Drawing Graphs in the Plane with High Resolution

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Abstract

In this paper, we study the problem of drawing a graph in the plane so that edges appear as straight lines and so that the minimum angle formed by any pair of incident edges is maximised. We define the *resolution of a layout* to be the sise of the minimum angle formed by incident edges of the graph, and the *resolution of a graph* to be the maximum resolution of any layout of the graph. We characterise the resolution R of a graph in terms of the maximum node degree d of the graph by proving that $\Omega(\frac{1}{d^2}) \leq R \leq \frac{2\pi}{d}$ for any graph. Moreover, we prove that $R = \Theta(\frac{1}{d})$ for many graphs including planar graphs, complete graphs, hypercubes, multidimensional meshes and tori, and other special networks. We also show that the problem of deciding if $R = \frac{2\pi}{d}$ for a graph is NP-hard for $d = 4$, and we use a counting argument to show that $R = O(\frac{\log d}{d^2})$ for many graphs.

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1. Introduction

Graph layout problems have been extensively studied in a wide variety of contexts. Examples include both linear [10,15] and planar [1,2,3,5,7,8,11,12,13,14] layout problems. Typically, nodes are represented by distinct points to be embedded in a line or plane, and they are sometimes restricted to be grid points. (Alternatively, nodes are sometimes represented by line segments [11].) Edges are often constrained to be drawn as straight lines [3,4,7,8,11] or as a contiguous set of line segments [1,5,12,14] (e.g., when bends are allowed). The objective is to find a layout for a graph that minimises some cost function, such as area [1,5], number of edge crossings [1], maximum edge length [1,2], number of bends [1,5,12,14], visual complexity [13], density[7,8] and so on.

In this paper, we consider straight line layouts of graphs in the plane. Specifically, we introduce a new cost function for such layouts, called *resolution*. We define the *resolution of a layout* of the graph to be the sise of the minimum angle formed by any two edges incident to a common node. We define the *resolution of a graph* to be the maximum resolution of any (straight line) layout for the graph in the plane. For example, the resolution of K_3 (the 3-node complete graph) is $\frac{\pi}{3}$. Our objective in this paper is to find layouts for graphs with the highest possible resolution.

An obvious upper bound on the resolution of a graph with maximum degree d is $\frac{2\pi}{d}$. Of course, this bound is not tight for many graphs. (For example, $d = 2$ for K_3 but the resolution is $\frac{\pi}{3}$). Unfortunately, we will show that the problem of determining whether or not a graph with maximum degree d has resolution $\frac{2\pi}{d}$ is NP-hard, at least in the case $d = 4$. Whether or not the resolution problem is in NP is still unknown. Determining the precise complexity of the problem is complicated by the fact that there are simple 11-node graphs (such as that shown in Figure 1) for which there is a layout with resolution $\frac{\pi}{3} - \epsilon$ for any $\epsilon > 0$, but for which there is no layout with resolution $\frac{\pi}{3}$. These difficulties can be overcome by restricting the problem so that the nodes of the graph are required to be placed at distinct grid points in a grid of a fixed size (e.g., $N \times N$), in which case the resolution of any N-node graph becomes a well defined NP-complete problem. We will consider both grid-based and unrestricted layouts in this paper.

On the positive side, we can prove nearly tight bounds on the resolution for many natural classes of graphs. For example, we prove that any planar graph with maximum degree d has resolution $\Theta(\frac{1}{d})$. We also prove similar bounds for special networks like the hypercube, torus, complete graph and others. We construct a layout for an arbitrary graph with maximum degree d that has resolution $\Omega(\frac{1}{d^2})$. Hence, the resolution of any bounded-degree graph is constant, independent of the number of nodes in the graph. Whether or not there exists a family of graphs with maximum degree d and resolution $\Theta(\frac{1}{d^2})$ is still unknown. Natural candidates for graphs with low resolution such as the (d^2-d+1)-point projective plane and the $(\frac{d}{2}+1)\times(\frac{d}{2}+1)$ mesh of cliques (both of which are d-regular) have resolution $\Theta(\frac{1}{d})$. We do not even know of any simply constructable family of graphs with maximum degree d that has resolution $o(\frac{1}{d})$, although, using a counting argument, we can prove the existence of many graphs with maximum degree d and resolution $O(\frac{\log d}{d^2})$. Hence, the $\Omega(\frac{1}{d^2})$ worst case lower bound is not to far from reality for many graphs.

Several of our constructions are based on the close relationship between the chromatic number of the square of a graph and its resolution. In particular, we will show that the resolution of any graph G is at least $\frac{\pi}{\chi(G^2)} - \epsilon$ for any $\epsilon > 0$, where $\chi(G^2)$ denotes the chromatic number of G^2. (The graph G^2 is formed from G by connecting pairs of nodes that are within distance 2 of each other in G.) Moreover, we will show that the resolution of any N-node graph G is $\Omega(\frac{1}{\chi(G^2)})$, even if we are restricted to position the nodes of G at distinct grid points of a square grid with area $O(\chi(G^2)^3 N)$. Hence, we can produce linear area layouts (in the sense of [1,5,14]) that have constant resolution for any bounded-degree graph.

