On the Perspectives Opened by Right Angle Crossing Drawings*

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Abstract. Right Angle Crossing (RAC) drawings are polyline drawings where each crossing forms four right angles. RAC drawings have been introduced because cognitive experiments provided evidence that increasing the number of crossings does not decrease the readability of the drawing if the edges cross at right angles. We investigate to what extent RAC drawings can help in overcoming the limitations of widely adopted planar graph drawing conventions, providing both positive and negative results. First, we prove that there exist acyclic planar digraphs not admitting any straight-line upward RAC drawing and that the corresponding decision problem is NP-hard. Also, we show digraphs whose straight-line upward RAC drawings require exponential area. Second, we study if RAC drawings allow us to draw bounded-degree graphs with lower curve complexity than the one required by more constrained drawing conventions. We prove that every graph with vertex-degree at most 6 (at most 3) admits a RAC drawing with curve complexity 2 (resp. 1) and with quadratic area. Third, we consider a natural non-planar generalization of planar embedded graphs. Here we give bounds for curve complexity and area different from the ones known for planar embeddings.

1 Introduction

In Graph Drawing, it is commonly accepted that *crossings* and *bends* can make the layout difficult to read and experimental results show that the human performance in path tracing tasks is negatively correlated to the number of edge crossings and to the number of bends along the edges [16,17,19]. However, further cognitive experiments in graph visualization show that increasing the number of crossings does not decrease the readability of the drawing if the edges cross at right angles [10,11]. These results provide evidence for the effectiveness of *orthogonal* drawings (in which edges are chains of horizontal and vertical segments) with few bends [4,12] and motivate the study of a

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new class of drawings, called *Right Angle Crossing drawings* (*RAC drawings*) [7]. A RAC drawing of a graph G is a polyline drawing D of G such that any two crossing segments in D are orthogonal. If D has curve complexity 0, D is a *straight-line RAC drawing*, where the *curve complexity* of D is the maximum number of bends along an edge of D.

This paper continues the study of RAC drawings initiated in [7] and investigates RAC drawings with low curve complexity for both directed and undirected graphs. For directed graphs (*digraphs*), a widely studied drawing standard is the *upward drawing* convention, where edges are monotone in the vertical direction. A digraph has an upward planar drawing if and only if it has a straight-line upward planar drawing [5]. However, not all planar digraphs have an upward planar drawing and straight-line upward planar drawings require exponential area for some families of digraphs [6].

We investigate *straight-line upward RAC drawings*, i.e. straight-line upward drawings with right angle crossings. In particular, it is natural to ask if all planar digraphs admit an upward RAC drawing and if all digraphs with an upward RAC drawing admit one with polynomial area. Both these questions have a negative answer (Sect. 3): (i) we prove that there exist acyclic planar digraphs that do not admit any straight-line upward RAC drawing, and that the problem of deciding whether an acyclic planar digraph admits such a drawing is NP-hard; (ii) we show that there exist upward planar digraphs whose straight-line upward RAC drawings require exponential area.

Concerning undirected graphs, it is known [7] that any n-vertex straight-line RAC drawing has at most $4n - 10$ edges, for every $n \geq 4$, and this bound is tight. Further, every graph admits a RAC drawing with at most three bends per edge, and this curve complexity is required in infinitely many cases. Indeed, RAC drawings with curve complexity 1 and 2 have $O(n^{4/3})$ and $O(n^{7/4})$ edges, respectively. Hence, we investigate families of graphs that can be drawn with curve complexity 1 or 2, proving the following results (Sect. 4): (i) every degree-6 graph admits a RAC drawing with curve complexity 2; (ii) every degree-3 graph admits a RAC drawing with curve complexity 1. Both types of drawings can be computed in linear time and require quadratic area. Observe that degree-4 graphs admit orthogonal drawings with curve complexity 2 [14], and that two bends on an edge are sometimes necessary even for degree-3 graphs.

In a *fixed embedding* setting, the input graph G is given with a (non-planar) *embedding*, i.e., a circular ordering of the edges incident to each vertex and an ordering of the crossings along each edge. A RAC drawing algorithm cannot change the embedding of G. For such a setting it has been proved in [7] that any n-vertex graph admits a RAC drawing with $O(kn^2)$ bends per edge, where k is the maximum number of crossings between any two edges. Also, there exist graphs whose RAC drawings require $\Omega(n^2)$ bends along some edges. In Sect. 5 we study the fixed embedding setting, namely: (i) we study non-planar graphs obtained by augmenting a plane triangulation with edges inside pairs of adjacent faces; we call these graphs *kite-triangulations* and prove that one bend per edge is always sufficient and sometimes necessary for a RAC drawing of a kite-triangulation; (ii) we study the area requirement of straight-line RAC drawings of kite-triangulations and prove that cubic area is sometimes necessary. Recall that every embedded planar graph admits a planar drawing with quadratic area [18].

