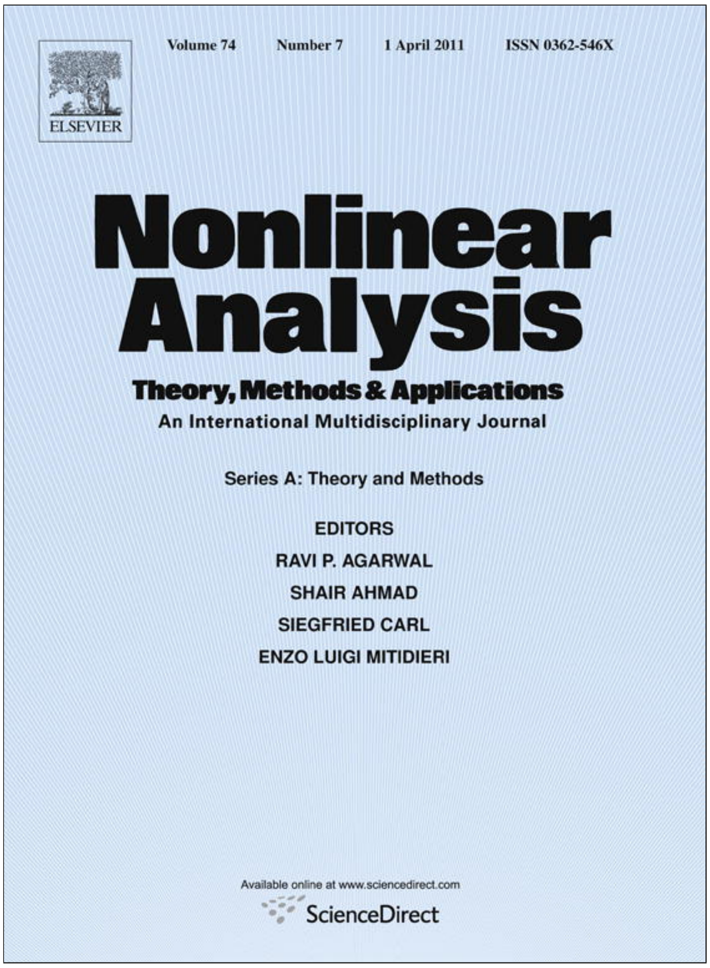
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# Global attractor for a system of Klein–Gordon–Schrödinger type in all $\mathbb{R}$

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ABSTRACT

In this paper we study the long time behavior of solutions for the following system of Klein–Gordon–Schrödinger type

$$\begin{aligned}
&i\psi_t + \kappa\psi_{xx} + i\alpha\psi = \phi\psi + f,\\
&\phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t = -\mathrm{Re}\,\psi_x + g,\\
&\psi(x,0) = \psi_0(x), \qquad \phi(x,0) = \phi_0(x), \qquad \phi_t(x,0) = \phi_1(x),\\
&\lim_{x\to\pm\infty}\psi(x,t) = \lim_{x\to\pm\infty}\phi(x,t) = 0, \quad t > 0,
\end{aligned}$$

where $x \in \mathbb{R}$, $\kappa > 0$, $\alpha > 0$, $\lambda > 0$. First, the existence, uniqueness and continuity of the solutions on the initial data are proved. Then the asymptotic compactness of the solutions and the existence of a global compact attractor are shown.



## 1. Introduction

The aim of this paper is to prove the existence of a global compact attractor for the following system

$$i\psi_t + \kappa\psi_{xx} + i\alpha\psi = \phi\psi + f, \tag{1.1}$$

$$\phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t = -\mathrm{Re}\,\psi_x + g, \tag{1.2}$$

$$\psi(x,0) = \psi_0(x), \qquad \phi(x,0) = \phi_0(x), \qquad \phi_t(x,0) = \phi_1(x), \tag{1.3}$$

$$\lim_{x\to\pm\infty}\psi(x,t) = \lim_{x\to\pm\infty}\phi(x,t) = 0, \quad t > 0, \tag{1.4}$$

where $x \in \mathbb{R}, t > 0, \kappa > 0, \alpha > 0, \lambda > 0$. Also $f, g$ are complex and real valued functions, respectively. The complex valued variable $\psi$ stands for the dimensionless low frequency electron field, whereas the real valued variable $\phi$ denotes the dimensionless low frequency density. The system (1.1)–(1.4) describes the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field, adapted to model the UHH plasma heating scheme. The dissipative mechanism of the system is introduced by the terms $i\alpha\psi$ and $\lambda\phi_t$.

Systems of Klein–Gordon–Schrödinger type have been studied for many years. To our knowledge, it seems that the first problem of this type is the so-called *Yukawa System*, which goes back to 1935. Another model which is of the same type is the

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so-called *Zakharov System*, which is formed by V. E. Zakharov in early seventies. Guo and Li [1] proved the existence of a strong global attractor in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ attracting bounded sets of $H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$ for a Klein–Gordon–Schrödinger system with Yukawa coupling. This was extended by Lu and Wang [2]. They established the existence of a strong global attractor in $H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$, $N = 1, 2, 3$, attracting bounded sets of $H^k(\mathbb{R}^N) \times H^k(\mathbb{R}^N)$, $k \geq 1$. For a dissipative system of Zakharov type Flahaut [3] proved the existence of a weak global attractor in $H_0^1((0, L)) \times H_0^1((0, L)) \bigcap H^2((0, L)) \times H_0^1((0, L)) \bigcap H^3((0, L))$ and obtained upper bounds for its Hausdorff and Fractal dimensions.

The model under consideration (1.1)–(1.4) appeared first in the work [4] (see also [5]), where for the undriven ($f \equiv 0, g \equiv 0$) and the bounded interval case, the global existence and uniqueness of the solutions were proved and necessary conditions were established for the system to manifest exponential energy decay. These results were extended by the authors to the more realistic driven case system (1.1)–(1.3) (see, [6]), where the driving terms $f, g \in L^2(\Omega)$. The existence of a global attractor was derived in the space $(H_0^1(\Omega)\cap H^2(\Omega))^2 \times H_0^1(\Omega)$, which attracts all bounded sets of $(H_0^1(\Omega)\cap H^2(\Omega))^2 \times H_0^1(\Omega)$ in the norm topology. Furthermore, in [7] the authors studied the finiteness of the dimension of the global attractor applying a general method based on the uniform Lyapunov exponents and found upper bounds for both Hausdorff and Fractal dimensions. Xanthopoulos and Zouraris [8] proposed a linearly implicit finite difference method to approximate the solution of the system (1.1)–(1.4). The numerical implementation of the method recovers known theoretical results for the behavior of the solution, while revealing additional nonlinear features.

The rest of the paper is divided into three parts. In Section 2, some useful estimates on the solutions of the system (1.1)–(1.4) are derived in $(H^1(\mathbb{R}) \cap H^2(\mathbb{R}))^2 \times H^1(\mathbb{R})$. Then, due to lack of compactness we approximate the whole space $\mathbb{R}$ by a bounded domain $\Omega_m = \{x \in \mathbb{R} : |x| \leq m\}$ for each $m > 0$ and prove that all solutions on the complement of $\Omega_m$ are uniformly small for large times. In Section 3, using the energy equation the continuous dependence of the solutions on the initial data in the space $(H^1(\mathbb{R}) \cap H^2(\mathbb{R}))^2 \times H^1(\mathbb{R})$ is proved. Finally, in Section 4, the asymptotic compactness of the dynamical system and the existence of a global attractor are established in the space $(H^1(\mathbb{R}) \cap H^2(\mathbb{R}))^2 \times H^1(\mathbb{R})$.

*Notation*: Denote by $H^s(\Omega)$ both the standard real and complex Sobolev spaces on $(\Omega)$. For *simplicity reasons* sometimes we use $H^s$, $L^s$ for $H^s(\mathbb{R})$, $L^s(\mathbb{R})$ and $\|.\|$, $(.,.)$ for the norm and the inner product of $L^2(\mathbb{R})$ respectively as well as $\int \mathrm{d}x$ denotes the integration over the domain $\mathbb{R}$. Finally, $C$ is a general symbol for any positive constant.

## 2. Global existence

In this section we derive a priori estimates for the solutions of the Klein–Gordon–Schrödinger system (1.1)–(1.4). Let us introduce the transformation $\theta = \phi_t + \delta\phi$ where $\delta$ is a small positive constant to be specified later. Then, system (1.1)–(1.2) takes the form

$$i\psi_t + \kappa\psi_{xx} + i\alpha\psi = \phi\psi + f, \tag{2.1}$$

$$\phi_t + \delta\phi = \theta, \tag{2.2}$$

$$\theta_t + (\lambda - \delta)\theta - \phi_{xx} + (1 - \delta(\lambda - \delta))\phi = -\mathrm{Re}\,\psi_x + g. \tag{2.3}$$

Also the new initial and boundary conditions related to (1.3)–(1.4) are

$$\psi(x, 0) = \psi_0(x), \qquad \phi(x, 0) = \phi_0(x), \qquad \theta(x, 0) = \theta_0(x) = \phi_0(x) + \delta\phi_1(x), \quad x \in \mathbb{R}, \tag{2.4}$$

$$\lim_{x\to\pm\infty} \psi(x, t) = \lim_{x\to\pm\infty} \phi(x, t) = \lim_{x\to\pm\infty} \theta(x, t) = 0, \quad t > 0. \tag{2.5}$$

**Lemma 2.1.** *Let $\|\psi_0(t)\| \leq R$, for some $R > 0$ and suppose that $f$ belongs to $L^2(\mathbb{R})$. Every solution of (2.1)–(2.5) satisfies*

$$\|\psi(t)\| \leq R^*, \quad t \geq t_1,$$

*where constant $R^*$ depends on $\alpha$, $\|f\|$; constant $t_1$ depends on $\alpha$, $\|f\|$ and $R$.*

