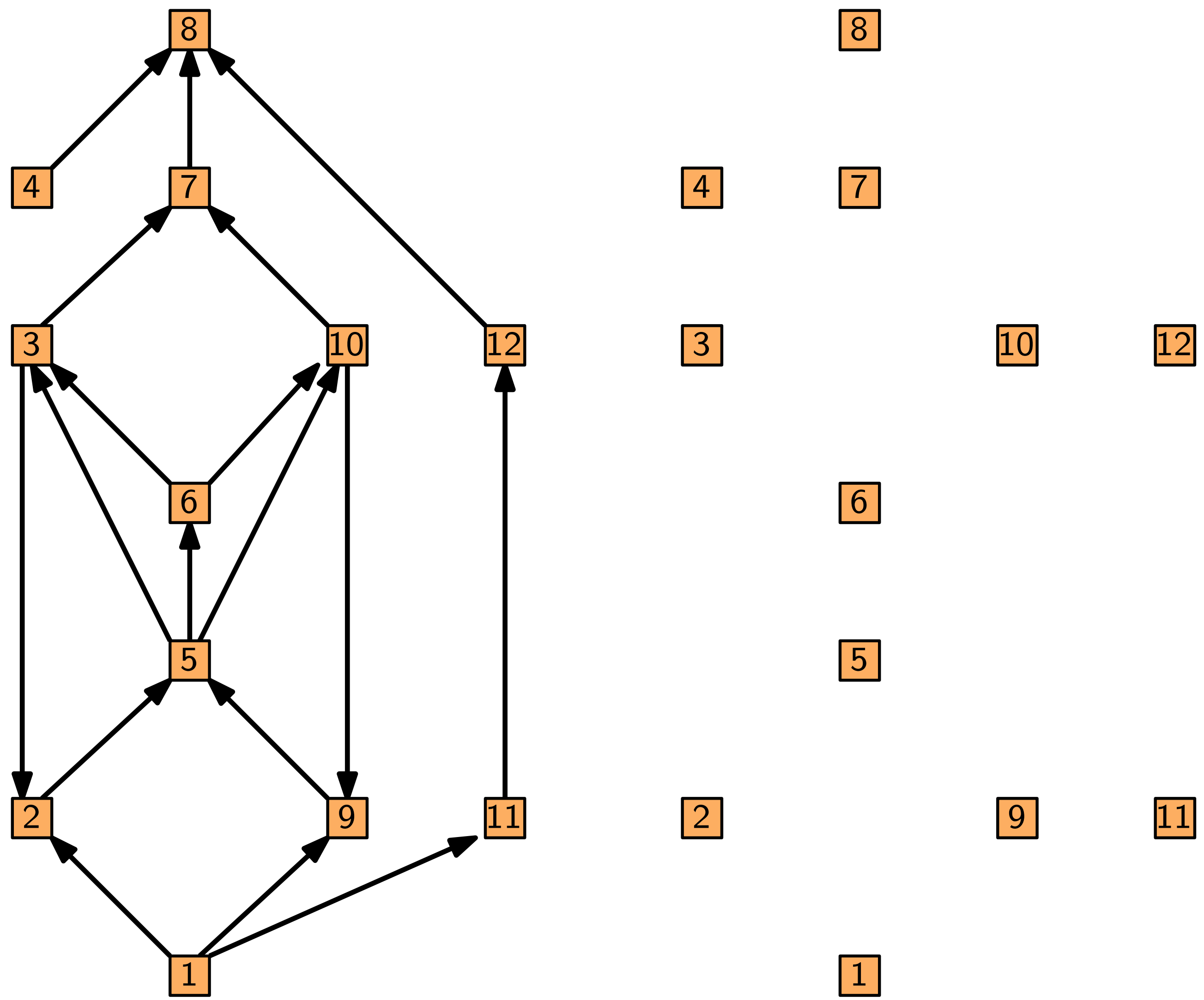

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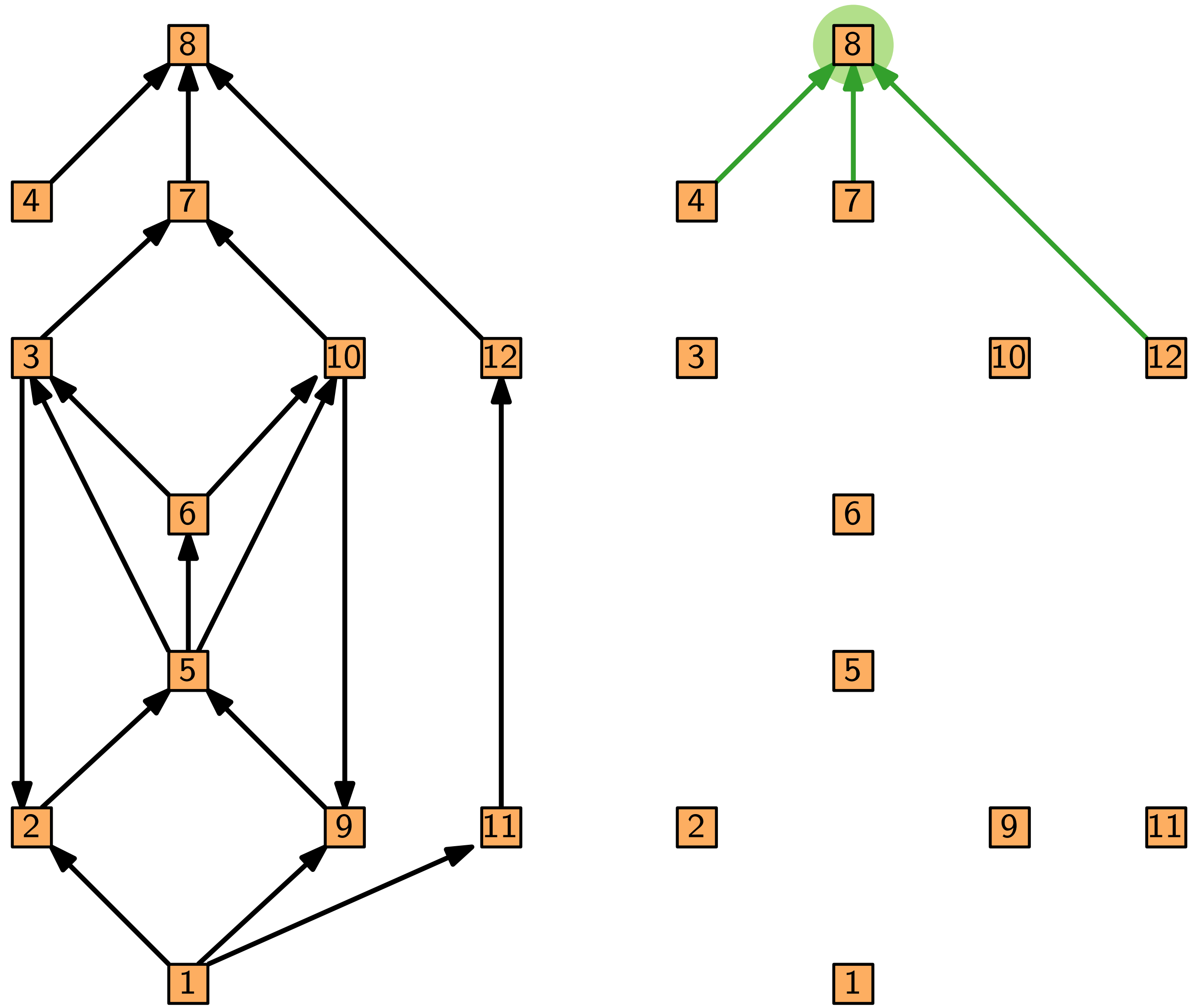

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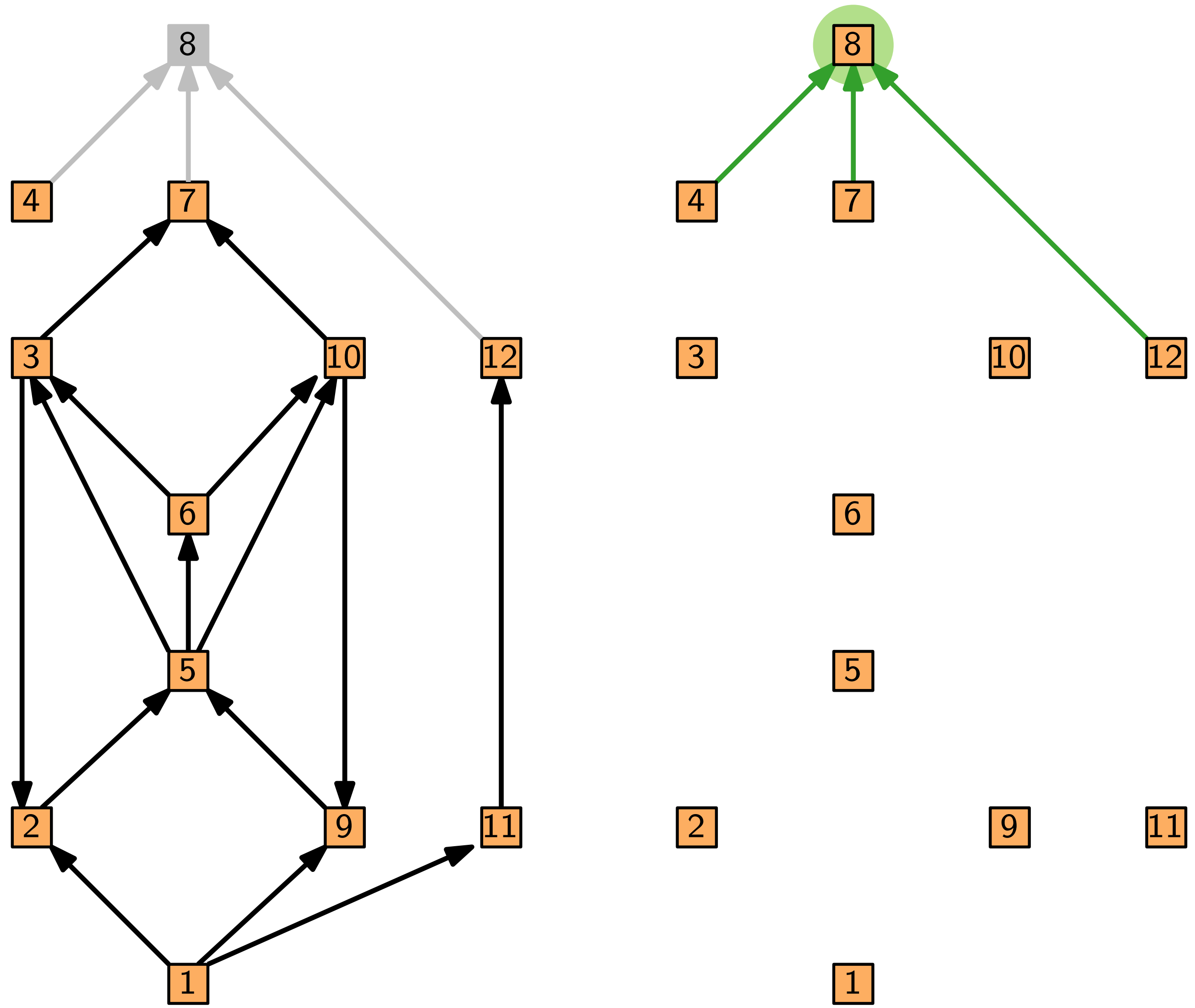

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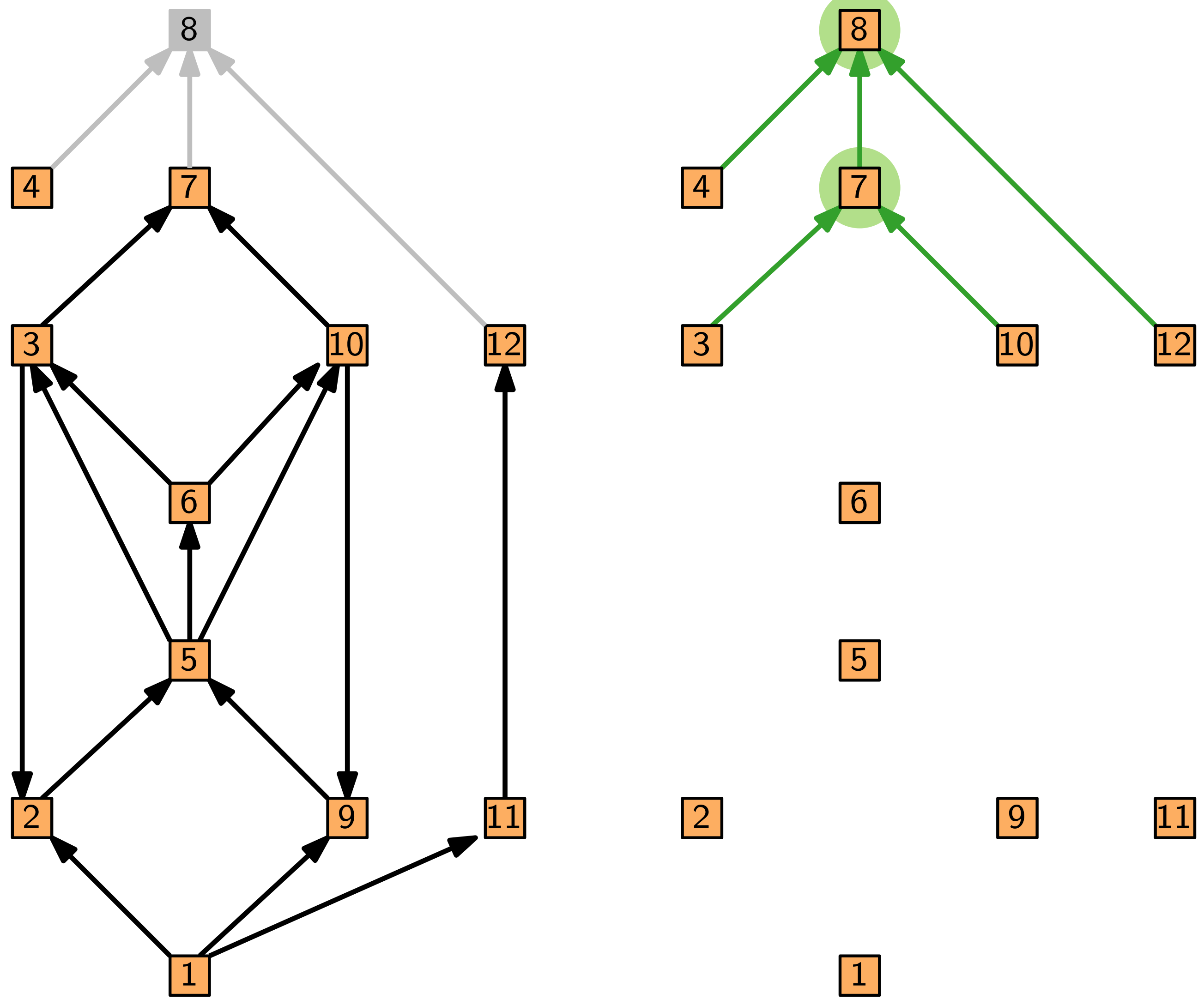

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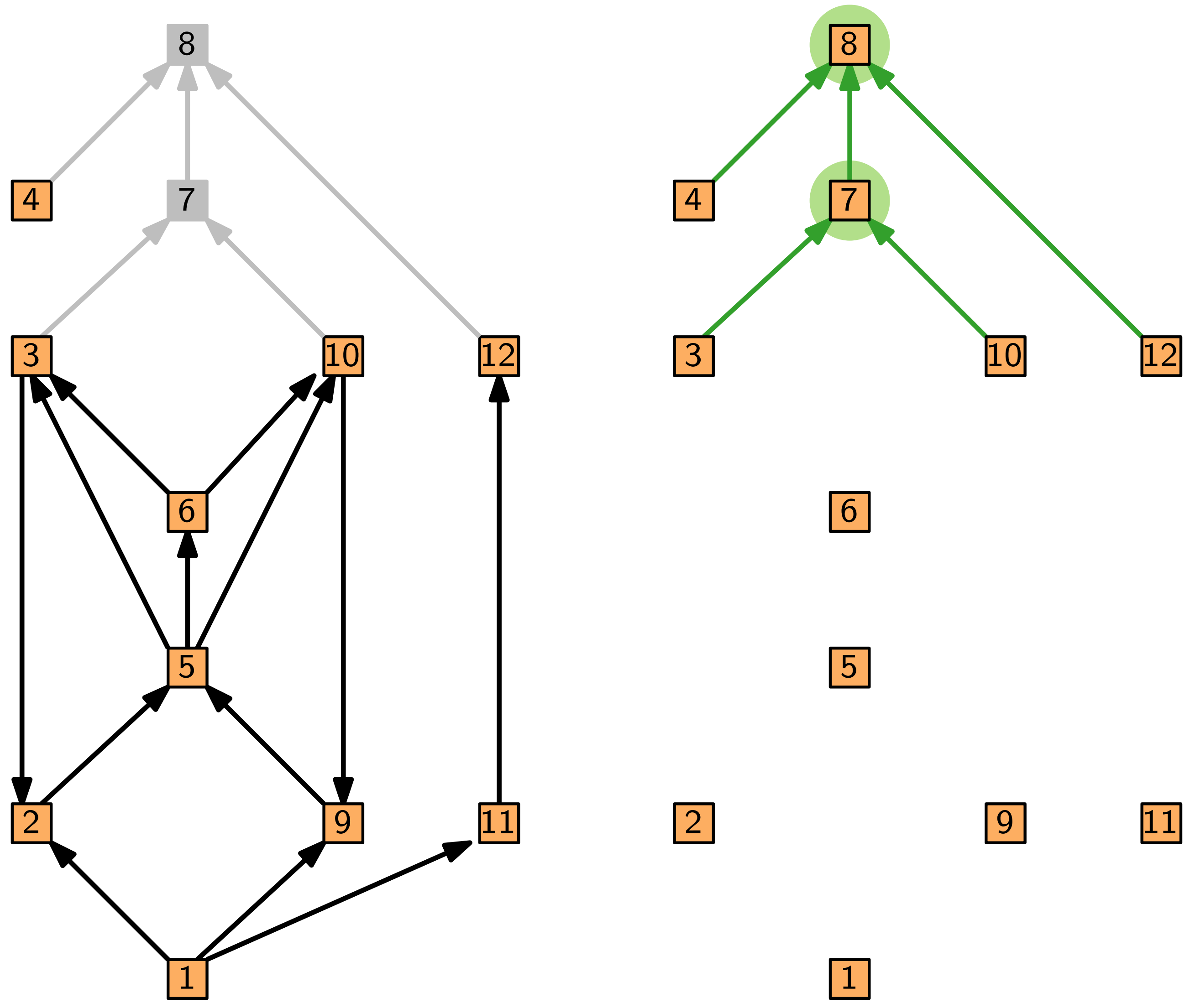

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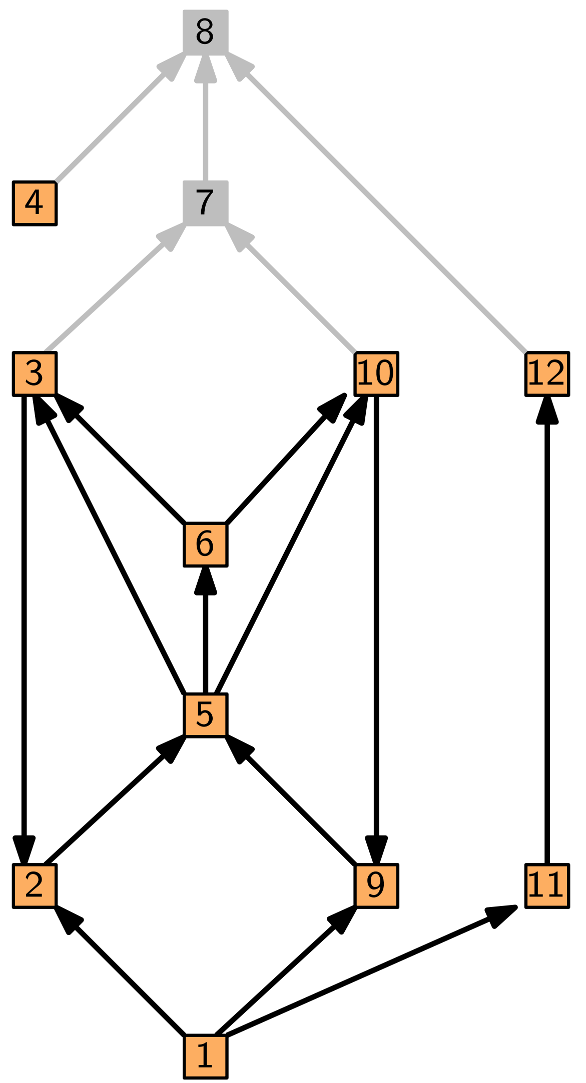

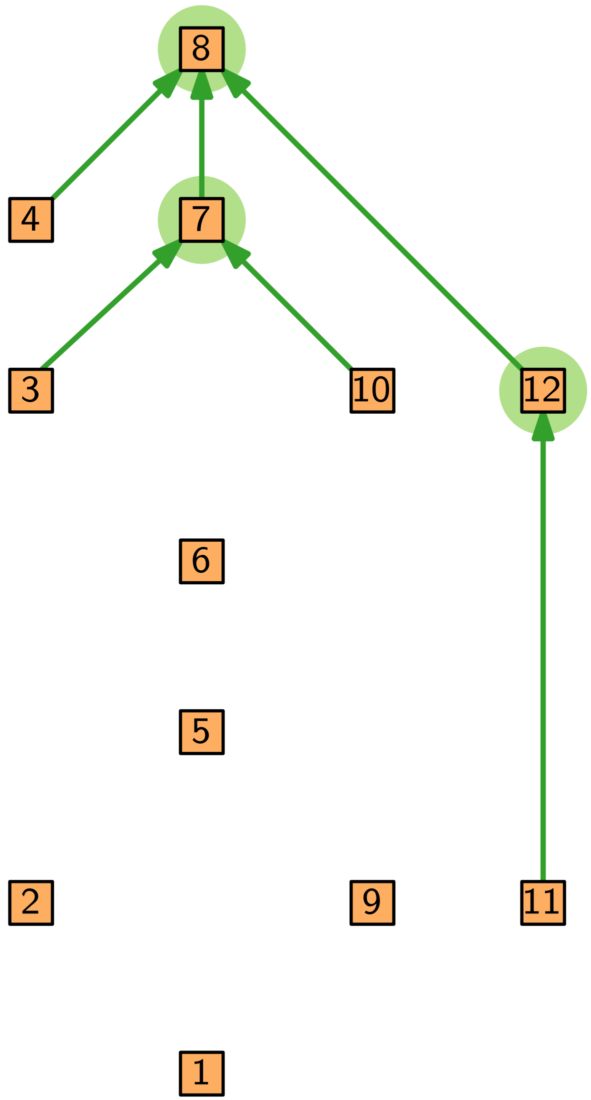

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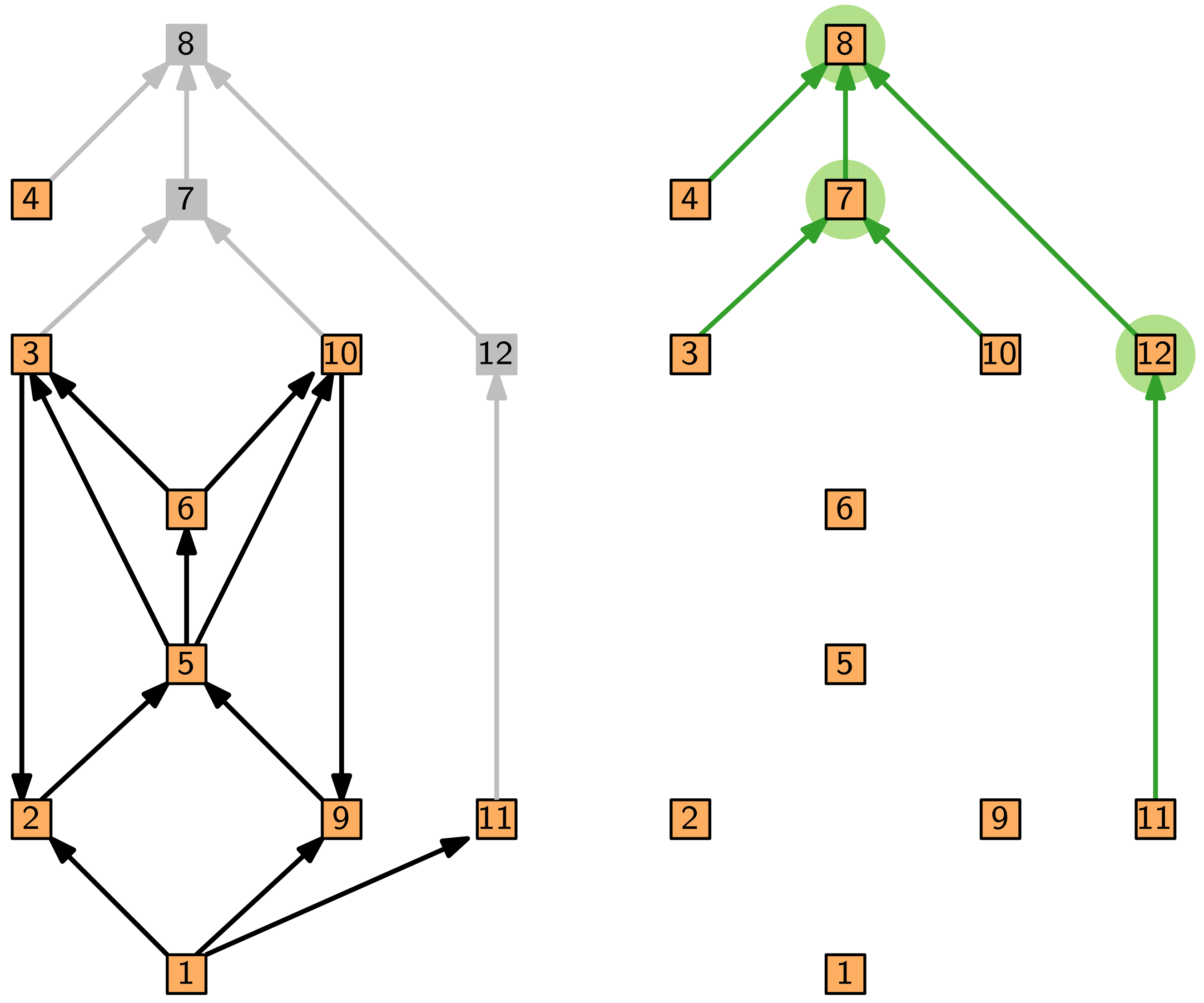

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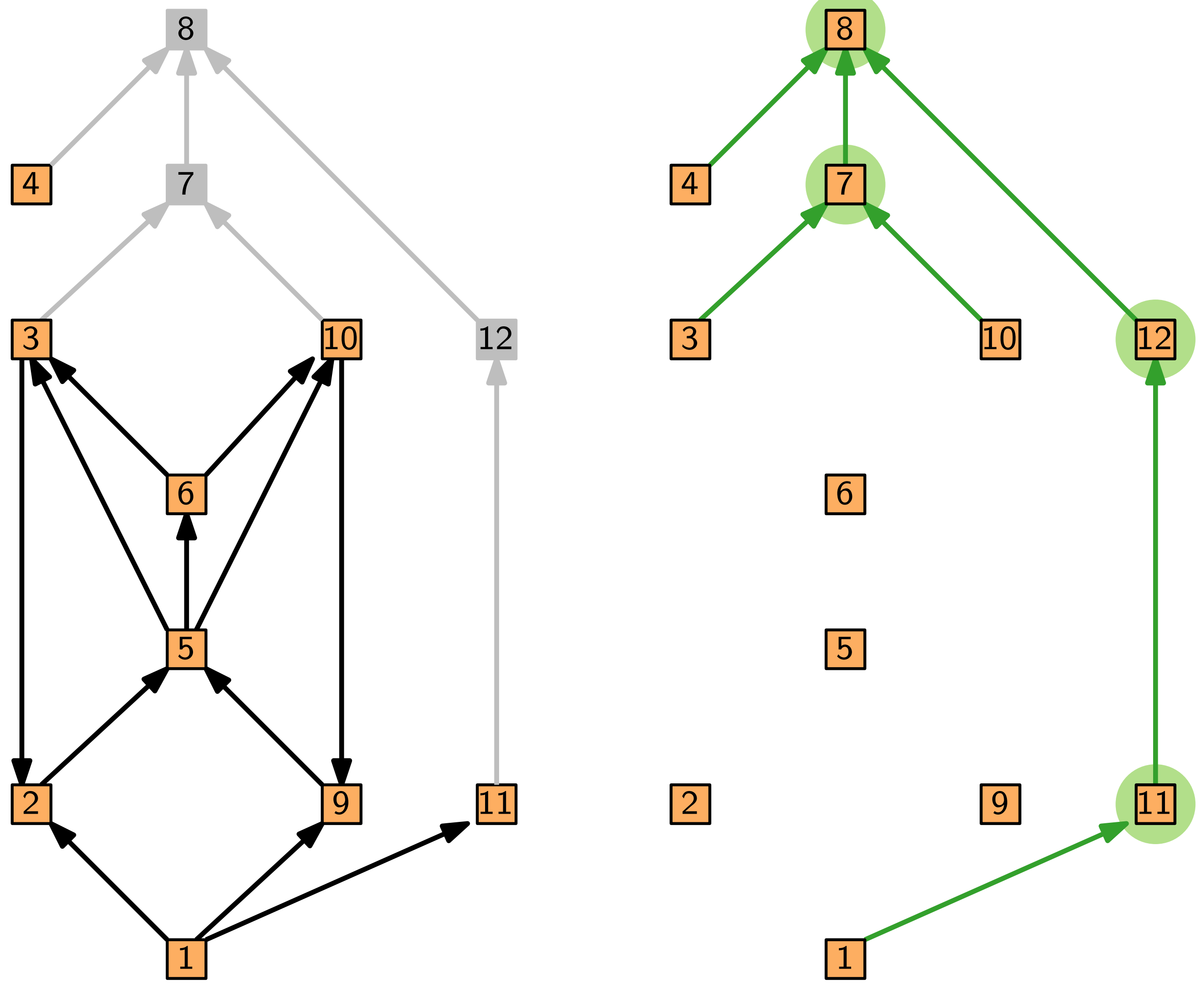

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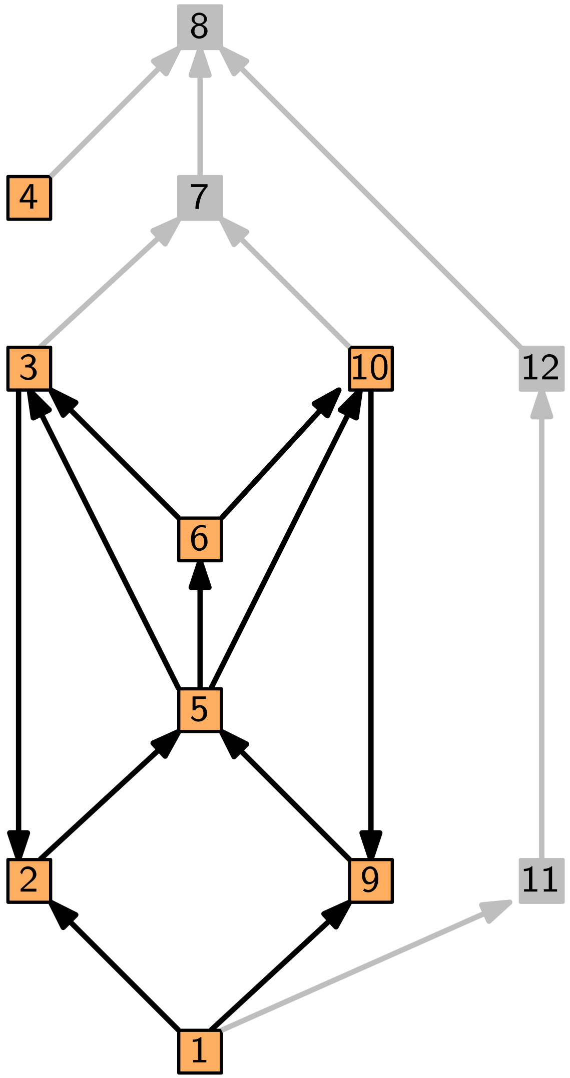