Sect. 6 concludes the paper with some open problems.

2 Preliminaries

We assume familiarity with graph drawing and planarity [4,12]. In the following, unless otherwise specified, all considered graphs are *simple*.

A *Right Angle Crossing drawing* (*RAC drawing*) of a graph G is a polyline drawing D of G such that any two crossing segments in D are orthogonal. The *curve complexity* of D is the maximum number of bends along an edge of D. If a RAC drawing D has curve complexity 0, D is a *straight-line RAC drawing*. A *fan* in a drawing D is a pair of edge segments incident to the same vertex. Two segments s_1 and s_2 crossing the same segment in D are parallel. This leads (Fig. 1) to the following properties (see also [7]).

Property 1. In a straight-line RAC drawing no edge can cross a fan.

Property 2. In a straight-line RAC drawing there cannot be a triangle $\triangle$ and two edges (a, b), (a, c) such that a lies outside $\triangle$ and b, c lie inside $\triangle$.

In an *upward drawing* all the edges are curves monotonically increasing in the upward direction. An *upward planar drawing* of a digraph G is an upward drawing of G without edge crossings. If G admits an upward planar drawing, G is an *upward planar* digraph. An *upward RAC drawing* of a digraph is a RAC drawing that is also upward.

3 Upward RAC Drawings

We now study straight-line upward RAC drawings (in this section, RAC drawings for short). We introduce an upward planar digraph H, shown in Fig. 2(a), that serves as a gadget for proving the main results of this section. The following lemmata show that two copies of H cannot cross each other in any RAC drawing. Let $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ be the embeddings of H shown in Fig. 2(a), 2(b), and 2(c), respectively.

Lemma 1. *In any RAC drawing of H, its embedding is one of $\mathcal{E}_1$, $\mathcal{E}_2$, or $\mathcal{E}_3$, up to a reversal of the adjacency lists of all the vertices.*

Let G be a digraph containing two copies H' and H'' of H, with vertex sets $\{u', v', w', z'\}$ and $\{u'', v'', w'', z''\}$, respectively, so that one vertex in $\{u', v'\}$ possibly coincides with one in $\{u'', v''\}$, while no other vertex is shared by the two graphs. A vertex of H'' that is coincident with a vertex of a 3-cycle of H' is considered both as internal and as external to the triangle representing the 3-cycle.

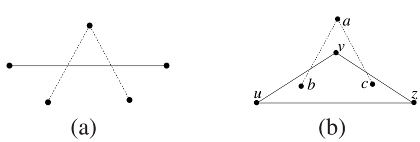


Fig. 1. Illustrations for (a) Property 1 and for (b) Property 2

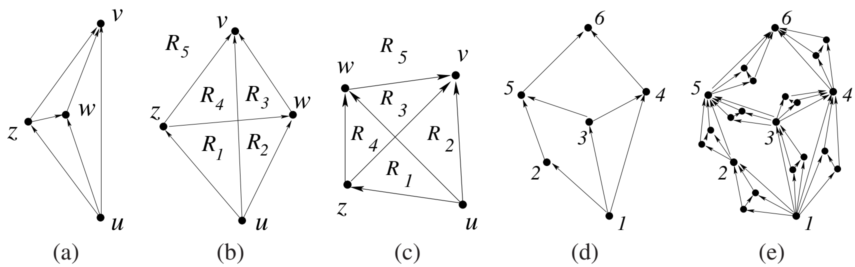


Fig. 2. (a) $\mathcal{E}_1$; (b) $\mathcal{E}_2$; (c) $\mathcal{E}_3$. (d) A planar digraph K. (e) The planar digraph K' obtained by replacing each edge of K with a copy of H.

Lemma 2. *Let D be a RAC drawing of G. For any 3-cycle (a', b', c') of H', which is represented in D by a triangle $\triangle'$, either all the vertices of H'' are inside $\triangle'$ or they are all outside it.*

Lemma 3. *In any RAC drawing D of G, no edge of H' crosses an edge of H''.*

Proof. Let D^* be D restricted to the edges of H' and H''. We show that in D^* there is no crossing among the edges of H' and the edges of H''.