**Proof.** The proof is analogue to the proof of Lemma 2.1 in [2]. □

**Lemma 2.2.** *Assume that $f$ and $g$ belong to $L^2(\mathbb{R})$, let $\|(\psi_0, \phi_0, \theta_0)\|_{H^1\times H^1\times L^2} \leq R$, where $R > 0$. Then, there exists a constant $\delta_1$ such that when $\delta \leq \delta_1$, every solution $(\psi, \phi, \theta)$ of problem (2.1)–(2.5) satisfies*

$$\|\psi(t)\|_{H^1} + \|\phi(t)\|_{H^1} + \|\theta(t)\| \leq M_1, \quad t \geq t_2,$$

*where $M_1$ depends on $\alpha, \kappa, \lambda, \delta, \|f\|, \|g\|$; $t_2$ on $\alpha, \kappa, \lambda, \delta, \|f\|, \|g\|$ and $R$.*

**Proof.** Multiplying Eq. (2.1) by $-\bar{\psi}_t$, integrating and taking the real part gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa\|\psi_x\|^2 + \int_{\mathbb{R}} \phi|\psi|^2 + 2\,\mathrm{Re}\int_{\mathbb{R}} f\bar{\psi}\right) + \kappa\alpha\|\psi_x\|^2 + \left(\alpha + \frac{\delta}{2}\right)\int_{\mathbb{R}} \phi|\psi|^2 = \frac{1}{2}\int_{\mathbb{R}} \theta|\psi|^2 + \alpha\,\mathrm{Re}\int_{\mathbb{R}} f\bar{\psi}. \tag{2.6}$$

Next, multiplying Eq. (2.3) by $\theta$ and substituting $\theta$ from Eq. (2.2) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta\|^2+\|\phi_x\|^2+(1-\delta(\lambda-\delta))\|\phi\|^2)+(\lambda-\delta)\|\theta\|^2+\delta\|\phi_x\|^2+\delta(1-\delta(\lambda-\delta))\|\phi\|^2$$
$$=-\mathrm{Re}\int_{\mathbb{R}}\theta\psi_x+\int_{\mathbb{R}}g\theta. \tag{2.7}$$

Adding relations $2\times$ (2.11) and $2\times$ (2.8) gives

$$F_1'(t)+\delta F_1(t)=G_1(t), \tag{2.8}$$

where, to simplify the notation, the following quantities are introduced

$$F_1:=\kappa\|\psi_x\|^2+\int_{\mathbb{R}}\phi|\psi|^2\mathrm{d}x+\|\theta\|^2+\|\phi_x\|^2+(1-\delta(\lambda-\delta))\|\phi\|^2+2\,\mathrm{Re}\int_{\mathbb{R}}f\bar{\psi},$$
$$G_1 := (\delta-2\kappa\alpha)\|\psi_x\|^2-2\alpha\int_{\mathbb{R}}\phi|\psi|^2+(3\delta-2\lambda)\|\theta\|^2-\delta(1-\delta(\lambda-\delta))\|\phi\|^2$$
$$-\delta\|\phi_x\|^2+\int_{\mathbb{R}}\theta|\psi|^2+2(\delta-\alpha)\mathrm{Re}\int_{\mathbb{R}}f\bar{\psi}-2\mathrm{Re}\int_{\mathbb{R}}\theta\psi_x+2\int_{\mathbb{R}}g\theta.$$

Taking $\delta$ small enough such that $\delta-2\kappa\alpha<0$, $3\delta-2\lambda<0$, $1-\delta(\lambda-\delta)>0$, one can render several terms of $G_1$ negative. Let us proceed by majorizing the integrals of $G_1$ as follows

$$\left|\int_{\mathbb{R}}\theta|\psi|^2\right|\le\|\theta\|\|\psi\|_4^2\le\|\theta\|\|\psi\|^{1/2}\|\psi\|^{3/2}\le\frac{\epsilon_1}{2}\|\theta\|^2+\frac{\epsilon_2}{2}\|\psi_x\|^2+C,$$
$$\left|2\alpha\int_{\mathbb{R}}\phi|\psi|^2\right|\le\epsilon_3\|\phi\|^2+\frac{\epsilon_2}{2}\|\psi_x\|^2+C,\quad\text{and}\quad\left|\int_{\mathbb{R}}\theta\psi_x\right|\le\|\psi_x\|\|\theta\|\le\frac{\epsilon}{2}\|\psi_x\|^2+\frac{1}{2\epsilon}\|\theta\|^2,$$
$$\left|2(\delta-\alpha)\int_{\mathbb{R}}f\bar{\psi}\right|\le C\|f\|\,\|\psi\|\le C,\quad\text{and}\quad\left|2\int_{\mathbb{R}}g\theta\right|\le2\|g\|\|\theta\|\le\frac{\epsilon_1}{2}\|\theta\|^2+C.$$

The next step is to estimate the arbitrary positive constants $\epsilon_1,\epsilon_2,\epsilon$, such that the following two inequalities hold simultaneously true $\epsilon_1+\frac{1}{2\epsilon}\le-(3\delta-2\lambda)$, $\epsilon_2+\frac{\epsilon}{2}\le-(\delta-2\kappa\alpha)$. Let $\nu>0$, $\nu\ne\frac{1}{2}$, $\bar{\alpha}=-(3\delta-2\lambda)$ and $\bar{\beta}=-(\delta-2\kappa\alpha)$. Setting $\epsilon_1=\frac{\bar{\alpha}}{2\nu}$, $\epsilon_2=\frac{\bar{\beta}}{2\nu}$ we have the following necessary condition: $\bar{\alpha}\bar{\beta}\ge\frac{\nu^2}{(2\nu-1)^2}$. Since $\bar{\alpha},\bar{\beta}>0$ the inequality is always true for sufficiently small $\nu$. Finally, taking $\epsilon_3$ small enough, so that $\epsilon_3<-\delta(1-\delta(\lambda-\delta))$ implies

$$F_1'(t)+\delta F_1(t)\le C.$$

The application of Gronwall's inequality completes the proof. □

**Lemma 2.3.** *Assume that $f$ and $g$ belong to $H^1(\mathbb{R})$, let $\|(\psi_0,\phi_0,\theta_0)\|_{(H^1\cap H^2)^2\times H^1}\le R$, where $R>0$. Then, every solution $(\psi,\phi,\theta)$ of the problem (2.1)–(2.5) satisfies*

$$\|\psi(t)\|_{H^1\cap H^2}+\|\phi(t)\|_{H^1\cap H^2}+\|\theta(t)\|_{H^1}\le M_2,\quad t\ge t_3,$$

*where $M$ depends on $\alpha,\kappa,\lambda,\delta,\|f\|_{H^1},\|g\|_{H^1}$ and $t_3$ depends on $\alpha,\kappa,\lambda,\delta,\|f\|_{H^1},\|g\|_{H^1}$ and $R$.*

**Proof.** The proof is analogue to the proof of Lemma 3 in [6]. □

Let $B_1,B_2$ denote the following balls of center zero and radius $M_1,M_2$ respectively

$$B_1=\{(\psi,\phi,\theta)\in H^1\times H^1\times L^2:\|\psi\|_{H^1}+\|\phi\|_{H^1}+\|\theta\|\le M_1\}, \tag{2.9}$$
$$B_2=\{(\psi,\phi,\theta)\in(H^1\cap H^2)^2\times H^1:\|\psi\|_{H^1\cap H^2}+\|\phi\|_{H^1\cap H^2}+\|\theta\|_{H^1}\le M_2\}, \tag{2.10}$$

where $M_1,M_2$ are the constants introduced in Lemmas 2.2 and 2.3, respectively. Therefore $B_1,B_2$ are bounded absorbing sets for (2.1)–(2.5). Since $B_1$ is bounded, we see that there exists a constant $T(B_1)$ depending on $B_1$ such that

$$S(t)B_1\subset B_1,\quad\text{for all } t\ge T(B_1). \tag{2.11}$$

Due to lack of compactness – because of the unboundedness of the domain – we approximate the whole space $\mathbb{R}$ by a bounded domain $\Omega_m=\{x\in\mathbb{R}:|x|\le m\}$, for each $m>0$. The following lemma states that all solutions on the complement of $\Omega_m$ are uniformly small for large times.

**Lemma 2.4.** *Let $u_0 \in B_1$, the bounded absorbing set in (2.9). Then for every $\epsilon > 0$, there exist $T(\epsilon)$ and $M(\epsilon)$ such that every solution $(\psi, \phi, \theta)$ of problem (2.1)–(2.4)*

$$\int_{|x|\geq m} (|\psi(t)|^2 + |\phi(t)|^2 + |\phi_x(t)|^2 + |\theta(t)|^2)\mathrm{d}x \leq \epsilon^2, \quad \textit{for all } t \geq T(\epsilon), m \geq M(\epsilon) \tag{2.12}$$

*where $T(\epsilon)$ and $M(\epsilon)$ depend on $\epsilon$.*

**Proof.** Let $\beta$ be a smooth function such that $0 \leq \beta(s) \leq 1$ for $s \in \mathbb{R}^+$, and

$$\beta(s) = 0, \quad \text{for } 0 \leq \beta(s) \leq 1, \qquad \beta(s) = 1 \quad \text{for } s \geq 2.$$

Then, there exists a constant $C$ such that $|\beta'(s)| \leq C$, for $s \in \mathbb{R}^+$. Taking the imaginary part of the inner product of Eq. (2.1) with $\beta\left(\frac{|x|^2}{m^2}\right)\bar{\psi}$ in $L^2$ get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 + \kappa \,\mathrm{Im} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\bar{\psi}\psi_{xx} + \alpha \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 \\ &\quad = \mathrm{Im} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\phi|\psi|^2 + \mathrm{Im} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\bar{\psi} f.\end{aligned} \tag{2.13}$$