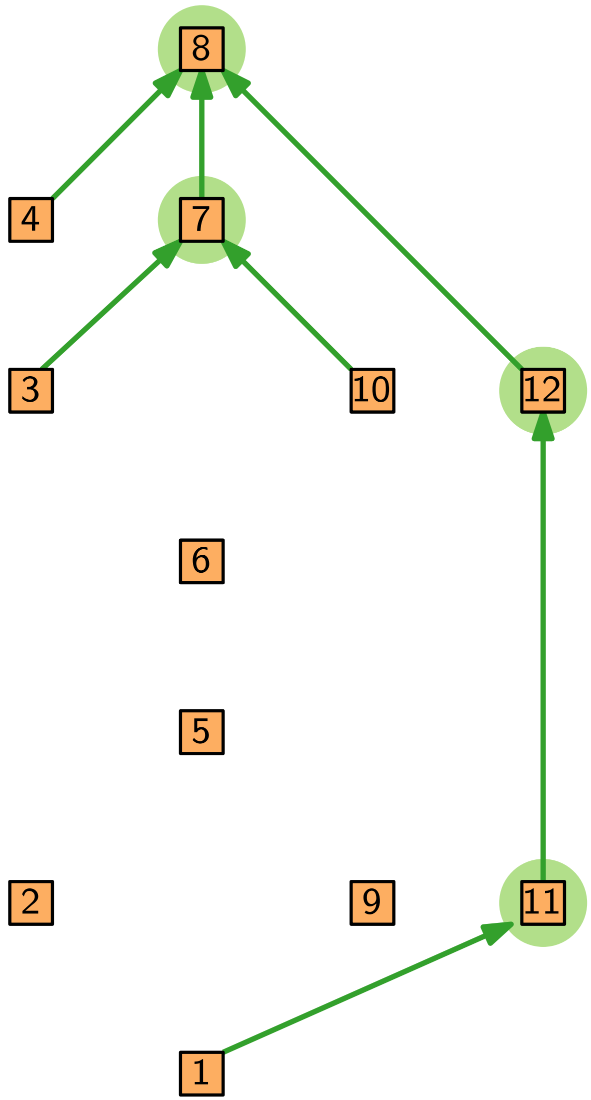

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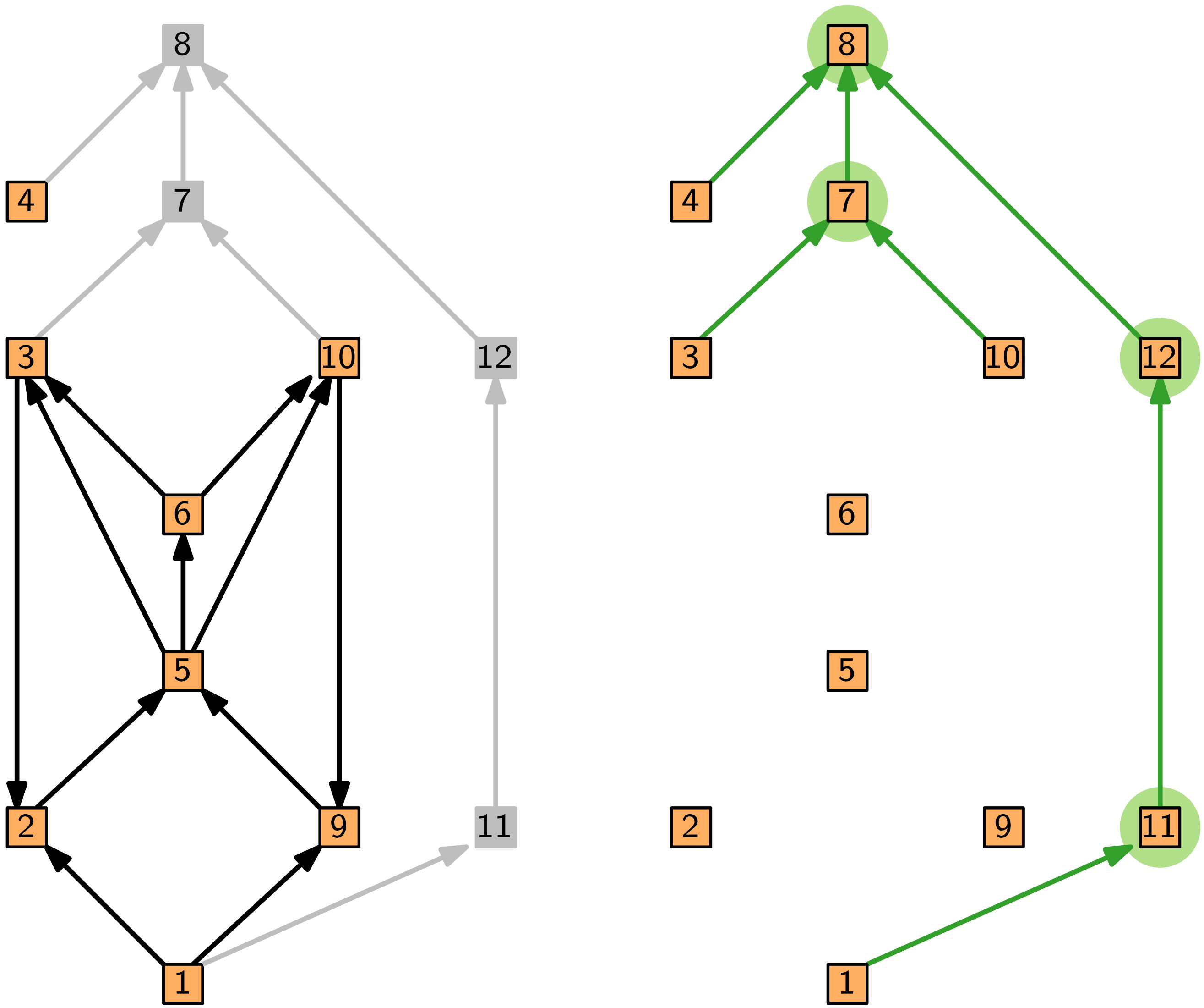

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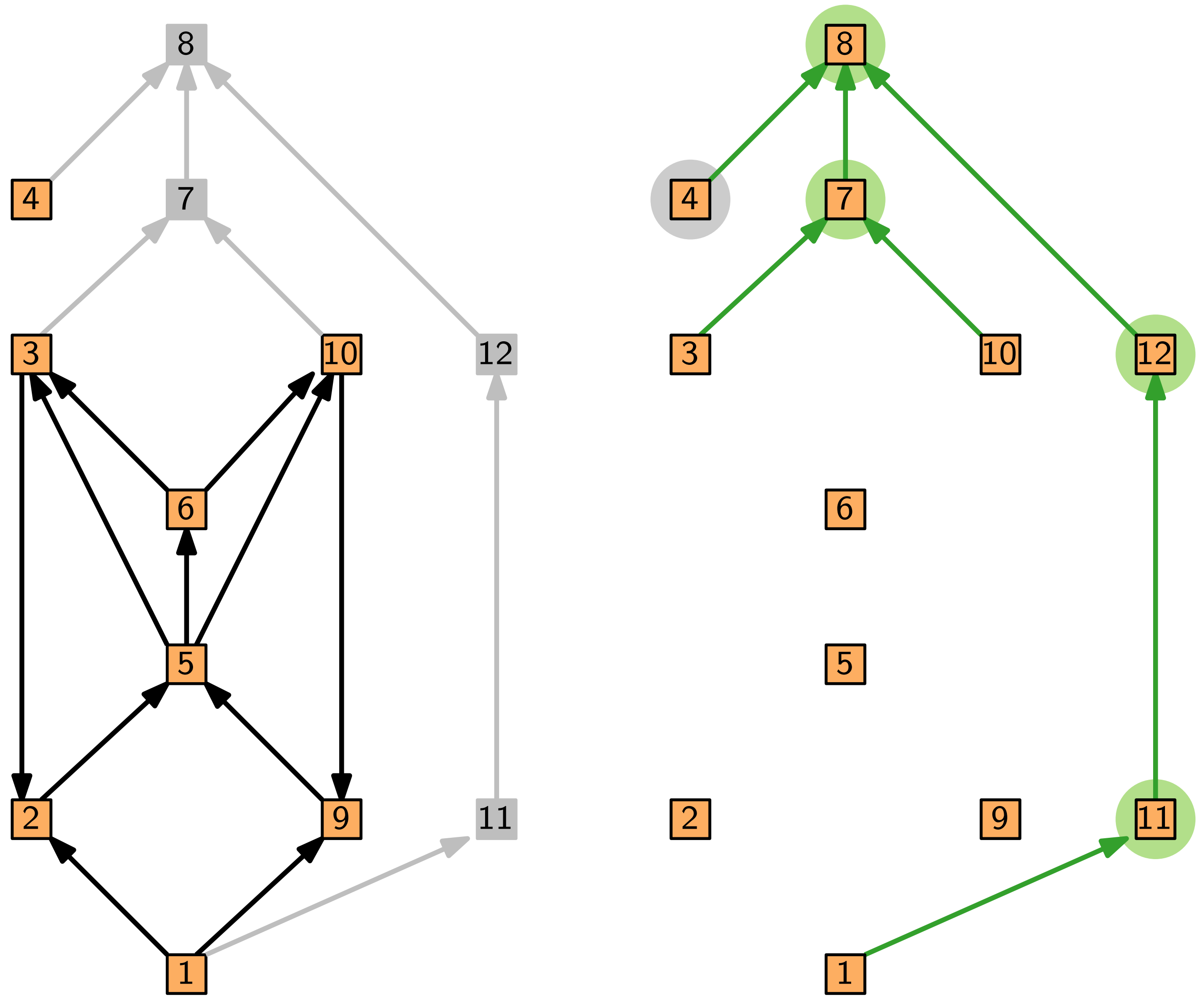

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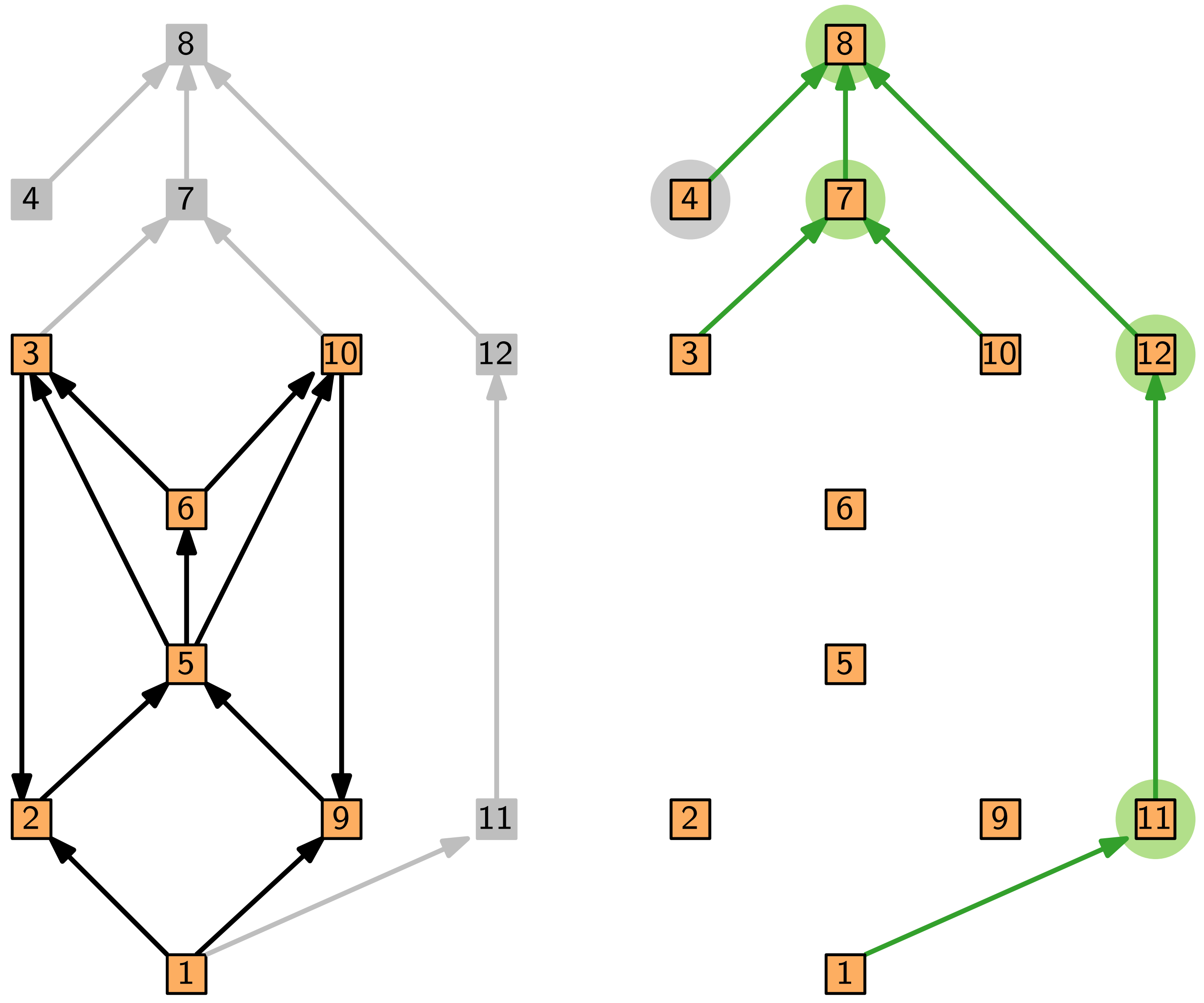

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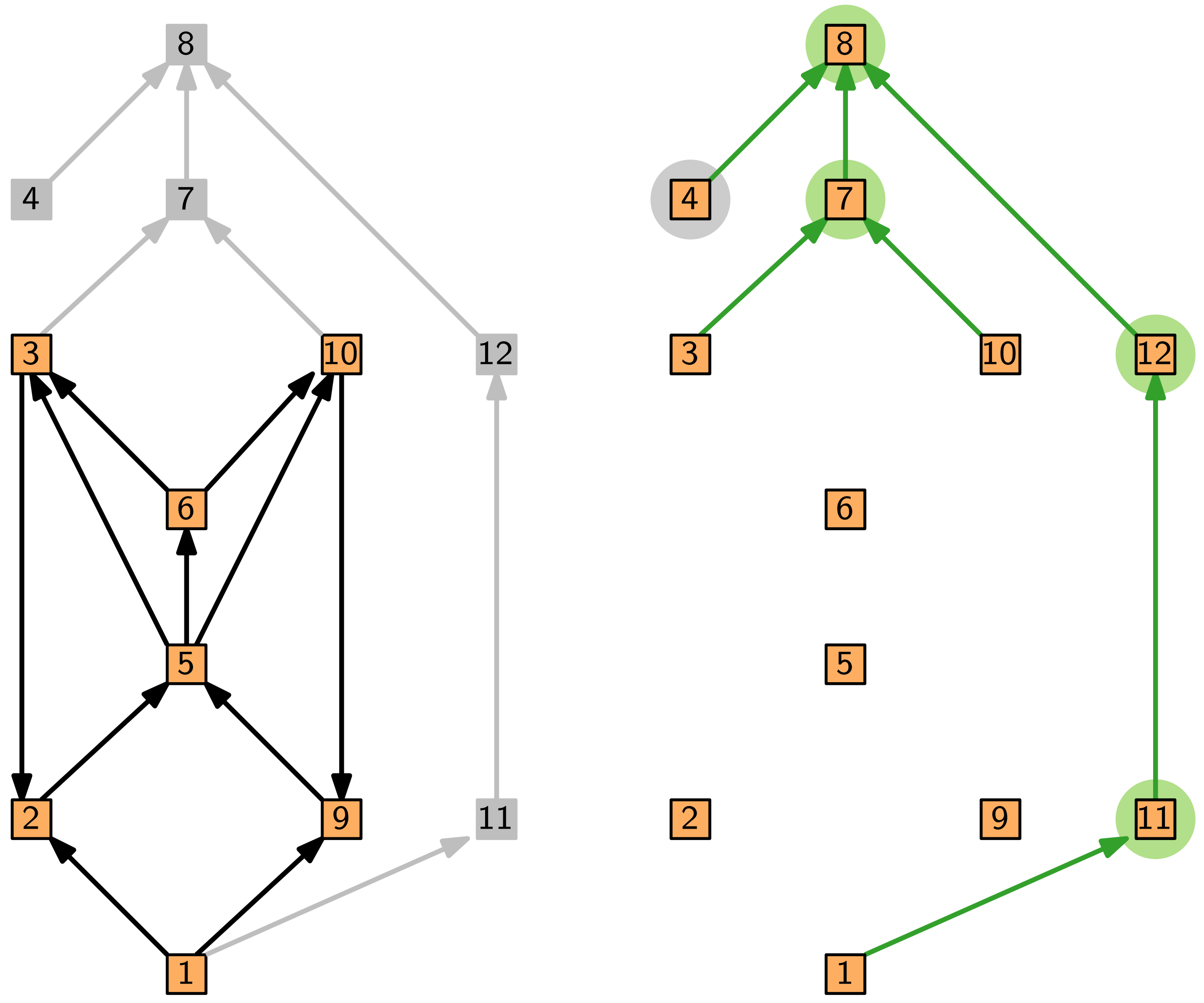

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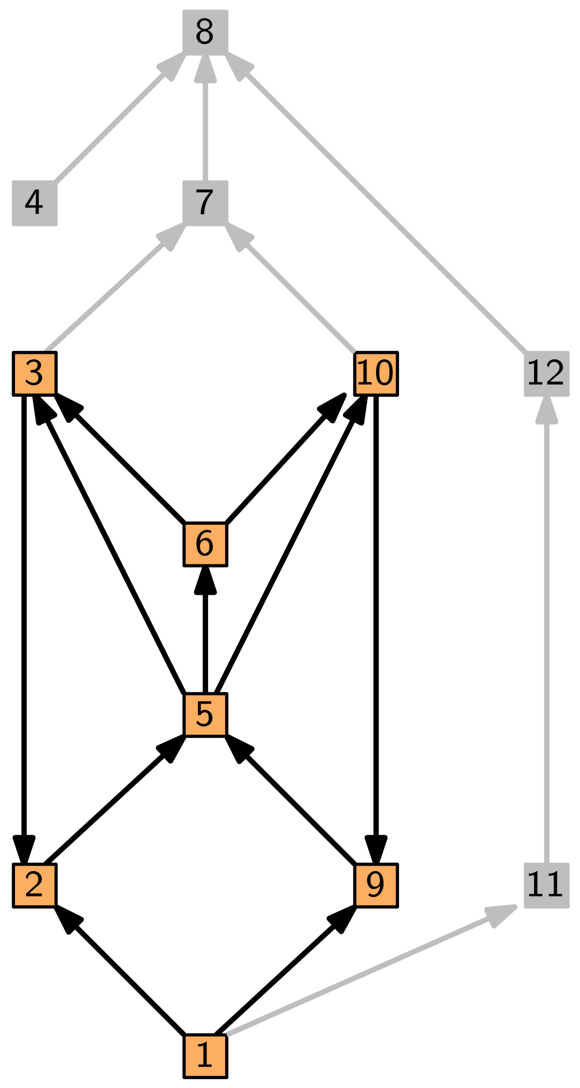

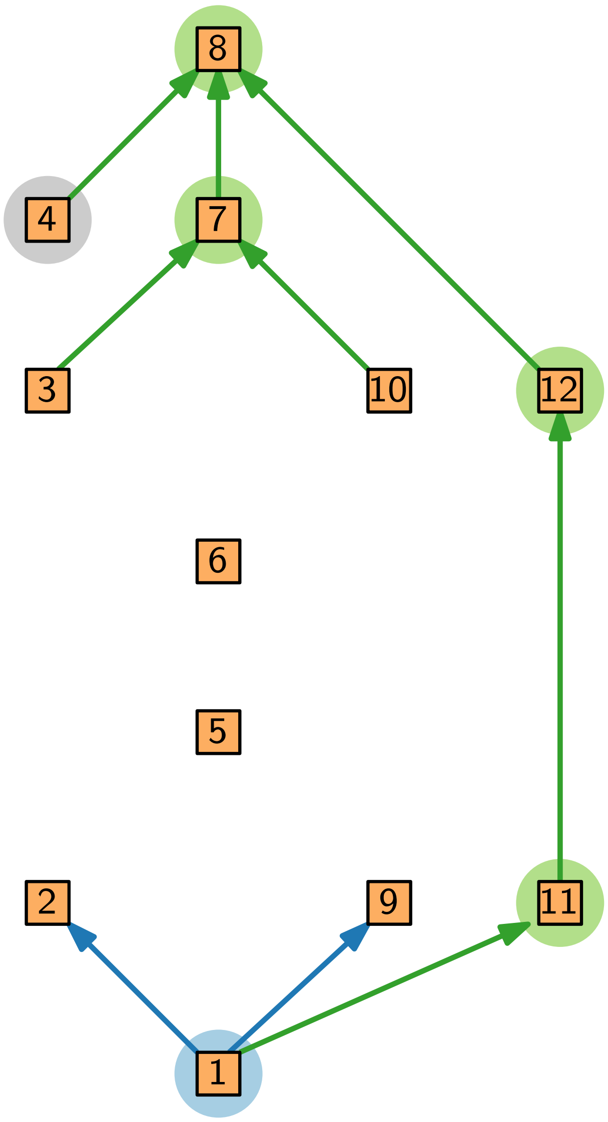

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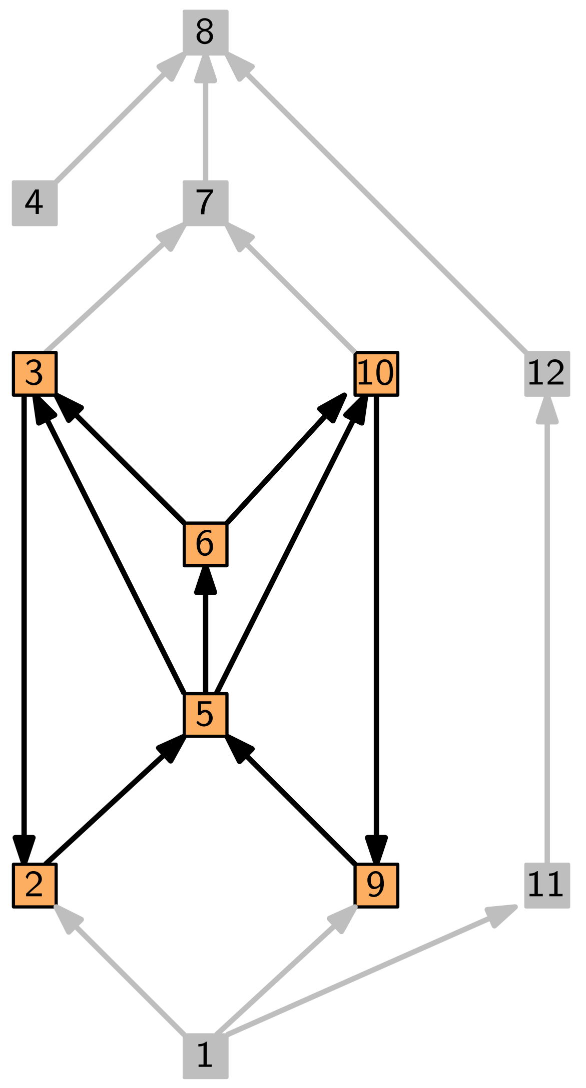

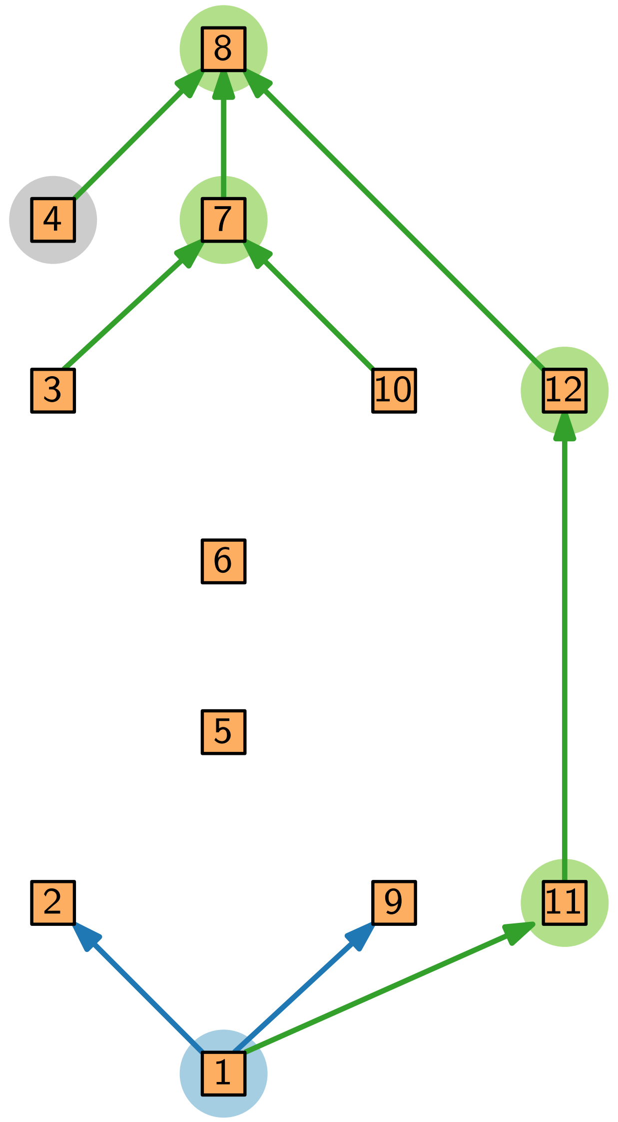

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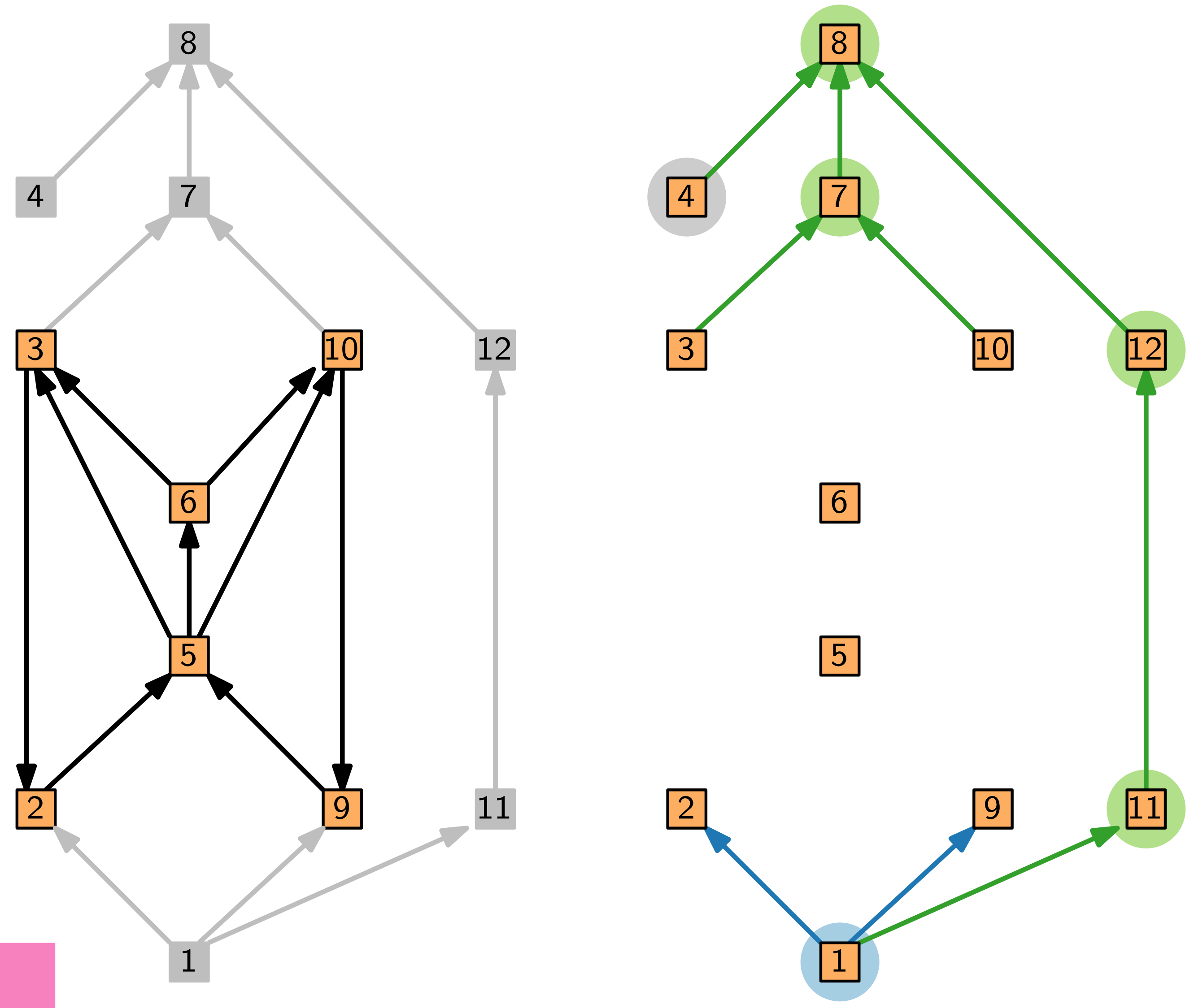

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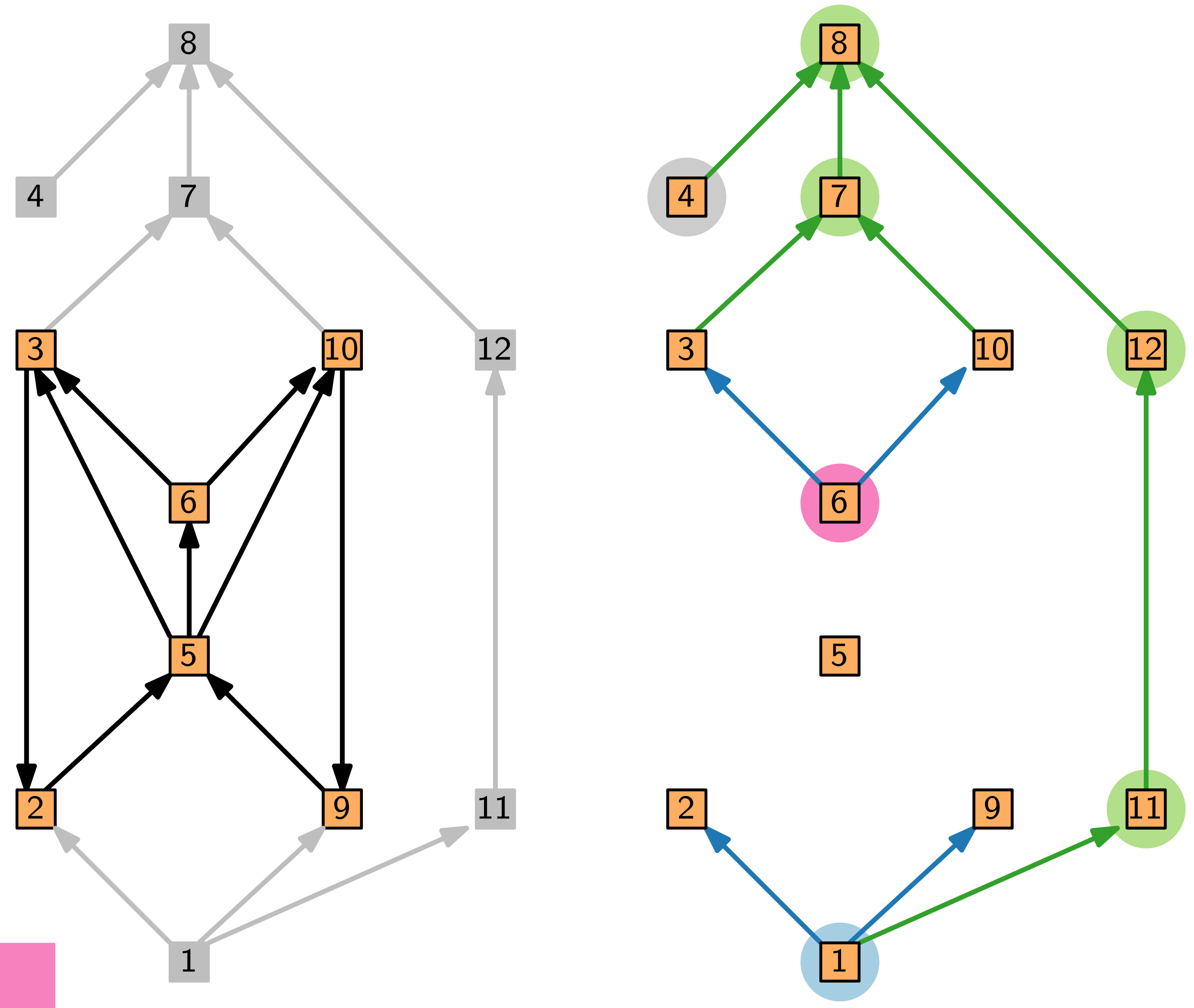