If the embedding of H' in D^* is $\mathcal{E}_1$, one of the four triangular faces of H', say (u', z', v'), is a triangle $\triangle'$ enclosing w' (see Fig. 2(a)). By Lemma 2, either all the vertices of H'' lie outside $\triangle'$ or they all lie inside it. In the former case, if there is a crossing between an edge of H' and an edge of H'', then such an edge of H'' cuts a fan composed of two edges of H', violating Property 1. In the latter case, the vertices of H'' lie in the faces of H' internal to $\triangle'$. By Lemma 2, all the vertices of H'' lie in one of the internal faces of H'. Hence, in both cases, no edge of H' crosses an edge of H''.

If the embedding of H' in D^* is $\mathcal{E}_2$, cycle (u', z', v', w') of H' is a convex quadrilateral with edges (u', v') and (z', w') crossing inside it, since $y(u') < y(z') < y(w') < y(v')$ and since (u', v') is a straight-line segment. Thus, connected regions $R_1, \ldots, R_5$ are created (see Fig. 2(b)). We prove that all the vertices of H'' are inside a region R_i. For every pair of regions R_i and R_j, with $j \neq i$, a 3-cycle (a', b', c') of H', with $a', b', c' \in \{u', z', v', w'\}$, exists containing R_i in its interior and R_j in its exterior, or vice versa. Suppose that vertices a'' and b'' exist such that: (i) $a'', b'' \in \{u'', z'', v'', w''\}$; (ii) a'' is inside R_i and b'' is inside R_j, with $i \neq j$; and (iii) a'' is outside R_j and b'' is outside R_i (notice that R_i and R_j can possibly share a vertex). Then a'' is inside the triangle representing (a', b', c') and b'' is outside such a triangle, or vice versa. However, by Lemma 2, D^* is not a RAC drawing. If all the vertices of H'' are in the same region R_i, with $1 \leq i \leq 4$, no edge of H' crosses an edge of H''. If all the vertices of H'' are in R_5, suppose that a crossing between an edge of H' and an edge of H'' exists. Then, such an edge of H'' cuts a fan composed of two edges of H'.

If the embedding of H' in D^* is $\mathcal{E}_3$, connected regions $R_1, \ldots, R_5$ are created by the edges of H' (see Fig. 2(c)). With the same argument as above, it can be proved that no edge of H' crosses an edge of H''. $\square$

We get the following:

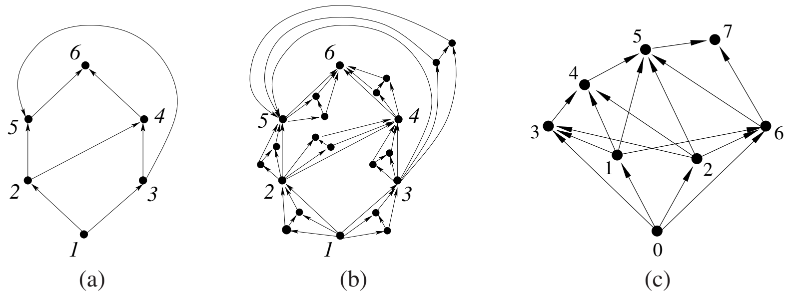


Fig. 3. (a) A planar acyclic digraph G that is not upward planar. (b) The planar acyclic digraph G' obtained by replacing each edge of G with a copy of H. (c) An 8-vertex planar digraph that does not admit any RAC drawing.

Lemma 4. *Consider a planar acyclic digraph K. Replace each edge (a, b) of K with a copy of H (see Figs. 2(d) and 2(e)), by identifying vertices a and b with vertices u and v of H, respectively. Let K' be the resulting planar digraph. Digraph K is upward planar if and only if K' is straight-line upward RAC drawable.*

Proof. First, suppose that K has an upward planar drawing. Then, by the results in [5], K admits a straight-line upward planar drawing D. Consider the drawing D' obtained by drawing each copy of H that replaces an edge (a, b) in such a way that: (i) the drawing of H is upward planar; (ii) the drawing of edge (u, v) of H in D' coincides with the drawing of (a, b) in D; and (iii) the drawing of the rest of H is arbitrarily close to (u, v). Since D is a straight-line upward planar drawing, D' is a straight-line upward planar drawing. Hence, D' is a RAC drawing of K'.

Second, suppose that K' has a RAC drawing D'. By construction, every edge of K' belongs to a copy of H, and, by Lemma 3, no two edges belonging to distinct copies of H cross in D'. Hence, since K is a subgraph of K' such that, for each copy of H belonging to K', only one edge, namely (u, v), belongs to K, the drawing D obtained as D' restricted to the edges of K is a straight-line upward planar drawing of K. □

We are ready to prove the first theorem of this section.