Our $\Theta(\frac{1}{d})$ bound on the resolution of any planar graph with maximum degree d, in particular, stems from the fact that the square of any planar graph with maximum degree d has chromatic number $O(d)$. In this paper, we show an upper bound of $\frac{13}{7}d+O(d^{2/3})$, which is close to the best known existential lower bound of $\frac{3}{2}d$. (The lower bound is provided by the graph shown in Figure 2.) The exact worst-case bound remains an unsolved problem.

Our interest in the problem of maximizing resolution stems largely from the fact that resolution seems to be a natural property of graphs that was previously (to our knowledge) unexplored. In addition, the resolution problem is related to problems that arise in network communication via optic beams [6,9]. For example, consider a network in which each node represents a processor that can communicate via optical beams with its neighbors in the graph. By maximizing the resolution of the layout, we simplify the task of designing the processor and the task of recognizing one's neighbors. (It is hard to send or receive at very tight angles for a unit size processor). Similar applications might arise in radio networks that make use of directional antennas.

The remainder of the paper is divided into sections as follows: In Section 2, we examine graphs of maximum degree 4 and prove that it is NP-hard to decide if we can draw them with resolution $\frac{\pi}{2}$. In Section 3, we first present an algorithm for general graph layouts. We then consider the case of planar graphs. Finally, because of their importance, we consider layouts of special networks. In Section 4, we present an upper bound on the resolution of random graphs. Section 5 contains some remarks and additional topics for research.

2. NP-hardness

Given a graph with maximum degree d, we know that its optimum embedding on the plane can have resolution at most $\frac{2\pi}{d}$. In what follows, we show that the problem of deciding whether a graph of maximum degree d has an embedding on the plane with resolution $\frac{2\pi}{d}$ is NP-hard in the case $d = 4$.

Theorem 1. *Given a graph G of maximum degree 4, the decision problem of whether or not G can be embedded in the plane with resolution $\frac{\pi}{2}$ is NP-hard.*

Proof: The proof is done by a reduction from 3-SAT. Let S be a formula in 3-CNF, let $U = \{x_1, x_2, x_3, ..., x_n\}$ be the variables occurring in S, and let $C = \{c_1, c_2, c_3, ..., c_k\}$ be the clauses in S, such that every clause $c \in C$ consists of exactly 3 literals. We construct a graph G of maximum degree 4 that is embeddable with resolution $\frac{\pi}{2}$ if and only if S is satisfiable.

The skeleton of G is given in Figure 3a. For each variable $x_i, 1 \leq i \leq n$, there is a node in the skeleton. The same holds for each clause $c_j, 1 \leq j \leq k$. Observe that up to reflections, rotations and stretchings, the embedding of G with resolution $\frac{\pi}{2}$ (if there is any) essentially has to look like the one in Figure 3a. Now we append at each node x_i the tower of Figure 3b. For each such tower, there are two possible embeddings (in relation to the skeleton): The negated nodes to the left and the nonnegated nodes to the right, or vice versa. Finally, for each clause $c_j = \{x_i, x_k, x_l\}$, we connect the node c_j to the nodes $x_{i,j}, x_{k,j}$ and $x_{l,j}$ (or to the corresponding negated nodes, if the literals are negated) by a path consisting of three edges (Figure 3c).

We claim that if there is an embedding of G with resolution $\frac{\pi}{2}$, then we can find a truth assignment for S. We make the following observations:

Observation 1: All nodes $c_j, 1 \leq j \leq k$, have to be embedded to the right of line L (see Figure 3a).

Observation 2: All nodes on the left side of a tower have to be embedded to the left of line L.

Observation 3: A path of length 3 that leaves c_j in the eastern direction can never reach any node on the left side of any tower (Figure 3d).

Observation 4: A path of length 3 that leaves c_j in the southern or the northern direction can reach any node, on the left or on the right, of any tower (Figure 3e).

Observation 5: A path of length 3 that leaves c_j in the eastern direction can reach any node on the right side of any tower (Figure 3f).

With the above observations in mind, the rest of the proof is obvious: If the negated nodes $\bar{x}_{i,j}, 1 \leq j \leq k$, are embedded on the left side of the tower at x_i, then x_i is set to TRUE. If the nonnegated nodes $x_{i,j}$, $1 \leq j \leq k$, are embedded on the left side, then x_i is set to FALSE. To see why the above assignment satisfies S, consider any embedded node c_j. There are three paths leaving c_j, one in the eastern, one in the northern and one in the southern direction. The eastern path can never reach a false value. Hence, the clause will contain at least one true literal.

Conversely if we are given a satisfying assignment for the 3-SAT problem, from the above discussion it is obvious how to embed the corresponding graph with resolution $\frac{\pi}{2}$. This completes the proof. ∎

3. Drawings with High Resolution

In this section, we describe how to draw graphs in the plane with high resolution. We start by establishing the connection between resolution and $\chi(G^2)$ in Section 3.1. As a consequence, we show that any N-node graph with maximum degree d has resolution at least $\Omega(\frac{1}{d^2})$, even if we are restricted to embedding nodes in distinct grid points of a square $O(min\{d^2, N\}^3 N)$-node grid. In Section 3.2, we show that $\chi(G^2) < \frac{13}{7}d + O(d^{2/3})$ for any planar graph G with maximum degree d, thereby obtaining a tight $\Theta(\frac{1}{d})$ bound on the resolution of any planar graph with maximum degree d. We conclude in Section 3.3 by constructing optimal-resolution layouts for a variety of special networks such as arrays, hypercubes, etc.