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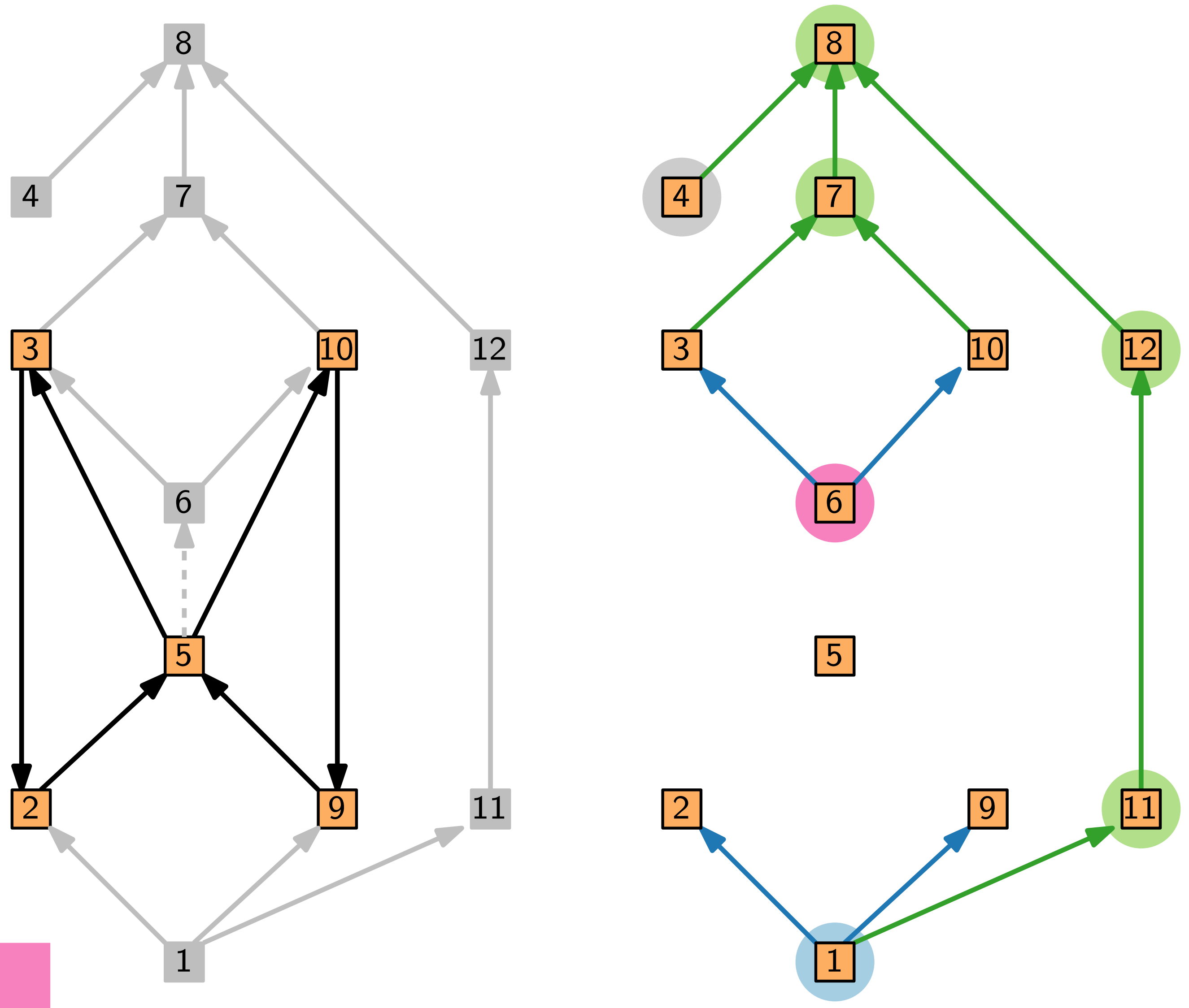

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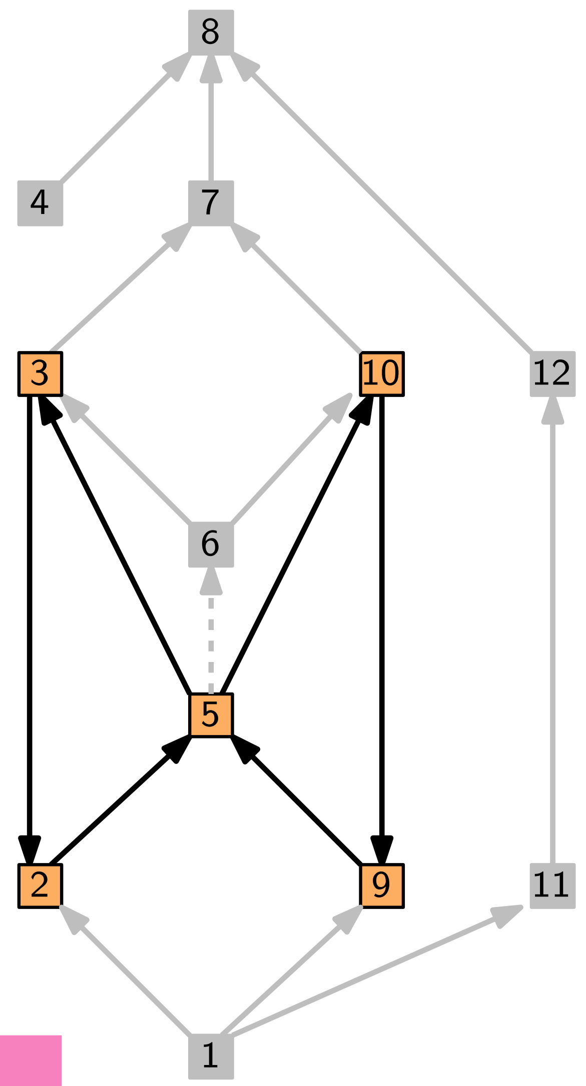

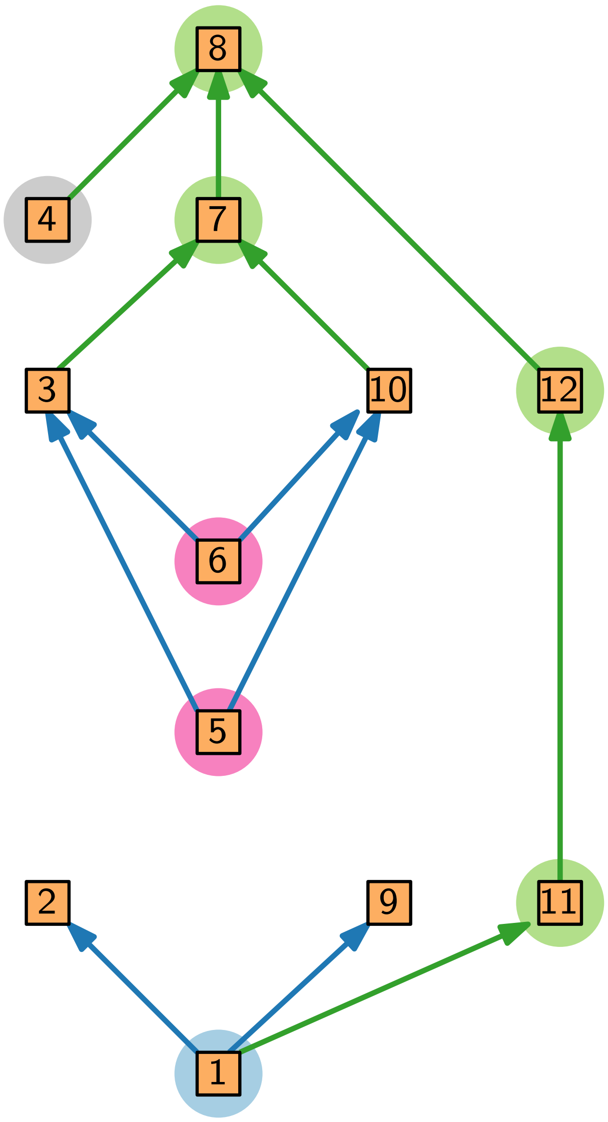

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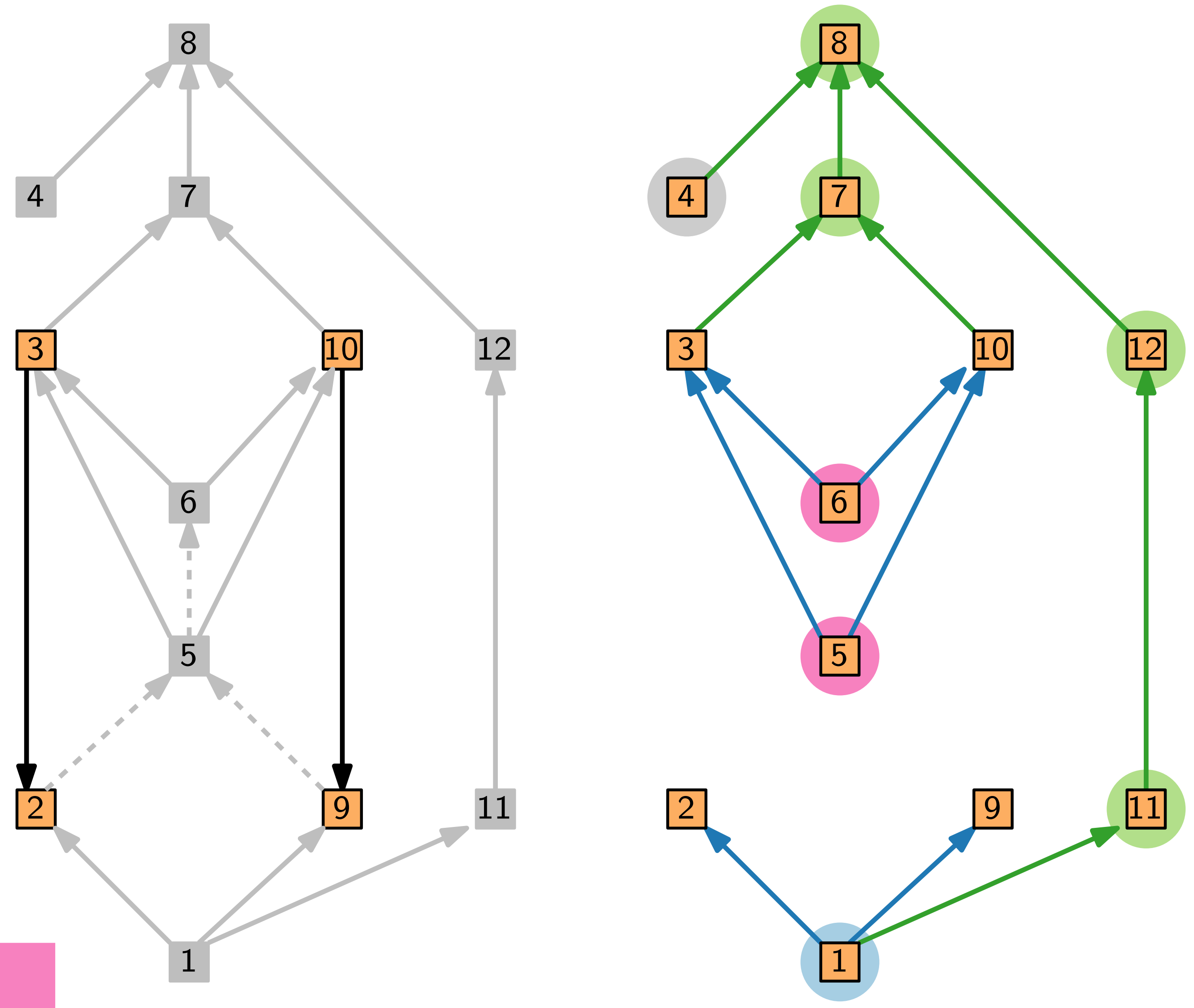

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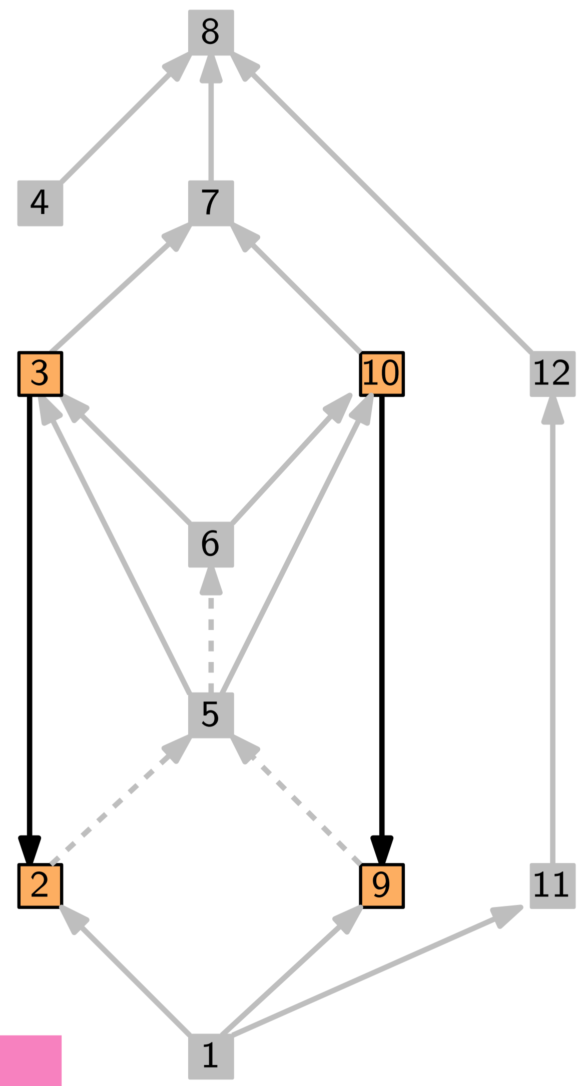

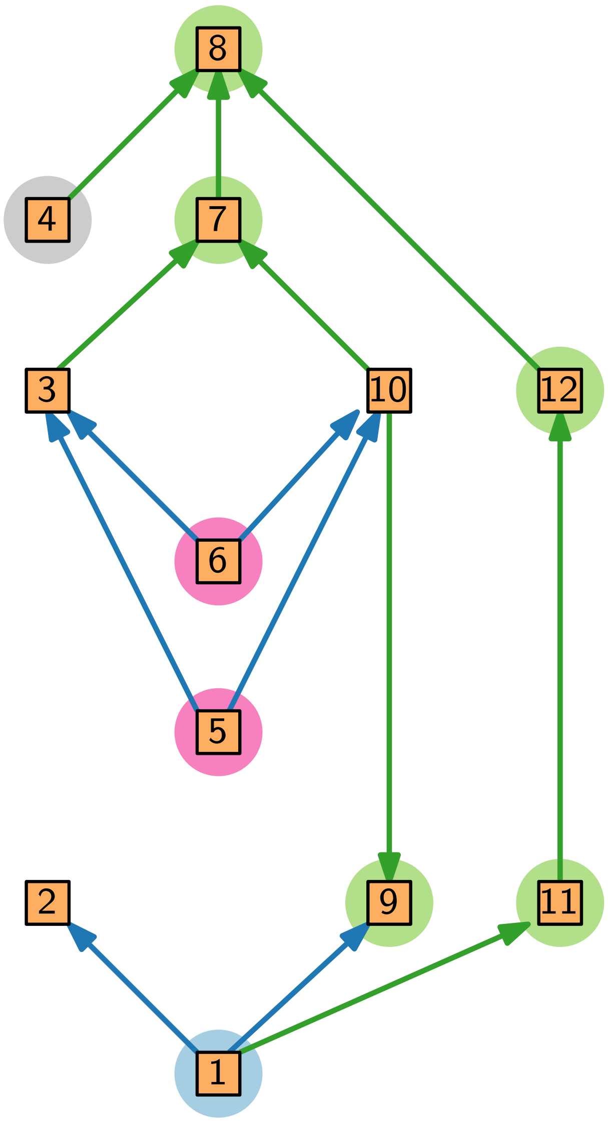

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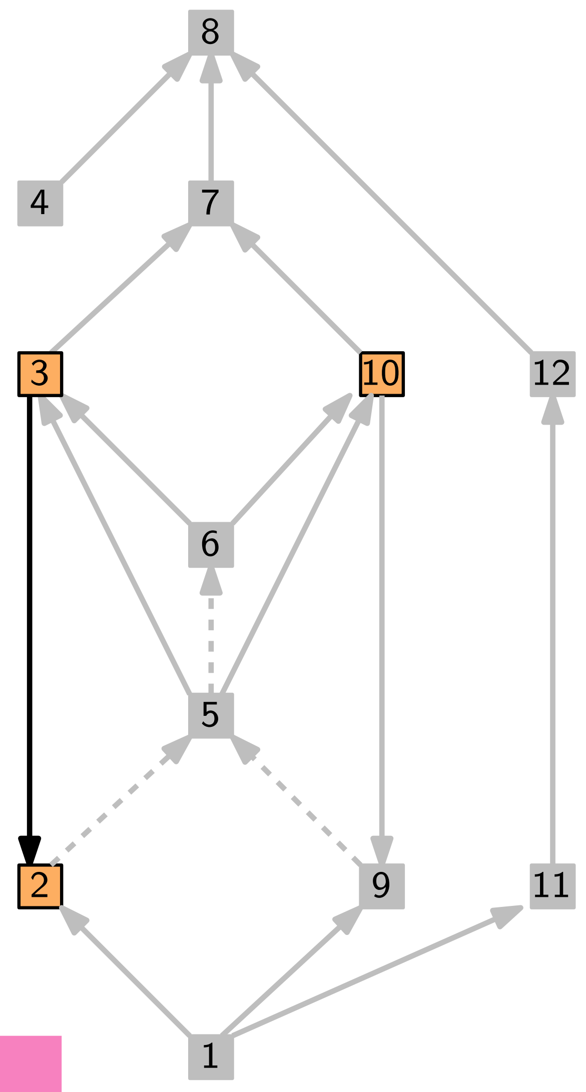

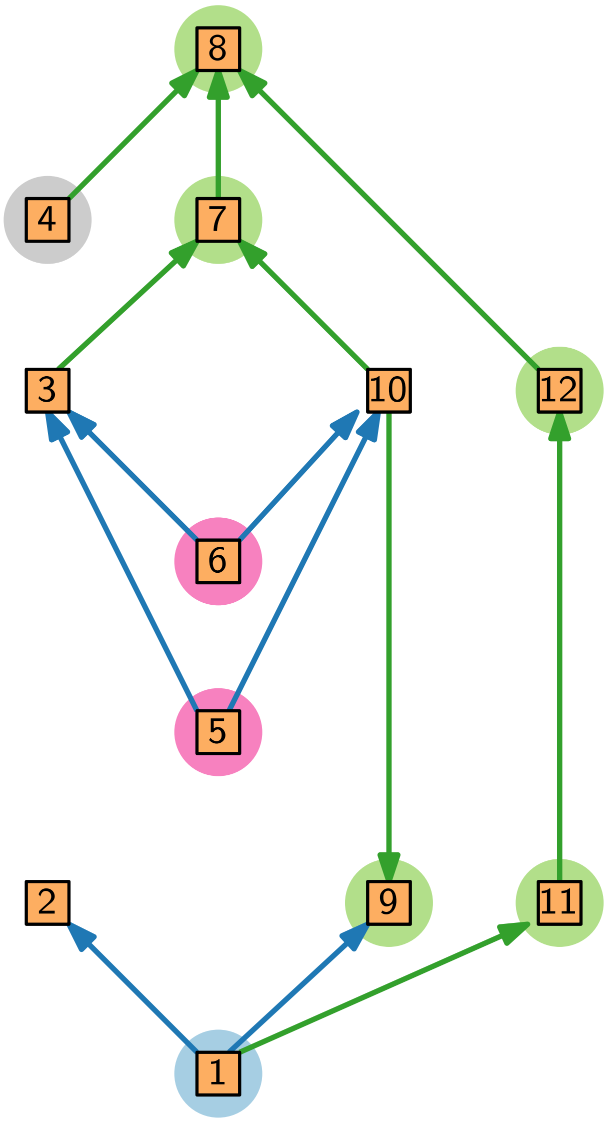

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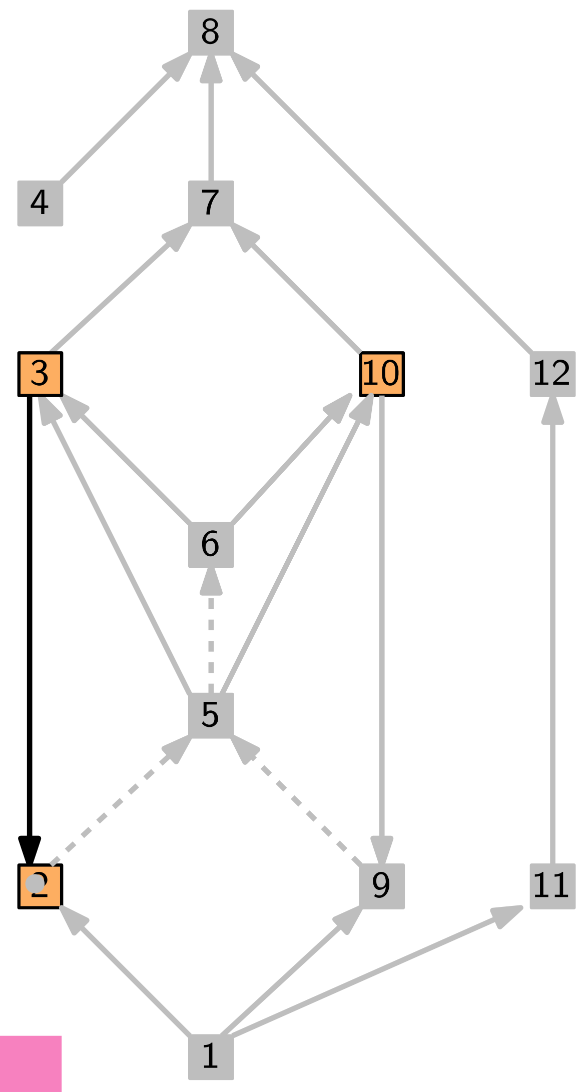

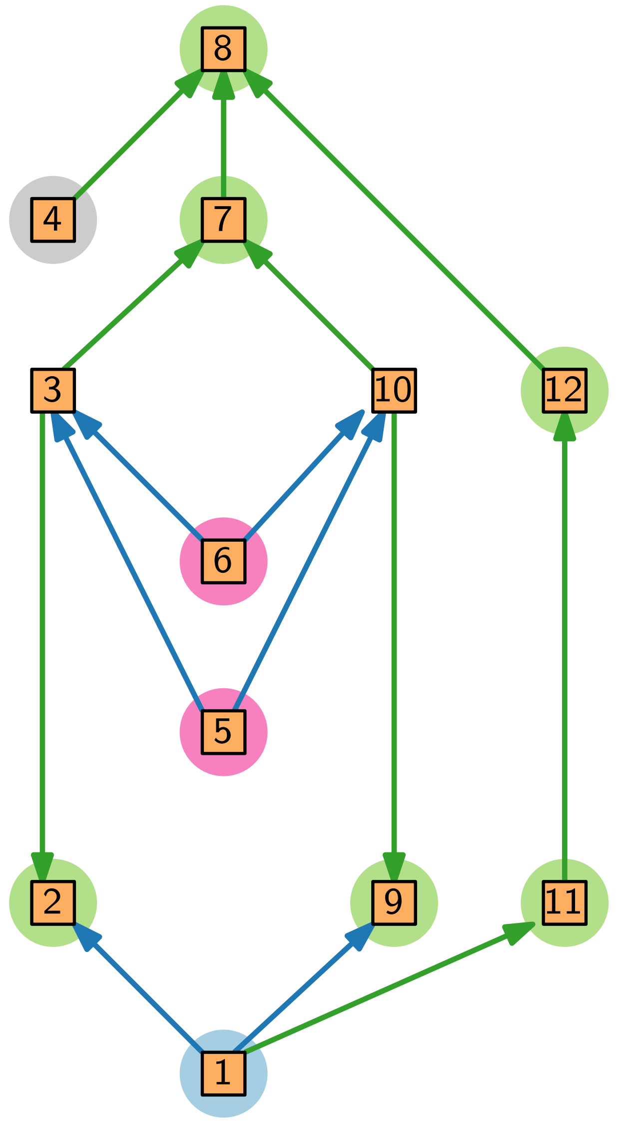

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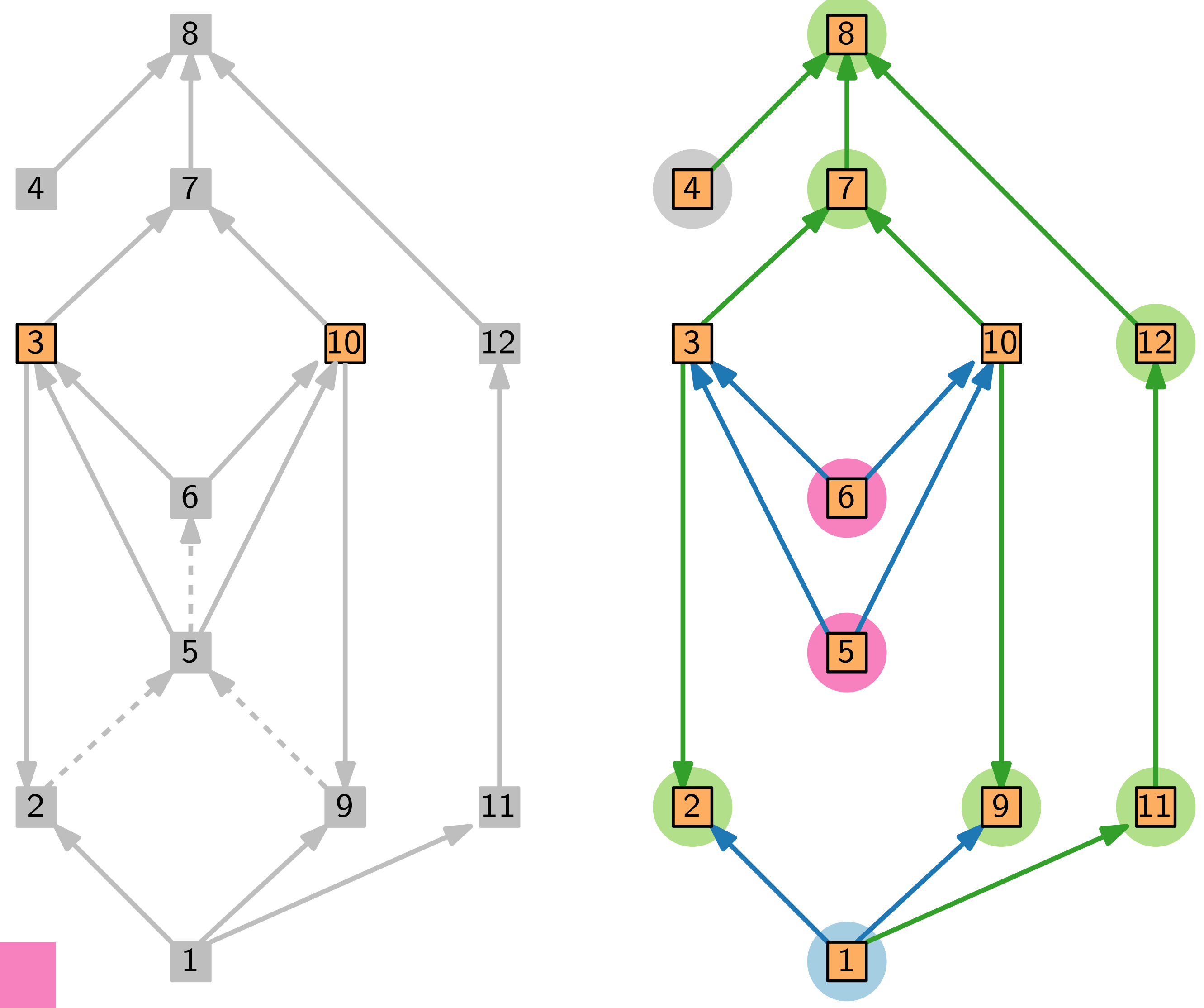

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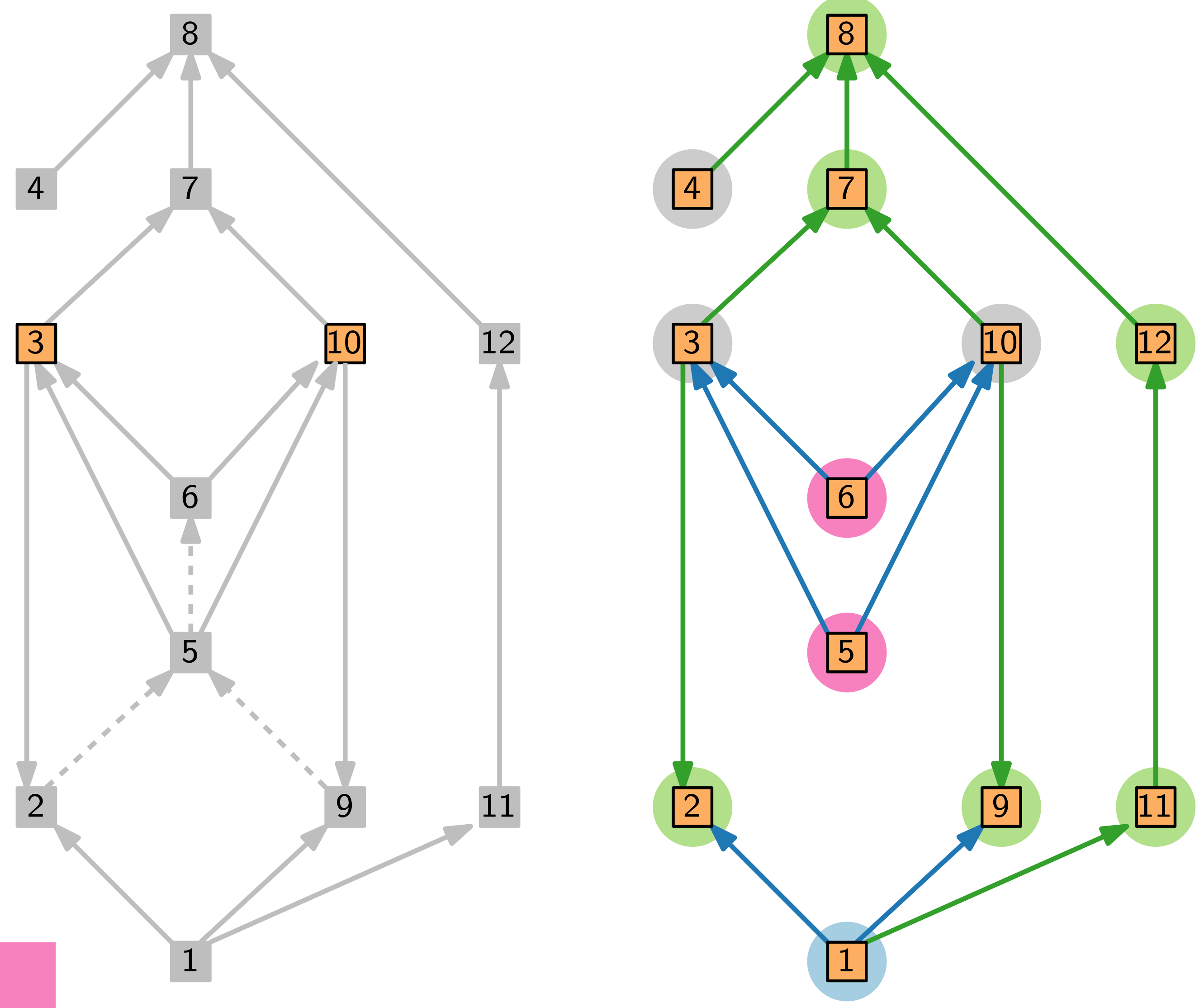

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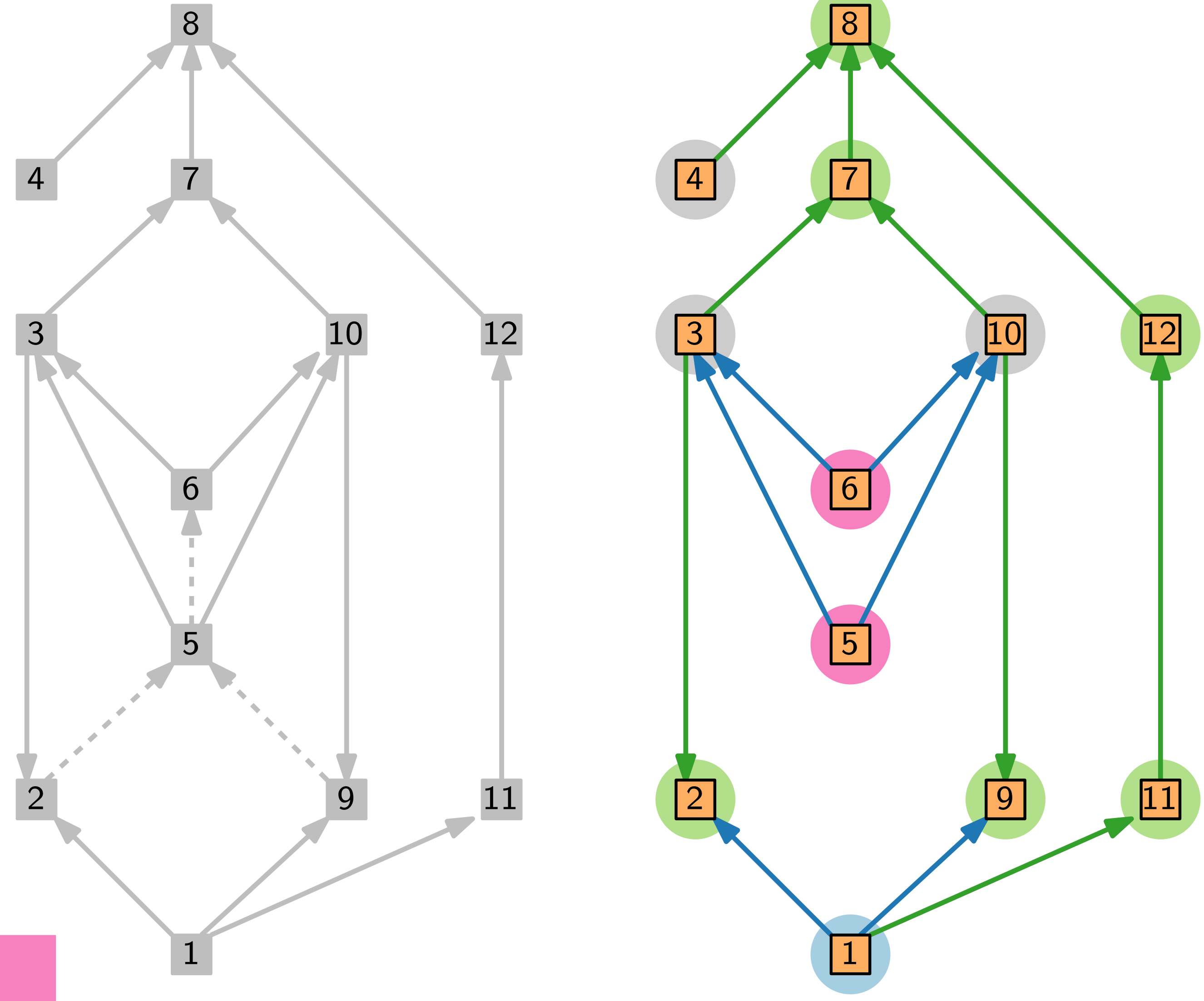