Evaluating the integrals we obtain

$$\begin{aligned}-\kappa \,\mathrm{Im} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\bar{\psi}\psi_{xx} &= \kappa \,\mathrm{Im} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi_x|^2 + \kappa \,\mathrm{Im} \int_{\mathbb{R}} \beta'\left(\frac{|x|^2}{m^2}\right)\bar{\psi}\frac{2x}{m^2}\psi_x \\ &= \kappa \,\mathrm{Im} \int_{m\leq|x|\leq\sqrt{2}m} \beta'\left(\frac{|x|^2}{m^2}\right)\bar{\psi}\frac{2x}{m^2}\psi_x \\ &\leq \frac{\kappa C}{m}\int_{m\leq|x|\leq\sqrt{2}m} |\psi||\psi_x| \leq \frac{\kappa C}{m}\int_{\mathbb{R}} |\psi||\psi_x| \\ &\leq \frac{\kappa C}{m}\|\psi\|\|\psi_x\| \leq \frac{\kappa C}{m}, \quad t \geq T(B_1),\end{aligned} \tag{2.14}$$

where $C$ is independent of $m$ and

$$\begin{aligned}\left|\mathrm{Im} \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\bar{\psi} f\right| &= \left|\mathrm{Im} \int_{|x|\geq m} \beta\left(\frac{|x|^2}{m^2}\right)\bar{\psi} f\right| \\ &\leq \left(\int_{|x|\geq m} |f|^2\right)^{1/2}\left(\int_{|x|\geq m} \beta^2\left(\frac{|x|^2}{m^2}\right)|\psi|^2\right)^{1/2} \\ &\leq \left(\int_{|x|\geq m} |f|^2\right)^{1/2}\left(\int_{\mathbb{R}} \beta^2\left(\frac{|x|^2}{m^2}\right)|\psi|^2\right)^{1/2} \\ &\leq \frac{1}{2\alpha}\int_{|x|\geq m} |f|^2 + \frac{\alpha}{2}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2.\end{aligned} \tag{2.15}$$

Substitution of the estimates (2.14)–(2.15) into Eq. (2.13) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 + \frac{\alpha}{2}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 \leq \frac{\kappa C}{m} + \frac{1}{2\alpha}\int_{|x|\geq m} |f|^2. \tag{2.16}$$

Since $f \in L^2$, given $\epsilon > 0$, there exists a constant $M_1(\epsilon) > 0$ such that for $m \geq M_1(\epsilon)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 + \alpha \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 \leq \epsilon, \quad \text{for all } t \geq T(B_1).$$

The use of Gronwall's inequality implies

$$\begin{aligned}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 &\leq e^{-\alpha(t-T(B_1))}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi(T(B_1))|^2 + \frac{\epsilon}{\alpha} \\ &\leq e^{-\alpha(t-T(B_1))}\|\psi(T(B_1))\|^2 + \frac{\epsilon}{\alpha} \leq M^2 e^{-\alpha(t-T(B_1))}\|\psi(T(B_1))\|^2 + \frac{\epsilon}{\alpha}.\end{aligned}$$

Setting $T_1(\epsilon) = \frac{1}{\alpha}\ln\left(\frac{\alpha M^2}{\epsilon}\right) + T(B_1)$ and $T_2(\epsilon) = \max\{T_1(\epsilon), T(B_1)+1\}$, then for $t \geq T_2(\epsilon)$ and $m \geq M_1(\epsilon)$ we have

$$\int_{|x|\geq 2m} |\psi(t)|^2 \leq \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\psi|^2 \leq \frac{2\epsilon}{\alpha}.$$

Next multiplying Eq. (2.3) with $\beta\left(\frac{|x|^2}{m^2}\right)\theta$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\theta|^2 + (\lambda-\delta)\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\theta|^2 + (1-\delta(\lambda-\delta))\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta\phi$$
$$= \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\phi_{xx}\theta - \mathrm{Re}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta\psi_x + \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta g. \tag{2.17}$$

Using Eq. (2.2) we have that

$$\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\phi_{xx}\theta = \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\phi_{xx}\phi_t + \delta\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\phi_{xx}\phi$$
$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\phi_x|^2 - \delta\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\phi_x|^2 - \int_{\mathbb{R}} \beta'\left(\frac{|x|^2}{m^2}\right)\theta\phi_x\frac{2x}{m^2}. \tag{2.18}$$

Using again Eq. (2.2) we get

$$(1-\delta(\lambda-\delta))\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta\phi = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1-\delta(\lambda-\delta))\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\phi|^2 + \delta(1-\delta(\lambda-\delta))\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\phi|^2. \tag{2.19}$$

Substitution of Eq. (2.18)–(2.19) into Eq. (2.17) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)((1-\delta(\lambda-\delta))|\phi|^2 + |\phi_x|^2 + |\theta|^2) + \delta\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)((1-\delta(\lambda-\delta))|\phi|^2$$
$$+ |\phi_x|^2 + |\theta|^2) + (\lambda-2\delta)\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\theta|^2$$
$$= -\mathrm{Re}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta\psi_x + \int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta g - \int_{\mathbb{R}} \beta'\left(\frac{|x|^2}{m^2}\right)\theta\phi_x\frac{2x}{m^2}. \tag{2.20}$$

Estimating the integrals on the right-hand side of the above equation, we get

$$\left|\int_{\mathbb{R}} \beta'\left(\frac{|x|^2}{m^2}\right)\theta\phi_x\frac{2x}{m^2}\mathrm{d}x\right| \leq \frac{c_2}{m}\|\theta\|\|\psi_x\|,$$
$$\left|\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta g\,\mathrm{d}x\right| \leq \frac{1}{2(\lambda-2\delta)}\int_{|x|\geq m}|g|^2 + \frac{(\lambda-2\delta)}{2}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\theta|^2,$$
$$\left|\mathrm{Re}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)\theta\psi_x\mathrm{d}x\right| \leq c_3\|\psi_x\|^2 + \frac{(\lambda-2\delta)}{2}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)|\theta|^2,$$

where $c_2, c_3$ are independent of $m$. Therefore choosing $\delta$ small enough such that $\lambda - 2\delta > 0$, for $m \geq M_2(\epsilon)$ and $t \geq T(B_1)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)((1-\delta(\lambda-\delta))|\phi|^2 + |\phi_x|^2 + |\theta|^2)\mathrm{d}x + 2\delta\int_{\mathbb{R}} \beta\left(\frac{|x|^2}{m^2}\right)((1-\delta(\lambda-\delta))|\phi|^2 + |\phi_x|^2 + |\theta|^2)\mathrm{d}x \leq \epsilon.$$

Gronwall's inequality implies the existence of a constant $T_3(\epsilon) > 0$, such that

$$\int_{|x|\geq 2m}((1-\delta(\lambda-\delta))|\phi|^2 + |\phi_x|^2 + |\theta|^2)\mathrm{d}x \leq \frac{2\epsilon}{\delta}, \quad \text{for } m \geq M_3(\epsilon) \text{ and } t \geq T_3(\epsilon),$$

which completes the proof. □

Repeating a similar procedure to the one used in the above lemmas we obtain the following results on a finite time interval.

**Lemma 2.5.** *Assume that $f$ and $g$ belong to $L^2(\mathbb{R})$, let $\|\psi_0, \phi_0, \theta_0\|_{H^1\times H^1\times L^2} \le R$, where $R > 0$. Then, every solution $(\psi, \phi, \theta)$ of the problem (2.1)–(2.5) satisfies*

$$\|\psi(t)\|_{H^1} + \|\phi(t)\|_{H^1} + \|\theta\| \le L_1, \quad 0 \le t \le T,$$

*where $L_1$ depends on $(\alpha, \kappa, \lambda, \delta, \|f\|, \|g\|$ and $T)$.*

**Lemma 2.6.** *Assume that $f$ and $g$ belong to $H^1(\mathbb{R})$, let $\|\psi_0, \phi_0, \theta_0\|_{(H^1\cap H^2)^2\times H^1} \le R$, where $R > 0$. Then, every solution $(\psi, \phi, \theta)$ of the problem (2.1)–(2.5) satisfies*

$$\|\psi(t)\|_{H^1\cap H^2} + \|\phi(t)\|_{H^1\cap H^2} + \|\theta\|_{H^1} \le L_2, \quad 0 \le t \le T,$$

*where $L_2$ depends on $(\alpha, \kappa, \lambda, \delta, \|f\|_{H^1}, \|g\|_{H^1}$ and $T)$.*

Therefore we are ready to state the main result of this section.