3.1 Drawings for General Graphs Based on $\chi(G^2)$

Given a graph $G = (V, E)$, the square of G (denoted by $G^2 = (V^2, E^2)$) is defined as follows: $V^2 = V$ and $E^2 = E \cup \{(i,j) \mid i,j \in V \text{ and } \exists\, k \in V \text{ such that } (i,k) \in E \text{ and } (k,j) \in E\}$. A simple argument reveals that if G has maximum degree d, then G^2 has maximum degree d^2 and thus that $\chi(G^2) \leq d^2 + 1$. In what follows, we will show how to draw G in the plane with resolution $\frac{\pi}{\chi(G^2)} - \epsilon$ for any $\epsilon > 0$.

Algorithm *DRAW*

- **step 1** Given G, construct G^2 and color the nodes of G^2 with u colors where $\chi(G^2) \leq u \leq d^2 + 1$. Adjacent nodes in G^2 should be assigned different colors.
- **step 2** Draw a unit circle on the plane and u equidistant points $P_1, \ldots, P_u$ on the circle.
- **step 3** Place the nodes of G that are assigned color i in G^2 into a ball of radius ϵ around P_i ($1 \leq i \leq u$).
- **step 4** Draw the edges of G as straight line segments.

Theorem 2. *Given a general graph G, a coloring of G^2 with u colors, and any ϵ, Algorithm DRAW embeds the graph on the plane with resolution $\frac{\pi}{u} - O(\epsilon)$.*

Proof: Let v_1 be an arbitrary node of G and $(v_1, v_2), (v_1, v_3)$ edges incident with v_1. In G^2, v_1, v_2 and v_3 are all adjacent and they are colored differently (say with colors c_1, c_2, c_3, respectively). Hence, they are placed within ϵ distance of three different points $P_{c_1}, P_{c_2}, P_{c_3}$, respectively, on the unit circle. The angles formed by edges connecting the u points P_i on the unit circle are all at least $\frac{\pi}{u}$. (In fact, they are all multiples of $\frac{\pi}{u}$.) Since $dist(v_i, P_{c_i}) \leq \epsilon$ and $dist(P_{c_i}, P_{c_j}) = \Omega(\frac{1}{u})$, for $1 \leq i,j \leq 3$, $i \neq j$, this means that the angle formed by v_1, v_2, v_3 must be at least $\frac{\pi}{u} - O(\epsilon)$, where the constant implicit in the $O(\epsilon)$ is independent of u and of the number of nodes in the graph. ∎

Corollary 1. For any graph G with maximum degree d, and any $\delta > 0$, we can draw G with resolution $\frac{\pi}{d^2+1} - \delta$.

In fact, the layout given by algorithm *DRAW* can be modified so that the nodes of the graph G are placed at distinct grid points of an $O(\sqrt{u^3 N})$-side grid. This can be done by first refining the coloring of G^2 so that each color group contains at most $\frac{N}{u}$ nodes. This step introduces at most $u - 1$ new colors for a total of $2u - 1$ overall. We next place the nodes of each color group arbitrarily in a $\sqrt{\frac{N}{u}} \times \sqrt{\frac{N}{u}}$ square, and then arrange the

$2u-1$ squares at equidistant points around the perimeter of a circle of radius $u^{\frac{3}{2}}\sqrt{N}$. This results in a layout of area $O(u^3N)$ and resolution $\Omega(\frac{1}{u})$. For graphs with bounded-degree, u is constant and the layout has linear area, which is optimal. For graphs with larger u, however, it seems likely that the bound on area can be improved without dramatically affecting the bound on resolution.

3.2 Drawing Planar and Outerplanar Graphs

We can substantially improve upon Corollary 1 in the case of planar and outerplanar graphs. In particular, we will prove that any planar graph of maximum degree d can be drawn with resolution $\Omega(\frac{1}{d})$, which matches the naive upper bound to within a constant factor. The proof is based on the fact that $\chi(G^2) = O(d)$ for any planar graph G with maximum degree d. Showing that $\chi(G^2) = O(d)$ is relatively straightforward. As the determination of the worst-case value of $\chi(G^2)$ may be of independent interest, we have included the details of the more complicated $\frac{13}{7}d+O(d^{2/3})$ bound in what follows.

Lemma 1. Let U and W be disjoint node sets in a planar graph and suppose that each node in U has at least 3 neighbors in W. Then $|U| \le 2|W| - 4$.

Proof: Remove all nodes not in $U \cup W$ and all edges except those with one endpoint in each of U and W. The resulting graph G is planar and bipartite. Denote by m and f the number of edges and faces of G, respectively. It is easy to see that $4f \le 2m$ and $m \ge 3|U|$. Hence, by Euler's formula,

$$|U| + |W| - 2 = m - f \ge m/2 \ge 3|U|/2$$

and $|U| \le 2|W| - 4$. ∎

Definition: For $k \ge 1$, denote by $\phi(k)$ the supremum, over all planar graphs G, of the proportion of nodes in G of degree $\ge k$.