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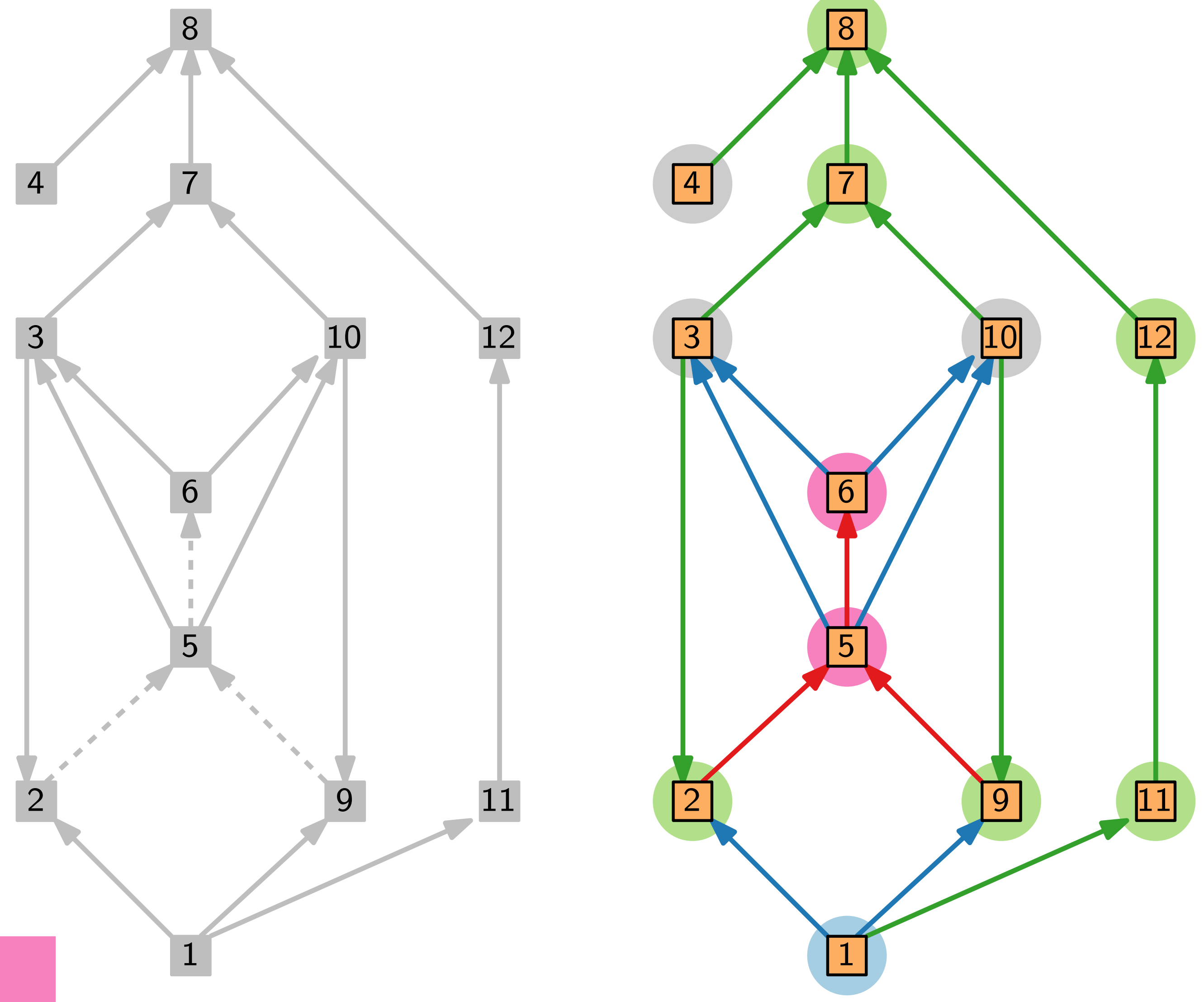

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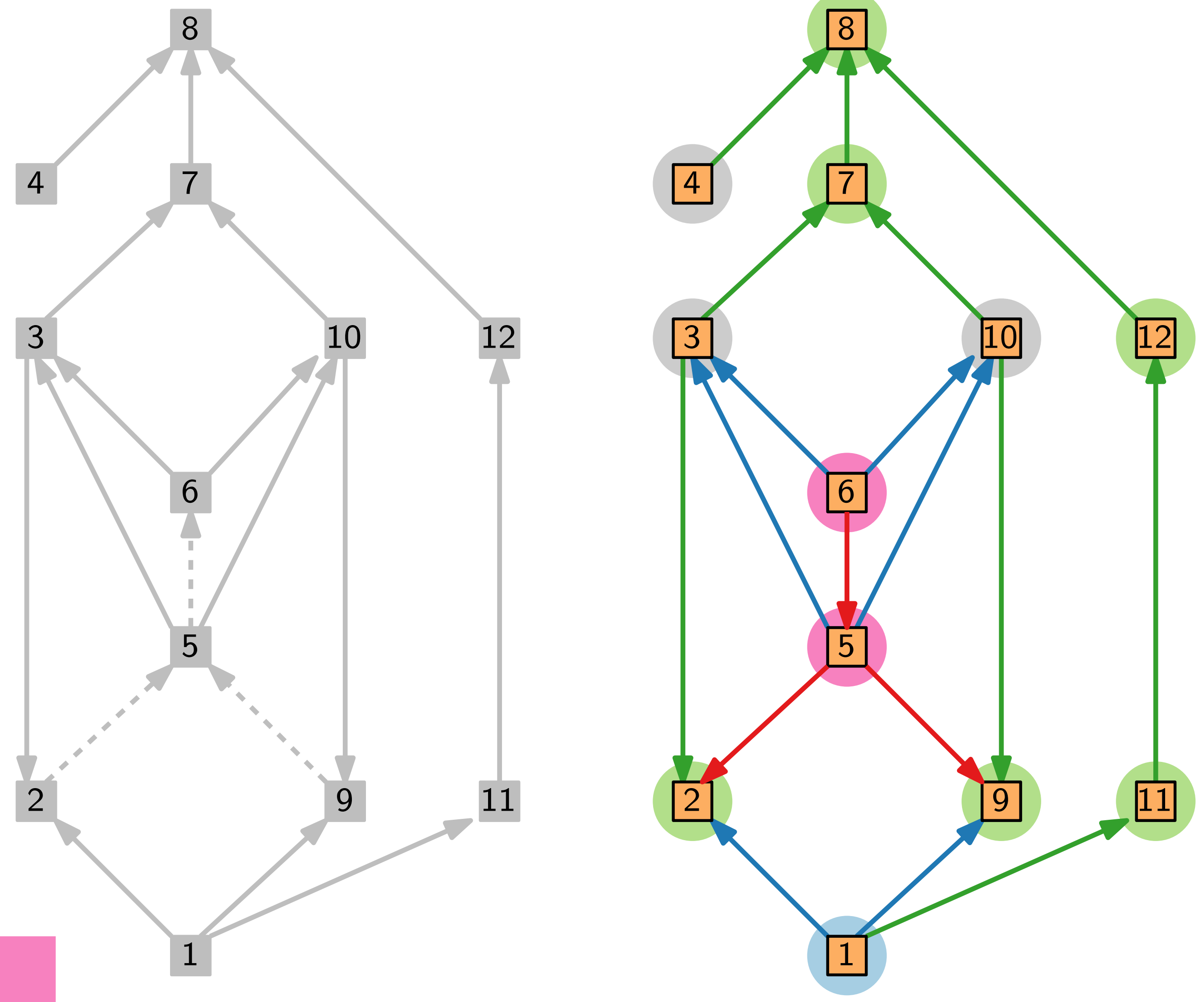

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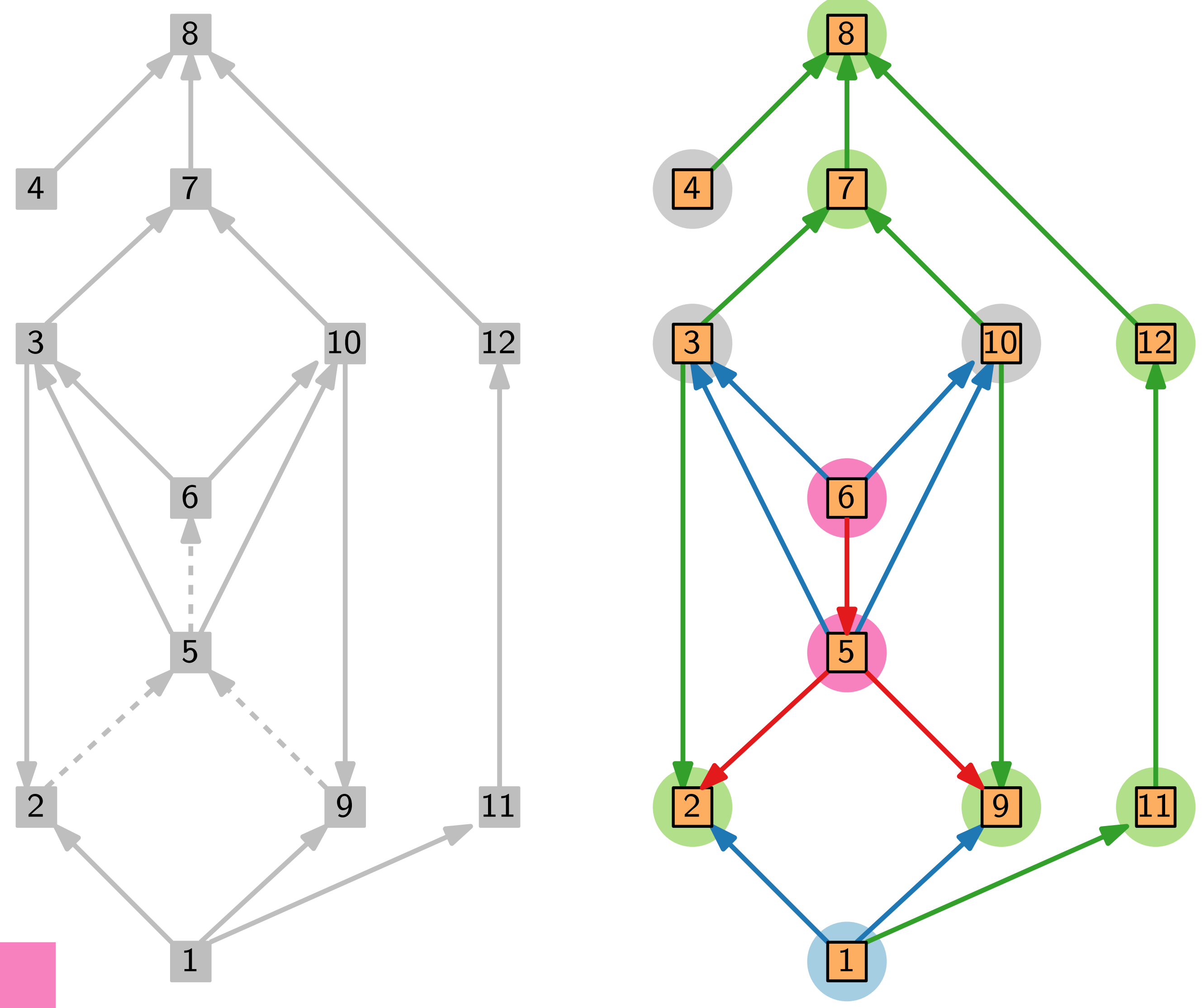
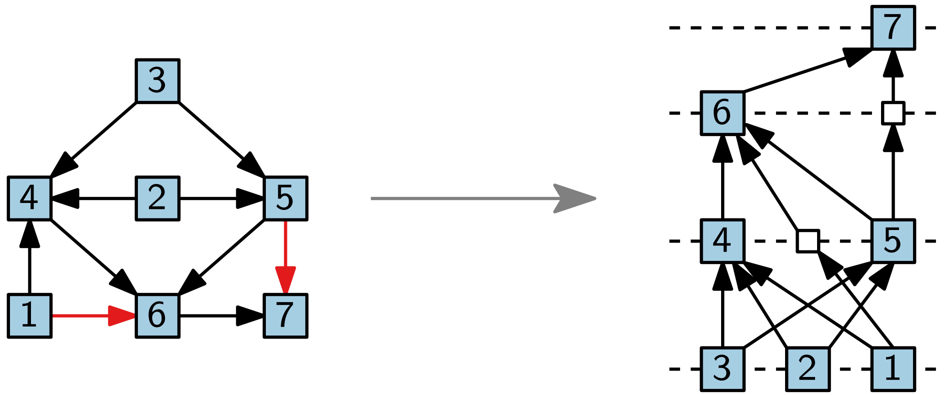


- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq |E|/2 + |V|/6$

Step 2: Leveling



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- Input: acyclic, digraph $G = (V, E)$
- Output: Mapping $y \colon V \to \{1, \ldots, |V|\}$,
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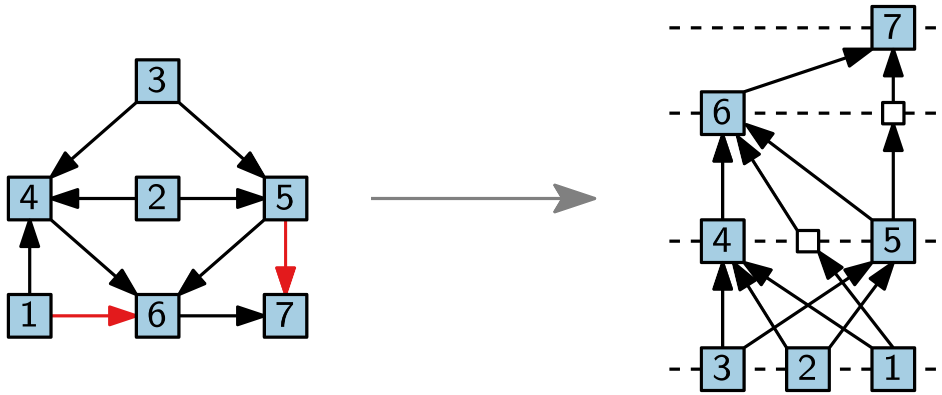

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Objective is to *minimize . . .*

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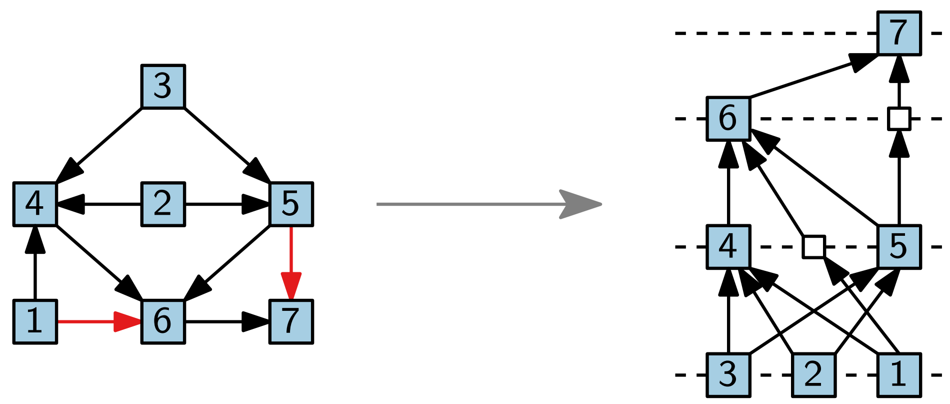


Problem.

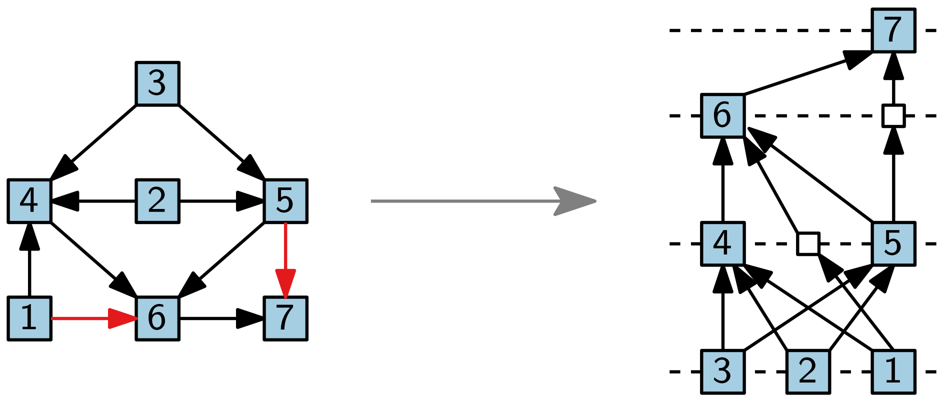
- Input: acyclic, digraph $G = (V, E)$
- Output: Mapping $y\colon V \to \{1, \ldots, |V|\}$, so that for every $uv \in A$, $y(u) < y(v)$.

Objective is to *minimize* . . .

- number of layers, i.e. $|y(V)|$
- length of the longest edge, i.e. $\max_{uv \in A} y(v) - y(u)$
- width, i.e. $\max\{|L_i| \mid 1 \leq i \leq h\}$
- total edge length, i.e. number of dummy vertices

Min number of layers

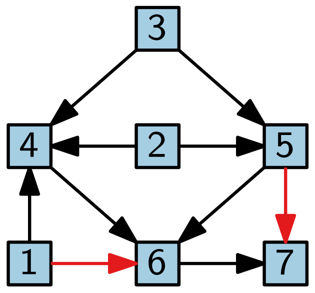
Algorithm.

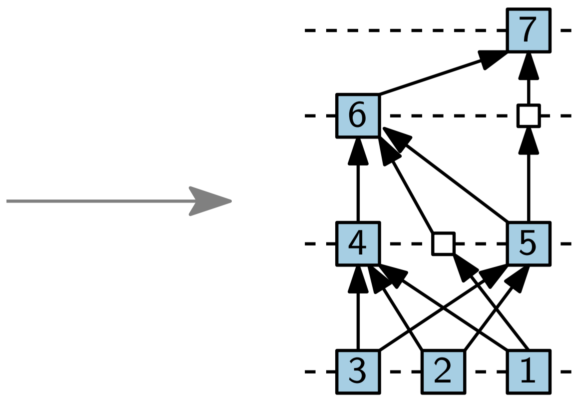


Min number of layers

Algorithm.

- for each source q
  set $y(q) := 1$

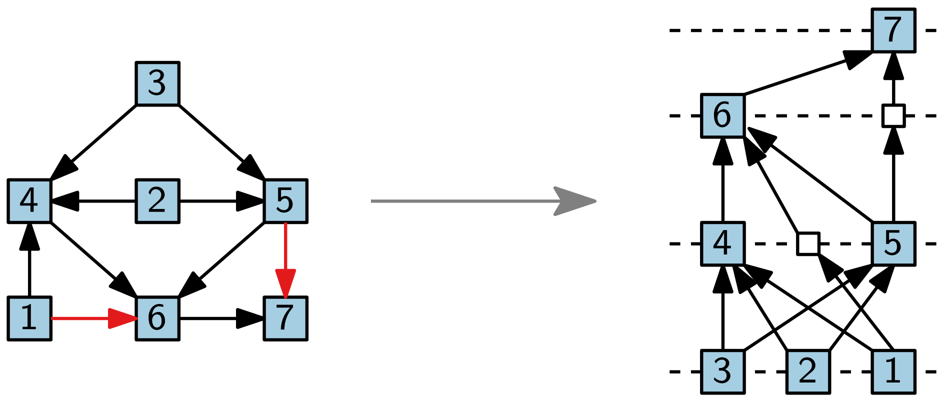




Min number of layers

Algorithm.

- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



Min number of layers

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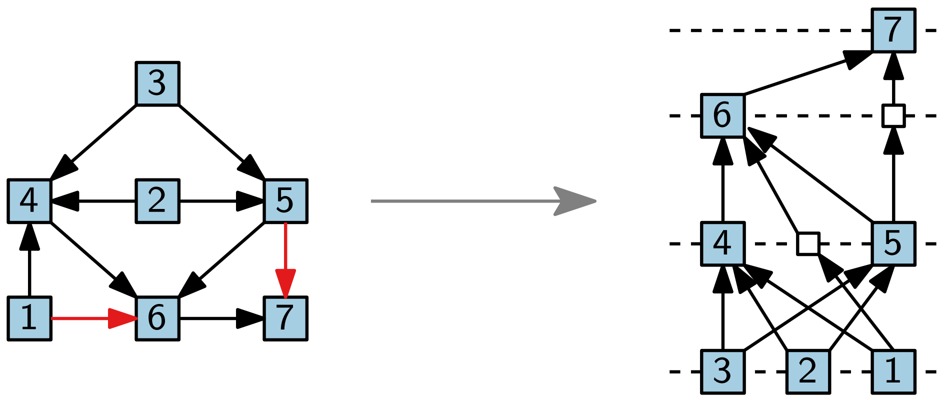


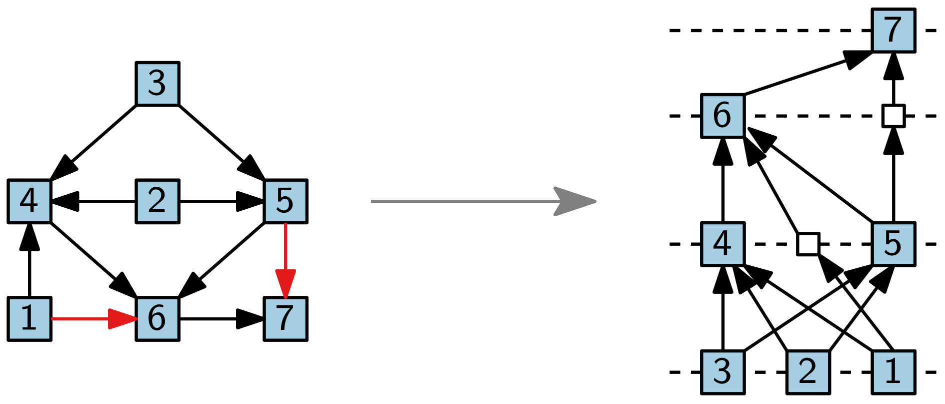
Observation.

- $y(v)$

Min number of layers

Algorithm.

- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.

Min number of layers

Algorithm.

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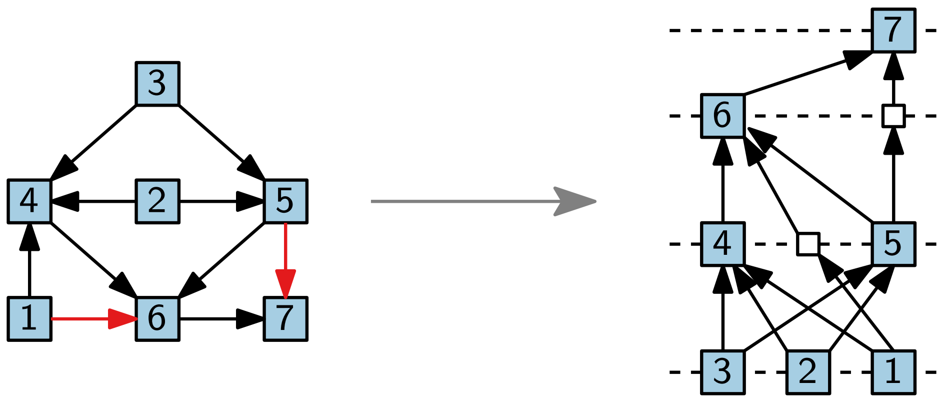
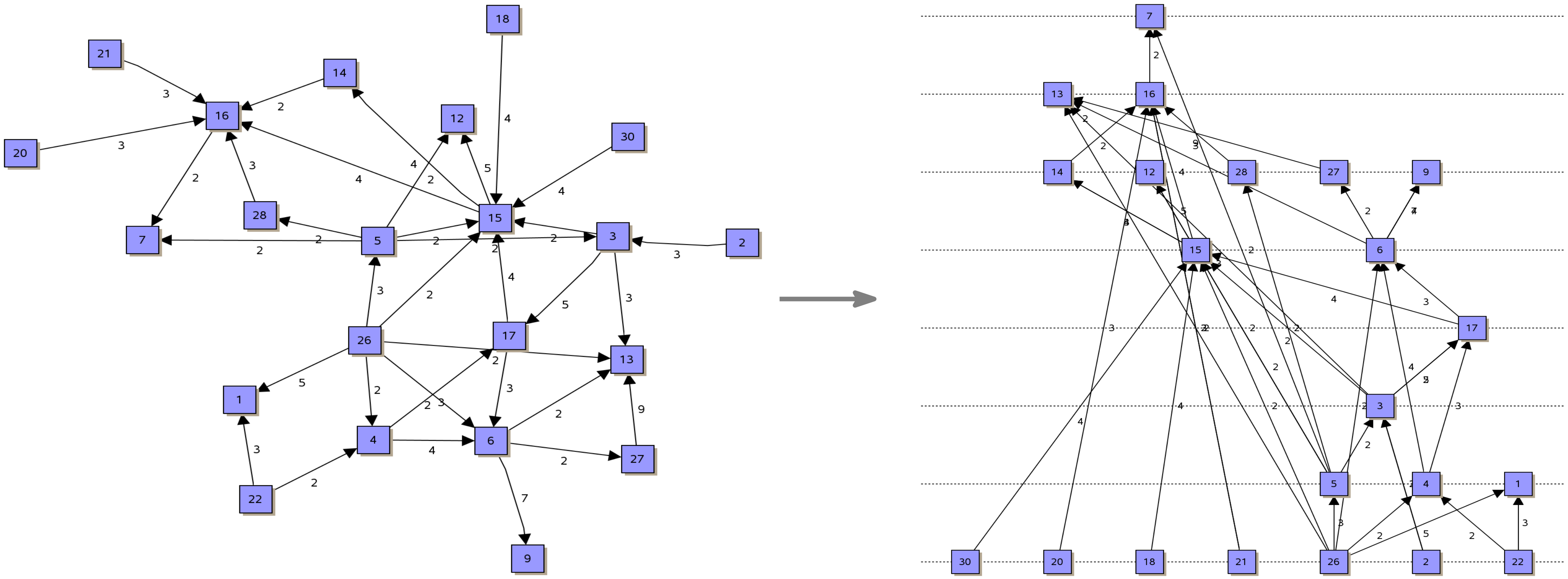
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



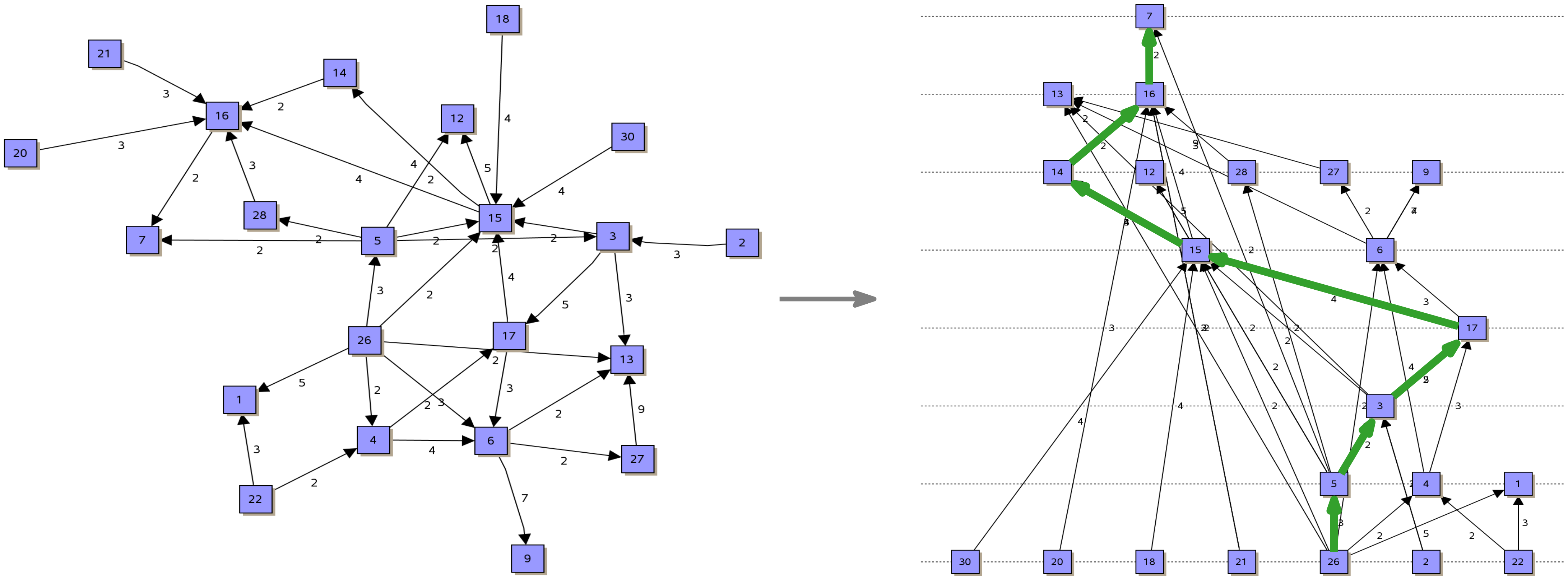
Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.
  . . . which is optimal!
- Can be implemented in linear time with recursive algorithm.

Example



Example



Total edge length – ILP

Can be formulated as an integer linear program:

$$
\begin{array}{rll}
\text{min} & \sum_{(u,v)\in E}(y(v)-y(u)) & \\
\text{subject to} & y(v)-y(u) \geq 1 & \forall (u,v) \in E \\
 & y(v) \geq 1 & \forall v \in V \\
 & y(v) \in \mathbb{Z} & \forall v \in V
\end{array}
$$

Total edge length – ILP

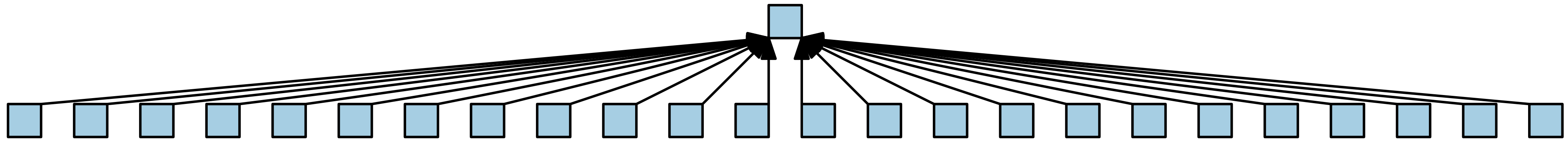
Can be formulated as an integer linear program:

$$\begin{array}{rll} \text{min} & \sum_{(u,v)\in E}(y(v)-y(u)) & \\ \text{subject to} & y(v)-y(u)\geq 1 & \forall(u,v)\in E \\ & y(v)\geq 1 & \forall v\in V \\ & y(v)\in\mathbb{Z} & \forall v\in V \end{array}$$

One can show that:

- Constraint-matrix is **totally unimodular**
  $\Rightarrow$ Solution of the relaxed linear program is integer
- The total edge length can be minimized in polynomial time

Width



Drawings can be very wide.

Narrower layer assignment

Problem: Leveling with a given width.

- Input: acyclic, digraph $G = (V, E)$, width $W > 0$
- Output: Partition the vertex set into a minimum number of layers such that each layer contains at most W elements.

Narrower layer assignment

Problem: Leveling with a given width.

- Input: acyclic, digraph $G = (V, E)$, width $W > 0$
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Problem: Precedence-Constrained Multi-Processor Scheduling

- Input: n jobs with unit (1) processing time, W identical machines, and a partial ordering $<$ on the jobs.
- Output: Schedule respecting $<$ and having minimum processing time.

Narrower layer assignment

Problem: Leveling with a given width.

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Problem: Precedence-Constrained Multi-Processor Scheduling

- Input: n jobs with unit (1) processing time, W identical machines, and a partial ordering $<$ on the jobs.
- Output: Schedule respecting $<$ and having minimum processing time.