Theorem 1. *There exist acyclic planar digraphs that do not admit any straight-line upward RAC drawing.*

Proof. Consider any planar acyclic digraph G (as the one of Fig. 3(a)) that is not upward planar. By Lemma 4, the planar acyclic digraph G' obtained by replacing each edge of G with a copy of H is not RAC drawable (see Fig. 3(b)). □

Note that there exist planar digraphs, as the one in Fig. 3(c), that do not admit any RAC drawing, that are not constructed using gadget H, and whose size is smaller than the one of the digraph in Fig. 3(b). However, proving that they are not RAC drawable could result in a complex case-analysis.

Motivated by the fact that there exist acyclic planar digraphs that do not admit any RAC drawing, we study the time complexity of the corresponding decision problem.

We show that the problem of testing whether a digraph admits a straight-line upward RAC drawing (UPWARD RAC DRAWABILITY TESTING) is NP-hard, by means of a reduction from the problem of testing whether a digraph admits a straight-line upward planar drawing (UPWARD PLANARITY TESTING), which is NP-complete [9].

Theorem 2. UPWARD RAC DRAWABILITY TESTING *is NP-hard.*

Proof. We reduce UPWARD PLANARITY TESTING to UPWARD RAC DRAWABILITY TESTING. Let G be an instance of UPWARD PLANARITY TESTING (see Fig. 2(d)). Replace each edge of G with a copy of H (see Fig. 2(e)). Let G' be the resulting planar digraph. By Lemma 4, G is upward planar if and only if G' admits a RAC drawing. □

As a final contribution of this section we show that there exists a class of planar acyclic digraphs that require exponential area in any RAC drawing.

Consider the class of digraphs G_n [6] which requires $\Omega(2^n)$ area in any straight-line upward planar drawing, under any resolution rule. Let G'_n be the class of digraphs obtained by replacing each edge (a, b) of G_n with a copy of H, so that vertices a and b are identified with vertices u and v of H, respectively.

Theorem 3. *Let G' be a digraph belonging to G'_n. Then, any straight-line upward RAC drawing of G' requires $\Omega(b^n)$ area, under any resolution rule, for some constant $b > 1$.*

Proof. Suppose, for a contradiction, that, for every constant $b > 1$, G' admits a RAC drawing D' with $o(b^n)$ area, under some resolution rule. Consider the digraph $G \in G_n$ corresponding to G'. By construction, G is a subgraph of G' containing only edge (u, v) for each copy of H in G'. By Lemma 3, no two edges belonging to distinct copies of H cross. Hence, the drawing D of G obtained as D' restricted to the edges of G is a straight-line upward planar drawing with $o(b^n)$ area, a contradiction. □

4 RAC-Drawings of Bounded-Degree Graphs

In this section, we present an algorithm for constructing RAC drawings of graphs with degree at most 6. The algorithm is based on the decomposition of a regular multigraph into cycle covers. A *cycle cover* of a directed graph is a spanning subgraph consisting of vertex-disjoint directed cycles. The decomposition into cycle covers follows from a classical result [15] stating that "a regular multigraph G of degree $2k$ has k edge-disjoint factors", where a *factor* is a spanning subgraph consisting of vertex-disjoint cycles (see also [2, pp.227]). A constructive proof of the following theorem was given in [8].

Theorem 4 ***(Eades,Symvonis,Whitesides [8]).*** *Let $G = (V, E)$ be an undirected graph of maximum degree Δ and let $d = \lceil \Delta/2 \rceil$. Then, there exists a directed multi-graph $G' = (V, E')$ such that: (i) each vertex of G' has indegree d and outdegree d; (ii) G is a subgraph of the underlying undirected graph of G'; and (iii) the edges of G' can be partitioned into d edge-disjoint cycle covers. Furthermore, for an n-vertex graph G, the directed graph G' and its d cycle covers can be computed in $O(\Delta^2 n)$ time.*

Let u be a vertex placed at a grid point. We say that an edge e exiting u uses the *Y-port* of u (resp. the *$-Y$-port* of u) if it exits u along the $+Y$ direction (resp. along the $-Y$ direction). In a similar way, we define the *X-port* and the *$-X$-port*.

Theorem 5. *Every n-vertex graph with degree at most 6 admits a RAC drawing with curve complexity 2 in $O(n^2)$ area. Such a drawing can be computed in $O(n)$ time.*

Proof. Let $G = (V, E)$ be a graph of maximum degree 6. Let $G' = (V, E')$ be the directed multigraph obtained from G as in Theorem 4, and let C_1, C_2, and C_3 be the edge-disjoint cycle covers of G'. We show how to obtain a RAC drawing of G'. Then, a RAC drawing of G can be obtained by removing from the drawing all the edges in $E' \setminus E$ and by ignoring the direction of the edges.