**Theorem 2.7.** *Let $f$ and $g$ belong to $L^\infty(\mathbb{R}^+, H^1(\mathbb{R}))$ and assume that*

$$(\psi_0, \phi_0, \theta_0) \in (H^1(\mathbb{R}) \cap H^2(\mathbb{R}))^2 \times H^1(\mathbb{R}).$$

*Then, there exists a unique solution for the system (2.1)–(2.5) such that*

$$\begin{aligned}
&\psi \in L^\infty(0, \infty; H^1(\mathbb{R}) \cap H^2(\mathbb{R})), \qquad \psi_t \in L^\infty(0, \infty; L^2(\mathbb{R})),\\
&\phi \in L^\infty(0, \infty; H^1(\mathbb{R}) \cap H^2(\mathbb{R})), \qquad \phi_t \in L^\infty(0, \infty; H^1(\mathbb{R})),\\
&\phi_{tt} \in L^\infty(0, \infty; L^2(\mathbb{R})),\\
&\psi(x, 0) = \psi_0(x), \qquad \phi(x, 0) = \phi_0(x), \qquad \phi_t(x, 0) = \phi_1(x), \quad x \in \mathbb{R}.
\end{aligned}$$

**Proof.** The proof follows the same basic steps as the one of Theorem 3.1 in [4] and is based on the result obtained in the above lemmas. □

## 3. The solution semigroup properties

The solution semigroup is endowed with the following useful properties

**Lemma 3.1.** *If $(\psi_n, \phi_n, \theta_n) \to (\psi, \phi, \theta)$ weakly in $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$, then for every $T > 0$, we have*

$$S(\cdot)(\psi_n, \phi_n, \theta_n) \to S(\cdot)(\psi, \phi, \theta), \quad \text{weakly in } L^2(0, T; H^1 \times H^1 \times L^2), \tag{3.1}$$

$$S(t)(\psi_n, \phi_n, \theta_n) \to S(t)(\psi, \phi, \theta), \quad \text{weakly in } H^1 \times H^1 \times L^2, \quad 0 \le t \le T. \tag{3.2}$$

**Proof.** The proof is analogue to the proof of Lemma 3.1 in [2]. □

**Theorem 3.2.** *Assume that $f$ and $g$ belong to $L^2(\mathbb{R})$. The solutions $(\psi, \phi, \theta) \in C(\mathbb{R}^+; H^1 \times H^1 \times L^2)$ of the problem (2.1)–(2.5) depend continuously on the initial data in $H^1 \times H^1 \times L^2$, and satisfy the energy equation*

$$F'(t) + \delta F(t) = H(t). \tag{3.3}$$

*where*

$$\begin{aligned}
F &:= \kappa\|\psi_x\|^2 + \int_{\mathbb{R}} \phi|\psi|^2 \mathrm{d}x + \|\theta\|^2 + \|\phi_x\|^2 + (1 - \delta(\lambda - \delta))\|\phi\|^2 + 2\,\mathrm{Re}\int_{\mathbb{R}} f\bar{\psi},\\
H &:= (\delta - 2\kappa\alpha)\|\psi_x\|^2 - 2\alpha\int_{\mathbb{R}} \phi|\psi|^2 + (3\delta - 2\lambda)\|\theta\|^2 - \delta(1 - \delta(\lambda - \delta))\|\phi\|^2\\
&\quad - \delta\|\phi_x\|^2 + \int_{\mathbb{R}} \theta|\psi|^2 + 2(\delta - \alpha)\mathrm{Re}\int_{\mathbb{R}} f\bar{\psi} - 2\,\mathrm{Re}\int_{\mathbb{R}} \theta\psi_x + 2\int_{\mathbb{R}} g\theta.
\end{aligned} \tag{3.4}$$

**Proof.** Given $(\psi_0, \phi_0, \theta_0)$ in $H^1 \times H^1 \times L^2$ we take a sequence $(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \in (H^1 \cap H^2)^2 \times H^1$ such that

$$(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \to (\psi_0, \phi_0, \theta_0) \in H^1 \times H^1 \times L^2. \tag{3.5}$$

We multiply Eq. (2.1) by $-\bar{\psi}_n$ and take the imaginary part

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_n\|^2 + 2\alpha\|\psi_n\|^2 = 2\mathrm{Im}\int_{\mathbb{R}} f\bar{\psi}_n.$$

Integration of the above equation implies

$$\|\psi_n(t)\|^2 = e^{-2\alpha t}\|\psi_n(0)\|^2 + 2\,\mathrm{Im}\int_{\mathbb{R}} e^{-2\alpha(t-s)}(f,\bar{\psi}_n(s))\mathrm{d}s. \tag{3.6}$$

Following the same procedure we may obtain

$$\|\psi(t)\|^2 = e^{-2\alpha t}\|\psi(0)\|^2 + 2\,\mathrm{Im}\int_{\mathbb{R}} e^{-2\alpha(t-s)}(f,\bar{\psi}(s))\mathrm{d}s. \tag{3.7}$$

Because of convergence (3.5) the sequence $(\psi_{0,n},\phi_{0,n},\theta_{0,n})$ is bounded in $H^1\times H^1\times L^2$ and by Lemma 2.6 we obtain

$$\|(\psi_n,\phi_n,\theta_n)\|_{H^1\times H^1\times L^2}\le C,\quad \text{for all } 0\le t\le T,\ n=1,\ldots,\text{for some } T>0, \tag{3.8}$$

where $(\psi_n,\phi_n,\theta_n)=S(t)(\psi_{0,n},\phi_{0,n},\theta_{0,n})$. Taking into consideration Lemma 3.1 and the convergence (3.5) implies

$$(\psi_n,\phi_n,\theta_n)\to(\psi,\phi,\theta),\quad \text{weakly in } H^1\times H^1\times L^2. \tag{3.9}$$

Then using relation (3.5) and Eq. (3.7), the limit of Eq. (3.6) gives

$$\|\psi_n(t)\|^2\to\|\psi(t)\|^2,\quad \text{as } n\to\infty. \tag{3.10}$$

Let $p_n(t)=\psi_n(t)-\psi(t)$, $q_n(t)=\phi_n(t)-\phi(t)$, $G_n(t)=\theta_n(t)-\theta(t)$ then system (2.1)–(2.3) becomes

$$ip_{n,t}+\kappa p_{n,xx}+i\alpha p_n=q_n\psi_n+p_n\phi_n, \tag{3.11}$$
$$q_{n,t}+\delta q_n=G_n, \tag{3.12}$$
$$G_{n,t}+(\lambda-\delta)G_n-q_{n,xx}+(1-\delta(\lambda-\delta))q_n=-\mathrm{Re}\,p_{n,x}. \tag{3.13}$$

Multiply Eq. (3.11) by $-\bar{p}_{n,t}$ in $L^2$ integrate and take the real part

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\kappa\|p_{n,x}\|^2+\kappa\alpha\|p_{n,x}\|^2+\alpha\int_{\mathbb{R}}\phi_n|p_n|^2+\alpha\,\mathrm{Re}\int_{\mathbb{R}}q_n\psi_n p_n=-\mathrm{Re}\int_{\mathbb{R}}p_n\phi_n\bar{p}_{n,t}-\mathrm{Re}\int_{\mathbb{R}}q_n\psi_n\bar{p}_{n,t}. \tag{3.14}$$

But $p_t=i(\kappa p_{xx}+i\alpha p-p\phi-q\psi)$. Therefore substituting into (3.14) produces

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\kappa\|p_{n,x}\|^2+\kappa\alpha\|p_{n,x}\|^2=\kappa\,\mathrm{Im}\int_{\mathbb{R}}\bar{p}_n\phi p_{n,xx}-\mathrm{Im}\int_{\mathbb{R}}\bar{p}_n\phi_n q_n\psi_n+\kappa\,\mathrm{Im}\int_{\mathbb{R}}\bar{q}_n\bar{\psi}_n p_{n,xx}-\mathrm{Im}\int_{\mathbb{R}}\bar{q}_n\bar{\psi}_n p_n\phi_n. \tag{3.15}$$

Analyzing integrals $\mathrm{Im}\int_{\mathbb{R}}\bar{p}_n\phi_n p_{n,xx}$, $\mathrm{Im}\int_{\mathbb{R}}\bar{q}_n\bar{\psi}_n p_{n,xx}$ Eq. (3.14) becomes

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\kappa\|p_{n,x}\|^2+\kappa\alpha\|p_{n,x}\|^2 &= -\kappa\,\mathrm{Im}\int_{\mathbb{R}}\bar{p}_n\phi_{n,x}p_{n,x}+\mathrm{Im}\int_{\mathbb{R}}\bar{p}_n\phi_n q_n\psi_n\\ &\quad-\kappa\,\mathrm{Im}\int_{\mathbb{R}}\bar{q}_{n,x}\bar{\psi}_n p_{n,x}-\kappa\,\mathrm{Im}\int_{\mathbb{R}}\bar{q}_n\bar{\psi}_{n,x}p_{n,x}+\mathrm{Im}\int_{\mathbb{R}}\bar{q}_n\bar{\psi}_n p_n\phi_n.\end{aligned} \tag{3.16}$$

Next multiplying Eq. (3.13) by $G_n$, produces

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|G_n\|^2+\|q_{n,x}\|^2+(1-\delta(\lambda-\delta))\|q_n\|^2)+(\lambda-\delta)\|G_n\|^2+\delta\|q_{n,x}\|^2+\delta(1-\delta(\lambda-\delta))\|q_n\|^2\\ &\quad=-\mathrm{Re}\int_{\mathbb{R}}G_nq_{n,x}.\end{aligned} \tag{3.17}$$

Addition of the formulas $2\delta\times$ (3.16) and $2\times$ (3.17) implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|G_n\|^2+\|q_{n,x}\|^2+(1-\delta(\lambda-\delta))\|q_n\|^2+\kappa\delta\|p_{n,x}\|^2)\\ &\quad+2\kappa\alpha\delta\|p_{n,x}\|^2+2(\lambda-\delta)\|G_n\|^2+2\delta\|q_{n,x}\|^2+2\delta(1-\delta(\lambda-\delta))\|q_n\|^2\\ &=-2\kappa\delta\,\mathrm{Im}\int_{\mathbb{R}}\bar{p}_n\phi_{n,x}p_{n,x}+2\delta\,\mathrm{Im}\int_{\mathbb{R}}\bar{p}_n\phi_n q_n\psi_n-2\kappa\delta\,\mathrm{Im}\int_{\mathbb{R}}\bar{q}_{n,x}\bar{\psi}_n p_{n,x}-2\kappa\delta\,\mathrm{Im}\int_{\mathbb{R}}\bar{q}_n\bar{\psi}_{n,x}p_{n,x}\\ &\quad+2\delta\,\mathrm{Im}\int_{\mathbb{R}}\bar{q}_n\bar{\psi}_n p_n\phi_n-2\,\mathrm{Re}\int_{\mathbb{R}}G_nq_{n,x}.\end{aligned} \tag{3.18}$$