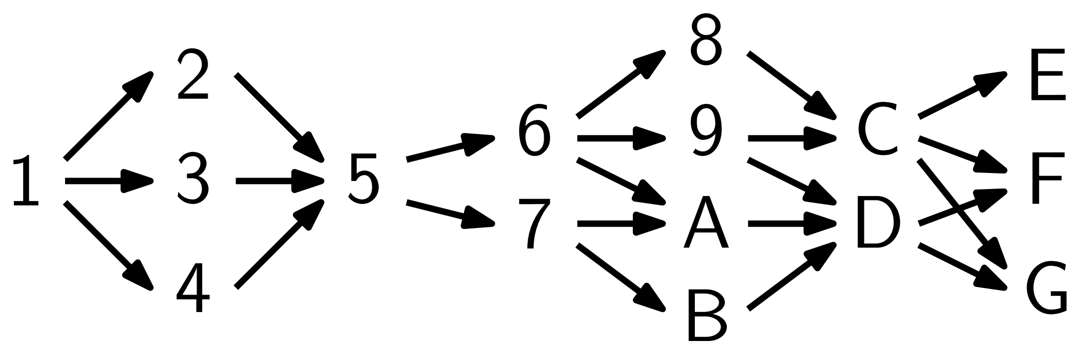
- NP-hard, $(2 - \frac{2}{W})$-Approx., no $(\frac{4}{3} - \varepsilon)$-Approx. $(W \geq 3)$.

Approximating PCMPS

- jobs stored in a list L
  (in any order, e.g., topologically sorted)
- for each time $t = 1, 2, \ldots$ schedule $\leq W$ available jobs
- a job in L is *available* when all its predecessors have been scheduled
- as long as there are free machines and available jobs, take the first available job and assign it to a free machine

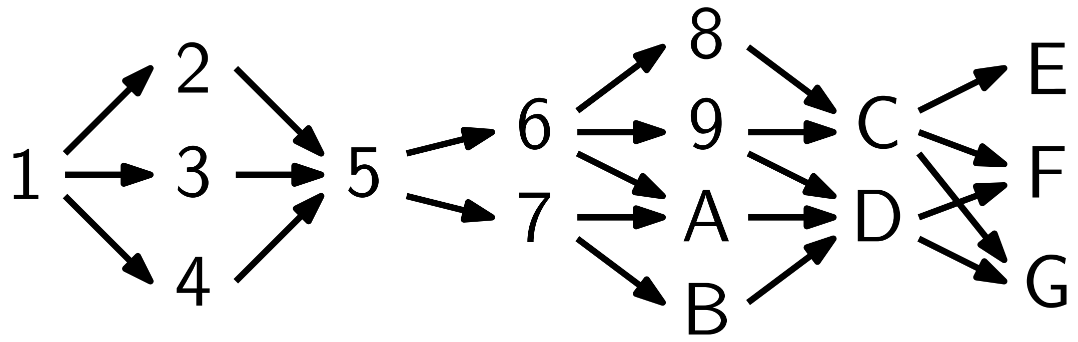
Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



Approximating PCMPS

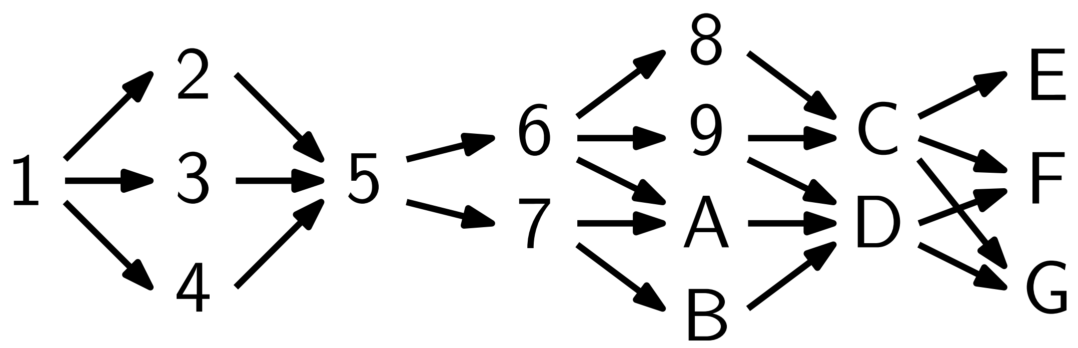
Input: Precedence graph (divided into layers of arbitrary width)



Number of Machines is $W = 2$.

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)

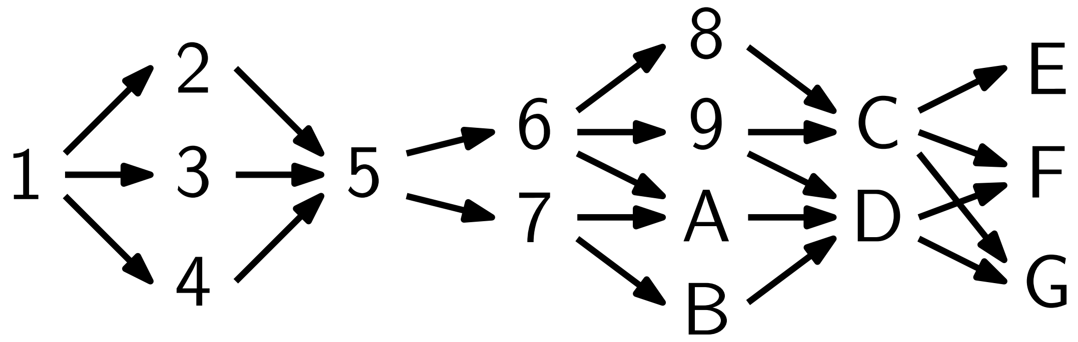


Number of Machines is $W = 2$.

Output: Schedule

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



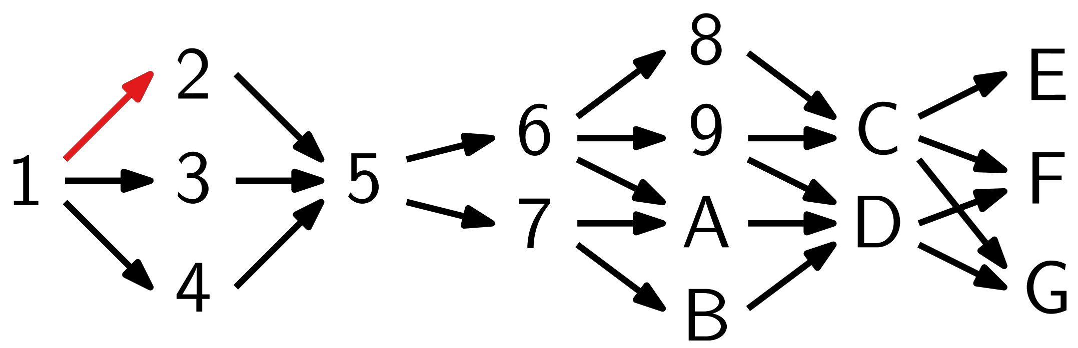
Number of Machines is $W = 2$.

Output: Schedule

M_1										
M_2										
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



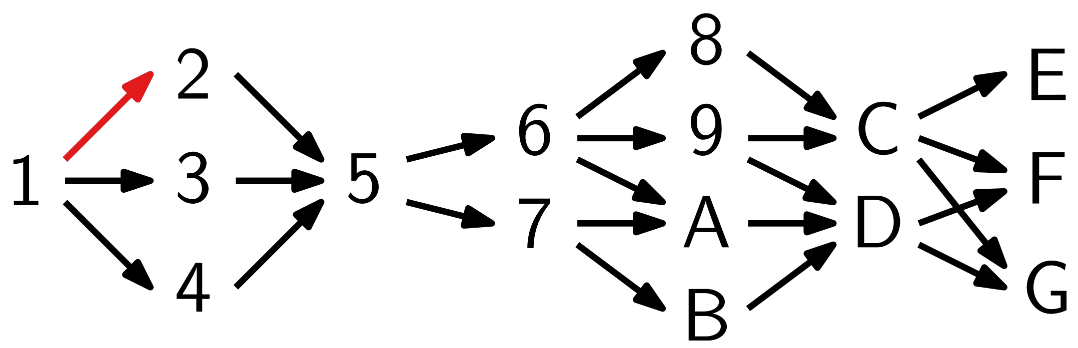
Number of Machines is $W = 2$.

Output: Schedule

M_1	1									
M_2	–									
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



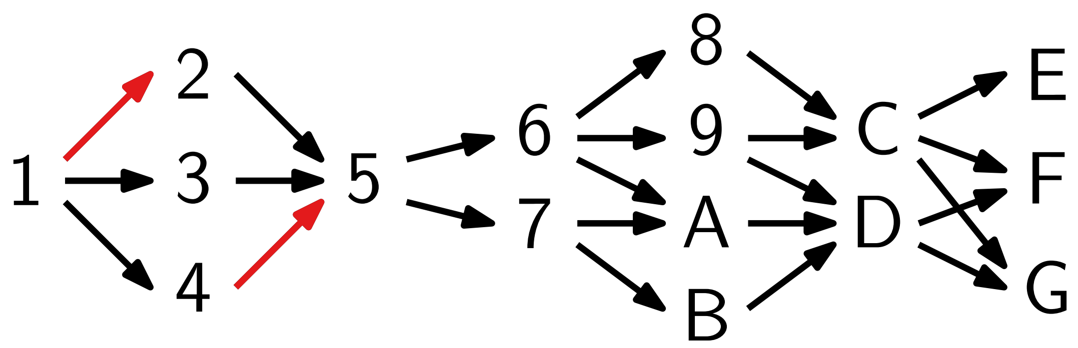
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2								
M_2	–	3								
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



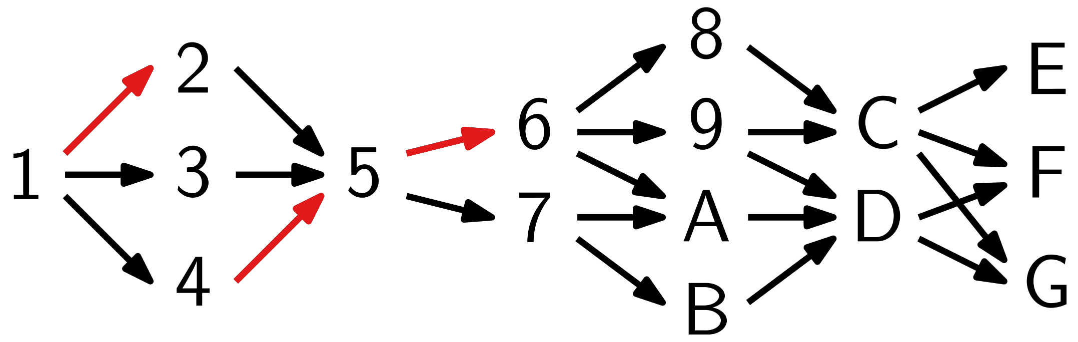
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4							
M_2	–	3	–							
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



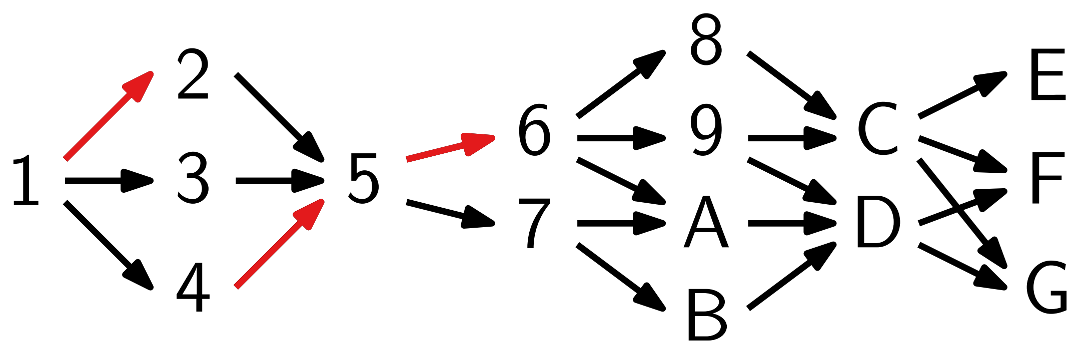
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5						
M_2	–	3	–	–						
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



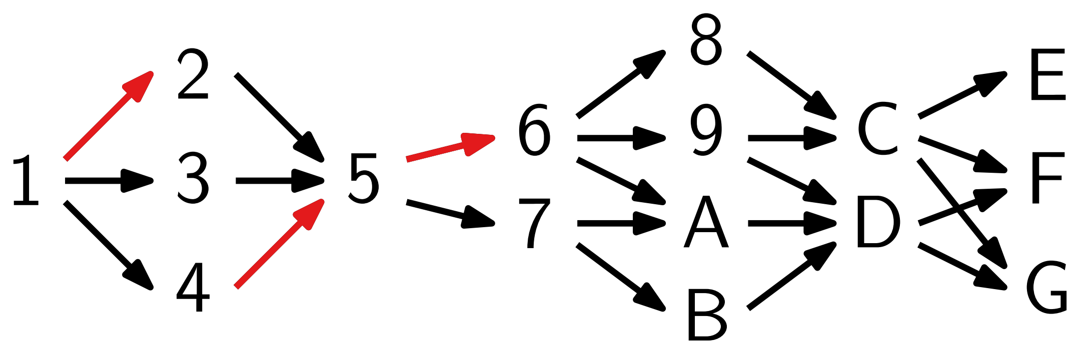
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6					
M_2	–	3	–	–	7					
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



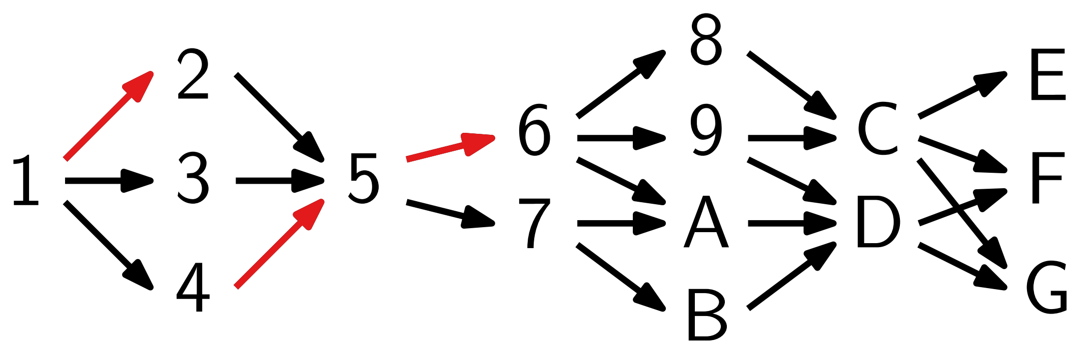
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8				
M_2	–	3	–	–	7	9				
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



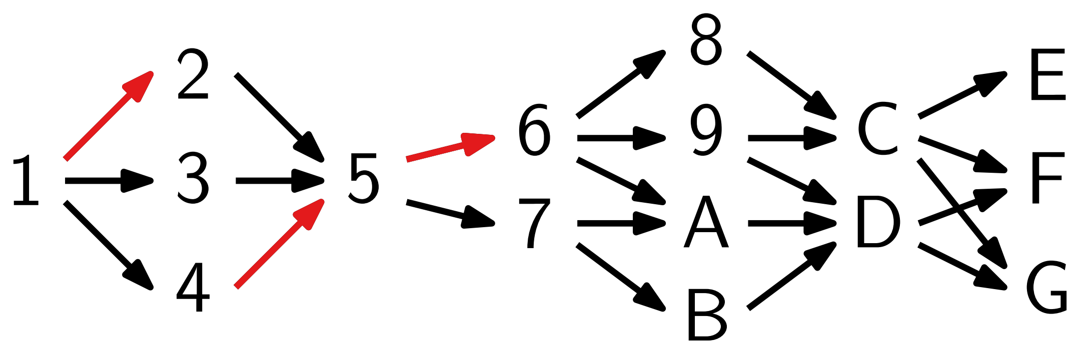
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A			
M_2	–	3	–	–	7	9	B			
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



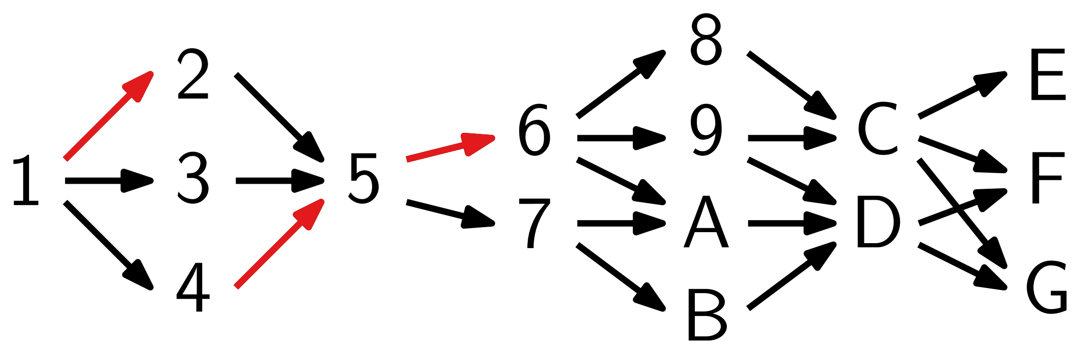
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C		
M_2	–	3	–	–	7	9	B	D		
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



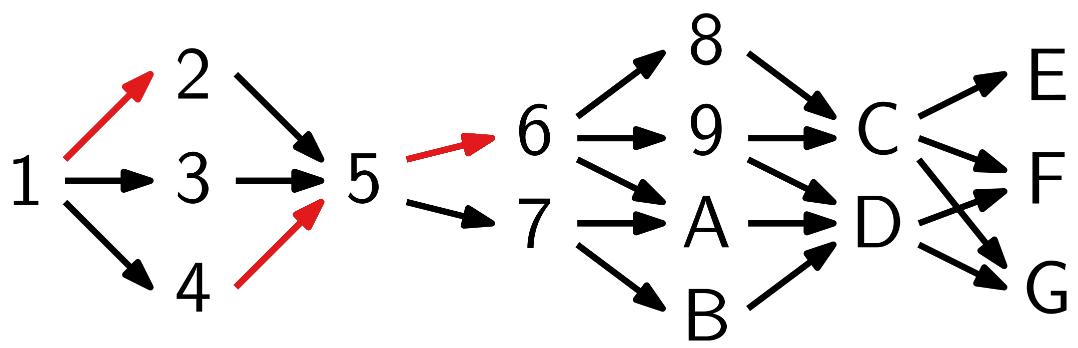
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	
M_2	–	3	–	–	7	9	B	D	F	
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



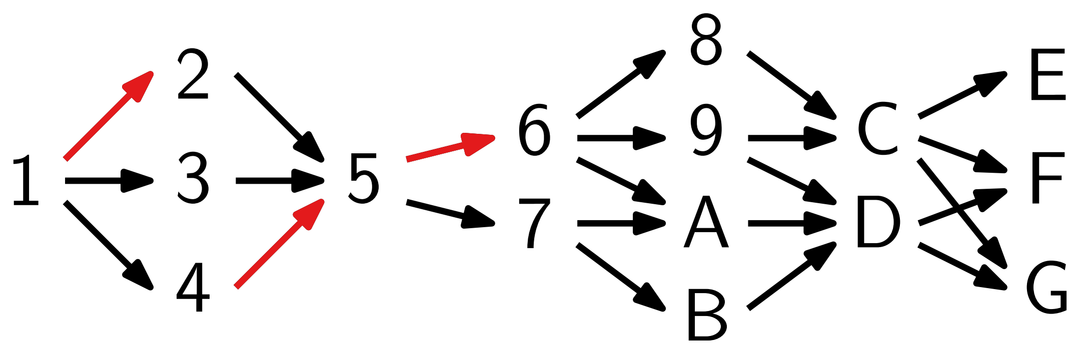
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



Number of Machines is $W = 2$.

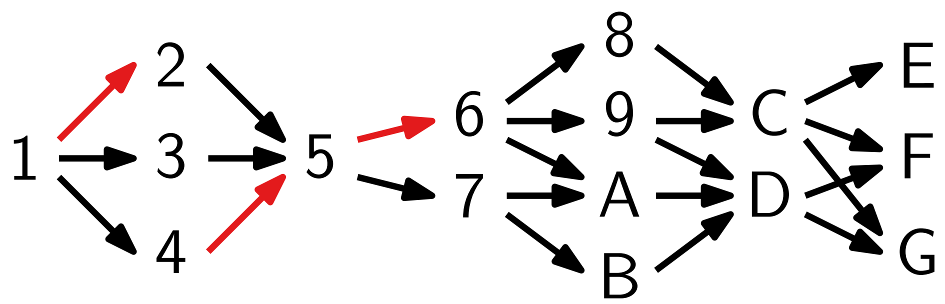
Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

Question: Good approximation factor?

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



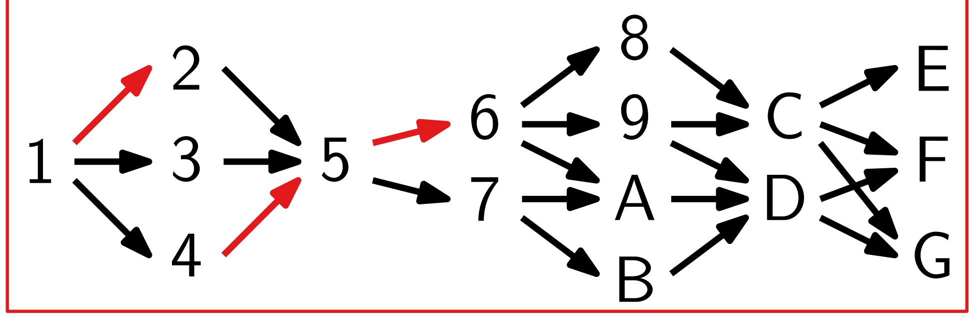
Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



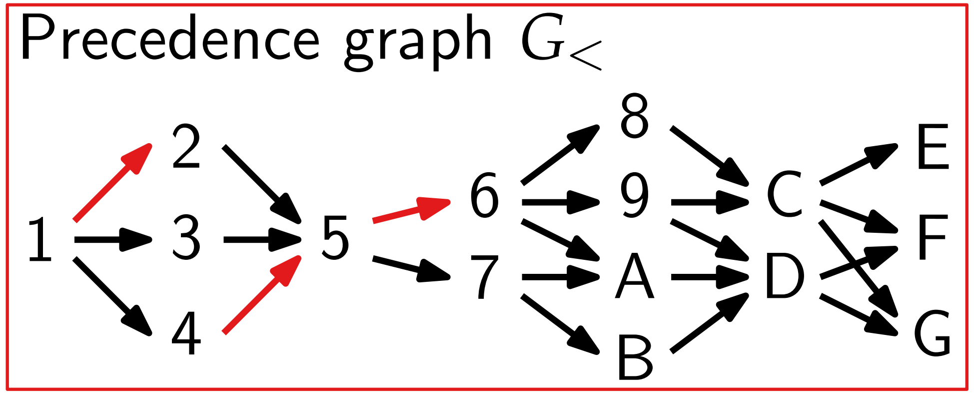
Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq$

Approximating PCMPS - analysis for $W = 2$



Schedule

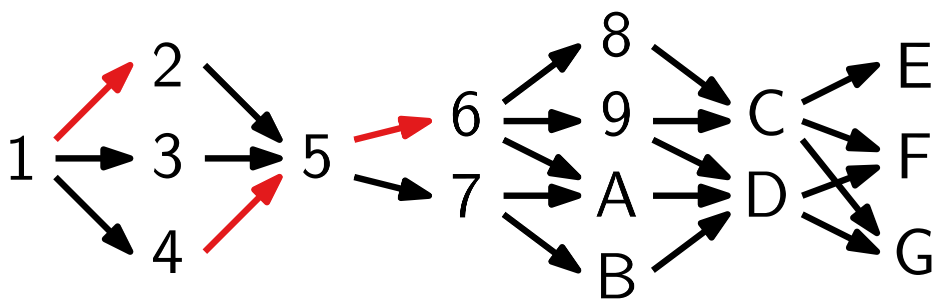
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

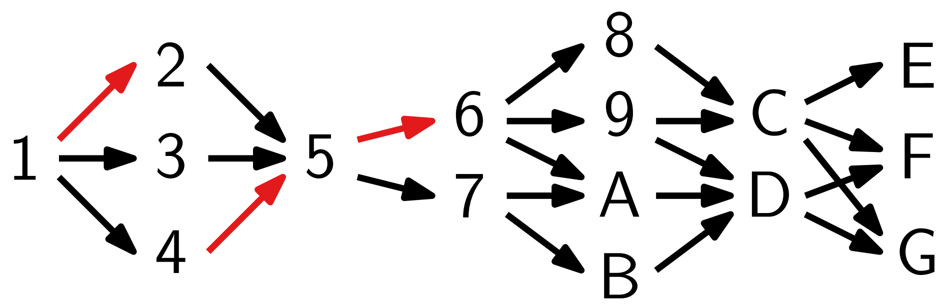
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

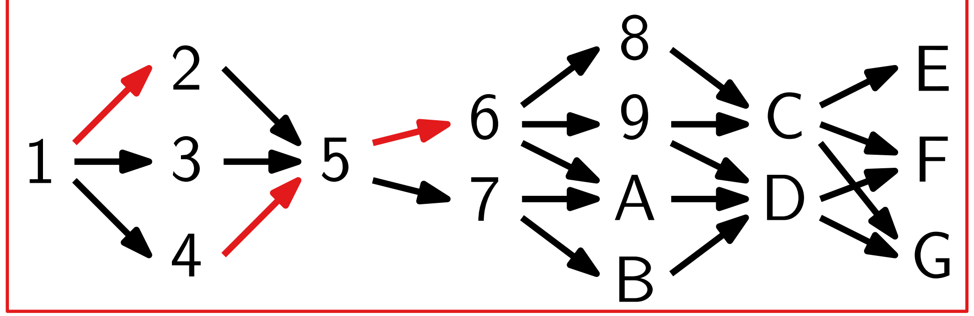
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

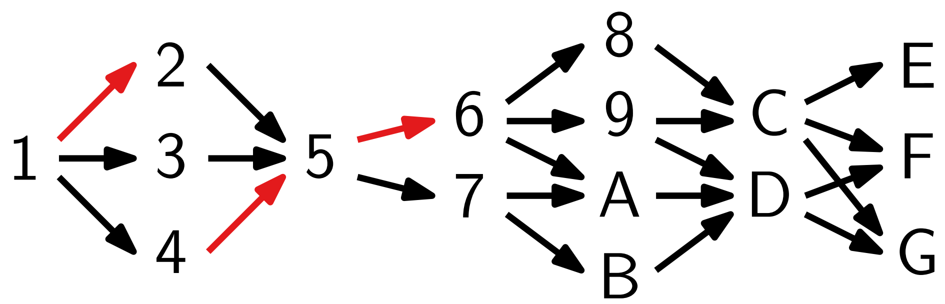
„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

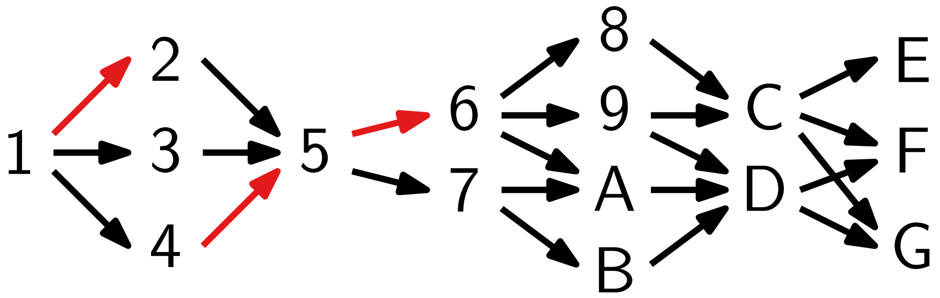
OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

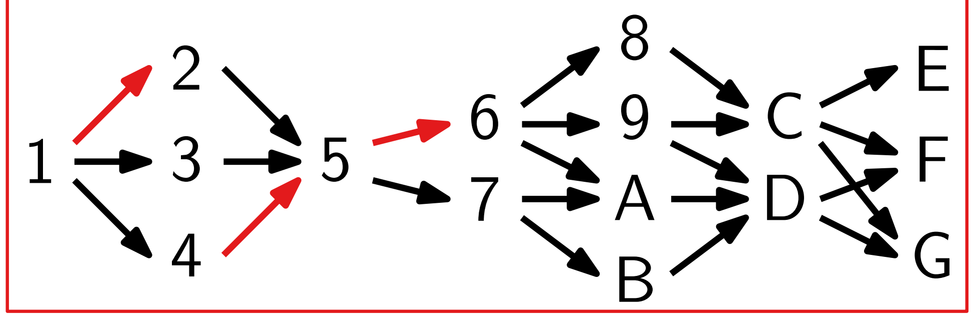
OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

$\mathrm{OPT} \geq \lceil n/2 \rceil$ and $\mathrm{OPT} \geq \ell :=$ Number of layers of $G_<$

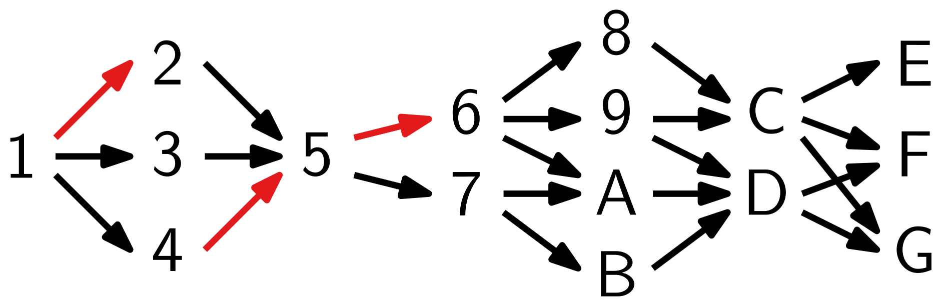
Goal: measure the quality of our algorithm using the lower bounds

Bound. $\mathrm{ALG} \leq$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

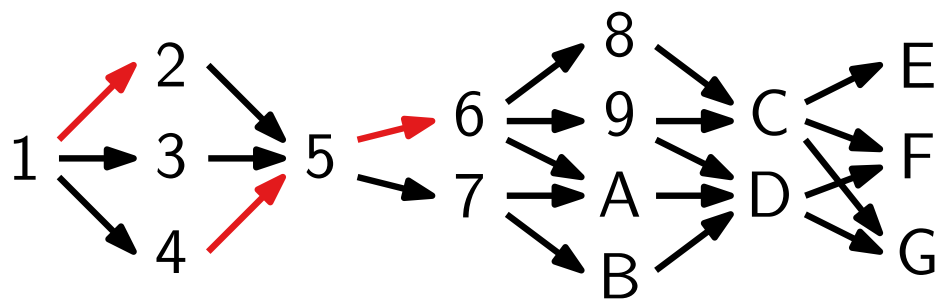
Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

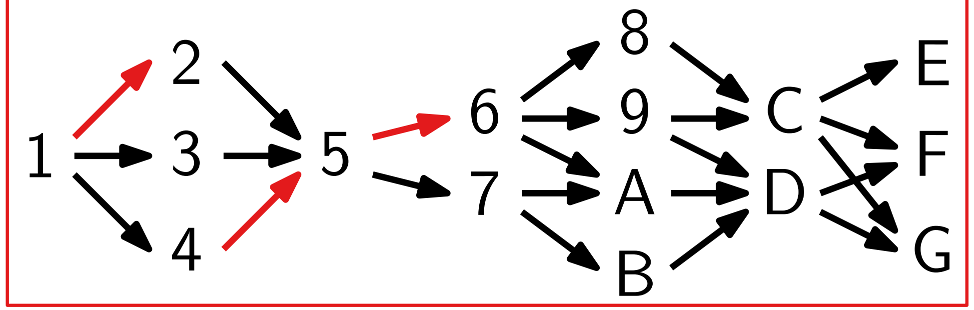
Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

$\mathrm{OPT} \geq \lceil n/2 \rceil$ and $\mathrm{OPT} \geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Bound. $\mathrm{ALG} \leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2$

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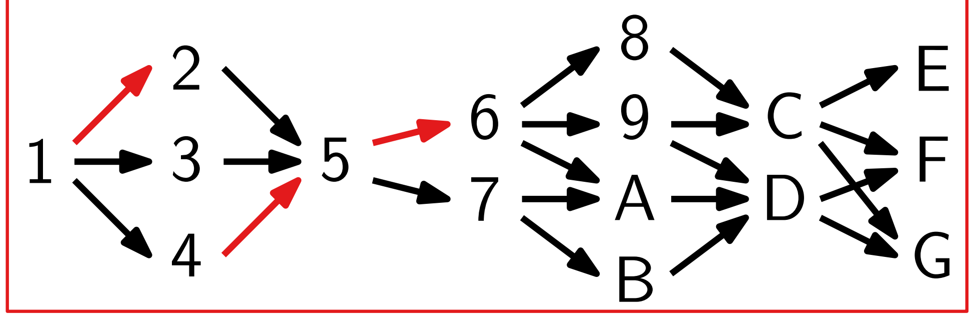

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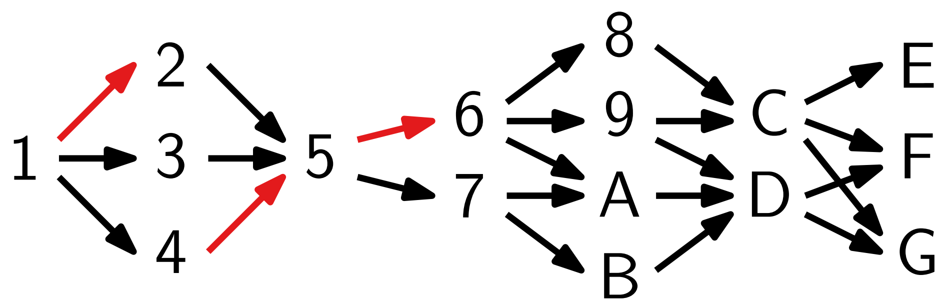

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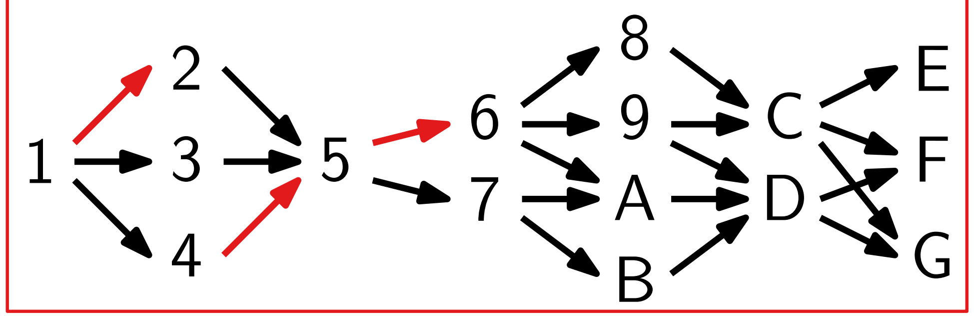

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$\leq (2 - 1/W) \cdot \mathrm{OPT}$ in general case

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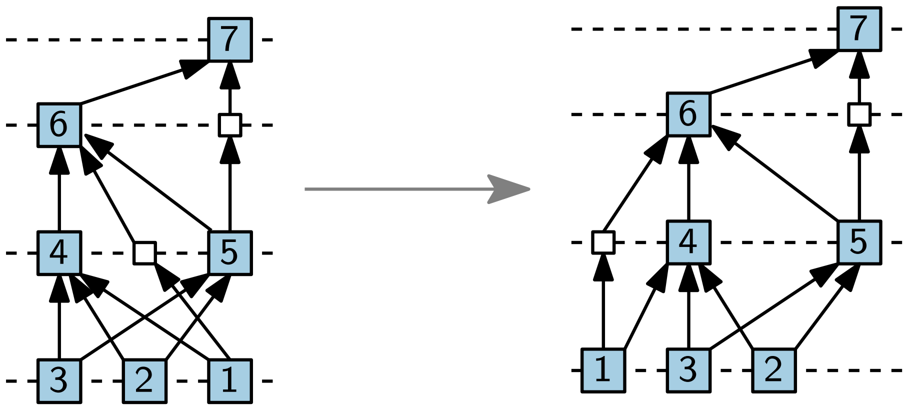
Step 3: Crossing minimization



Problem.