The algorithm places the vertices of V on the main diagonal of an $n \times n$ grid, in an order determined by one of the cycle covers, say C_1. Most of the edges of C_1 are drawn as straight lines along the diagonal while the edges of C_2 and C_3 are drawn as 3-segment lines above and below the diagonal, respectively. Finally, the remaining "closing" edges of C_1 (i.e., the edges that cannot be drawn as straight lines on the diagonal) are drawn without creating any overlap with other edges.

We first describe how to place the vertices of the graph along the main diagonal. Arbitrarily name the cycles $c_1, c_2, \ldots, c_k$ of C_1. Consider a cycle c_i, $1 \leq i \leq k$. If there exists a vertex $u \in c_i$ and an edge $(u, z) \in C_2$ or C_3 such that z belongs to a cycle c_j of C_1 with $j > i$ (note that there could be several of such vertices and edges), then let u be the *topmost vertex* of c_i and let the vertex following u in c_i be the *bottommost vertex* of c_i. Otherwise, if there exists a vertex $v \in c_i$ and an edge $(v, w) \in C_2$ or C_3 such that w belongs to a cycle c_j of C_1 with $j < i$, then let v be the bottommost vertex of c_i and let the vertex preceding v in c_i be the topmost vertex of c_i. If no such vertices exist, all the edges of C_2 and C_3 originating from vertices of c_i are also directed to vertices of c_i. In this case, let an arbitrary vertex w of c_i be the bottommost vertex of c_i and let the vertex preceding w in c_i be the topmost vertex of c_i. Then, for $i = 1, \ldots, k$, place the vertices of c_i in the order they appear as traversing the cycle, with the bottommost vertex placed at the bottommost free grid point.

Fig. 4(a) shows a regular directed multigraph G' of indegree and outdegree 3 and its cycle covers (C_1: thin, C_2: thick, C_3: dashed). C_1 consists of cycles $c_1 : (5, 1, 2, 3, 4, 5)$ and $c_2 : (6, 7, 8, 9, 6)$. We set 4 as the topmost vertex of c_1 since edge $(4, 6)$ of C_2 has vertex 6 of c_2 as its destination. Similarly, we set 6 as the bottommost vertex of c_2 since

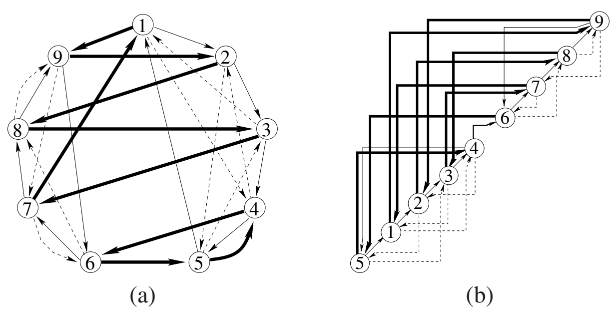


Fig. 4. (a) A regular directed multigraph G' with indegree and outdegree equal to 3 and its cycle covers. (b) A RAC drawing of G' with two bends per edge. Vertex 4 is the topmost vertex of c_1 and vertex 6 is the bottommost vertex of c_2.

edge $(6, 5)$ of C_2 has vertex 5 of c_1 as its destination. Fig. 4(b) shows the RAC drawing of G' of curve complexity 2 produced by the algorithm described in this proof.

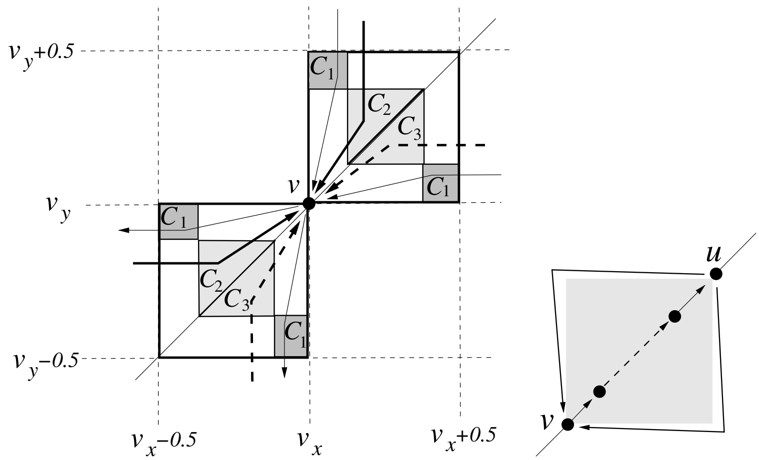
Having placed the vertices on the grid, we turn our attention to draw the edges of G'. No edge overlaps are allowed and each edge is drawn either as a 1-segment edge along the diagonal, or as a 2/3-segment polyline above or below the diagonal. We draw the edges so that all the crossing line segments are parallel to the axes and, consequently, all the crossings are at right angles. In our drawings, every line segment s that is not parallel to the axes is incident to a vertex v_s of the graph; further, the other end-point of s is confined to a dedicated area within a unit-side rectangle centered at v_s (see Fig. 5(a)).

We first describe how to draw the edges of cycle cover C_2 above the diagonal. Consider an edge (u, v) of C_2 and let u and v be placed at grid points (u_x, u_y) and (v_x, v_y), respectively. If u is placed below v (i.e., $u_y < v_y$), then edge (u, v) is drawn as a 3-segment line exiting vertex u from the Y-port and being defined by bend-points $(u_x,\ v_y - \frac{3}{8} + \epsilon_2)$ and $(v_x - \frac{3}{8} + \epsilon_1,\ v_y - \frac{3}{8} + \epsilon_2),\ 0 < \epsilon_1 < \epsilon_2 < \frac{1}{4}$. Note that the second bend-point is located within the lightly-shaded region (above the diagonal) of the south-west quadrant of the square centered at vertex v (see Fig. 5(a)). If u is placed above v (i.e., $u_y > v_y$), then edge (u, v) is drawn as a 3-segment line exiting vertex u from the $-X$-port and being defined by bend-points $(v_x + \frac{1}{8} + \epsilon_1,\ u_y)$ and $(v_x + \frac{1}{8} + \epsilon_1,\ v_y + \frac{1}{8} + \epsilon_2),\ 0 < \epsilon_1 < \epsilon_2 < \frac{1}{4}$. Note that, in this case, the second bend-point is located within the lightly-shaded region (above the diagonal) of the north-east quadrant of the square centered at v (see Fig. 5(a)). It is easy to observe that the only line segments that belong to edges of C_2 and that cross other line segments are parallel to the axes, hence they cross at right angles. In a symmetric fashion, the edges of C_3 are drawn below the diagonal. Fig. 4(b) shows the routing of cycle covers C_2 and C_3 for graph G' of Fig. 4(a).

Consider now the edges of C_1. All such edges, except those closing the cycles of C_1, are drawn as straight-line segments along the diagonal. As all the edges of C_2 (resp. C_3) are drawn above (resp. below) the diagonal, these edges are not involved in any edge crossing. To complete the drawing of G', we describe how to draw the edges connecting the topmost vertex to the bottommost vertex of each cycle of C_1. Consider an arbitrary cycle c_i of C_1 and let (u, v) be its closing edge. We consider 3 cases:

Case 1: *u was selected to be the topmost vertex of c_i due to the existence of an edge (u, z) of C_2 or C_3 with vertex z being placed higher on the diagonal than u.* This implies that after drawing the edges of C_2 and C_3, vertex u has not used either its $-X$-port or its $-Y$-port, or both. Assume that the $-X$-port is free (the case where the $-Y$-port is free is treated symmetrically). Edge (u, v) is drawn above the diagonal as a 3-segment line exiting vertex u from the $-X$-port and being defined by bend-points $(v_x + \epsilon_1,\ u_y)$ and $(v_x + \epsilon_1,\ v_y + \frac{3}{8} + \epsilon_2),\ 0 < \epsilon_1,\ \epsilon_2 < \frac{1}{8}$. Note that, in this case, the second bend-point is located within the dark-shaded region (above the diagonal) of the north-east quadrant of the square centered at vertex v (see Fig. 5(a)).

Case 2: *v was selected to be the bottommost vertex of c_i due to the existence of an edge (v, w) of C_2 or C_3 with vertex w being placed lower on the diagonal than v.* This implies that after drawing the edges of C_2 and C_3, vertex v has not used either its X-



(a) (b)

Fig. 5. (a) The area around a vertex. Each specific subarea hosts a single bend-point of an edge belonging to a specific cycle cover. (b) Two alternatives for the drawing of a closing edge of a cycle that has only edges originating and destined for its own vertices.

port or its Y-port, or both. The drawing of the closing edge is done in a way similar to the above case.