Lemma 2.

$$\phi(k) = \begin{cases} 1, & \text{for } k \le 6; \\ \frac{3}{k-3}, & \text{for } 6 \le k \le 12; \\ \frac{2}{k-6}, & \text{for } k \ge 12. \end{cases}$$

Proof: We prove in detail only the assertion needed in the following, namely $\phi(k) \le \frac{2}{k-6}$ for $k \ge 12$. For each value of k, the lower bound is realised by the infinite hexagonal grid, augmented by a suitable independent set of nodes of degree 2 or 3.

Fix $k \ge 12$, let $G = (V, E)$ be a planar graph and define W as the set of nodes in G of degree $\ge k$ and U as the set of nodes in $V \backslash W$ with at least 3 neighbors in W. By Lemma 1, $|U| < 2|W|$.

By planarity, the subgraph of G induced by $U \cup W$ has at most $3(|U| + |W|)$ edges, at most $3|W|$ of which have both endpoints in W. Hence at least $k|W| - 3|U| - 6|W|$ edges in G join a node in W with a node in $V\backslash(U \cup W)$. It follows that

$$|V\backslash(U \cup W)| \ge \tfrac{1}{2}(k|W| - 3|U| - 6|W|)$$

and hence that

$$|V| = |W| + |U| + |V\backslash(U \cup W)|$$

$$\ge |W| + |U| + \tfrac{1}{2}(k|W| - 3|U| - 6|W|)$$

$$= \tfrac{1}{2}k|W| - 2|W| - \tfrac{1}{2}|U|$$

$$\ge \tfrac{1}{2}k|W| - 2|W| - |W| = (\tfrac{k}{2} - 3)|W|.$$

Finally, the proportion of nodes in G of degree $\ge k$ is

$$\frac{|W|}{|V|} \le \frac{|W|}{(\frac{k}{2}-3)|W|} = \frac{2}{k-6}. \quad ∎$$

Theorem 3. The square of any planar graph G with maximum degree d can be colored using at most $\frac{13}{7}d + O(d^{2/3})$ colors.

Proof: Let k, l, and d be any integers for which $k \ge l \ge 12$ and $d \ge k + l$. We will prove by induction on the number of nodes that the square of any planar graph with maximum degree at most d can be colored using at most $\Delta = d + (l-3)(l-1) + max\{\, k, d - \lfloor \frac{l-12}{l-6}\frac{k-6}{6} \rfloor + 1\}$ colors. By setting $k = \lceil \frac{6}{7}d \rceil$ and $l = \Theta(d^{1/3})$, this will produce the desired asymptotic bound as d becomes large. As can be easily seen, the constant factors associated with the low order terms will not be large.

Define W as the set of nodes in G of degree $\ge k$ and U as the set of nodes in G of degree $< l$ and with at most 2 neighbors of degree $\ge l$. By Lemmas 1 and 2, $|W| \le \frac{2}{k-6}|V|$ and

$$|U| > |V| - 3 \cdot \tfrac{2}{l-6}|V| = \tfrac{l-12}{l-6}|V| \ge 0.$$

Case 1: Some node v in U has at most one neighbor in W. Contract v into a neighbor w of v of minimal degree, i.e., add an edge between w and each node other than w that is a neighbor of v, but not of w, and subsequently remove v. Since $w \notin W$ unless v is of degree 1, the new degree of w is bounded by $max\{d, k + l - 3\} = d$, and the inductive hypothesis implies that the square of the resulting graph can be colored with at most Δ colors. Furthermore, the colors assigned can be retained in a valid coloring of G^2, the only remaining problem being to color v. Since the number of nodes in G at distance

1 or 2 from v is at most $d+(k-1)+(l-3)(l-1)$, the indicated number of colors suffices.

Case 2: Every node in U has exactly 2 neighbors in W. By planarity, at most $3|W|$ pairs of nodes in W can have a common neighbor in U. Hence some pair of nodes in W has at least

$$\frac{|U|}{3|W|} > \frac{l-12}{l-6} \cdot \frac{k-6}{6}$$

common neighbors in U. Choose v as one of these common neighbors and contract v into a neighbor of minimal degree as above. Again, this does not increase the maximum degree, and the inductive assumption applies to the resulting graph. Finally note that the number of nodes in G at distance 1 or 2 from v is less than

$$2d - \frac{l-12}{l-6} \cdot \frac{k-6}{6} + (l-3)(l-1),$$

and we can find an acceptable color for v. ∎

Theorem 4. Any planar graph with maximum node degree d has resolution $\Theta(\frac{1}{d})$.

Proof: The upper bound is trivial. The lower bound follows from Theorems 2 and 3. ∎

For outerplanar graphs, the bounds on $\chi(G^2)$ are much tighter, as we show in what follows.

Lemma 3. Every biconnected outerplanar graph on at least three nodes contains a node of degree 2 with a neighbor of degree 2 or with adjacent neighbors, one of which is of degree at most 4.

Proof: Let T be the dual of an outerplanar embedding $\mathcal{E}$ of the given graph, with (the node representing) the outer face removed. As is well-known, T is a free tree, i.e., it is connected and acyclic. A face of $\mathcal{E}$ is a leaf, i.e., of degree 1, in T if and only if exactly one of its boundary edges does not bound the outer face. Let F_r and F_v be faces of $\mathcal{E}$ whose distance from each other in T is maximal. Root T at F_r, let v be any node of degree 2 on the boundary of F_v, and let v_α and v_β be the neighbors of v (Figure 4). If v_α or v_β is of degree ≤ 3, we are done. Otherwise define F, F_α and F_β as shown in the figure. At least one of F_α and F_β, say F_α, is a child of F in T and hence, by the choice of F_r and F_v, a leaf in T. But then v_α is of degree 4. ∎

Theorem 5. The square of any outerplanar graph G of maximum degree d can be colored using at most $d+3$ colors.