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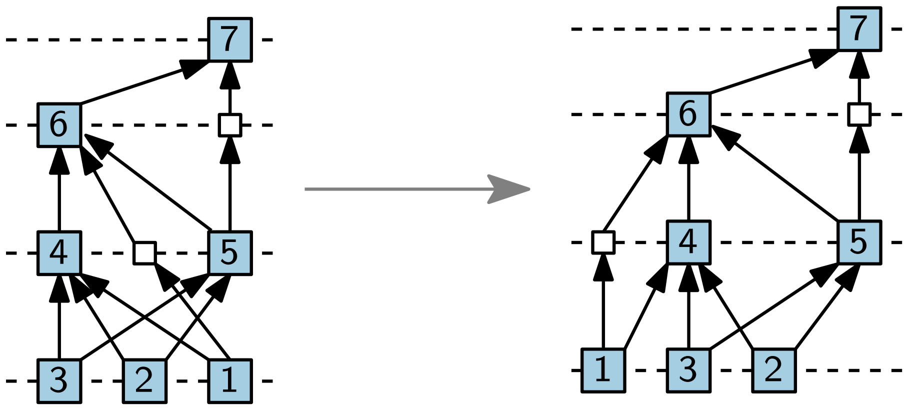

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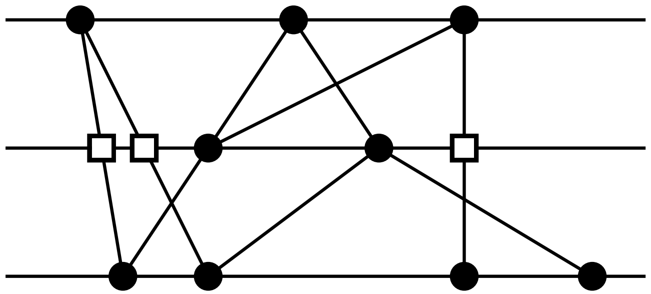
- NP-hard, even for 2 layers [Garey & Johnson '83]

Iterative crossing reduction – idea

Observation.

The number of crossings only depends on permutations of adjacent layers.



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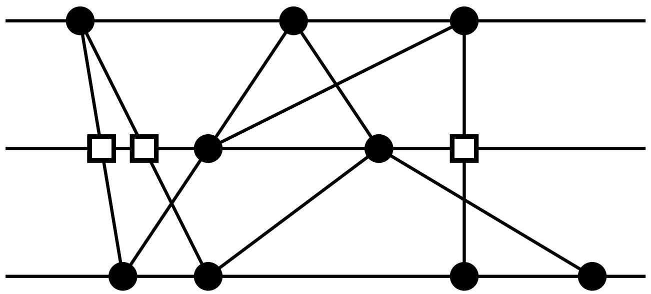

- Add dummy-vertices for edges connecting "far" layers.
- Consider adjacent layers $(L_1, L_2), (L_2, L_3), \ldots$ bottom-to-top.
- Minimize crossings by permuting L_{i+1} while keeping L_i fixed.

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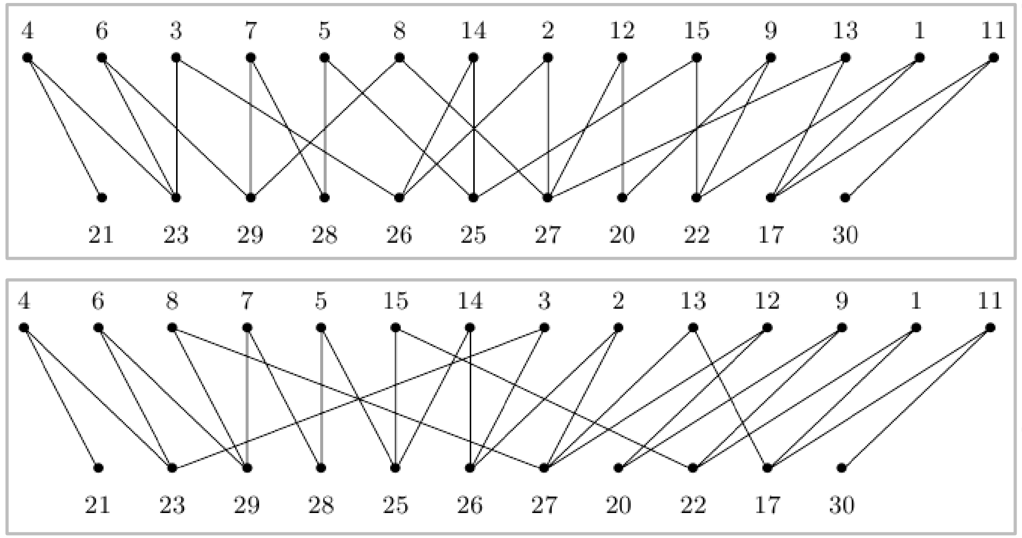

Abb. aus [Kaufmann und Wagner: Drawing Graphs] (c) Springer-Verlag

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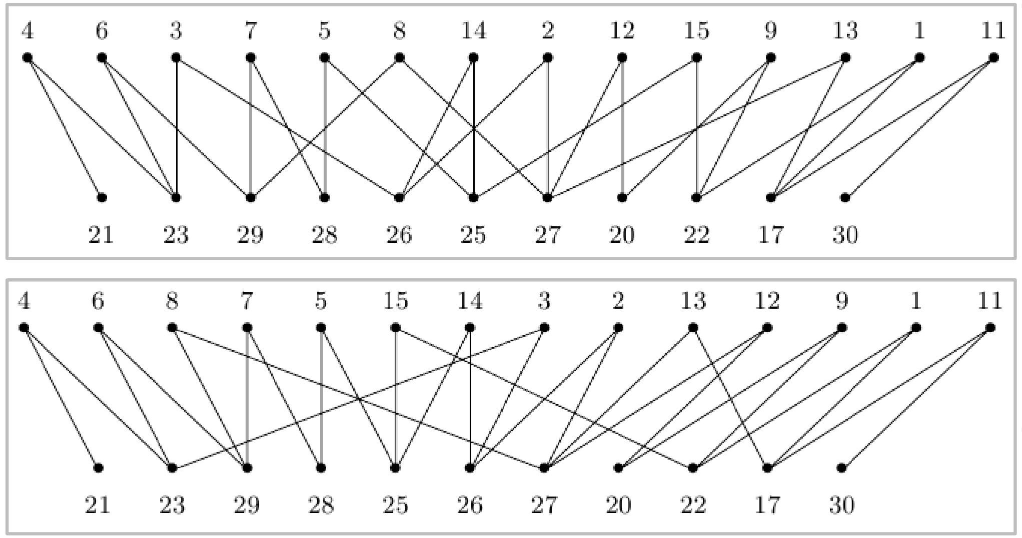

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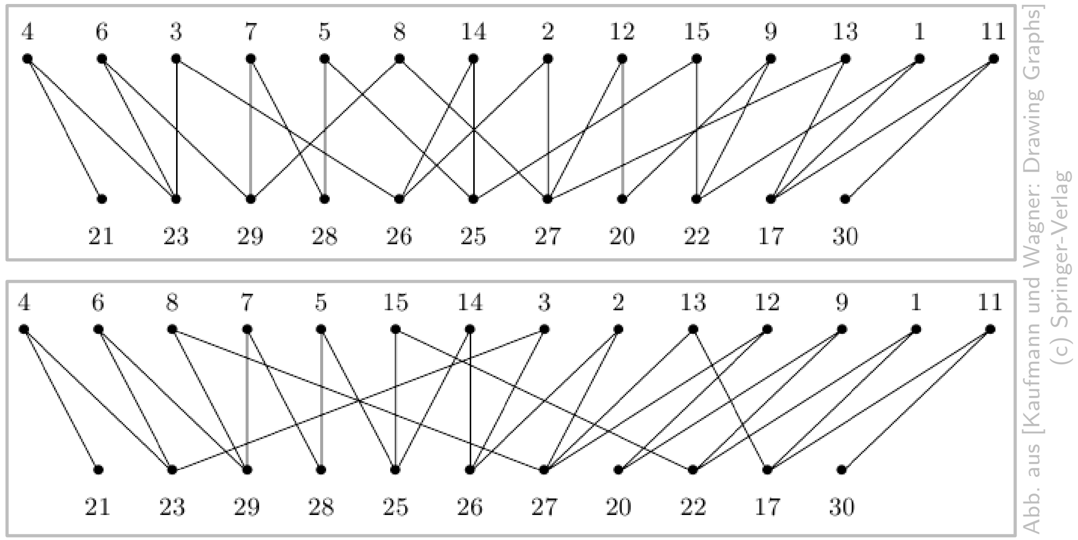
Algorithms.

- barycenter heuristic
- median heuristic
- Greedy-Switch
- ILP

...



Barycentre heuristic

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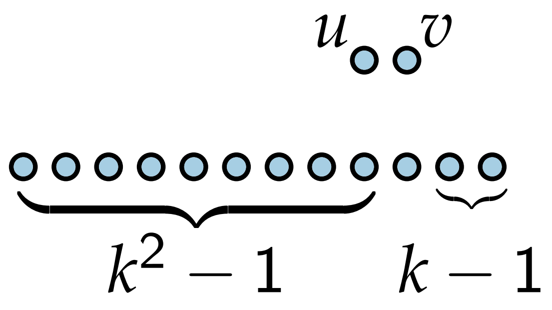

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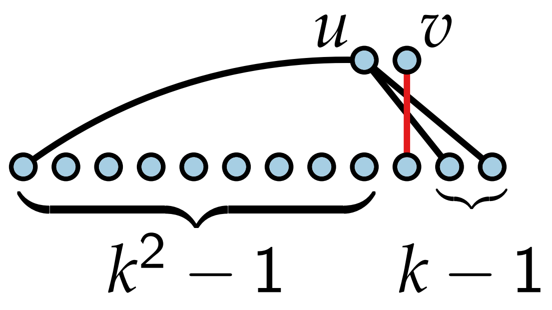

Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
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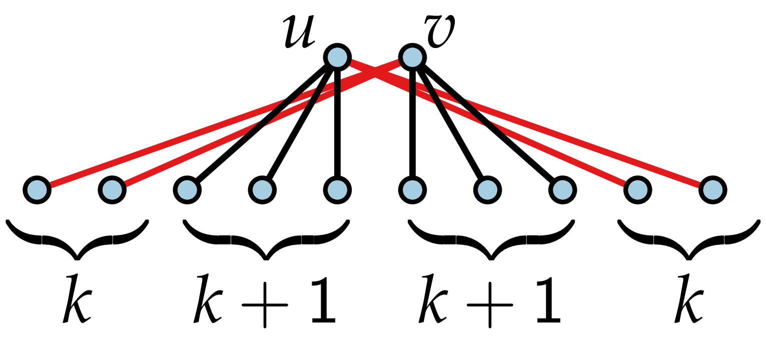

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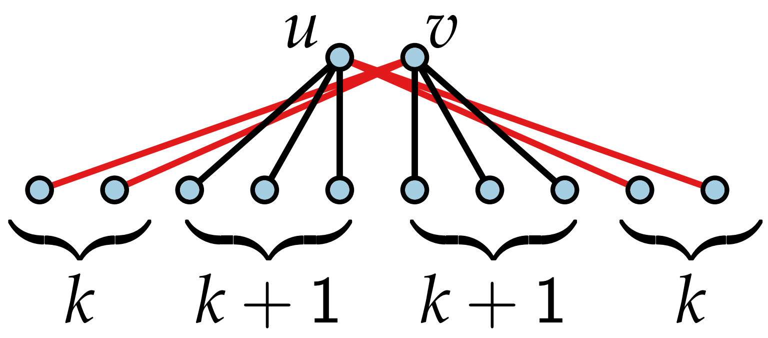

$2k(k+1) + k^2$ vs. $(k+1)^2$

Greedy-switch heuristic

- iteratively swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
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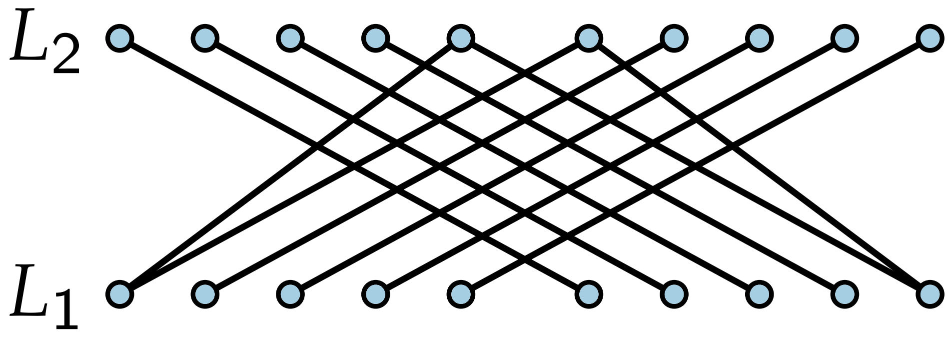

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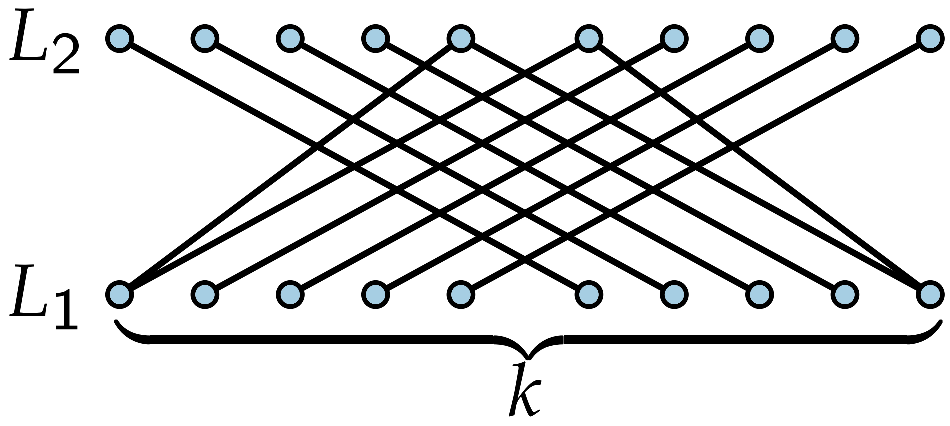

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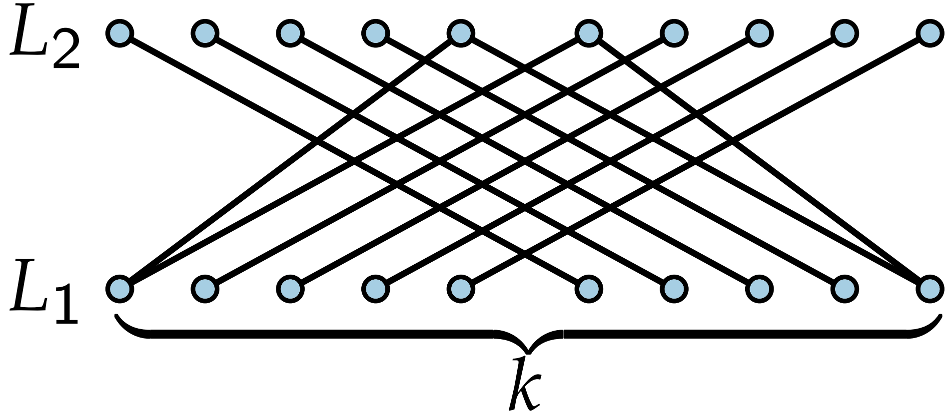

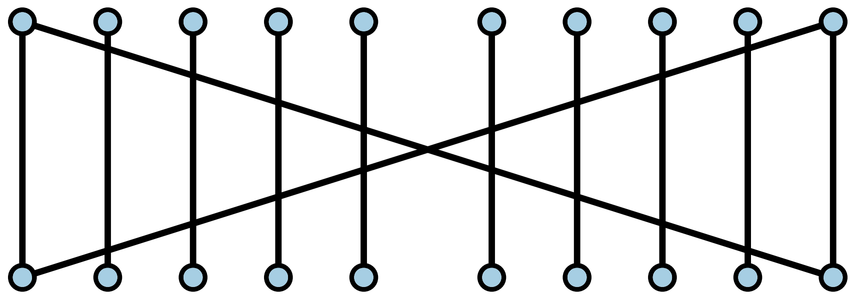

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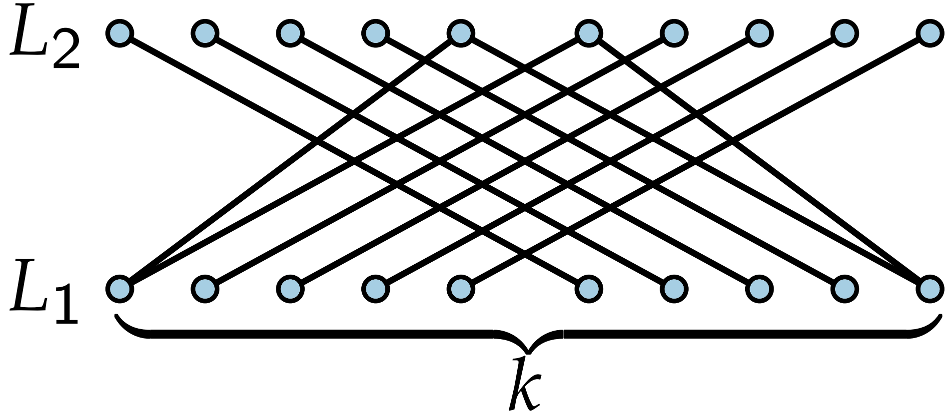
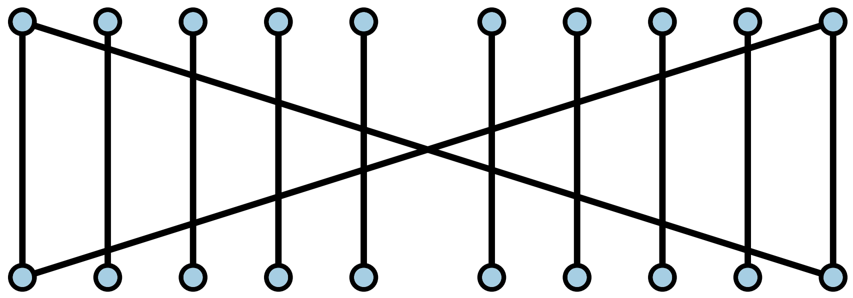




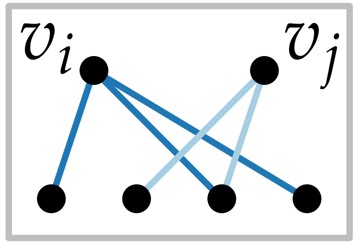
$\approx k^2/4$

$\approx 2k$

Integer linear program

[Jünger & Mutzel, '97]

- Constant $c_{ij} :=$ # crossings between edges incident to v_i or v_j when $\pi_2(v_i) < \pi_2(v_j)$



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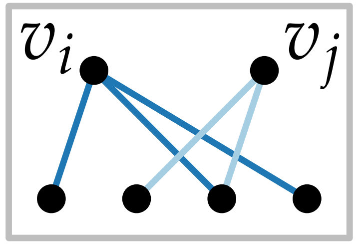

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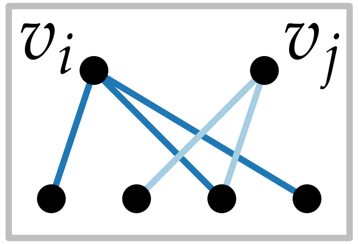

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- The number of crossings of a permutations π_2

$$\text{cross}(\pi_2) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij} + \underbrace{\sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} c_{ji}}_{\text{constant}}$$

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- Transitivity constraints:

$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \qquad \text{for } 1 \leq i < j < k \leq n_2$$

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

- Transitivity constraints:

$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \qquad \text{for } 1 \leq i < j < k \leq n_2$$

i.e., if $x_{ij} = 1$ and $x_{jk} = 1$, then $x_{ik} = 1$

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$\phantom{i.e., if x_{ij} =}$ 0 $\qquad$ 0 $\qquad$ 0

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

- Transitivity constraints:

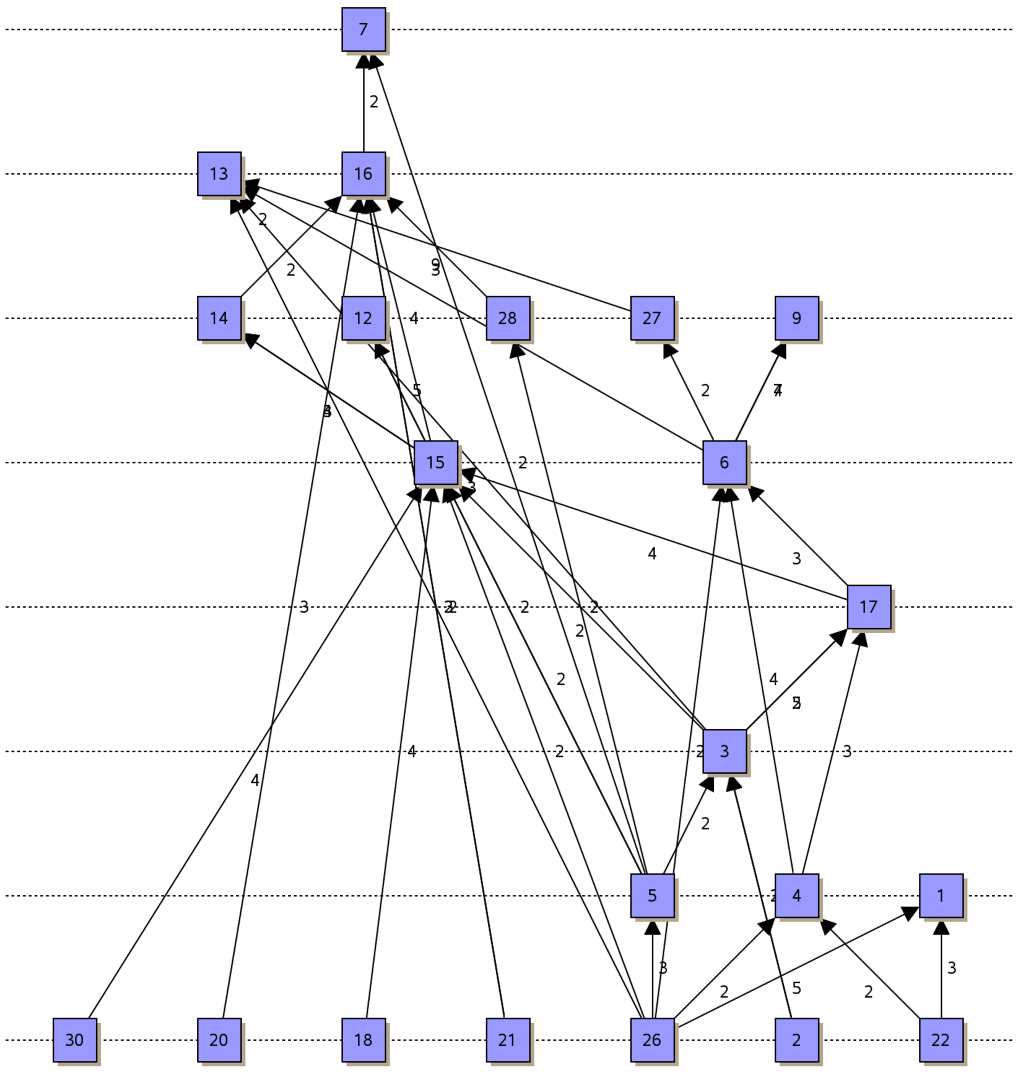
$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \qquad \text{for } 1 \leq i < j < k \leq n_2$$

i.e., if $x_{ij} = \genfrac{}{}{0pt}{}{1}{0}$ and $x_{jk} = \genfrac{}{}{0pt}{}{1}{0}$, then $x_{ik} = \genfrac{}{}{0pt}{}{1}{0}$

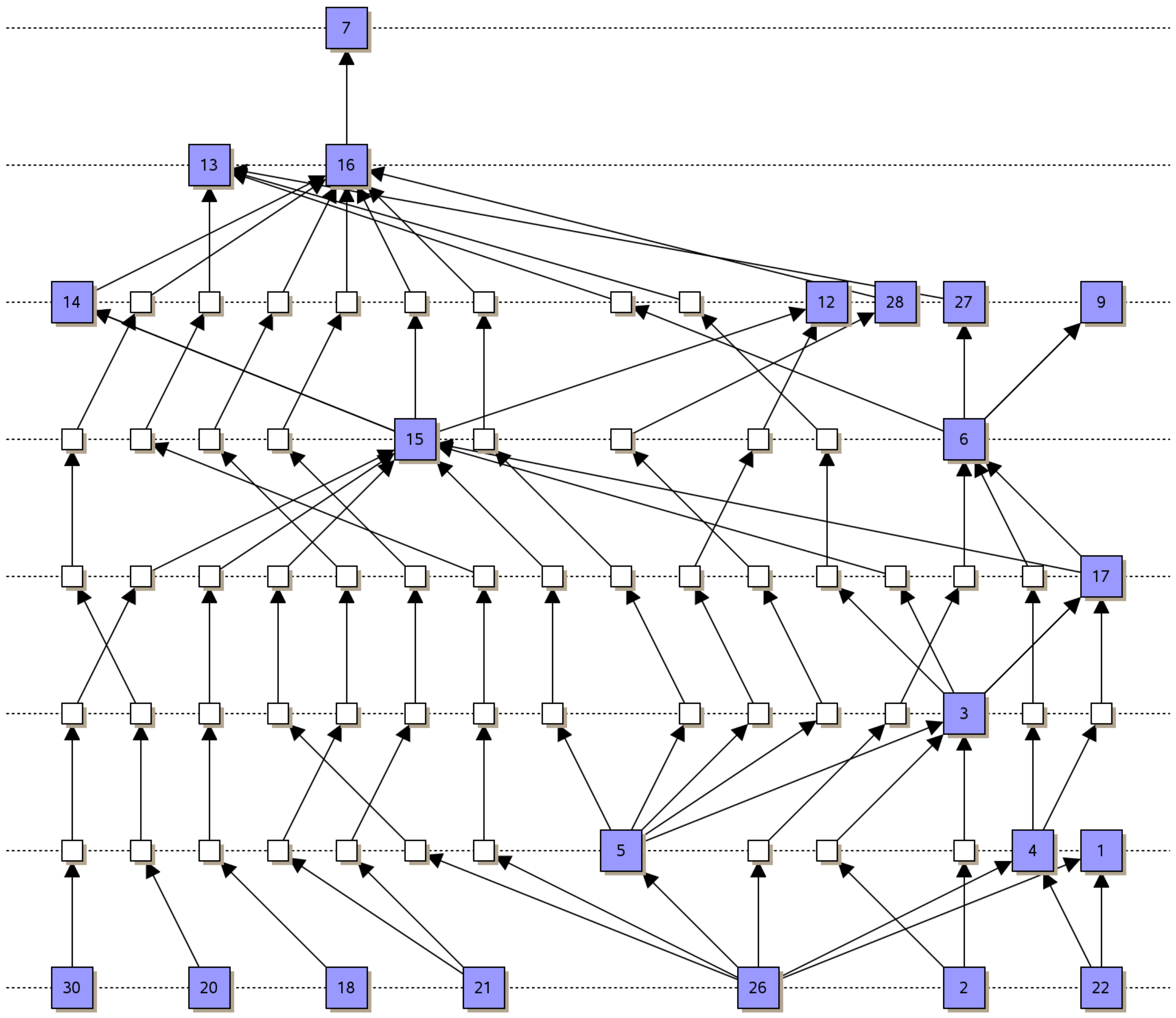
Properties.