Case 3: *None of the above cases applies.* In this case, all the edges of C_2 and C_3 originating from vertices of cycle c_i are also directed to vertices of c_i and the topmost vertex u and the bottommost vertex v of c_i were selected arbitrarily. Then, both the $-X$-port and the $-Y$-port of u and both the X-port and the Y-port of v are already used. Also, the drawing of the edges of C_2 and C_3 connecting vertices of c_i takes place entirely within the square having points (v_x, v_y) and (u_x, u_y) as its opposite corners (the shaded square in Fig. 5(b)). Hence, the closing edge can be easily drawn as a 2-segment line connecting u and v just outside the boundary of the square, either above or below the diagonal (see Fig. 5(b)).

Given the three cycle covers, it is easy to see that the placement of the vertices along the diagonal and the routing of all the edges can be completed in linear time. By Theorem 4, the three cycle covers can be also computed in linear time, resulting in a linear time algorithm. Also, the produced RAC drawing requires $O(n^2)$ area. □

With similar techniques we can prove the following:

Theorem 6. *Every n-vertex graph with degree at most 3 admits a RAC drawing with curve complexity 1 in $O(n^2)$ area. Such a drawing can be computed in $O(n)$ time.*

5 RAC Drawings of Kite-Triangulations

In this section we study the impact of admitting orthogonal crossings on the drawability of the non-planar graphs obtained by adding edges to maximal planar graphs, in a fixed embedding scenario. We show that such graphs always admit RAC drawings with curve complexity 1 and that such a curve complexity is sometimes required.

Let G' be a triangulation and let (u, z, w) and (v, z, w) be two adjacent faces of G' sharing edge (z, w). We say that $[u, v]$ is a *pair of opposite vertices* with respect to (z, w). Let $E^+ = \{[u_i, v_i] | i = 1, 2, \cdots, k\}$ be a set of pairs of opposite vertices of G', where $[u_i, v_i]$ is a pair of opposite vertices with respect to (z_i, w_i) and edge (u_i, v_i) does not belong to G'. Suppose that (z_p, w_p) and (z_q, w_q) do not share a face of G', for any $1 \leq p, q \leq k$ and $p \neq q$. Let G be the embedded non-planar graph obtained by adding an edge (u_i, v_i) to G', for each pair $[u_i, v_i]$ in E^+, so that edge (u_i, v_i) crosses edge (z_i, w_i) and does not cross any other edge of G. We say that G is a *kite-triangulation* and that G' is its *underlying triangulation*.

Theorem 7. *Every kite-triangulation admits a RAC drawing with curve complexity 1.*

Proof sketch. Consider any kite-triangulation G and its underlying triangulation G'. Remove from G' all the edges (z_i, w_i), for $i = 1, \ldots, k$, obtaining a new planar graph G'' whose faces contain at most four vertices. Construct any straight-line planar drawing Γ'' of G''. Construct a RAC drawing Γ of G inserting in Γ'', for each $i = 1, \ldots, k$, edges (u_i, v_i) and (z_i, w_i). Two cases are possible, either face (u_i, w_i, v_i, z_i) is strictly convex in Γ'' (Fig. 6(a)) or it is not (Fig. 6(b)). $\square$

Theorem 8. *There exist kite-triangulations that do not admit straight-line RAC drawings.*

Proof. Consider the graph H in Fig. 6(c). Triangle (u, a, z) and vertices v, x, y create the forbidden structure of Property 2. Hence, every kite-triangulation G containing H as a subgraph requires one bend in any RAC drawing. $\square$

Planar graphs are a proper subset of straight-line RAC drawable graphs. However, while straight-line planar drawings can always be realized on a grid of quadratic size (see, e.g., [3,18]), straight-line RAC drawings may require larger area, as shown in the following.

Theorem 9. *There exists an n-vertex kite-triangulation that requires $\Omega(n^3)$ area in any straight-line grid RAC drawing.*

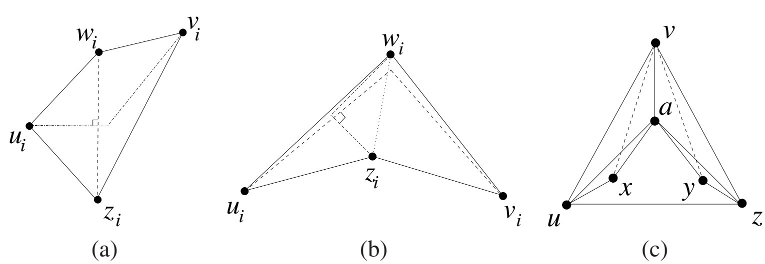


Fig. 6. (a) Drawing (u_i, v_i) and (z_i, w_i) inside (u_i, w_i, v_i, z_i), if (u_i, w_i, v_i, z_i) is strictly convex. (b) Drawing (u_i, v_i) and (z_i, w_i) inside (u_i, w_i, v_i, z_i), if (u_i, w_i, v_i, z_i) is not strictly convex. (c) An embedded graph that is a subgraph of infinitely many kite-triangulations with curve complexity 1 in any RAC drawing.

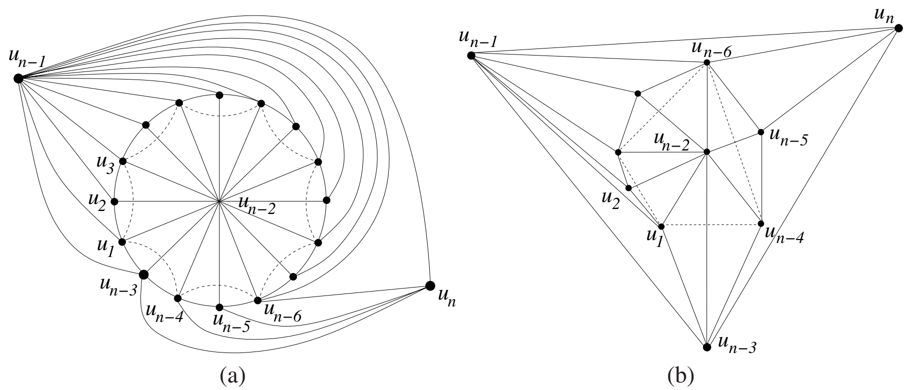


Fig. 7. (a) A kite triangulation G requiring $\Omega(n^3)$ area in any straight-line grid RAC drawing. (b) A straight-line RAC drawing of G.

Proof. Consider a triangulation G' defined as follows (see Fig. 7(a)). Let $C = (u_1, u_2, \ldots, u_{n-4}, u_{n-3})$ be a simple cycle, for some odd integer n. Insert a vertex u_{n-2} inside C and connect it to u_i, with $i = 1, 2, \ldots, n-3$. Insert two vertices u_{n-1} and u_n outside C. Connect u_{n-1} to u_i, with $i = 1, 2, \ldots, n-6$ and to u_{n-3}; connect u_n to u_{n-6}, u_{n-5}, u_{n-4}, u_{n-3}, and u_{n-1}. Let G be the kite-triangulation obtained from G' by adding edges (u_i, u_{i+2}), for $i = 1, 3, 5, \ldots, n-6$, and edge (u_1, u_{n-4}), so that (u_i, u_{i+2}) crosses edge (u_{i+1}, u_{n-2}) of G', and so that (u_1, u_{n-4}) crosses edge (u_{n-3}, u_{n-2}) of G'.

In the following we prove that, in any straight-line RAC drawing of G, cycle $C' = (u_1, u_3, \ldots, u_{n-6}, u_{n-4}, u_1)$ is a strictly-convex polygon. This claim, together with the observation that G admits a straight-line RAC drawing (see Fig. 7(b)), clearly implies the lemma, since any strictly-convex polygon needs cubic area if its vertices have to be placed on a grid (see, e.g., [1]).

Suppose, for a contradiction, that there exists a straight-line RAC drawing Γ of G with an angle $u_i\widehat{u_{i+2}}u_{i+4} \geq 180°$ inside C'. Then, any two segments orthogonally crossing $\overline{u_iu_{i+2}}$ and $\overline{u_{i+2}u_{i+4}}$, respectively, meet at a point outside C', possibly at infinity, while they should meet at u_{n-2}, which is inside C'. Thus, either $\overline{u_{n-2}u_{i+1}}$ is not orthogonal to $\overline{u_iu_{i+2}}$ or $\overline{u_{n-2}u_{i+3}}$ is not orthogonal to $\overline{u_{i+2}u_{i+4}}$, hence contradicting the assumption that Γ is a RAC drawing. □

6 Conclusions and Open Problems

When a graph G does not admit any planar drawing in some desired drawing convention, requiring that all the crossings form right angles can be considered as an alternative solution for the readability of a drawing of G. In this direction, this paper has shown negative results for digraphs that must be drawn upward with straight-line edges, and positive results for classes of non-planar undirected graphs that must be drawn with curve complexity 1 or 2. We list some open research directions that are related to the results of this paper: (i) It is known that a digraph is upward planar iff it is a subgraph of a planar st-digraph. Is it possible to characterize digraphs admitting straight-line upward RAC drawings? (ii) There exist outerplanar digraphs that are not upward planar and that admit upward straight-line RAC drawings [13]. Does every acyclic outerplanar digraph admit a straight-line upward RAC drawing? (iii) What are the exact bounds for the curve complexity of RAC drawings of bounded degree graphs?

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