Proof: We can assume that G is biconnected and contains at least 3 nodes. Let V be the set of nodes in G of degree 2 with at least one neighbor of degree 2. If $V \neq \emptyset$, remove all nodes in V and, if any nodes are left, color the remaining graph inductively, using $d+3$ colors. Then obtain a coloring of the original graph by adding back the nodes in V and coloring them in an arbitrary order. Since at most $d+2$ nodes have distance 1 or 2 from each fixed node in V, this can be done using $d+3$ colors.

If $V = \emptyset$, remove a node v of degree 2 with adjacent neighbors, one of which is of degree ≤ 4, and color the remaining graph inductively. Since there are at most $d+2$ nodes at distance 1 or 2 from v, the proof again is easily completed. ∎

It is worth noting that the bound in Theorem 5 is nearly tight since the $(d+1)$-node star graph is an outerplanar graph G with maximum node degree d for which $\chi(G^2) = d+1$.

3.3 Special Networks

In this section, we examine the resolution of some special networks. We present optimal or nearly optimal layouts for the complete graph, the hypercube, multi-dimensional arrays and tori, the mesh of cliques, and the projective plane. The first four of these networks are important because of their uses as processor interconnection networks. The last two networks are interesting because they would seem to be good candidates for graphs with resolution $\Theta(\frac{1}{d^2})$ since the chromatic number of the square of a d-regular mesh of cliques and projective plane is $\Theta(d^2)$. Somewhat surprisingly, however, we show that the resolution of all of the special networks mentioned is $\Theta(\frac{1}{d})$.

3.3.1 The Complete Graph

Theorem 6. *The complete graph of N nodes has resolution $\frac{\pi}{N}$.*

Proof: For the lower bound, draw the N nodes of the graph at equidistant points on a circle. Then the angles formed by incident edges will have size at least $\frac{\pi}{N}$.

For the upper bound, consider three consecutive nodes u, v, and w on the convex hull of some layout. If the angle formed by edges (u,v) and (v,w) has size greater than $\frac{\pi-2\pi}{N}$, then one of the other angles in the triangle formed by u, v, and w will have size less than $\frac{\pi}{N}$, and we are done. Hence, we can assume that the angle formed by edges (u,v) and (v,w) has size at most $\frac{\pi-2\pi}{N}$. Since u, v, and w are consecutive nodes on the convex hull of the layout, all the other $N-3$ nodes are contained within the angle formed by (u,v) and (v,w). Hence, there is an angle of size at most $\frac{(\pi-\frac{2\pi}{N})}{(N-2)} = \frac{\pi}{N}$ at node v. ∎

Corollary 2. *The resolution of any d-regular graph is at most $\frac{\pi}{d-1}$.*

3.3.2 Hypercubes and Multidimensional Meshes

In this section, we consider hypercubes and multidimensional meshes. The k-hypercube has 2^k nodes, each one represented by a k-tuple $(i_1, i_2, ..., i_k)$ for $0 \leq i_1, i_2, ..., i_k \leq 1$. Edges occur between nodes that differ in precisely one bit. By Corollary 2, we know that any layout of a k-hypercube has resolution at most $\frac{\pi}{k-1}$. In what follows, we present an algorithm that draws the k-hypercube with resolution $\frac{\pi}{k}$. We then extend this algorithm to derive an optimal layout for the k-dimensional mesh.

ALGORITHM *HYPERCUBE(k)*

- step 1 Design on the plane an angle ϕ of size $\pi - \frac{\pi}{k}$. Divide ϕ into k-1 equal angles which define a k-axes system.
- step 2 Initialization: Create a 1-hypercube on the 1^{st} axis.
- step 3 for $j = 2$ to $k - 1$ do
  - 3.1 Create a copy of the $(j-1)$-hypercube.
  - 3.2 Translate the copy parallel to the j^{th} axis.
  - 3.3 Create connections between the corresponding nodes of the two $(j-1)$-hypercubes.

Since each line segment drawn by algorithm *HYPERCUBE(k)* is parallel to one of the k axes, we have the following theorem:

Theorem 7. *Algorithm HYPERCUBE draws the k-hypercube in the plane with resolution* $\frac{\pi}{k}$.

Remark. It can be shown that Algorithm *Hypercube* produces an optimal layout for the 3-hypercube.

An algorithm to embed a k-dimensional mesh with resolution $\frac{\pi}{k}$ can be obtained by extending algorithm *HYPERCUBE*. Note that a k-hypercube is the basic unit component of a k-dimensional mesh. The maximum degree for any internal node of the mesh is $2k$. Since $\frac{2\pi}{d}$ is an obvious upper bound for the resolution of any graph of maximum degree d, the extended algorithm will produce an optimal embedding.

3.3.3 Tori

The m-dimensional torus network $T(m)$ is actually an m-dimensional mesh with wrap-around connections.