- branch-and-cut technique for DAGs of limited size
- useful for graphs of small to medium size
- finds optimal solution
- solution in polynomial time is not guaranteed

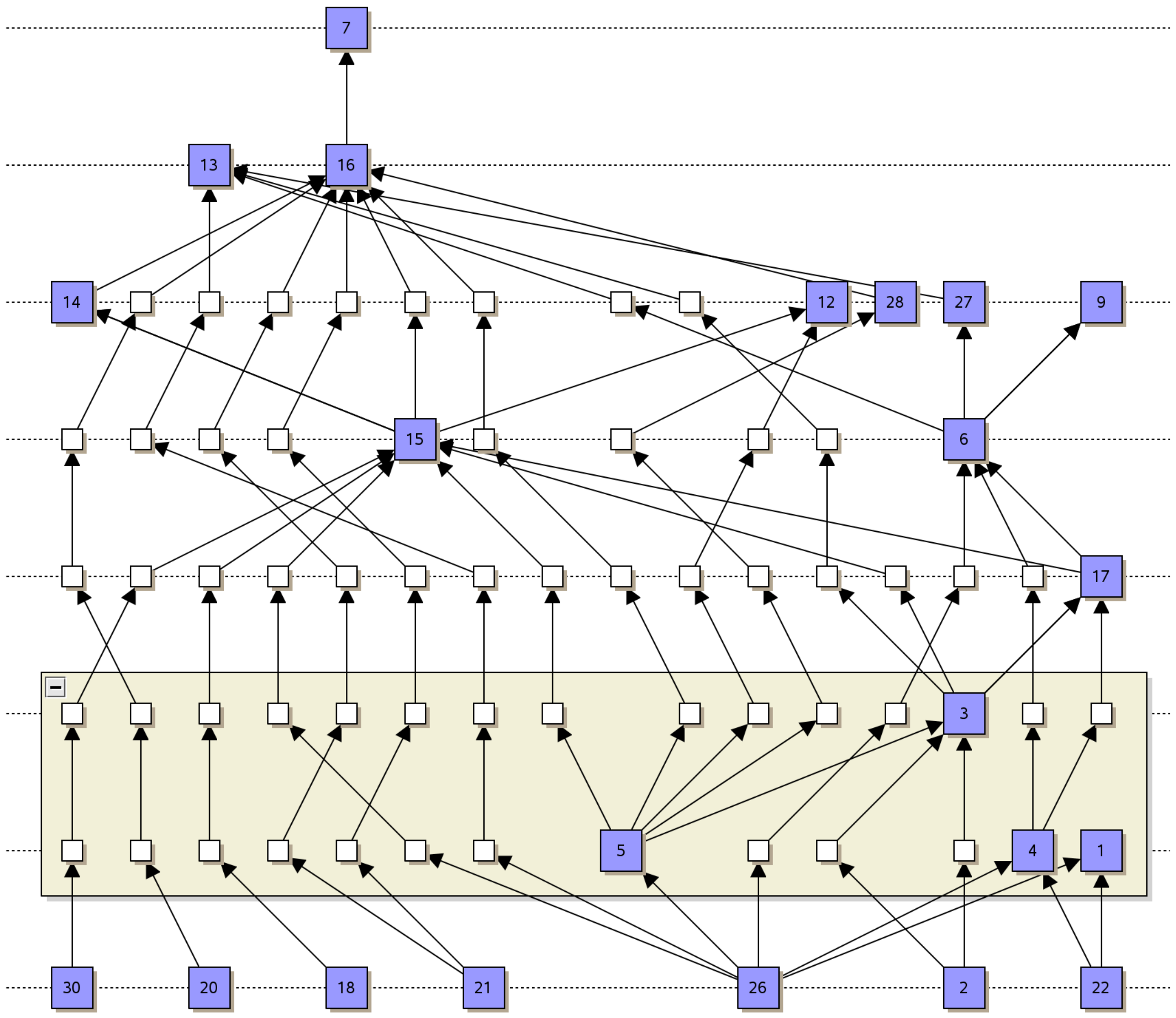
Iterations on example



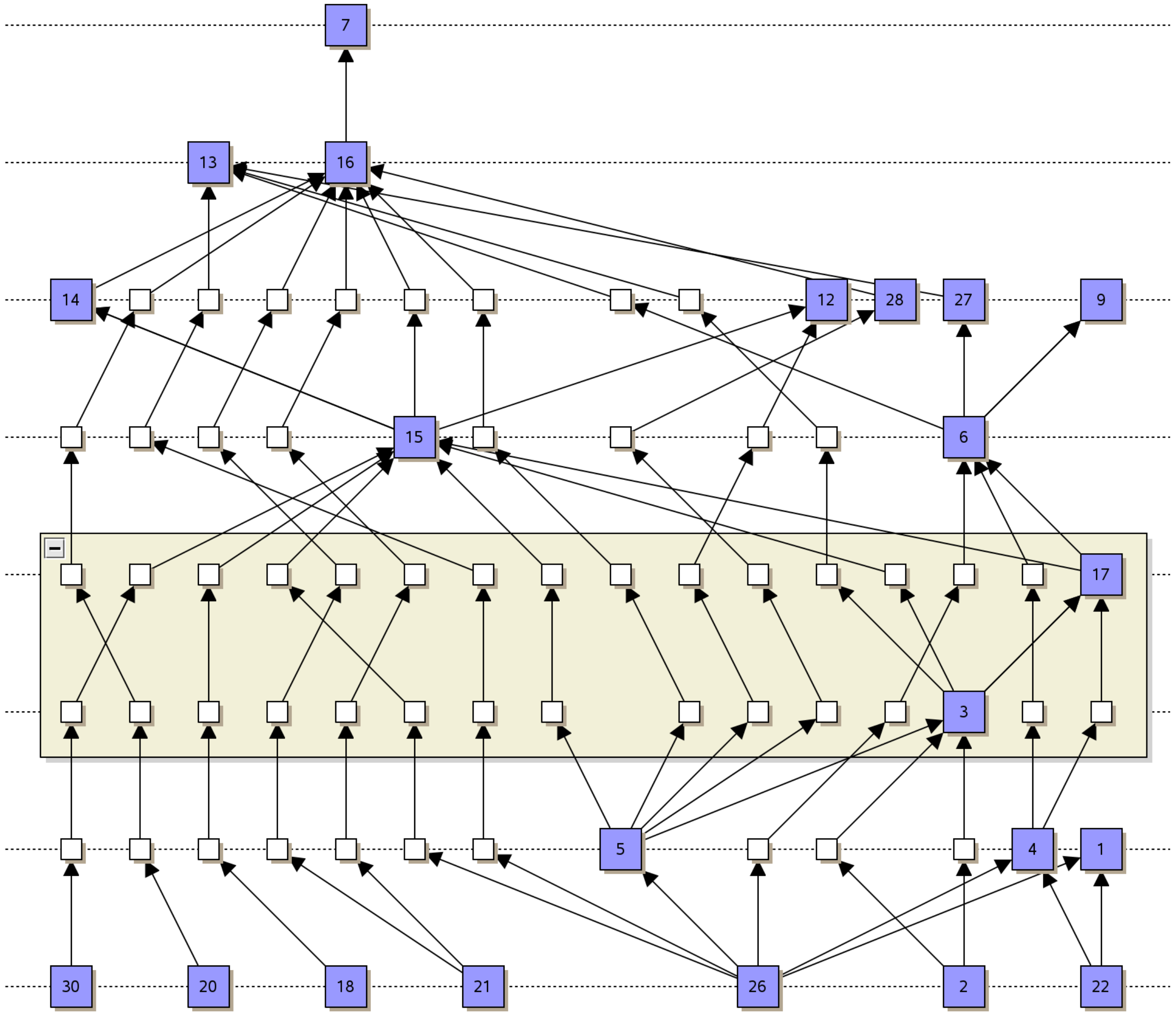
Iterations on example



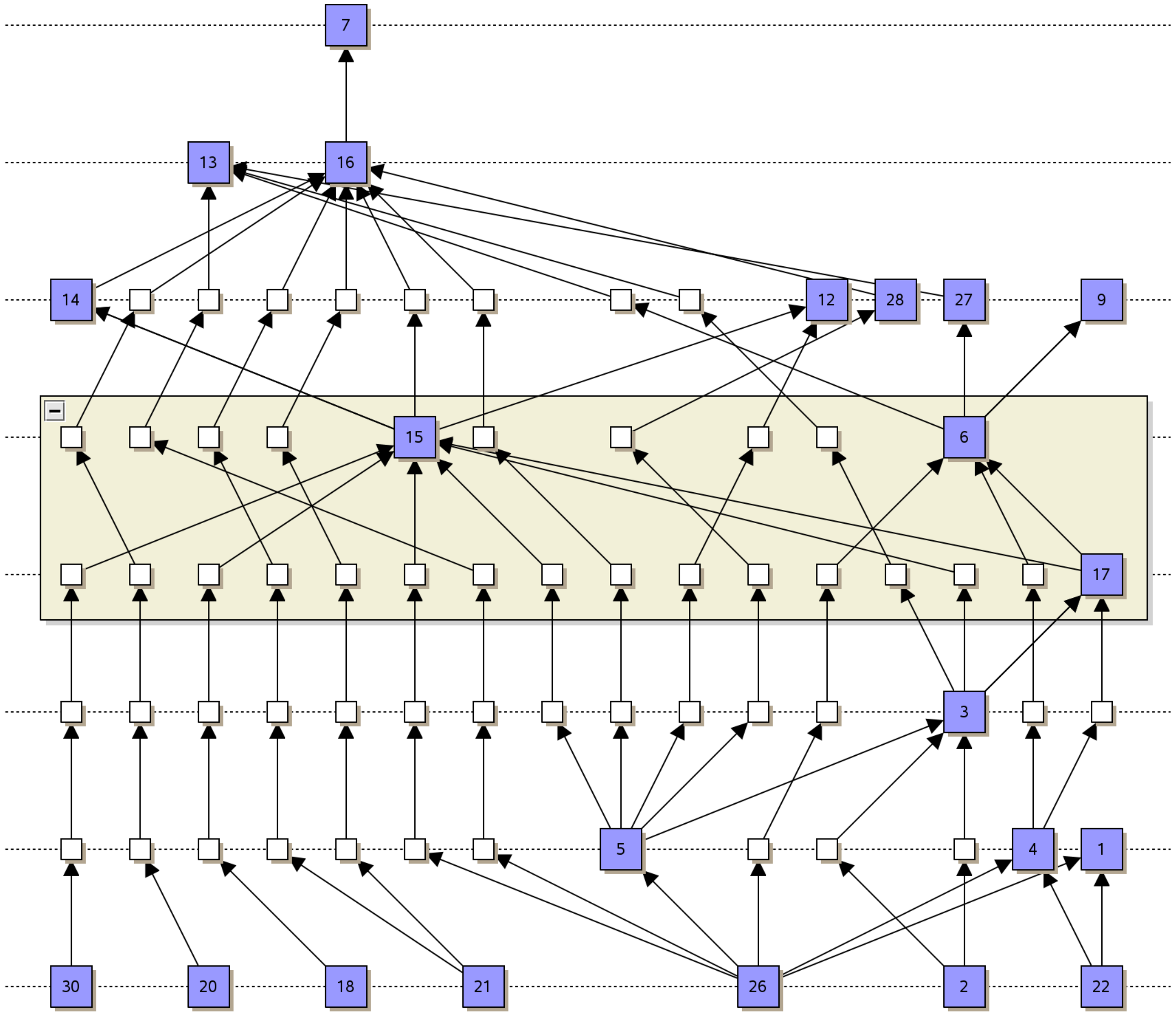
Iterations on example



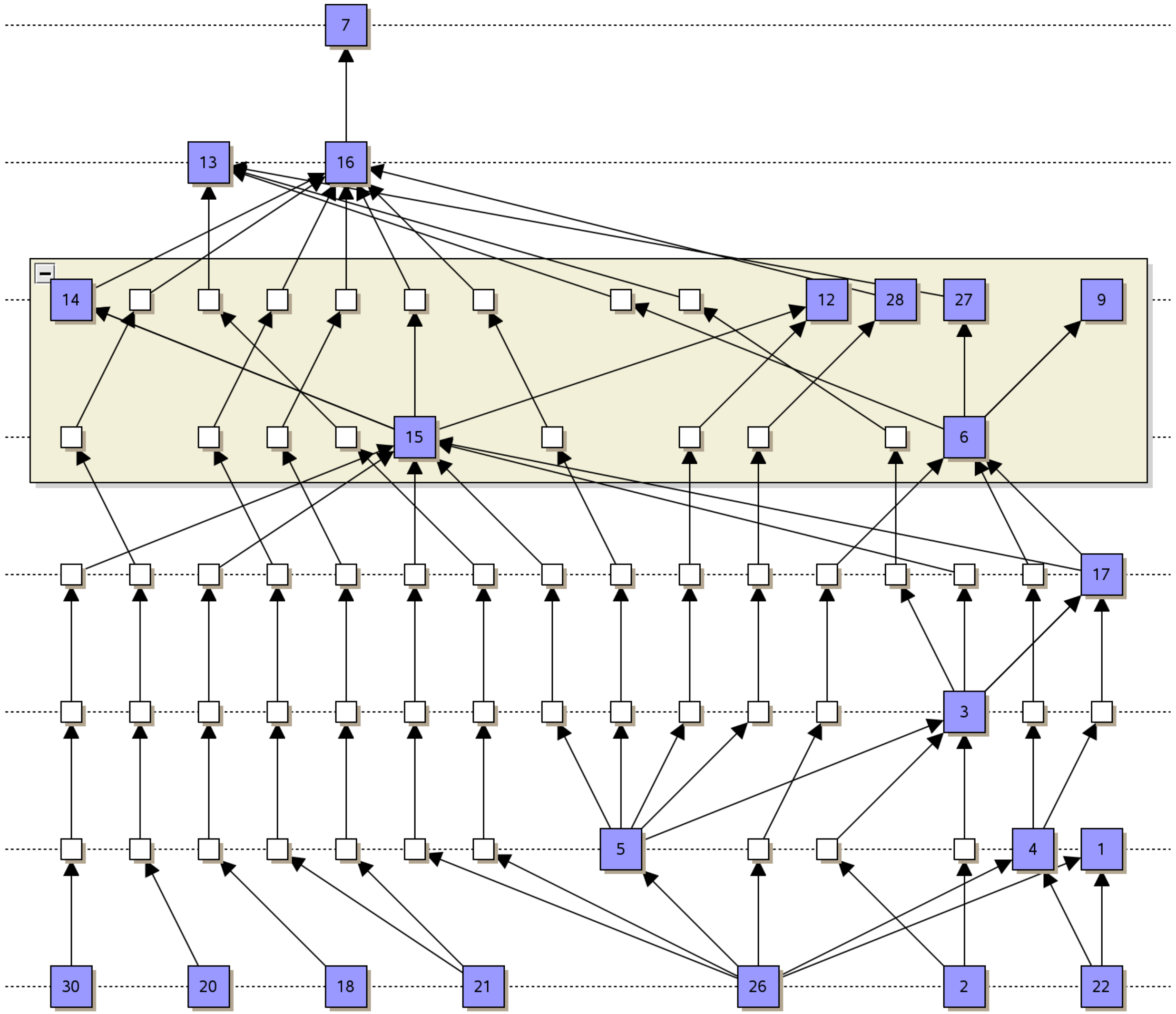
Iterations on example



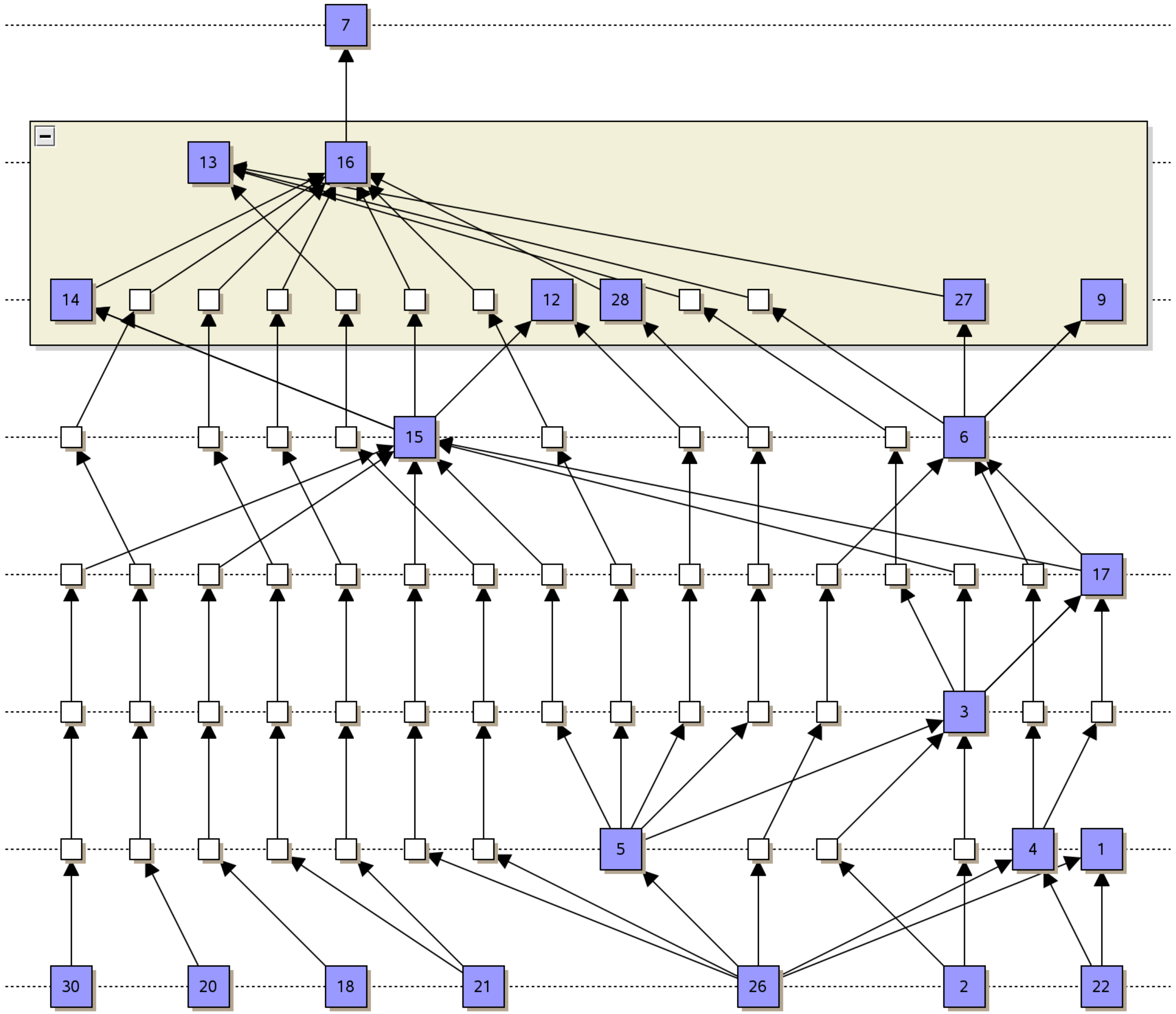
Iterations on example



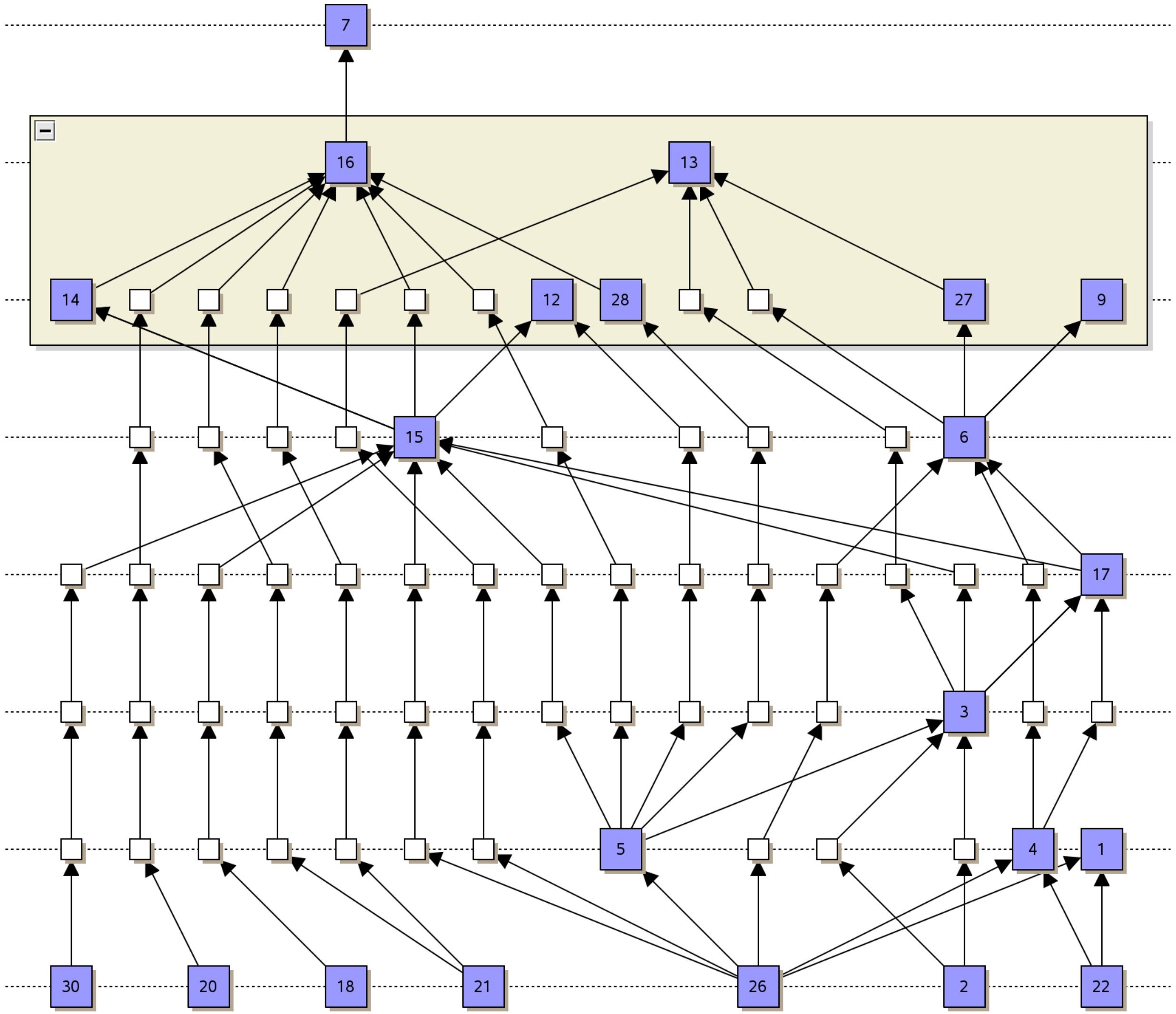
Iterations on example



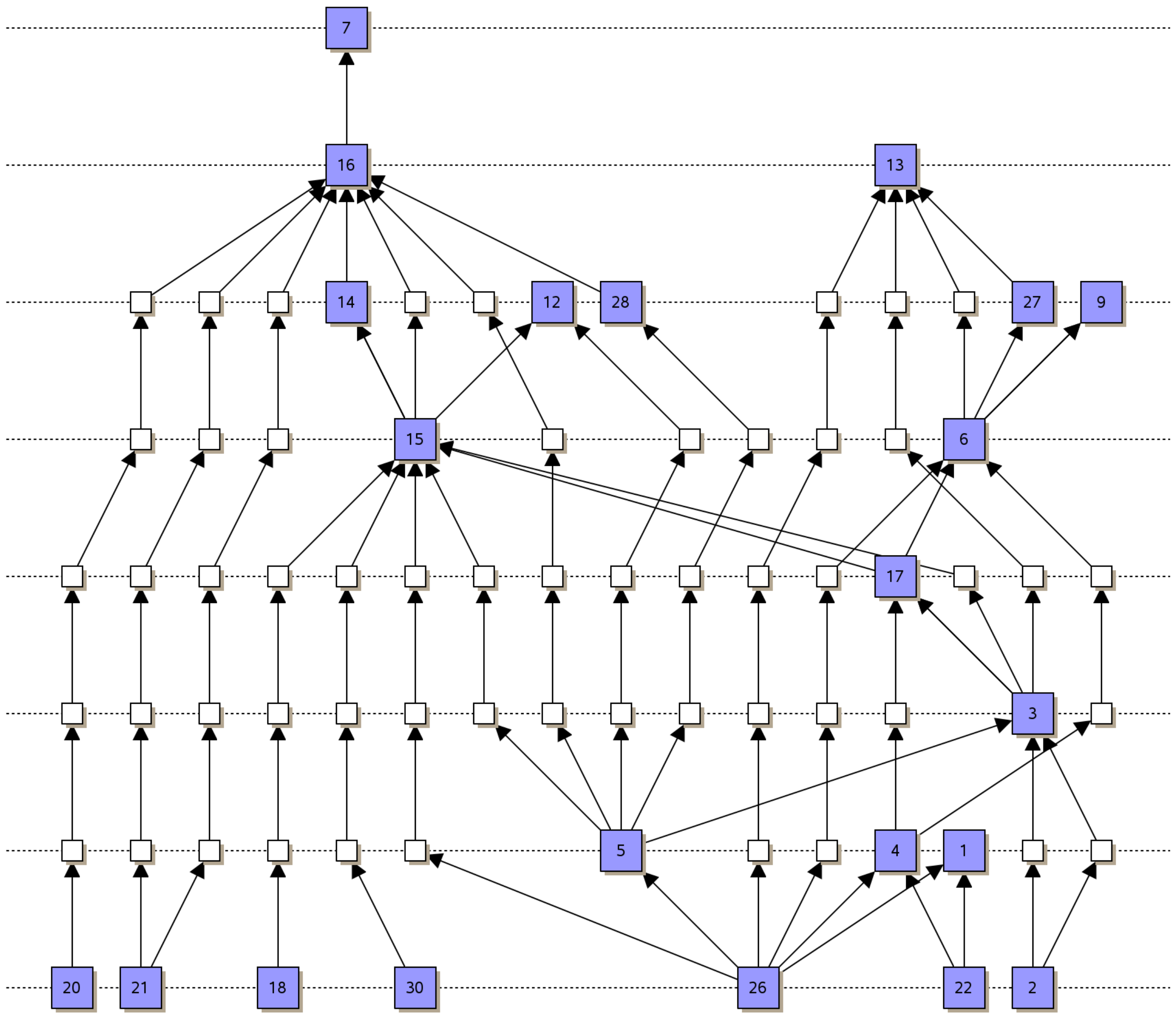
Iterations on example



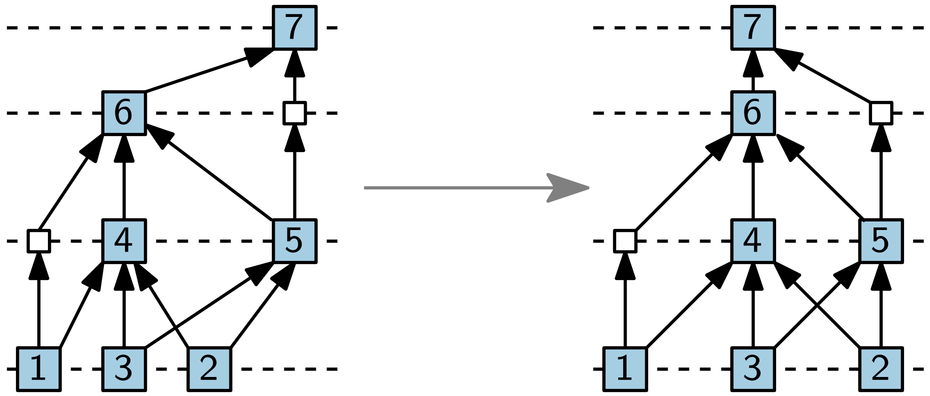
Iterations on example



Iterations on example



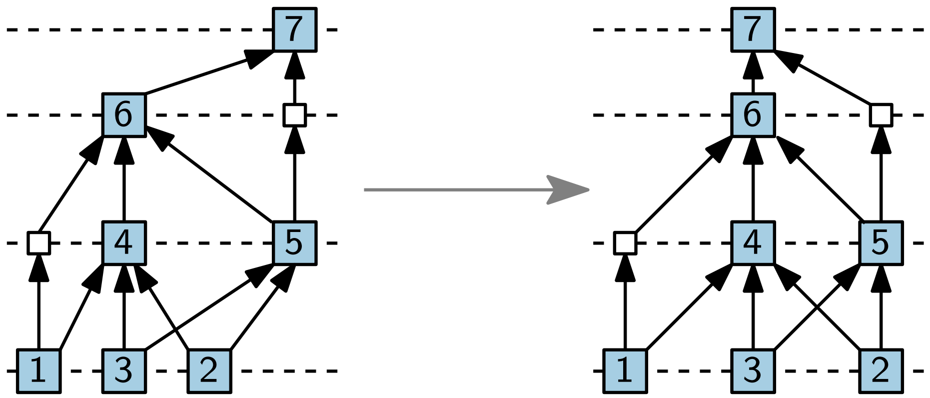
Step 4: Vertex positioning



Goal.

paths should be close to straight, vertices evenly spaced

Step 4: Vertex positioning



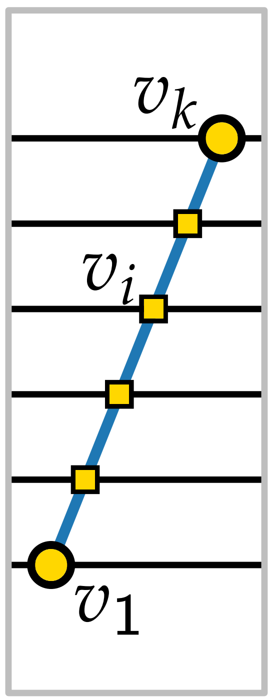
Goal.

paths should be close to straight, vertices evenly spaced

- **Exact:** Quadratic Program (QP)
- **Heuristic:** iterative approach

Quadratic Program

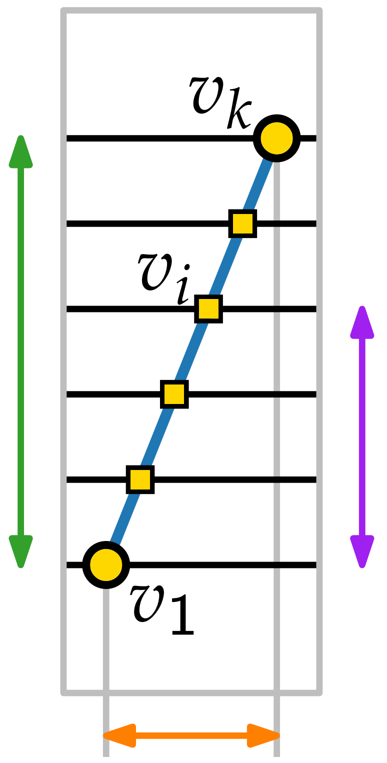
- Consider the path $p_e = (v_1, ..., v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, ..., v_{k-1}$



Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$



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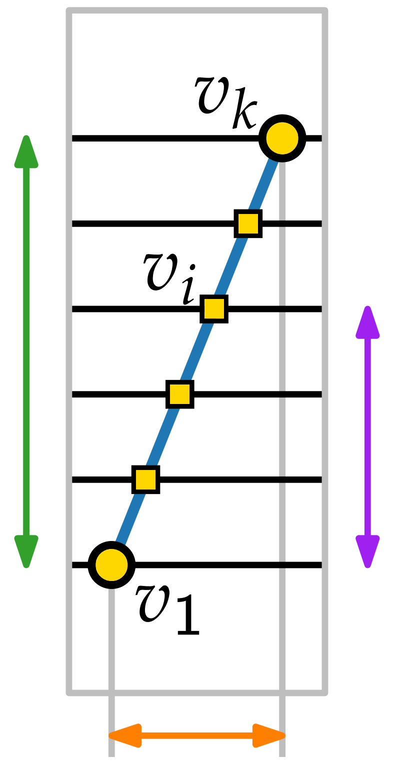
- define the deviation from the line

$$\mathrm{dev}(p_e) := \sum_{i=2}^{k-1}\left(x(v_i) - \overline{x(v_i)}\right)^2$$



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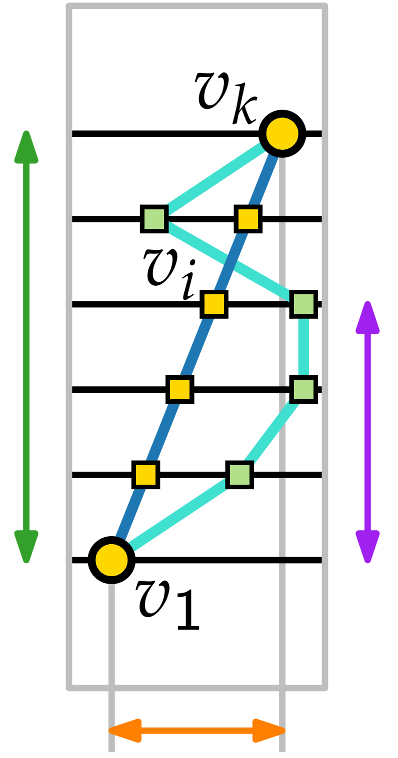

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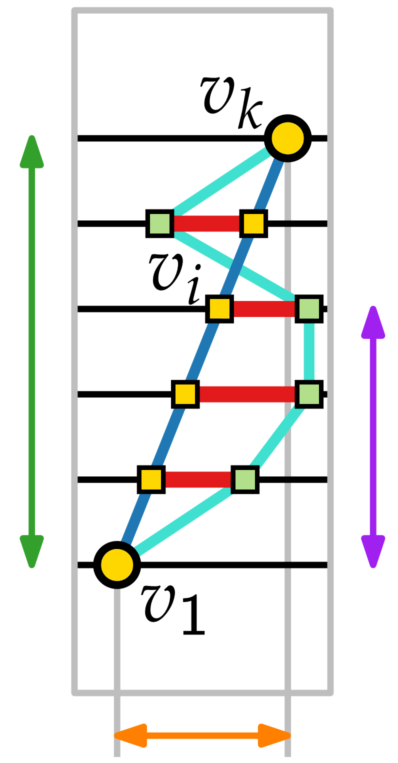

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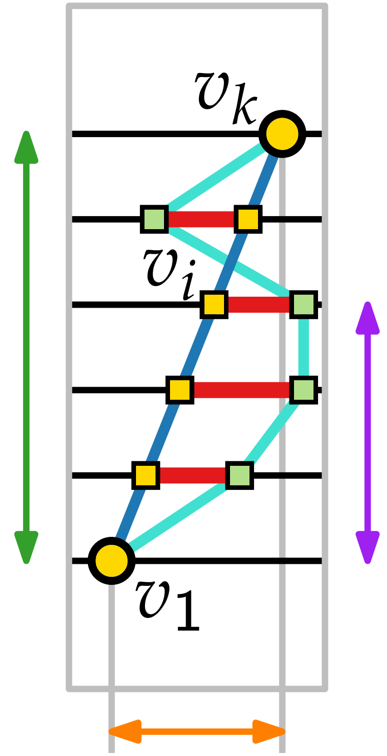

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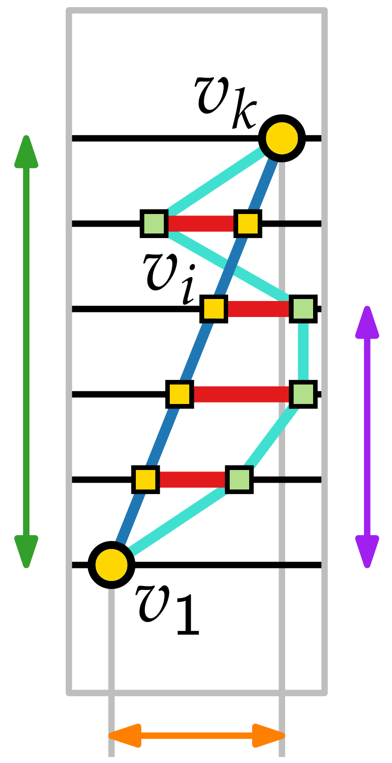
- Constraints for all vertices v, w in the same layer with w right of v: $x(w) - x(v) \geq \rho(w, v)$ ← min. horizontal distance