Majorizing the right-hand side of the above equation obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|G_n\|^2+\|q_{n,x}\|^2+(1-\delta(\lambda-\delta))\|q_n\|^2+\kappa\delta\|p_{n,x}\|^2)\\&\quad+2\kappa\alpha\delta\|p_{n,x}\|^2+2(\lambda-\delta)\|G_n\|^2+2\delta\|q_{n,x}\|^2+2\delta(1-\delta(\lambda-\delta))\|q_n\|^2\\&\quad\le 2\kappa\delta\|\phi_n\|_6\|p_n\|_3\|p_{n,x}\|+2\delta\|\phi_n\|_\infty\|\psi_n\|_\infty\|p_n\|\|q_n\|+2\kappa\delta\|\psi_n\|_\infty\|q_{n,x}\|\|p_{n,x}\|+2\kappa\delta\|q_n\|_\infty\|\psi_x\|\|p_x\|\\&\quad+2\delta\|\psi_n\|_\infty\|\phi_n\|_\infty\|q_n\|\|p_n\|+2\|G_n\|\|q_x\|.\end{aligned}\tag{3.19}$$

Because of the embeddings $H^1(\mathbb{R})\hookrightarrow L^6(\mathbb{R})$, $H^1(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$, $\|p_n\|_3\le C\|p_n\|^{1/2}\|p_n\|_{H^1}^{1/2}$ and Eq. (3.8), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|G_n\|^2+\|q_{n,x}\|^2+(1-\delta(\lambda-\delta))\|q_n\|^2+\kappa\|p_{n,x}\|^2)\le C\|p_n\|^2.\tag{3.20}$$

Gronwall' inequality implies

$$\begin{aligned}&\|G_n(t)\|^2+\|q_{n,x}(t)\|^2+(1-\delta(\lambda-\delta))\|q_n(t)\|^2+\kappa\delta\|p_{n,x}(t)\|^2\\&\quad\le\|G_n(0)\|^2+\|q_{n,x}(0)\|^2+(1-\delta(\lambda-\delta))\|q_n(0)\|^2+\kappa\delta\|p_{n,x}(0)\|^2+C\int_0^t\|p_n\|^2\mathrm{d}\tau.\end{aligned}\tag{3.21}$$

Taking into account the limit of the last inequality and the result (3.10) obtain

$$\|p_n(t),q_n(t),G_n(t)\|_{H^1\times H^1\times H}\to 0,\quad\text{as } n\to\infty.$$

Therefore

$$(\psi_n(t),\phi_n(t),\theta_n(t))\to(\psi(t),\phi(t),\theta(t)),\quad\text{as } n\to\infty,\ \text{in } H^1\times H^1\times H.\tag{3.22}$$

Hence it follows that, for all $0\le t\le T$,

$$S(t)(\psi_{0,n},\phi_{0,n},\theta_{0,n})\to S(t)(\psi_0,\phi_0,\theta_0),\quad\text{as } n\to\infty,\ \text{in } H^1\times H^1\times H,$$

where $(\psi_{0,n},\phi_{0,n},\theta_{0,n})\to(\psi_0,\phi_0,\theta_0)$ in $H^1\times H^1\times H$ and $(\psi_{0,n},\phi_{0,n},\theta_{0,n})\in(H^1\cap H^2)^2\times H^1$. In order to prove that the equation above holds when $(\psi_{0,n},\phi_{0,n},\theta_{0,n})\in H^1\times H^1\times H$ we follow similar argument as the one in [2, Theorem 3.1].

Since $(\psi_{0,n},\phi_{0,n},\theta_{0,n})\in(H^1\cap H^2)^2\times H^1$, by Lemma 2.6 we know that for fixed $n=1,\dots,$ $(\psi_n,\phi_n,\theta_n)\in L^\infty(0,T;H^1\cap H^2)^2\times(H^1)$. Therefore from Lemma 2.6 we have that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa\|\psi_{x,n}\|^2+\int_{\mathbb{R}}\phi_n|\psi_n|^2+2\,\mathrm{Re}\int_{\mathbb{R}}f\bar{\psi}_n\right)+\kappa\alpha\|\psi_{x,n}\|^2+\left(\alpha+\frac{\delta}{2}\right)\int_{\mathbb{R}}\phi_n|\psi_n|^2\\&\quad=\frac{1}{2}\int_{\mathbb{R}}\theta_n|\psi_n|^2+\alpha\,\mathrm{Re}\int_{\mathbb{R}}f\bar{\psi}_n.\end{aligned}\tag{3.23}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}F_1(\psi_n(t),\phi_n(t),\theta_n(t))+2\alpha F_1(\psi_n(t),\phi_n(t),\theta_n(t))=2H_1(\psi_n(t),\phi_n(t),\theta_n(t)).$$

Integration of the last formula implies

$$F_1(\psi_n(t),\phi_n(t),\theta_n(t))=e^{-2\alpha t}F_1(\psi_n(0),\phi_n(0),\theta_n(0))+\int_0^t e^{-2\alpha(t-s)}H_1(\psi_n(s),\phi_n(s),\theta_n(s))\mathrm{d}s.\tag{3.24}$$

Evaluating the (nonlinear) integrals of (3.23) we have

$$\begin{aligned}&\left|\int_{\mathbb{R}}\phi_n|\psi_n|^2-\int_{\mathbb{R}}\phi|\psi|^2\right|=\left|\int_{\mathbb{R}}(\phi_n-\phi)|\psi_n|^2-\int_{\mathbb{R}}\phi(|\psi_n|^2-|\psi|^2)\right|\\&\quad\le\|\phi_n-\phi\|\|\psi_n\|_{H^1}^2+\|\phi\|_{H^1}\|\psi_n-\psi\|(\|\psi_n\|_{H^1}+\|\psi\|_{H^1})\\&\quad\le C\|\phi_n-\phi\|+C\|\psi_n-\psi\|\to 0,\quad\text{as } n\to\infty\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}}\theta_n|\psi_n|^2-\int_{\mathbb{R}}\theta|\psi|^2\right|=\left|\int_{\mathbb{R}}(\theta_n-\theta)|\psi_n|^2-\int_{\mathbb{R}}\theta(|\psi_n|^2-|\psi|^2)\right|\\&\quad\le\|\theta_n-\theta\|\|\psi_n\|_{H^1}^2+\|\theta\|\|\psi_n-\psi\|_3(\|\psi_n\|_{H^1}+\|\psi\|_{H^1})\\&\quad\le C\|\theta_n-\theta\|+C\|\psi_n-\psi\|_3\to 0,\quad\text{as } n\to\infty,\end{aligned}$$

where

$$\|\psi_n-\psi\|_3\le C\|\psi_n-\psi\|^{1/2}\|\psi_n-\psi\|_{H_1}^{1/2}\le C\|\psi_n-\psi\|^{1/2}\to 0,\quad \text{as } n\to\infty.$$

From the convergence (3.5) we have that

$$F_1(\psi_n(0),\phi_n(0),\theta_n(0))\to F_1(\psi(0),\phi(0),\theta(0)),\quad \text{as } n\to\infty.$$

Taking into consideration the results above it also follows that

$$H_1(\psi_n(t),\phi_n(t),\theta_n(t))\to H_1(\psi(t),\phi(t),\theta(t)),\quad \text{for all } 0\le t\le T. \tag{3.25}$$

Finally, using (3.8), (3.25) and the Lebesgue dominated convergence theorem we get

$$\int_0^t e^{-2\alpha(t-s)}H_1(\psi_n(s),\phi_n(s),\theta_n(s))\mathrm{d}s\to\int_0^t e^{-2\alpha(t-s)}H_1(\psi(s),\phi(s),\theta(s))\mathrm{d}s, \tag{3.26}$$

as $n\to\infty$. Taking the limit of Eq. (3.24) we get

$$\kappa\|\psi_x\|^2+\int_{\mathbb{R}}\phi|\psi|^2+2\,\mathrm{Re}\int_{\mathbb{R}}f\bar{\psi}=e^{-2\alpha t}F_1(\psi_0,\phi_0,\theta_0)+\int_0^t e^{-2\alpha(t-s)}H_1(\psi(s),\phi(s),\theta(s))\mathrm{d}s, \tag{3.27}$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}F_1(\psi(t),\phi(t),\theta(t))+2\alpha F_1(\psi(t),\phi(t),\theta(t))=2H_1(\psi(t),\phi(t),\theta(t)). \tag{3.28}$$

On the other hand, since $(\psi_n,\phi_n,\theta_n)\in L^\infty(0,T;(H^1\cap H^2)^2\times H^1)$, for every $n=1,2,3,\ldots$, by Lemma 2.6 it can be deduced that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta_n\|^2+\|\phi_{n,x}\|^2+(1-\delta(\lambda-\delta))\|\phi_n\|^2)+2(\lambda-\delta)\|\theta_n\|^2+2\delta\|\phi_{n,x}\|^2+2\delta(1-\delta(\lambda-\delta))\|\phi_n\|^2\\&\quad=-2\,\mathrm{Re}\int_{\mathbb{R}}\theta\psi_{n,x}+2\int_{\mathbb{R}}g\theta_n.\end{aligned} \tag{3.29}$$

Then by the convergence (3.22), the formula $2\times$ (3.28) $+\,2\times$ (3.29) implies the energy equation (3.3). Now, since we have proved that

$$S(t)(\psi_{0,n},\phi_{0,n},\theta_{0,n})\to S(t)(\psi_0,\phi_0,\theta_0),\quad \text{as } n\to\infty,\ \text{in } H^1\times H^1\times H, \tag{3.30}$$

it remains to show that Eq. (3.29) holds when $(\psi_{0,n},\phi_{0,n},\theta_{0,n})\in H^1\times H^1\times H$. In order to prove this argument we follow similar steps with [2, Theorem 3.1]. □

**Remark 3.3.** To achieve the above results we had to overcome difficulties arising from the fact that our system has only one nonlinearity $(\phi\psi)$. Instead, Youkawa's and Zakharov's models are endowed with two nonlinearities, which "cancel" one another from the beginning of the above procedure.

**Theorem 3.4.** *Assume that $f$ and $g$ belong to $H^1$. The solutions $(\psi,\phi,\theta)\in C(\mathbb{R}^+,(H^1\cap H^2)^2\times H^1)$ of the problem (2.1)–(2.5) depend continuously on the initial data in $(H^1\cap H^2)^2\times H^1$ and it satisfies the energy equation*

$$F_2'(t)+\delta F_2(t)=H_2(t) \tag{3.31}$$

*where*

$$\begin{aligned}F_2&:=\kappa\|\psi_{xx}\|^2-2\,\mathrm{Re}\int_{\mathbb{R}}\phi\psi\bar{\psi}_{xx}+2\mathrm{Re}\int_{\mathbb{R}}f_x\bar{\psi}_x+\|\theta_x\|^2+\|\phi_{xx}\|^2+(1-\delta(\lambda-\delta))\|\phi_x\|^2,\\H_2&:=(\delta-2\kappa\alpha)\|\psi_{xx}\|^2+(3\delta-2\lambda)\|\theta_x\|^2-\delta\|\phi_{xx}\|^2-\delta(1-\delta(\lambda-\delta))\|\phi_x\|^2\\&\quad+4\alpha\,\mathrm{Re}\int_{\mathbb{R}}\phi\psi\bar{\psi}_{xx}-2\,\mathrm{Re}\int_{\mathbb{R}}\theta_x\psi_{xx}-2\,\mathrm{Re}\int_{\mathbb{R}}\theta\psi\bar{\psi}_{xx}\mathrm{d}x-2\,\mathrm{Im}\int_{\mathbb{R}}\phi^2\psi\bar{\psi}_{xx}+2(\delta-\alpha)\int_{\mathbb{R}}f_x\bar{\psi}_x+2\int_{\mathbb{R}}g_x\theta_x.\end{aligned}$$

**Proof.** Consider two solutions $(\psi_1,\phi_1,\theta_1)$ and $(\psi_2,\phi_2,\theta_2)$ of the problem (2.1)–(2.5), then the differences $\psi=\psi_2-\psi_1$, $\phi=\phi_2-\phi_1$, $\theta=\theta_2-\theta_1$ satisfy the following system

$$i\psi_t+\kappa\psi_{xx}+i\alpha\psi=\phi\psi_1+\phi_2\psi, \tag{3.32}$$

$$\phi_t+\delta\phi=\theta, \tag{3.33}$$

$$\theta_t+(\lambda-\delta)\theta-\phi_{xx}+(1-\delta(\lambda-\delta))\phi=-Re\psi_x. \tag{3.34}$$

Multiplying Eq. (3.32) by $\psi_{xx,t} + \alpha\psi_{xx}$, taking the real parts, and evaluating the terms get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|^2_{H^1\cap H^2} \leq C(\|\psi\|^2_{H^1\cap H^2} + \|\phi\|^2_{H^1\cap H^2}). \tag{3.35}$$

Next, multiplication of Eq. (3.34) by $-\theta_{xx}$ and integration gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta\|^2_{H^1} + \|\phi\|^2_{H^1\cap H^2} + (1-\delta(\lambda-\delta))\|\phi\|^2_{H^1}\right) \leq C(\|\theta\|^2_{H^1} + \|\phi\|^2_{H^1\cap H^2} + (1-\delta(\lambda-\delta))\|\phi\|^2_{H^1}). \tag{3.36}$$

By adding (3.35) and (3.36) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta\|^2_{H^1} + \|\phi\|^2_{H^1\cap H^2} + (1-\delta(\lambda-\delta))\|\phi\|^2_{H^1} + \|\psi\|^2_{H^1\cap H^2}\right)$$
$$\leq C(\|\theta\|^2_{H^1} + \|\phi\|^2_{H^1\cap H^2} + (1-\delta(\lambda-\delta))\|\phi\|^2_{H^1} + \|\psi\|^2_{H^1\cap H^2}).$$

Gronwall's lemma implies the continuity of the solutions in the space $(H^1\cap H^2)^2 \times H^1$ and therefore the energy equation (3.31) holds. □

## 4. Existence of a global attractor

The aim of this section is to prove the existence of a global attractor for the dynamical system $S(t)$ in the space $(H^1\cap H^2)^2 \times H^1$. First, it is necessary to prove the asymptotic compactness of the solutions.

**Theorem 4.1.** *Let $f, g$ belong to $L^2$. Then the dynamical system $S(t)$ is asymptotically compact in $H^1 \times H^1 \times L^2$. That is, if $(\psi_n, \phi_n, \theta_n)$ is bounded in $H^1 \times H^1 \times L^2$ and $t_n \to \infty$, then $S(t)(\psi_n, \phi_n, \theta_n)$ is precompact in the same space.*

**Proof.** Ideas developed in [9] will be used. Since the sequence $(\psi_n, \phi_n, \theta_n)$ is bounded, there exists $R$ such that $\|(\psi_n, \phi_n, \theta_n)\|_{H^1\times H^1\times L^2} \leq R$. Therefore Lemma 2.2 implies the existence of a constant $T(R)$ such that

$$S(t)(\psi_n, \phi_n, \theta_n) \in B_1, \quad \text{for all } t \geq T(R), \tag{4.1}$$

where $B_1$ is the absorbing set given by relation (2.11). Taking $t_n \to \infty$, there exists $N_1(R)$ such that if $n \geq N_1(R)$, then $t_n \geq T(R)$, and thereby

$$S(t_n)(\psi_n, \phi_n, \theta_n) \in B_1, \quad \text{for all } n \geq N_1(R). \tag{4.2}$$

Hence there exists $(\psi, \phi, \theta) \in B_1$ such that

$$S(t_n)(\psi_n, \phi_n, \theta_n) \to (\psi, \phi, \theta), \quad \text{as } n \to \infty, \text{ weakly in } H^1 \times H^1 \times L^2. \tag{4.3}$$

Since $t_n \to \infty$, for every $T > 0$, there exists $N_2(R, T)$ such that, when $n \geq N_2(R, T)$ one has $t_n - T \geq T(R)$. Therefore by relation (4.1)

$$S(t_n - T)(\psi_n, \phi_n, \theta_n) \in B_1 \quad \text{for all } n \geq N_2(R, T). \tag{4.4}$$

This implies that there exists $(\psi_T, \phi_T, \theta_T) \in B_1$ such that

$$S(t_n - T)(\psi_n, \phi_n, \theta_n) \to (\psi_T, \phi_T, \theta_T), \quad \text{as } n \to \infty, \text{ weakly in } H^1 \times H^1 \times L^2. \tag{4.5}$$

By relation (3.2) it follows that

$$S(T)(S(t_n - T)(\psi_n, \phi_n, \theta_n)) \to S(T)(\psi_T, \phi_T, \theta_T), \quad \text{as } n \to \infty, \text{ weakly in } H^1 \times H^1 \times L^2, \tag{4.6}$$

and from the uniqueness of the solution we get that

$$(\psi, \phi, \theta) = S(T)(\psi_T, \phi_T, \theta_T). \tag{4.7}$$

Now from relation (4.3) we have

$$\liminf_{n\to\infty} \|S(t_n)(\psi_n, \phi_n, \theta_n)\|_{H^1\times H^1\times L^2} \geq \|(\psi, \phi, \theta)\|_{H^1\times H^1\times L^2}. \tag{4.8}$$

Every solution satisfies Eq. (3.3), therefore for every $t \geq s \geq 0$, we find that

$$F_1(S(t)(\psi_0, \phi_0, \theta_0)) = e^{-\delta(t-s)}F_1(S(s)(\psi_0, \phi_0, \theta_0)) + \int_s^T e^{-\delta(t-\tau)}G_1(S(\tau)(\psi_0, \phi_0, \theta_0))\mathrm{d}\tau. \tag{4.9}$$

On the other hand from Eq. (4.7)

$$F_1(\psi,\phi,\theta)=F_1(S(T)(\psi_T,\phi_T,\theta_T))=e^{-\delta(T-T_0)}F_1(S(T_0)(\psi_T,\phi_T,\theta_T))+\int_{T_0}^{T}e^{-\delta(T-\tau)}G_1(S(\tau)(\psi_T,\phi_T,\theta_T))\mathrm{d}\tau. \quad (4.10)$$

Let $T_0(\epsilon)$ be a constant such that $T_0(\epsilon)\geq \max\{T(\epsilon),T(B_1)\}$. Let $S(t)(S(t_n-T)(\psi_n,\phi_n,\theta_n))$ be a solution of the system (2.1)–(2.5), then taking $T\geq T_0(\epsilon)$ we know that every solution satisfies (2.9) hence relations (4.9) will give for $n\geq N_2(R,T)$

$$\begin{aligned}F_1(S(t_n)(\psi_n,\phi_n,\theta_n)) &= e^{-\delta(T-T_0)}F_1(S(T_0)(S(t_n-T)(\psi_n,\phi_n,\theta_n)))\\ &\quad+\int_{T_0}^{T}e^{-\delta(T-\tau)}G_1(S(\tau)(S(t_n-T)(\psi_n,\phi_n,\theta_n)))\mathrm{d}\tau \end{aligned} \quad (4.11)$$

where $s=T_0$ and $t=T$. Next we treat each term. Since $T_0\geq T(B_1)$

$$e^{-\delta(T-T_0)}F_1(S(T_0)(S(t_n-T)(\psi_n,\phi_n,\theta_n)))\leq Ce^{-\delta(T-T_0)},\quad \text{for all } n\geq N_2(R,T),$$

where $C$ is independent of $T$. Analyzing the last term of (4.11) and taking into consideration Eq. (3.3) we have that

$$\begin{aligned}\int_{T_0}^{T}&e^{-\delta(T-\tau)}G_1(S(\tau)(S(t_n-T)(\psi_n,\phi_n,\theta_n)))\mathrm{d}\tau=(\delta-2\kappa\alpha)\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)((S(t_n-T)\psi_{x,n}))\|^2\\ &-2\alpha\int_{T_0}^{T}e^{-\delta(T-\tau)}\int_{\mathbb{R}}S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T)\psi_n))|^2\\ &+(3\delta-2\lambda)\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)(S(t_n-T)\theta_n)\|^2-\delta(1-\delta(\lambda-\delta))\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)(S(t_n-T)\phi_n)\|^2\\ &-\delta\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)(S(t_n-T)\phi_{x,n})\|^2+\int_{T_0}^{T}e^{-\delta(T-\tau)}\int_{\mathbb{R}}S(\tau)(S(t_n-T)\theta_n)|S(\tau)(S(t_n-T))\psi_n|^2\\ &+2(\delta-\alpha)\int_{T_0}^{T}e^{-\delta(T-\tau)}\mathrm{Re}\int_{\mathbb{R}}f\overline{S(\tau)(S(t_n-T)\psi_n)}\\ &-2\int_{T_0}^{T}e^{-\delta(T-\tau)}\mathrm{Re}\int_{\mathbb{R}}S(\tau)(S(t_n-T)\theta_n)S(\tau)(S(t_n-T)\psi_{x,n})+2\int_{T_0}^{T}e^{-\delta(T-\tau)}\int_{\mathbb{R}}gS(\tau)(S(t_n-T)\theta_n).\end{aligned} \quad (4.12)$$

By convergence relations (3.2) and (4.5) we have, as $n\to\infty$

$$\begin{aligned}&e^{-\delta(T-\tau)}S(\tau)((S(t_n-T)\psi_n))\to e^{-\delta(T-\tau)}S(\tau)\psi_T,\quad \text{weakly in } L^2(T_0,T;H^1),\\ &e^{-\delta(T-\tau)}S(\tau)((S(t_n-T)\theta_n))\to e^{-\delta(T-\tau)}S(\tau)\theta_T,\quad \text{weakly in } L^2(T_0,T;H),\\ &e^{-\delta(T-\tau)}S(\tau)((S(t_n-T)\phi_n))\to e^{-\delta(T-\tau)}S(\tau)\phi_T,\quad \text{weakly in } L^2(T_0,T;H^1).\end{aligned}$$

Therefore

$$\liminf_{n\to\infty}\|e^{-\delta(T-\tau)}S(\tau)(S(t_n-T)\psi_n)\|_{L^2(T_0,T;H^1)}\geq\|e^{-\delta(T-\tau)}S(t)\psi_\tau\|_{L^2(T_0,T;H^1)}, \quad (4.13)$$

$$\liminf_{n\to\infty}\|e^{-\delta(T-\tau)}S(\tau)(S(t_n-T)\theta_n)\|_{L^2(T_0,T;H)}\geq\|e^{-\delta(T-\tau)}S(t)\theta_\tau\|_{L^2(T_0,T;H)}, \quad (4.14)$$

$$\liminf_{n\to\infty}\|e^{-\delta(T-\tau)}S(\tau)(S(t_n-T)\phi_n)\|_{L^2(T_0,T;H^1)}\geq\|e^{-\delta(T-\tau)}S(t)\phi_\tau\|_{L^2(T_0,T;H^1)}. \quad (4.15)$$

Then, taking $\delta$ small enough and taking into consideration that the following inequalities hold $\delta-2\kappa\alpha<0$, $3\delta-2\lambda<0$, $1-\delta(\lambda-\delta)>0$ produces

$$\begin{aligned}&\limsup_{n\to\infty}(\delta-2\kappa\alpha)\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)((S(t_n-T)\psi_{x,n}))\|^2\\ &\quad+\limsup_{n\to\infty}(3\delta-2\lambda)\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)(S(t_n-T)\theta_n)\|^2\\ &\quad+\limsup_{n\to\infty}-\delta(1-\delta(\lambda-\delta))\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)(S(t_n-T)\phi_n)\|^2\\ &\quad+\limsup_{n\to\infty}-\delta\int_{T_0}^{T}e^{-\delta(T-\tau)}\|S(\tau)(S(t_n-T)\phi_{x,n})\|^2\end{aligned}$$

$$\le \limsup_{n\to\infty} \int_{T_0}^{T} e^{-\delta(T-\tau)}((\delta - 2\kappa\alpha)\|S(\tau)\psi_{x,T}\|^2 + (3\delta - 2\lambda)\|S(\tau)\theta_T\|^2 - \delta(1-\delta(\lambda-\delta))\|S(\tau)\phi_T\|^2 - \delta\|S(\tau)\phi_{x,T}\|^2). \tag{4.16}$$

Following the same arguments we also have

$$2(\delta-\alpha)\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{Re}\int_{\mathbb{R}} f\overline{S(\tau)(S(t_n-T)\psi_n)} + 2\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{\mathbb{R}} gS(\tau)(S(t_n-T)\theta_n) \to 2(\delta-\alpha)\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{Re}\int_{\mathbb{R}} f\overline{S(\tau)\psi_T} + 2\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{\mathbb{R}} gS(\tau)\theta_T. \tag{4.17}$$

The next step is to evaluate the nonlinear terms of Eq. (4.12). In order to overcome the difficulty of the noncompactness of Sobolev embedding in $\mathbb{R}$ we will approach the whole space by bounded intervals. That is

$$\begin{aligned}&\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{\mathbb{R}} S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T)\psi_n))|^2 \mathrm{d}\tau\,\mathrm{d}x \\ &= \int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\ge m} S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T))\psi_n)|^2 \mathrm{d}\tau\,\mathrm{d}x \\ &\quad + \int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\le m} S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T)\psi_n))|^2 \mathrm{d}\tau\,\mathrm{d}x.\end{aligned} \tag{4.18}$$

For a given $\epsilon > 0$, it follows from estimate (2.12) that, for all $n \ge N_2(R,T)$ and $m \ge M(\epsilon)$,

$$\begin{aligned}&\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\ge m} S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T)\psi_n))|^2 \mathrm{d}\tau\,\mathrm{d}x \\ &\le \int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\left(\int_{|x|\ge m}|S(\tau)(S(t_n-T)\phi_n)|^3\mathrm{d}x\right)^{1/3} \\ &\quad\times\left(\int_{|x|\ge m}|S(\tau)((S(t_n-T)\psi_n))|^6\mathrm{d}x\right)^{1/6}\left(\int_{|x|\ge m}|S(\tau)((S(t_n-T)\psi_n))|^2\mathrm{d}x\right)^{1/2} \\ &\le \epsilon\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\left(\int_{\mathbb{R}}|S(\tau)(S(t_n-T)\phi_n)|^3\mathrm{d}x\right)^{1/3}\left(\int_{\mathbb{R}}|S(\tau)((S(t_n-T))\psi_n)|^6\mathrm{d}x\right)^{1/6} \\ &\le \epsilon\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\,\|S(\tau)(S(t_n-T)\phi_n)\|_{H^1}\|S(\tau)(S(t_n-T)\psi_n)\|_{H^1} \\ &\le \frac{\epsilon C}{\delta}, \quad \text{for all } n \ge N_2(R,T),\ m \ge M(\epsilon).\end{aligned}$$

Next, we deal with the second term of the right-hand side of Eq. (4.18). By convergence Eqs. (3.2) and (4.5) we have, for every fixed $\tau \in [T_0, T]$,

$$S(\tau)(S(t_n-T)\psi_n), S(\tau)(S(t_n-T)\phi_n), S(\tau)(S(t_n-T)\theta_n) \to S(\tau)(\psi_T,\phi_T,\theta_T), \quad \text{weakly in } H^1\times H^1\times H. \tag{4.19}$$

Let $\Omega_m = \{x \in \mathbb{R} : |x| \le m\}$. Then

$$S(\tau)(S(t_n-T)\psi_n), S(\tau)(S(t_n-T)\phi_n), S(\tau)(S(t_n-T)\theta_n) \to S(\tau)(\psi_T,\phi_T,\theta_T) \quad \text{strongly in } L^2(\Omega_m)\times L^2(\Omega_m). \tag{4.20}$$

Eq. (2.7), relation (4.4) and the Lebesgue dominated convergence theorem imply

$$\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\le m} S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T)\psi_n))|^2 \mathrm{d}\tau\,\mathrm{d}x \to \int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\le m} S(\tau)\phi_T|S(\tau)\psi_T|^2\mathrm{d}\tau\,\mathrm{d}x. \tag{4.21}$$

Following similar arguments we get

$$\begin{aligned}&\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{\mathbb{R}} S(\tau)(S(t_n-T)\theta_n)|S(\tau)(S(t_n-T))\psi_n|^2\mathrm{d}\tau\,\mathrm{d}x\\
&=\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\geq m} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T)\psi_n))|^2\mathrm{d}\tau\,\mathrm{d}x\\
&\quad+\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\leq m} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T))\psi_n)|^2\mathrm{d}\tau\,\mathrm{d}x. \qquad (4.22)\end{aligned}$$

Hence

$$\begin{aligned}&\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\geq m} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T)\psi_n))|^2\mathrm{d}\tau\,\mathrm{d}x\\
&\leq C\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\left(\int_{|x|\geq m}|S(\tau)(S(t_n-T)\theta_n)|^2\mathrm{d}x\right)^{1/2}\\
&\quad\times\left(\int_{|x|\geq m}|S(\tau)((S(t_n-T)\psi_{x,n}))|^2\mathrm{d}x\right)\left(\int_{|x|\geq m}|S(\tau)((S(t_n-T)\psi_n))|^2\mathrm{d}x\right)^2\\
&\leq C\epsilon\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\left(\int_{|x|\geq m}|S(\tau)(S(t_n-T)\theta_n)|^2\mathrm{d}x\right)^{1/2}\left(\int_{|x|\geq m}|S(\tau)((S(t_n-T)\psi_{x,n}))|^2\mathrm{d}x\right)^2\\
&\leq C\epsilon\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\,\|S(\tau)(S(t_n-T)\theta_n)\|\,\|S(\tau)(S(t_n-T)\psi_n)\|_{H^1}^{1/2}\\
&\leq\frac{\epsilon C}{\delta},\quad\text{for all } n\geq N_2(R,T),\ m\geq M(\epsilon),\end{aligned}$$

and

$$\begin{aligned}&\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\leq m} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T)\psi_n))|^2\mathrm{d}\tau\,\mathrm{d}x\\
&\quad\to\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\leq m} S(\tau)\theta_T|S(\tau)\psi_T|^2\mathrm{d}\tau\,\mathrm{d}x.\end{aligned}$$

Also

$$\begin{aligned}&\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{Re}\int_{\mathbb{R}} S(\tau)(S(t_n-T)\theta_n)S(\tau)(S(t_n-T)\psi_{x,n})\mathrm{d}\tau\,\mathrm{d}x\\
&=\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\geq m} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T)\psi_{x,n}))|^2\mathrm{d}\tau\,\mathrm{d}x\\
&\quad+\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\leq m} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T)\psi_{x,n}))|^2\mathrm{d}\tau\,\mathrm{d}x. \qquad (4.23)\end{aligned}$$

Similarly, for the last nonlinear term, we obtain

$$\begin{aligned}&\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{|x|\geq m} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T)\psi_{x,n}))|^2\mathrm{d}\tau\,\mathrm{d}x\\
&\quad\leq\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\left(\int_{|x|\geq m}|S(\tau)(S(t_n-T)\theta_n)|^2\mathrm{d}x\right)^{1/2}\left(\int_{|x|\geq m}|S(\tau)((S(t_n-T)\psi_{x,n}))|^2\mathrm{d}x\right)^{1/2}\\
&\quad\leq\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{d}\tau\,\|S(\tau)(S(t_n-T)\theta_n)\|\,\|S(\tau)(S(t_n-T)\psi_{x,n})\|_{H^1}\\
&\quad\leq\frac{\epsilon C}{\delta},\quad\text{for all } n\geq N_2(R,T),\ m\geq M(\epsilon).\end{aligned}$$

Therefore

$$\limsup_{n\to\infty}\int_{T_0}^{T} e^{-\delta(T-\tau)}\left(\int_{\mathbb{R}} S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T))\psi_n)|^2\mathrm{d}\tau\,\mathrm{d}x\right.$$

$$
\begin{aligned}
&+\int_{\mathbb{R}} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T)\psi_n))|^2 \mathrm{d}\tau \mathrm{d}x \\
&+\int_{\mathbb{R}} S(\tau)((S(t_n-T)\theta_n))|S(\tau)((S(t_n-T))\psi_{x,n})|^2 \mathrm{d}\tau \mathrm{d}x\Big) \\
&\le \epsilon C+\int_{T_0}^{T} e^{-\delta(T-\tau)}\Big(\int_{|x|\le m} S(\tau)\phi_T|S(\tau)\psi_T|^2 \mathrm{d}\tau \mathrm{d}x \\
&+\int_{|x|\le m} S(\tau)\theta_T|S(\tau)\psi_T|^2 \mathrm{d}\tau \mathrm{d}x+\int_{|x|\le m} S(\tau)\theta_T|S(\tau)\psi_{x,T}|^2 \mathrm{d}\tau \mathrm{d}x\Big).
\end{aligned}
\tag{4.24}
$$

Hence

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_{T_0}^{T} e^{-\delta(T-\tau)}\Big(\int_{\mathbb{R}} S(\tau)((S(t_n-T)\phi_n))|S(\tau)((S(t_n-T)\psi_n))|^2 \mathrm{d}\tau \mathrm{d}x \\
&+\int_{\mathbb{R}} S(\tau)((S(t_n-T)\theta_n))|S(\tau)(S(t_n-T)\psi_n)|^2 \mathrm{d}\tau \mathrm{d}x \\
&+\int_{\mathbb{R}} S(\tau)(S(t_n-T)\theta_n)|S(\tau)((S(t_n-T)\psi_{x,n}))|^2 \mathrm{d}\tau \mathrm{d}x\Big) \\
&\le \epsilon C+\int_{T_0}^{T} e^{-\delta(T-\tau)}\Big(\int_{\mathbb{R}} S(\tau)\phi_T|S(\tau)\psi_T|^2 \mathrm{d}\tau \mathrm{d}x \\
&+\int_{\mathbb{R}} S(\tau)\theta_T|S(\tau)\psi_T|^2 \mathrm{d}\tau \mathrm{d}x+\int_{\mathbb{R}} S(\tau)\theta_T|S(\tau)\psi_{x,T}|^2 \mathrm{d}\tau \mathrm{d}x\Big), \quad \text{as } m\to\infty.
\end{aligned}
$$

Therefore Eq. (4.12) becomes

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_{T_0}^{T} e^{-\delta(T-\tau)} G_1(S(\tau)(S(t_n-T)(\psi_n,\phi_n,\theta_n)))\mathrm{d}\tau \\
&\le (\delta-2\kappa\alpha)\int_{T_0}^{T} e^{-\delta(T-\tau)}\|S(\tau)\psi_{x,T}\|^2-2\alpha\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{\mathbb{R}} S(\tau)\phi_T|S(\tau)\psi_T|^2 \\
&+(3\delta-2\lambda)\int_{T_0}^{T} e^{-\delta(T-\tau)}\|S(\tau)\theta_T\|^2-\delta(1-\delta(\lambda-\delta))\int_{T_0}^{T} e^{-\delta(T-\tau)}\|S(\tau)\phi_T\|^2 \\
&-\delta\int_{T_0}^{T} e^{-\delta(T-\tau)}\|S(\tau)\phi_{x,T}\|^2+\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{\mathbb{R}} S(\tau)\theta_T|S(\tau)\psi_T|^2 \\
&+2(\delta-\alpha)\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{Re}\int_{\mathbb{R}} f\overline{S(\tau)\psi_T}-2\int_{T_0}^{T} e^{-\delta(T-\tau)}\mathrm{Re}\int_{\mathbb{R}} S(\tau)\theta_T S(\tau)\psi_{x,T} \\
&+2\int_{T_0}^{T} e^{-\delta(T-\tau)}\int_{\mathbb{R}} gS(\tau)\theta_T+\epsilon C=\int_{T_0}^{T} e^{-\delta(T-\tau)} G_1(S(\tau)(\psi_T,\phi_T,\theta_T))\mathrm{d}\tau+\epsilon C.
\end{aligned}
\tag{4.25}
$$

We finally obtain

$$
\limsup_{n\to\infty} F_1(S(t_n)(\psi_n,\phi_n,\theta_n))\le Ce^{-\delta(T-T_0)}+\int_{T_0}^{T} e^{-\delta(T-\tau)} G_1(S(\tau)(\psi_T,\phi_T,\theta_T))\mathrm{d}\tau+\epsilon C. \tag{4.26}
$$

But using Eq. (4.10) and the above results we get

$$
\limsup_{n\to\infty} F_1(S(t_n)(\psi_n,\phi_n,\theta_n))\le F_1(\psi,\phi,\theta)+Ce^{-\delta(T-T_0)}-e^{-\delta(T-T_0)} G_1(S(T_0)(\psi_T,\phi_T,\theta_T))+\epsilon C. \tag{4.27}
$$

Hence, since $(\psi_T,\phi_T,\theta_T)\in B$ and $T_0\ge T(B)$ we have

$$
\limsup_{n\to\infty} F_1(S(t_n)(\psi_n,\phi_n,\theta_n))\le F_1(\psi,\phi,\theta)+Ce^{-\delta(T-T_0)}+\epsilon C. \tag{4.28}
$$

Taking the limit as $T\to\infty$ and $\epsilon\to 0$ we obtain

$$
F_1(S(t_n)(\psi_n,\phi_n,\theta_n))\le F_1(\psi,\phi,\theta). \tag{4.29}
$$

Finally, in order to give an estimate of the norm of the solutions in $H^1\times H^1\times H$ we need to evaluate the nonlinear terms of $F$ of Eq. (3.4). The rest of the proof follows similar steps as those in Theorem 4.1 of [2]. □

**Remark 4.2.** Remark 3.3 applies also in the proof of the above Theorem 4.1.

**Theorem 4.3.** *Assume that $f$ and $g$ belong to $H^1(\Omega)$. Then, the dynamical system $S(t)$ is asymptotically compact in $(H^1 \cap H^2)^2 \times H^1$ that is, if $(\psi_n, \phi_n, \theta_n)$ is bounded in $(H^1 \cap H^2)^2 \times H^1$ and $t_n \to \infty$, then $S(t)(\psi_n, \phi_n, \theta_n)$ is precompact in the same space.*

**Proof.** The proof is omitted as it follows the ideas of Theorem 3.4. □

For the completion of the work it is necessary to state the following result [see [10]].

**Proposition 4.4.** *Assume that $X$ is a metric space and $\{S(t)\}_{t\geq 0}$ is a semigroup of continuous operators in $X$. If $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set and is asymptotically compact, then $\{S(t)\}_{t\geq 0}$ possesses a global attractor which is a compact invariant set and attracts every bounded set in $X$.*

**Theorem 4.5.** *Assume that $f$ and $g$ belong to $H^1(\Omega)$. Then the problem (2.1)–(2.5) possesses a strong compact global attractor in $(H^1 \cap H^2)^2 \times H^1$, which is a compact invariant subset and attracts every bounded set of $(H^1 \cap H^2)^2 \times H^1$, with respect to the norm topology.*

**Proof.** Taking into consideration the asymptotic compactness of $S(t)$ in the space $(H^1 \cap H^2)^2 \times H^1$, Theorem 4.3 and Proposition 4.4, the proof is completed. □

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