Theorem 8. *The m-dimensional torus can be embedded on the plane with resolution* $\frac{\pi}{2m}$, *provided that all dimensions have size greater than 3.*

Proof: Consider the embedding of the 4×4 torus (Figure 5). Observe that it replicates the embedding of the 4-hypercube. We use this embedding as a base layout of any two-dimensional torus. We can extend the embedding of the 4×4 torus to any $a \times b$ torus by inserting extra nodes in regular intervals of the edges of the respective dimensions. By a similar argument, the embedding of an m-dimensional $4 \times 4 \times ... \times 4$ torus can be used as a base layout of any m-dimensional torus, since it is isomorphic to the $2m$-hypercube. From Theorem 7 we know that the k-hypercube can be embedded with resolution $\frac{\pi}{k}$. Therefore, the proof follows. ∎

3.3.4 Mesh of Cliques

The *m-regular Mesh of Cliques* is defined to be the regular graph with m^2 nodes arranged as an $m \times m$ mesh. All nodes on the same row of the mesh are connected in a clique. The same holds for nodes in the same column. Obviously, the m-regular Mesh of Cliques has degree $2m-2$. It also has the property that between any two of its nodes there exists a path of length 2. Thus, its square graph is a clique of size m^2. In the following we present a layout of the m-regular Mesh of Cliques which has resolution $O(\frac{1}{m})$.

Theorem 9. *The m-regular Mesh of Cliques can be embedded on the plane with resolution* $O(\frac{1}{m})$.

Proof (by construction): We group the cliques that correspond to rows into m canonical m-gons. Then we place these m-gons on the plane such that their centers are located on the boundary of a circle and, also form a canonical m-gon. Observe that the angles formed by any two edges that belong to the same "row (column) clique" have resolution at least $\frac{\pi}{m}$ degrees. Thus, we only need to consider the angles formed by an edge that belongs to a "column clique" with an edge that belongs to a "row clique". Notice that for any canonical m-gon the slopes of the lines in the m-gon all differ from each other by a multiple of $\frac{\pi}{m}$. Hence, if we rotate the m-gons that correspond to "row cliques" by $\frac{\pi}{2m}$ relative to m-gons that corresponds to "column cliques", then the embedding has resolution $\frac{\pi}{2m}$. ∎

3.3.5 Projective Plane

The *Projective Plane* consists of a set of objects called *points*, a second set of objects called *lines*, and a notion of when a point lies on a line, so that the following three conditions are satisfied:

(C1) Two distinct points lie on one and only one common line.

(C2) Two distinct lines pass through one and only one point.

(C3) There are four distinct points, no three of which lie on the same line.

In a *Projective Plane of order d-1*, every point lies on exactly d lines and every line passes through exactly d points. A projective plane of order $d-1$ exists if $d-1$ is a prime power. It has exactly $d^2 - d + 1$ points and $d^2 - d + 1$ lines. The projective plane of order $d - 1$ can be represented as a bipartite graph $G = (A, B, E)$

where the set of nodes A corresponds to points, the set of nodes B corresponds to lines, and an edge (u, v), $u \in A$ and $v \in B$, belongs to E if and only if point u lies on line v. Note that in the square graph of G the nodes of A and B form two cliques, and thus $\chi(G^2) = \Theta(d^2)$.

As in the case of the mesh of cliques, however, the projective plane can be drawn with resolution $\Theta(\frac{1}{d})$. The embedding is roughly described as follows. Arrange the nodes of A into a square grid so that most of the lines consist of one node in each column (a few of the lines will consist of an entire column). Then arrange the nodes of B in a square grid so that most of the nodes in A are linked to at most one node in each column of B. (Again, a few nodes in A will be linked to every node in some column of B.) This can be accomplished by the properties of the projective plane. Next embed one square above the other in the plane. The few nodes that are connected to every node in a column (of A or B) are embedded to the side of the appropriate square. The bound on the resolution then follows from the fact that every angle connects nodes in different columns (or rows, for the nodes embedded on the side). Thus, we have:

Theorem 10. *The Projective Plane of order $d-1$ can be embedded on the plane with resolution $O(\frac{1}{d})$.*

4. An Upper Bound on the Resolution of a Random Graph

In this section, we prove that many graphs with maximum degree d have resolution at most $O(\frac{\log d}{d^2})$, for any d. We will prove this result with a counting argument. For simplicity we will consider directed graphs with outdegree d and we will restrict our attention to the angles formed by the outgoing edges at each node. (Angles with incoming edges are ignored.) We will show that almost all such graphs with N nodes ($N \geq d^2$) have resolution $O(\frac{\log d}{d^2})$. Since almost all graphs with outdegree d have indegree $O(d + \log N)$, this means that many undirected graphs with degree $O(d)$ have resolution $O(\frac{\log d}{d^2})$. It is probably also true that almost all d-regular graphs have resolution $O(\frac{\log d}{d^2})$ but the proof appears to be more complicated.

The proof will make use of the following combinatorial facts.

Fact 1. For all a, b, $\binom{a}{b} \leq (\frac{ae}{be^{b/2a}})^b$

Proof: Standard asymptotic analysis. ∎

Fact 2. For $a > 2b$, $\binom{a}{b} \geq \frac{a^b e^b}{\sqrt{2\pi b}\, b^b e^{\frac{b^2}{2a} + O(b^3/a^2)}}$

Proof: Standard asymptotic analysis. ∎

Fact 3. Given m boxes containing $n_1, n_2, ..., n_m$ labelled balls (respectively), the number of ways of choosing j balls from the boxes so that at most one ball is chosen from each box is at most

$$\binom{m}{j}(\frac{n}{m})^j$$

where $n = n_1 + n_2 + ... + n_m$.

Proof sketch: The worst case is when each box has the same number of balls. ∎

Fact 4. Given $m+1$ boxes containing $n_0, n_1, ..., n_m$ labelled balls, respectively, the number of ways of choosing d balls from the boxes so that at most one ball is chosen from boxes 1, 2,..., m (any number can be taken from box 0) is at most

$$(d+1)(\frac{ne}{de^{(1-x)^2 d/4m}})^d$$

where $x = \frac{n_0}{n}$ and $n = n_0 + n_1 + ... + n_m$.

Proof sketch: Derived from Facts 1, 2 and 3 using asymptotic analysis. ∎

Fact 5. Given any placement of N points on the plane, it is possible to find concentric circles with radii r_1 and r_2 so that at least $\frac{N}{5}$ points are inside or on the boundary of the inner circle, and so that at least $\frac{N}{5}$ points are outside or on the boundary of the outer circle, where $r_2 \geq \sqrt{2}\, r_1$.

Proof: Find a smallest circle that contains at least $\frac{N}{5}$ points. This will be the inner circle; it has radius r_1. Let r_2 be the radius of the largest concentric outer circle that leaves $\frac{N}{5}$ points outside. Since the concentric circle with radius $\sqrt{2}\, r_1$ can be covered with four circles of radius r_1 (Figure 6), there are at least $\frac{N}{5}$ points outside of the concentric circle with radius $\sqrt{2}\, r_1$. Hence, $r_2 \geq \sqrt{2}\, r_1$. ∎

We can now bound the resolution of a random graph with outdegree d.

Theorem 11. *Given a random N-node directed graph G in which every node has outdegree d, with high probability, every embedding of G has resolution $O(\frac{\log d}{d^2})$.*

Proof: We will count all graphs with N labelled nodes and outdegree d that can be constructed so that there is an embedding in which every angle formed by outgoing edges is at least $\frac{\log d}{cd^2}$, where c is a sufficiently small constant. We will show that this number is far less than $\binom{N-1}{d}^N$, which is the number of N-node outdegree d graphs, thereby implying the theorem.

Given any embedding of any graph, we know by Fact 4 that there are concentric circles with radii r_1 and r_2 so that $\frac{N}{5}$ points are contained in the inner circle, $\frac{N}{5}$ points are outside the outer circle, and $r_2 \geq \sqrt{2}\, r_1$. Partition the region outside the outer circle into m equal slices, as shown in Figure 7. Notice that if any node within the inner circle is connected by outgoing edges to two or more nodes in the same slice, then there must be an angle of size $O(\frac{1}{m})$.

In what follows, we will show that if $m = \frac{cd^2}{\log d}$, then there are far less than $\binom{N-1}{d}^N$ ways of constructing such a graph. The counting proceeds as follows:

a) The number of ways to pick $\frac{N}{5}$ nodes to be inside the inner circle is:

$$\binom{N}{N/5} \leq (5e)^{\frac{N}{5}}$$

b) The number of ways to pick $\frac{N}{5}$ nodes to be outside the outer circle is:

$$\binom{4N/5}{N/5} \leq (4e)^{\frac{N}{5}}$$

c) The number of ways to assign the $\frac{N}{5}$ selected nodes outside to slices is:

$$m^{\frac{N}{5}}$$

d) The number of ways to connect $\frac{4N}{5}$ nodes outside the inner circle to other nodes is:

$$\binom{N-1}{d}^{\frac{4N}{5}}$$

e) The number of ways to connect $\frac{N}{5}$ nodes inside the inner circle to other nodes (Fact 4) is:

$$\leq \left((d+1)\Big(\frac{(N-1)e}{de^{(1-4/5)^2 d/4m}}\Big)^d\right)^{\frac{N}{5}}$$

Thus, the total number of graphs that do not have resolution $O(\frac{1}{m})$ divided by the total number of graphs overall is:

$$\leq \frac{(5e)^{\frac{N}{5}}(4e)^{\frac{N}{5}}m^{\frac{N}{5}}\binom{N-1}{d}^{\frac{4N}{5}}\left((d+1)(\frac{(N-1)e}{de^{(1-4/5)^2 d/4m}})^d\right)^{\frac{N}{5}}}{\binom{N-1}{d}^N}$$

$$= \left(\frac{20e^2 m(d+1)(\frac{(N-1)e}{de^{d/100m}})^d}{\binom{N-1}{d}}\right)^{\frac{N}{5}}$$

$$\leq \left(\frac{20e^2 m(d+1)\frac{(N-1)^d e^d}{d^d e^{d^2/100m}}}{\frac{(N-1)^d e^d}{\sqrt{2\pi d}\, d^d e^{d^2/2(N-1)+O(d^3/N^2)}}}\right)^{\frac{N}{5}}$$

$$\leq \left(\frac{20\sqrt{2\pi}\, e^2 m(d+1)^{3/2} e^{d^2/2(N-1)+O(d^3/N^2)}}{e^{d^2/100m}}\right)^{\frac{N}{5}}$$

By choosing $N \geq d^2$ and $m \leq \frac{cd^2}{\log d}$ for some small constant c, the value in the brackets can be made smaller than $\frac{1}{d^5}$. Hence, the probability of getting a graph with resolution $O(\frac{\log d}{d^2})$ is at least $1 - \frac{1}{d^N}$. ∎

5. Remarks

There are several questions left open in this paper. We list some of them below.

1. Is the problem of determining the resolution of a graph in NP ?
2. Are there interesting tradeoffs between the resolution of a layout and its area for graphs with large maximum d ? Can the area bound in Section 3 be improved ?
3. What is the worst case value of $\chi(G^2)$ if G is a planar graph with maximum degree d ?
4. What happens to the resolution of planar graphs if we restrict the layout to be planar ? Does every degree 3 planar graph have a planar embedding with constant (independent of the number of nodes) resolution ?
5. Is there a meaningful relationship between the resolution of a graph and its density (see [7,8] for definitions)? (In the case of planar graphs, the two quantities appear to be very similar.)

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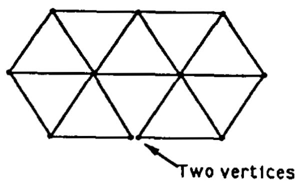

Figure 1. An 11-node graph for which there is a layout with resolution $\frac{\pi}{3} - \epsilon$ for any $\epsilon > 0$, but for which there is no layout with resolution $\frac{\pi}{3}$

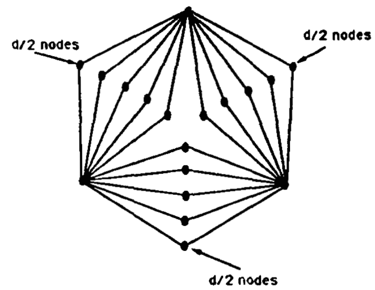

Figure 2. Example of a planar graph with maximum degree d for which $\chi(G^2) \geq \frac{3d}{2}$.

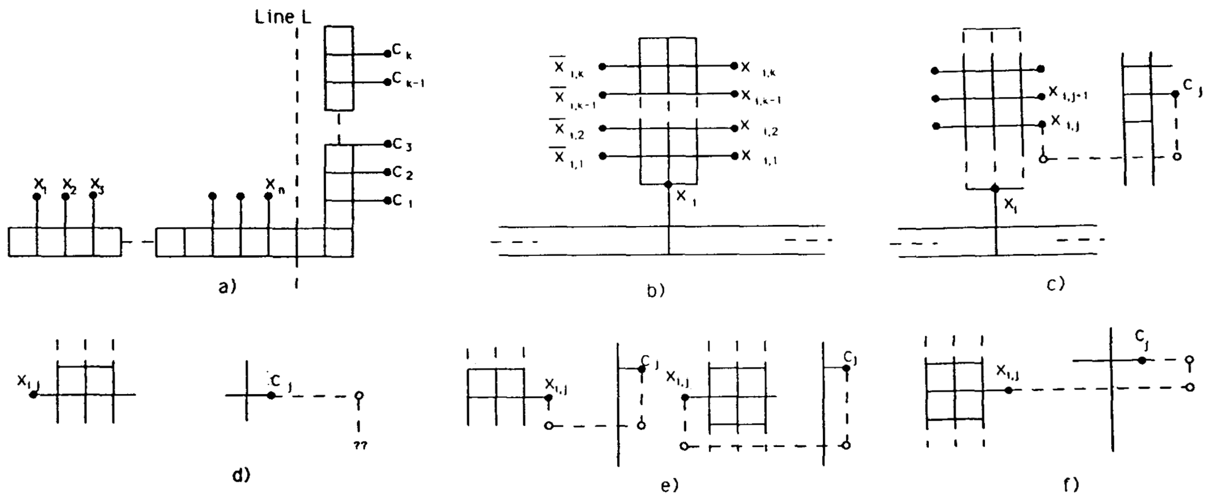

Figure 3. The components used in the NP-hardness proof.

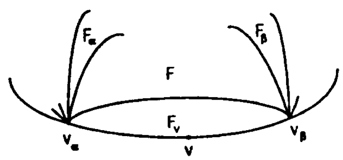

Figure 4. The faces of the planar graph used in the proof of Lemma 3.

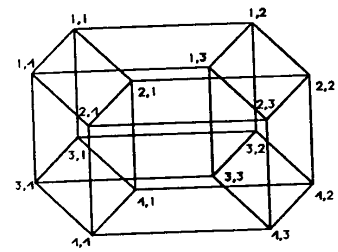

Figure 5. The embedding of the 4 × 4 Torus.

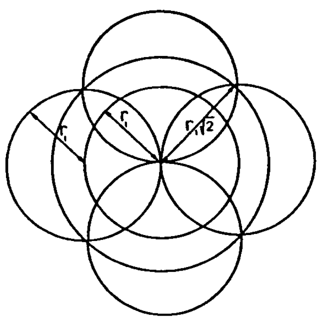

Figure 6. Covering a circle of radius $r_1\sqrt{2}$ with 4 circles of radius r_1.

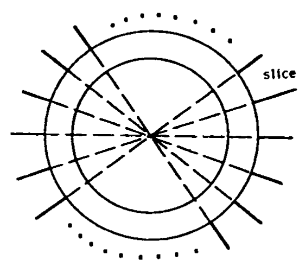

Figure 7. The partition of the plane used in the proof of Theorem 11.