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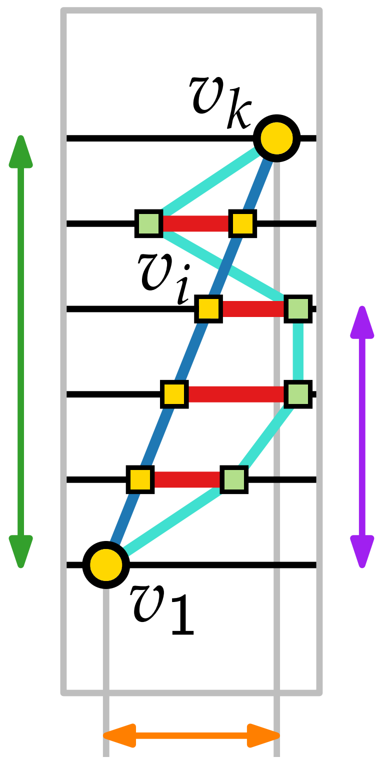

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- QP is time-expensive
- width can be exponential

Iterative heuristic

- compute an initial layout

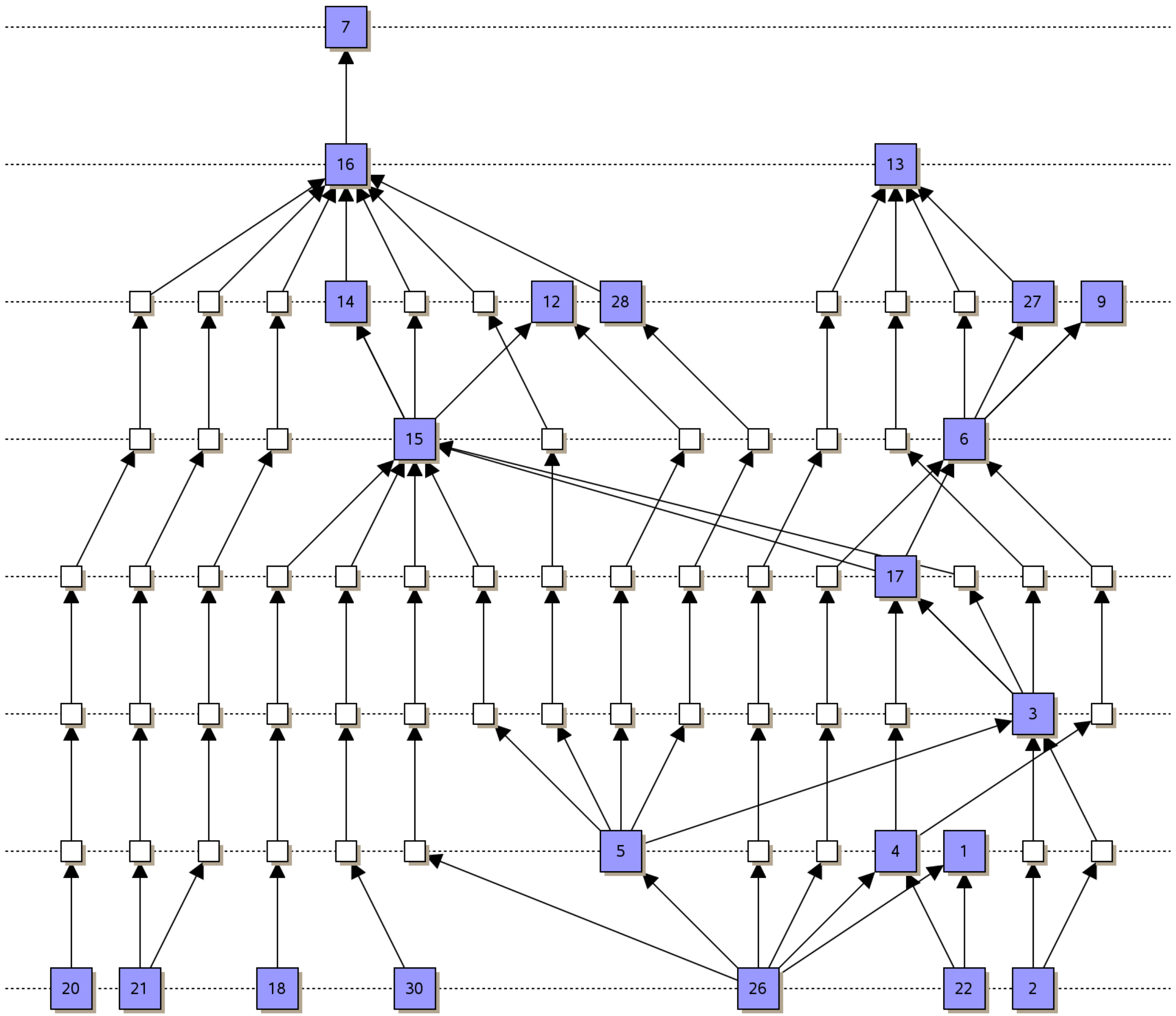
Iterative heuristic

- compute an initial layout
- apply the following steps as long as improvements can be made:

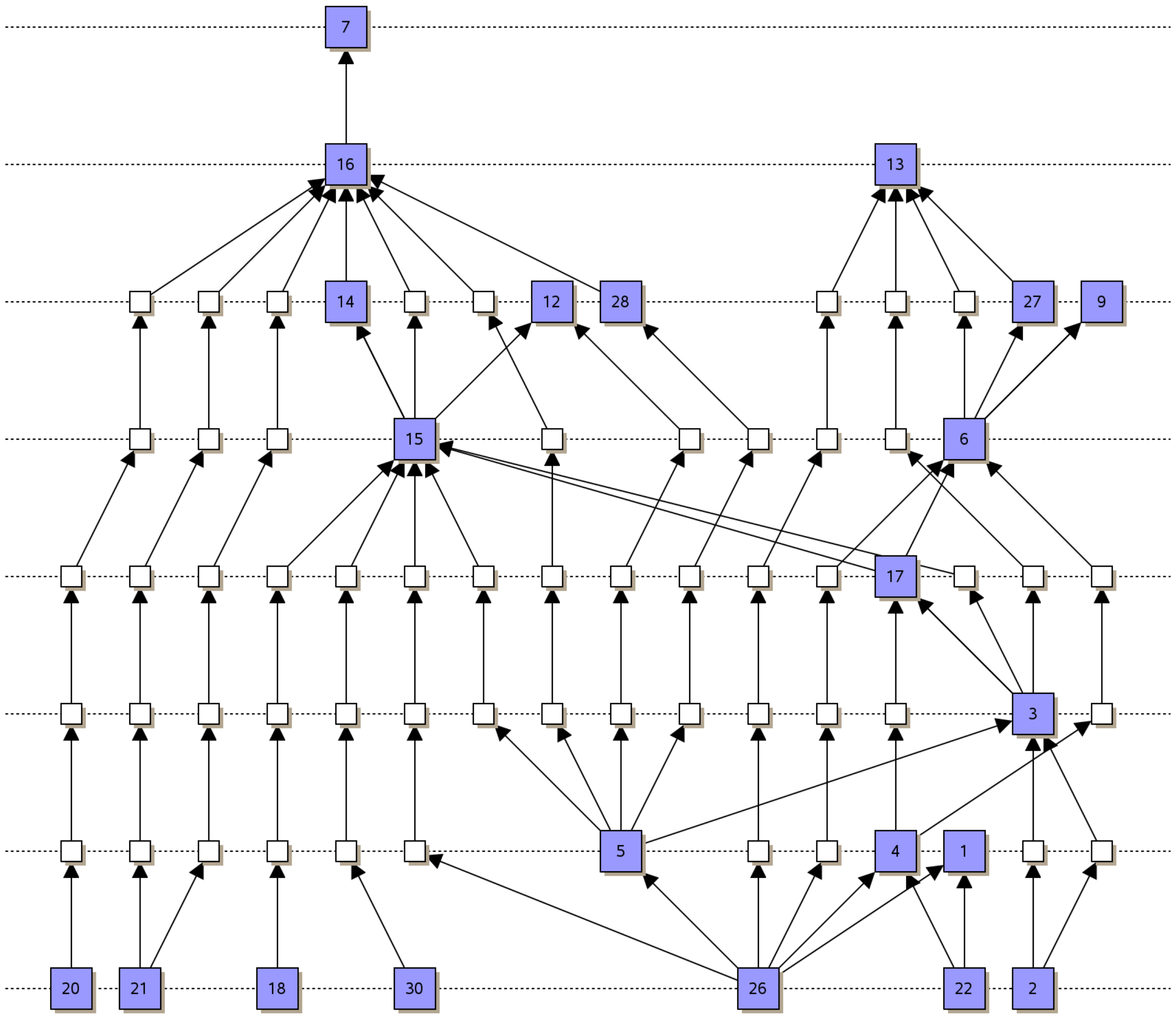
Iterative heuristic

- compute an initial layout
- apply the following steps as long as improvements can be made:
  1. vertex positioning,
  2. edge straightening,
  3. compactifying the layout width.

Example



Example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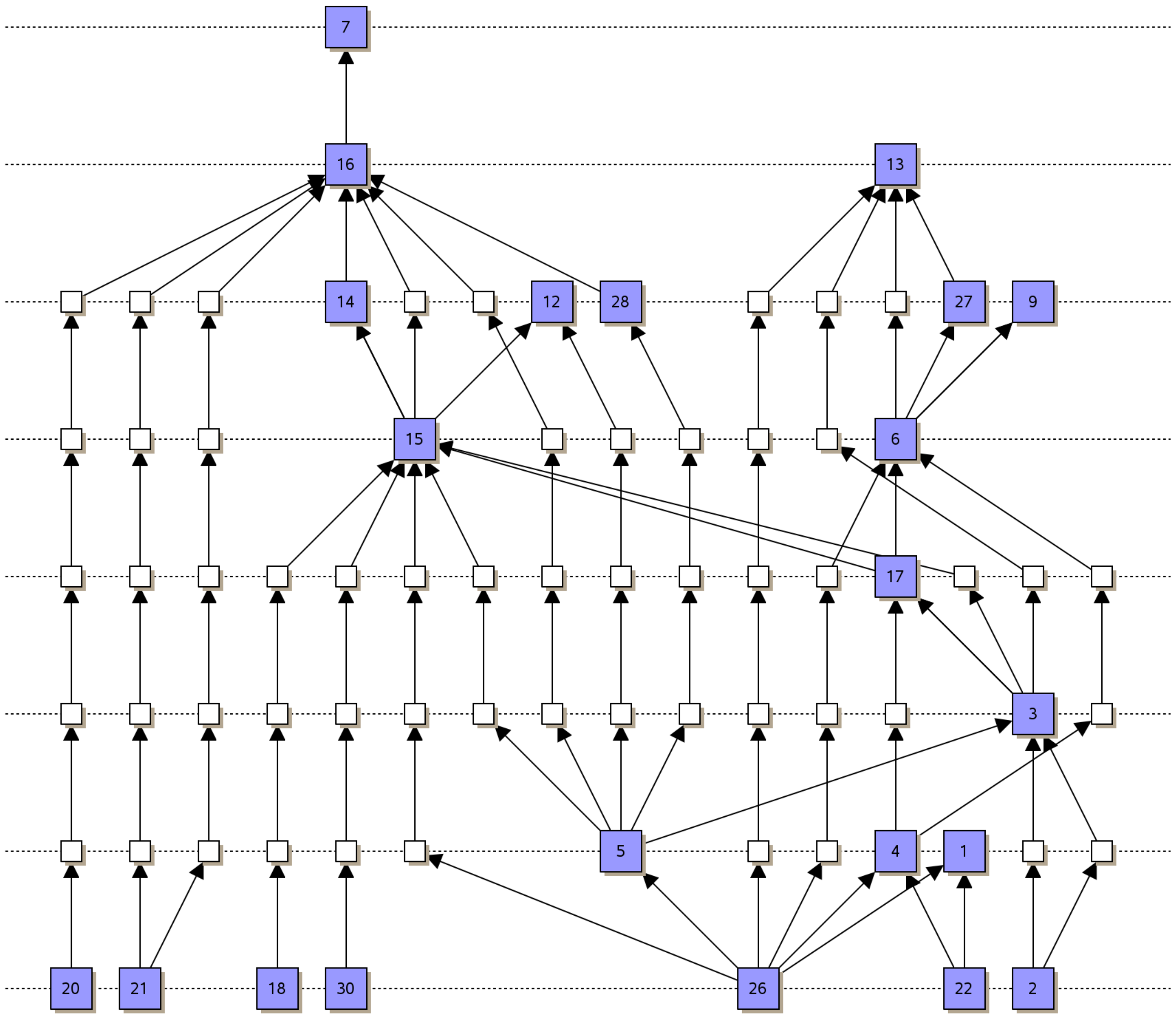
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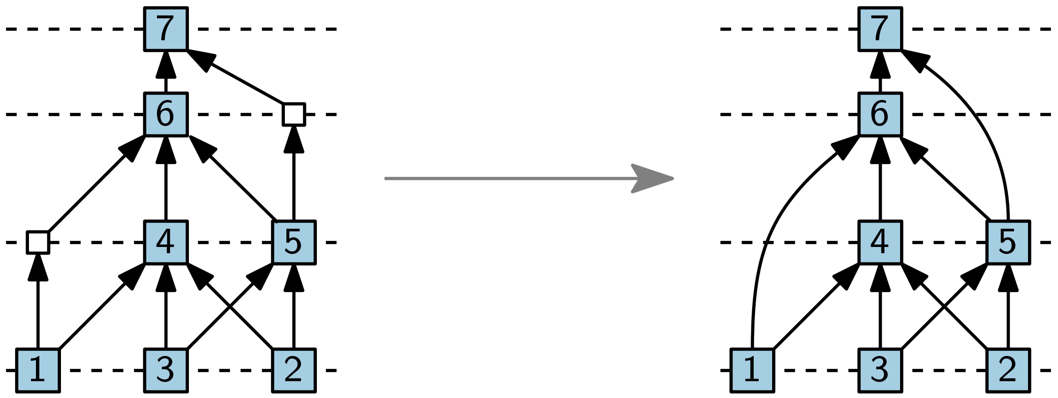
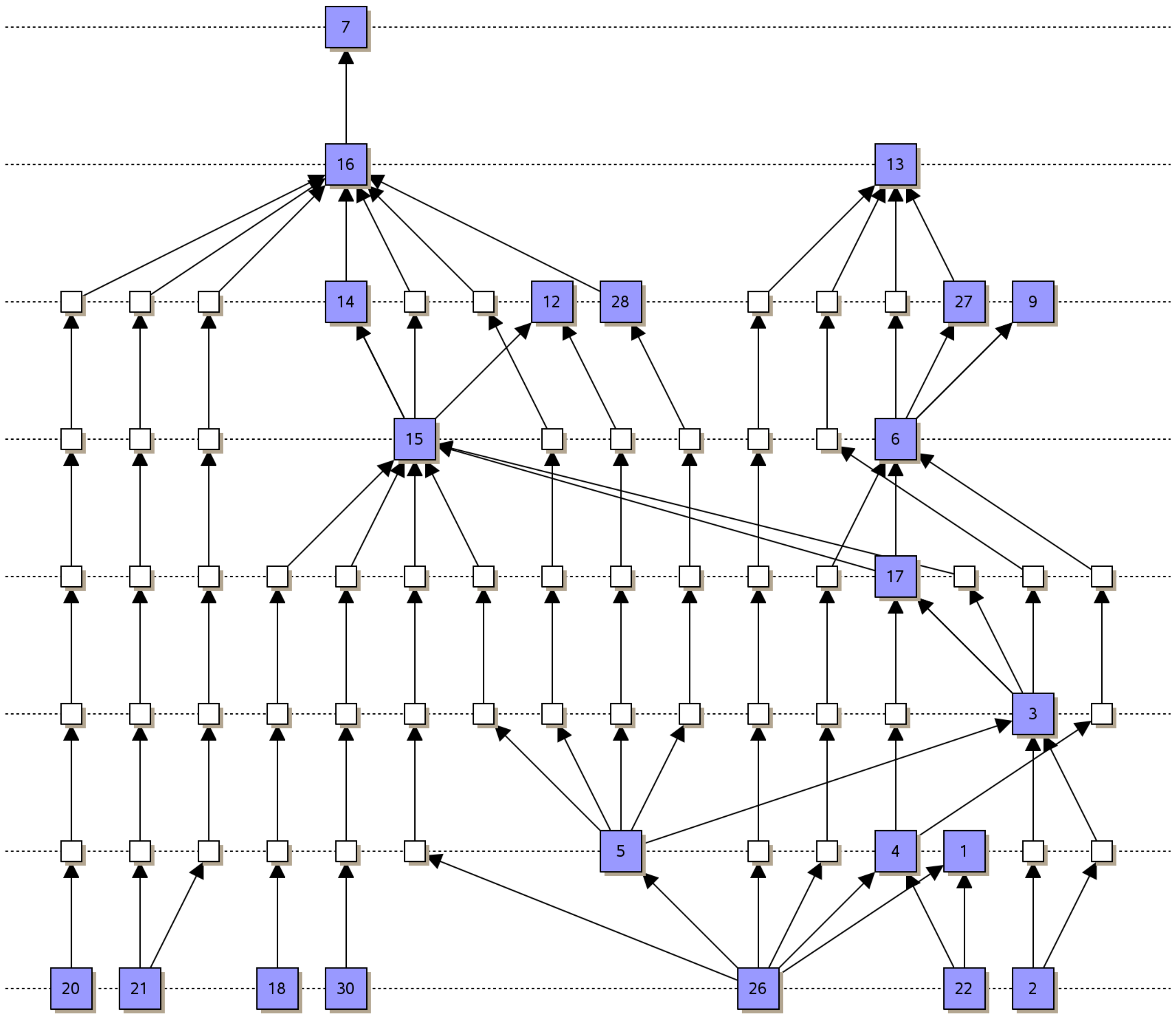
Step 5: Drawing edges



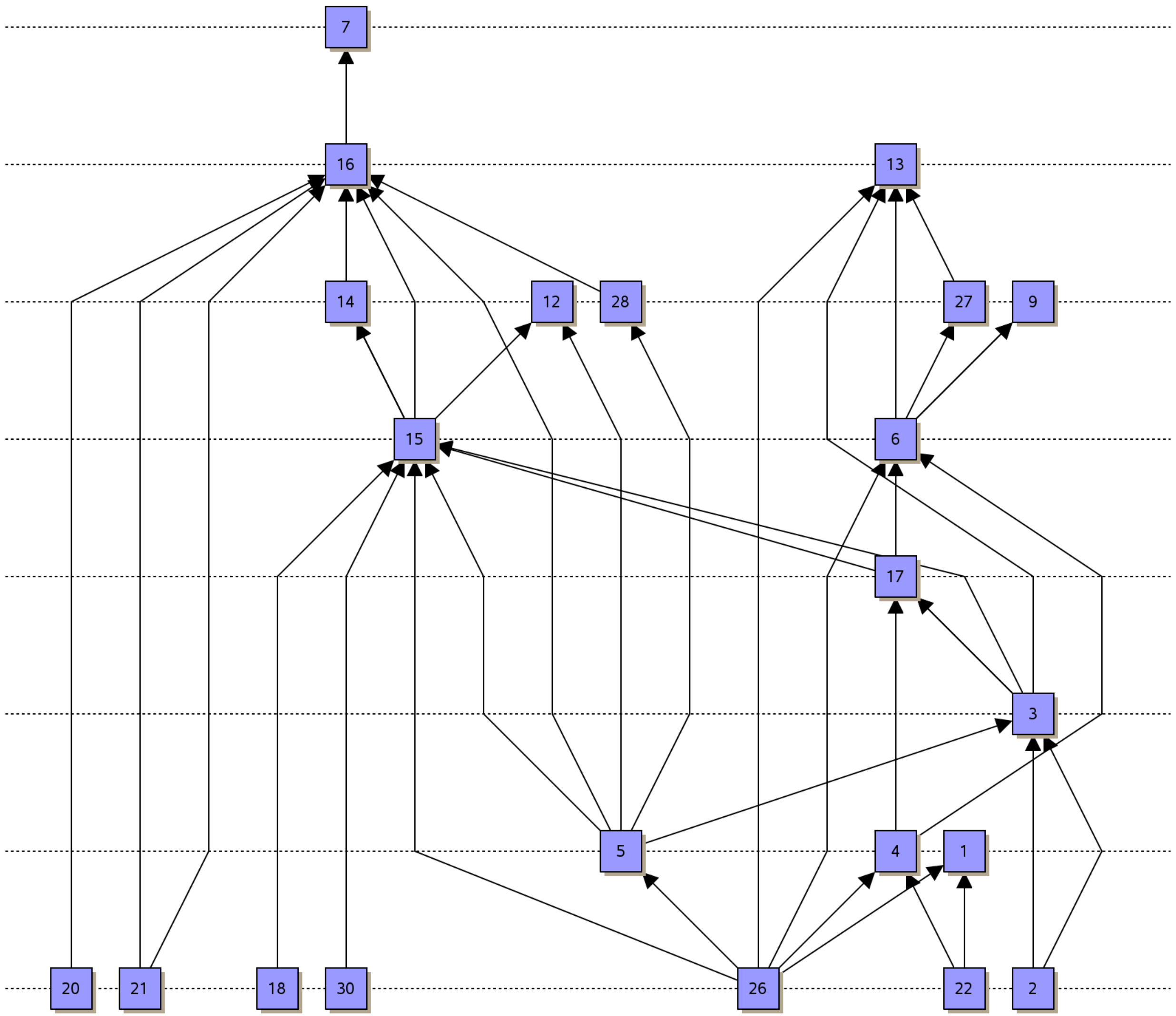
Possibility.

Substitute polylines by Bézier curves

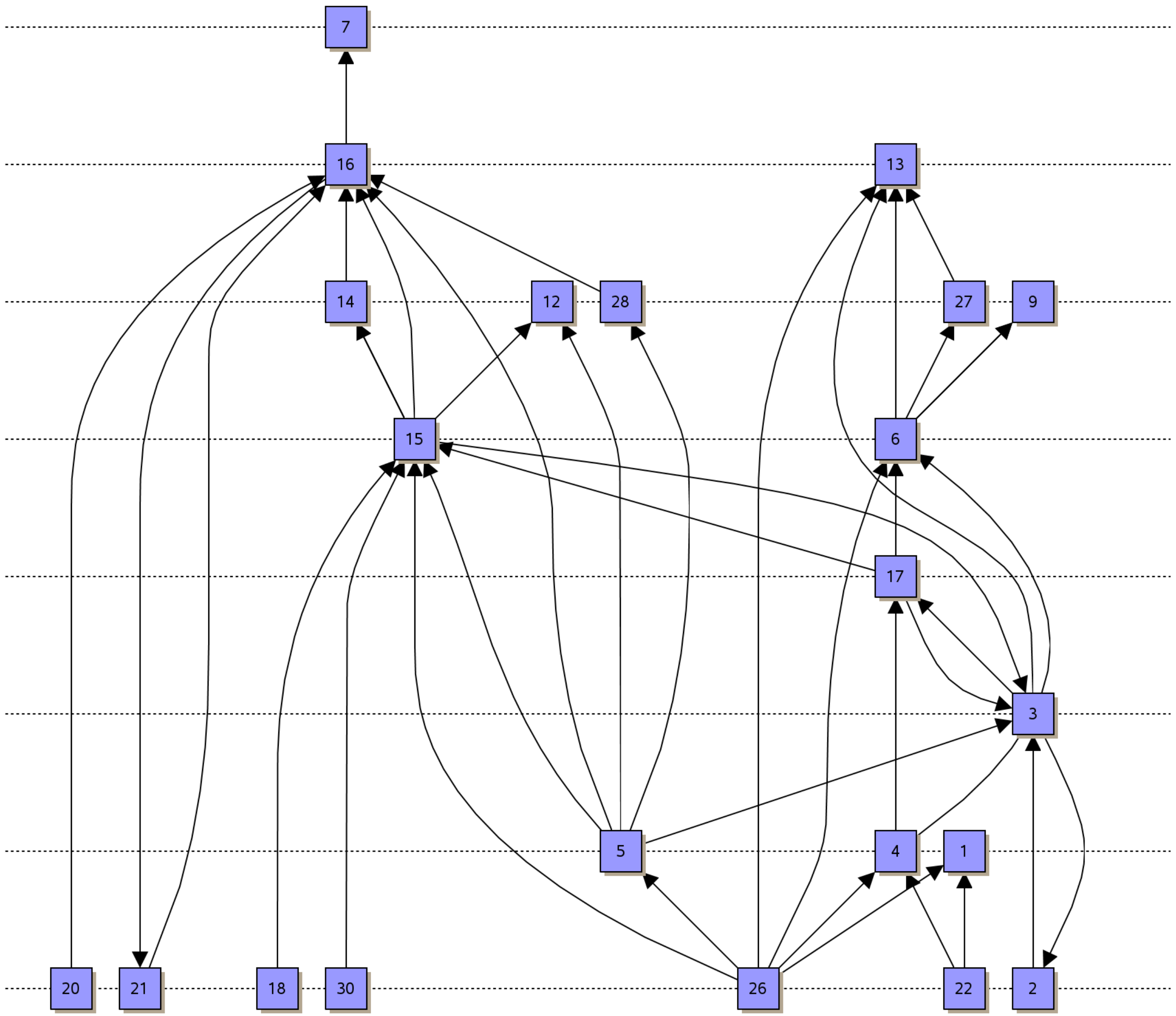
Example



Example

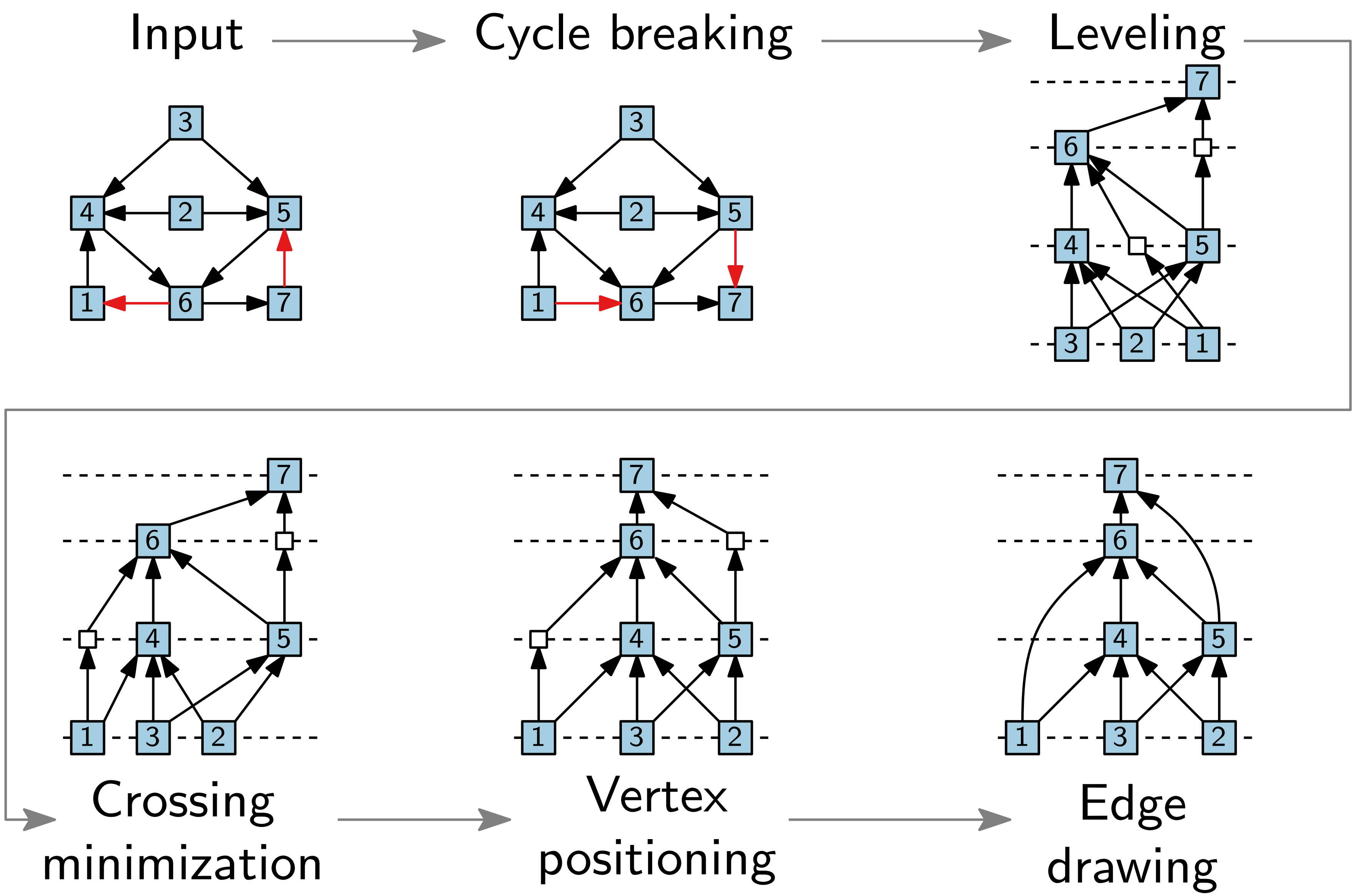


Example



Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]



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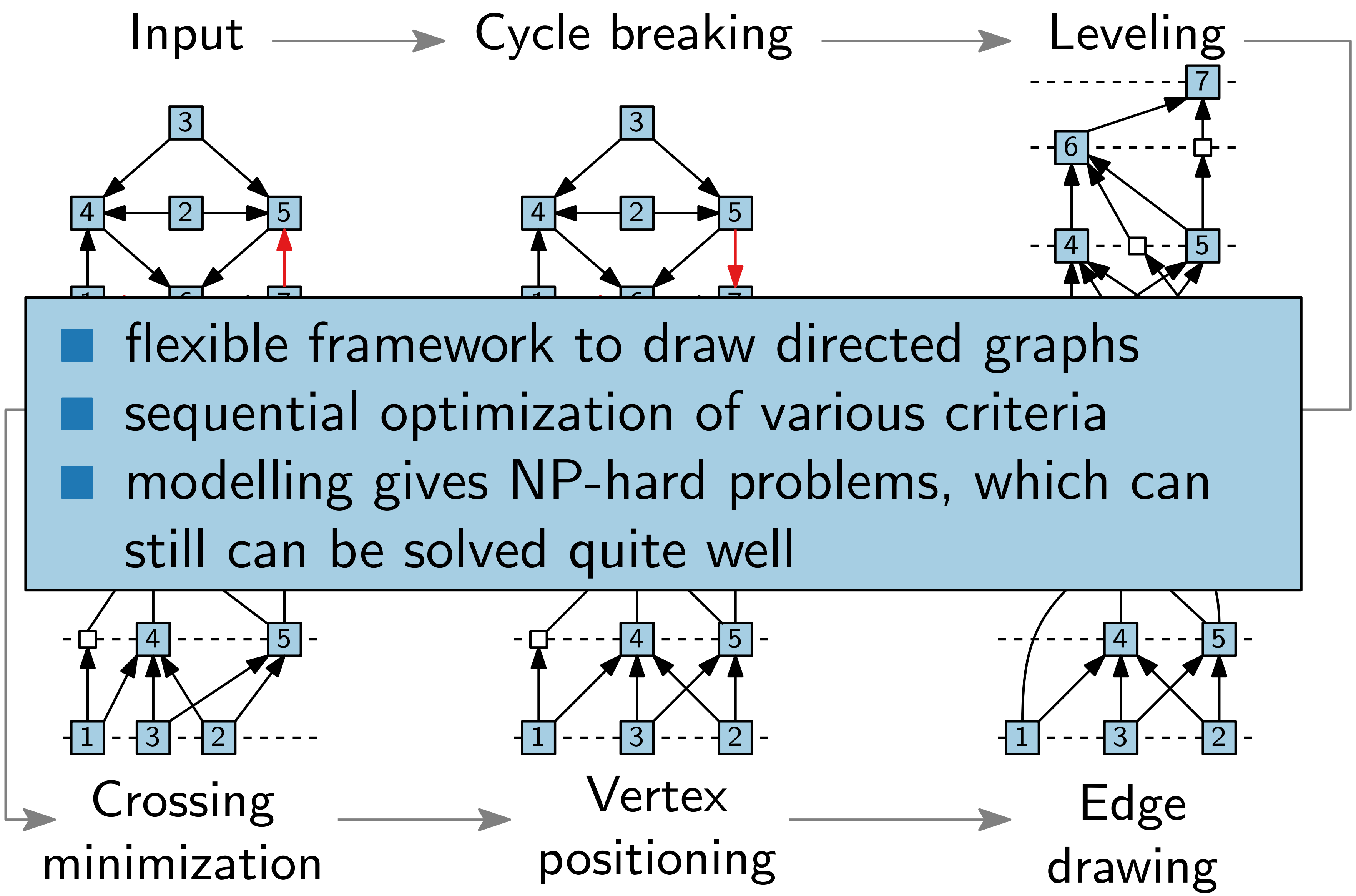

- flexible framework to draw directed graphs
- sequential optimization of various criteria
- modelling gives NP-hard problems, which can still can be solved quite well

Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81] Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation alogorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97] 2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms