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GLOBAL ATTRACTOR FOR SOME WAVE EQUATIONS OF $p-$ AND $p(x)-$LAPLACIAN TYPE

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Abstract. We study the existence of solutions for the equation $u_{tt} - \Delta_{p(x)}u - \Delta u_t + g(u) = f(x,t),\ x \in \Omega$ (bounded) $\subset \mathbb{R}^n,\ t > 0$ in both the isotropic case ($p(x) \equiv p$, a constant) and the anisotropic case ($p(x)$ a measurable function). Furthermore, in the isotropic case we obtain results concerning the asymptotic behavior of solutions. Since uniqueness for this type of problem seems rather difficult, a method implementing generalized semiflows is being used to prove the existence of a global attractor in the phase space $W_0^{1,p}(\Omega) \times L^2(\Omega)$, when $p \geq n$.

1. Introduction

The study of quasilinear wave equations involving the $p-$Laplacian operator in the principal part and viscosity damping originates from the nonlinear Voight model of longitudinal motion of a rod made from a viscoelastic material. Specifically, for the so-called Ludwick materials, it can be shown that they obey the following equations, under the effect of an external force f. For the Euler rod

$$\rho\frac{\partial^2 u}{\partial t^2} = K\frac{\partial}{\partial x}\Big(\Big|\frac{\partial u}{\partial x}\Big|^{n-1}\frac{\partial u}{\partial x}\Big) + f,$$

for the Euler beam

$$\rho I'\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\Big(KI_n\Big|\frac{\partial^2 u}{\partial x^2}\Big|^{n-1}\frac{\partial^2 u}{\partial x^2}\Big) + f,$$

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and finally, for the plate

$$\rho h \frac{\partial^2 u}{\partial t^2} = d_n \Big[\frac{\partial^2}{\partial x^2} \Big(|D(u)|^{n-1} \Big(\frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} \Big) \Big) +$$

$$\frac{\partial^2}{\partial y^2} \Big(|D(u)|^{n-1} \Big(\frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} \Big) \Big) + (1-\nu) \frac{\partial^2}{\partial x \partial y} \Big(|D(u)|^{n-1} \frac{\partial^2 u}{\partial x \partial y} \Big) \Big] + f,$$

where ρ is the density of the material, A the cross-sectional area of the rod, h the thickness of the plate, d_n a characteristic parameter of the material and

$$D(u) = \sqrt{ \Big(\frac{\partial^2 u}{\partial x^2} \Big)^2 + \Big(\frac{\partial^2 u}{\partial x \partial y} \Big)^2 + \Big(\frac{\partial^2 u}{\partial y^2} \Big)^2 }$$

(see [18]). Motivated by these models, Biazutti studied thoroughly an abstract evolution wave equation in a bounded domain, applying the existence and uniqueness results in examples of PDEs, when the principal part is of $p-$Laplacian type and of Kirchhoff type. She generalized previous results by Yamada, Zuazua and Dinh, concerning specific quasilinear wave equations (see [4] and the references therein). However, the uniqueness argument of the evolution equation is based on a rather special assumption on the principal part and cannot be applied in our case.

To obtain uniqueness results one must either consider specific nonlinearities, or reduce the equation to a semilinear one. Implementing nonlinear dissipative terms, principal parts and viscous damping make it difficult for one to obtain any uniqueness result. As a result, existence of attractors has barely been studied in the above context, since the classical semigroup theory for dynamical systems demands a unique solution. Biazutti's work initiated a sequence of papers in such hyperbolic equations, where, in general, uniqueness is not present. Ma and Soriano in [11] study the existence and exponential decay of an evolution equation with a $p-$Laplacian principal part and viscous damping. Park et al., in [15], extend the work of Ma and Soriano, by assuming a discontinuous nonlinearity. On the other hand, Melnik and Valero, in [13], deal with parabolic $p-$Laplacian inclusions, using multivalued semiprocesses.

An attractor is obtained through that method and it seems that, using the theory of multivalued functions, one can study the existence of attractors in equations with nonuniqueness. Another method, presented by Chepyzhov and Vishik in [6], using the so-called trajectory attractors, can be applied in equations with nonuniqueness, but it is quite complicated and practically inapplicable in quasilinear equations with complicated terms. Ball in [1] deploys another method, using the notion of a generalized semiflow, an extension of the classical single-valued semigroup, to obtain a global attractor. However, his method is based on estimates of the system as a whole and, as a result, it is also inapplicable in cases where the presence of nonlinearities is quite strong. In his most recent paper [2] though, he develops a method extending ideas from the classical semigroup decomposition to the generalized semiflow theory. This is the method that we will use in the present work. Furthermore, we study the equivalent problem with the $p(x)-$Laplacian operator

$$-\Delta_{p(x)}u = -\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big).$$

Such a generalization models the same mechanical phenomena, involving structures made of anisotropic materials, rather than making the assumption that the various parameters of the system are space-invariant. The study of the corresponding Lebesgue - Sobolev spaces is quite old (see [7] and references therein), but there is still little knowledge concerning elliptic and evolutionary problems (see [5], [8] and references therein).

The rest of the paper is organized as follows. In Section 2, first we state an existence theorem for the (isotropic) $p-$Laplacian problem. Next, we deploy the theorems needed to apply our method and do the necessary estimations. Then, we use the generalized semiflow theorems to obtain the existence of the global attractor. Finally, in Section 3, an existence result is presented for the (anisotropic) $p(x)-$Laplacian problem.

2. The Isotropic Case

2.1. **Existence of Solutions. The Generalized Semiflow.** The $p-$Laplacian operator

$$-\Delta_p = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big), \quad p > 1$$

is nonlinear, monotone, bounded and hemicontinuous from the space $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Moreover, if $u(t) \in W_0^{1,p}(\Omega)$, the following relations are true (e.g., see the references [10], [11], [15]):

$$\langle -\Delta_p u(t), u'(t)\rangle = \frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p \qquad \text{and} \qquad \langle -\Delta_p u(t), u(t)\rangle = \|\nabla u(t)\|_p^p.$$

Let Ω be a bounded subset of $\mathbb{R}^n$. Then the problem under consideration may be stated as follows:

$$\begin{aligned} &u''(t) - \Delta_p u(t) - \Delta u'(t) + g(u) = f(x,t), \quad x \in \Omega,\ t > 0,\ p \geq n,\\ &u(x,0) = u_0(x),\ u'(x,0) = u_1(x), \quad x \in \Omega, \end{aligned} \tag{2.1}$$

$$u(x,t)=0, \quad x\in\partial\Omega,\ t>0,\ \text{where } (u_0,u_1)\in W_0^{1,p}(\Omega)\times L^2(\Omega).$$

Also, for the functions g, f we assume that g is continuous and for each $b>0$ there exists a $C_b>0$, such that

$$|g(s)|\le C_b\, exp(b|s|^{\frac{p}{p-1}}), \quad \text{for all } s\in\mathbb{R}, \qquad \text{and} \tag{2.2}$$

$$f\in L^2((0,\infty);L^2(\Omega)). \tag{2.3}$$

Under the above assumptions, the following existence result is well known.

Theorem 2.1. *Let $p=n$. Then there exists a function $u:\ (0,\infty)\times\Omega\to\mathbb{R}$, where $u\in L^\infty((0,\infty);W_0^{1,p}(\Omega))$ and $u'\in L^\infty((0,\infty);L^2(\Omega))\cap L^2((0,\infty);W_0^{1,p}(\Omega))$, satisfying system* (2.1).

Proof. For the proof we refer to [11]. □

Uniqueness results are not known for the problem (2.1), even in the special case of $p=n$. Now we proceed to prove the analogue of the above result for the more general case $p\ge n$. To this end one needs the following estimate.

Lemma 2.2. *Let $p\ge n$ and $u\in W_0^{1,p}(\Omega)$. Then the following inequality is true:*

$$\int_\Omega exp\Big(a|u|^{\frac{p}{p-1}}\Big)dx<\infty.$$

Proof. (i) $(p=n)$. This case is the well-known Trudinger's inequality (see [3]).

(ii) $(p>n)$. Since $u\in W_0^{1,p}(\Omega)$ Morrey's theorem implies that $u\in L^\infty(\Omega)$. Thus, $u\in L^{\frac{pn}{p-1}}(\Omega)$. Then, the following estimates are implied by Orlicz' inequality (see [3]):

$$\begin{aligned}
&\int_\Omega \exp\left(a|u|^{\frac{p}{p-1}}\right)dx\le c\cdot\exp\left(a\int_\Omega|u|^{\frac{pn}{p-1}}dx+b\int_\Omega\left|\nabla|u|^{\frac{p}{p-1}}\right|^n dx\right)\\
&\le c\cdot\exp\left(a\int_\Omega|u|^{\frac{pn}{p-1}}dx+\frac{bp^n}{(p-1)^n}\int_\Omega\left||u|^{\frac{1}{p-1}}\frac{u}{|u|}\nabla u\right|^n dx\right)\\
&= c\cdot\exp\left(a\int_\Omega|u|^{\frac{pn}{p-1}}dx+\frac{bp^n}{(p-1)^n}\int_\Omega|u|^{\frac{n}{p-1}}|\nabla u|^n dx\right)\\
&\le c\cdot\exp\left(a\int_\Omega|u|^{\frac{pn}{p-1}}dx+\frac{bp^n}{(p-1)^n}\left(\text{ess sup}_{x\in\Omega}|u|^{\frac{n}{p-1}}\int_\Omega|\nabla u|^n dx\right)\right)<\infty
\end{aligned}$$

and this completes the proof. □

Lemma 2.3. *Suppose $p\ge n$, g satisfies hypothesis* (2.2) *and $\{u_m\}$ is a bounded sequence in $L^\infty((0,T);\ W_0^{1,p}(\Omega))$. If $u_m\to u$ almost everywhere* *in* $(0,T)\times\Omega$, *for some* $u\in L^\infty((0,T);W_0^{1,p}(\Omega))$, *then* $g(u_m)\rightharpoonup g(u)$ *in* $L^2((0,T);L^2(\Omega))$.

Proof. By Lemma 2.2 the boundedness of $\{u_m\}$ implies that $\{g(u_m)\}$ is a bounded sequence in $L^2((0,T);L^2(\Omega))$. In addition, since $u_m\to u$, almost everywhere in $(0,T)\times\Omega$, it follows from the continuity of g that $g(u_m)\to g(u)$, almost everywhere in $(0,T)\times\Omega$. Then, the proof is completed by a result from [10, Lemma 1.3]. □

The main existence result is now stated as follows.

Theorem 2.4. *Let hypotheses* (2.2)-(2.3) *be satisfied and* $p>n$. *Then, there exists a function* $u:(0,\infty)\times\Omega\to\mathbb{R}$, *with* $u\in L^\infty((0,\infty);W_0^{1,p}(\Omega))$ *and* $u'\in L^\infty((0,\infty);L^2(\Omega))\cap L^2((0,\infty);W_0^{1,2}(\Omega))$, *satisfying system* (2.1).

Proof. Define $G(u)=\int_0^u g(u)du$. Using Fubini's theorem and Lemma 2.2, one can show that $G(u)\in W_0^{1,p}(\Omega)$, for all $u\in W_0^{1,p}(\Omega)$. Next, one may follow the standard Galerkin scheme. For any integer $r>\frac{p}{2}$, the embedding $W_0^{r,2}(\Omega)\hookrightarrow W_0^{1,p}(\Omega)$ is continuous. Let $\{w_1,w_2,...\}$ be an orthonormal basis of $W_0^{r,2}(\Omega)$. For any $m\in\mathbb{N}$, consider the space $V_m=span\{w_1,w_2,...,w_m\}$. Since $W_0^{r,2}(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, one may choose two sequences $\{u_{0m}\}$ and $\{u_{1m}\}$ in V_m, such that they converge strongly to the initial data of the equation under consideration. Thus, the approximate o.d.e. system in the spaces V_m, can be defined and is known to possess a local solution $u_m(t)$ in some interval $[0,t_m)$, $0<t_m<T$. Next, following the ideas presented in [11], multiply the o.d.e. system by $u_m'(t)$ and integrate over $(0,t)$ to obtain

$$\frac{1}{2}\|u_n'(t)\|_2^2+\frac{1}{p}\|\nabla u_n(t)\|_p^p+\int_0^t\|\nabla u_n'(s)\|_2^2ds+\frac{1}{2}\int_\Omega G(u_n(t))dx\le C. \quad (2.4)$$

Using Lemma 2.2 and the fact that the $p-$Laplacian is a bounded operator, the approximate solutions can be extended to the whole [0,T] and the sequences $\{u_m\}$, $\{u_m'\}$, $\{-\Delta_p u_m\}$ are bounded in the appropriate spaces. Moreover, using a standard projection argument on the approximated equation, we obtain that $\{u_m''\}$ is bounded in $L^2((0,T);W^{-r,2}(\Omega))$. Applying the Aubin-Lions compactness lemma, one sees that $u_m\to u$ and $u_m'\to u'$, strongly in $L^2((0,T);L^2(\Omega))$. Moreover, $-\Delta_p u_m\overset{*}{\rightharpoonup}\xi$ (weak-$*$) in $L^\infty((0,T);W^{-1,p'}(\Omega))$. So, from Lemma 2.2, $g(u_m)\rightharpoonup g(u)$. Thus, passing to the limit of the approximated o.d.e. problem and using monotonicity arguments, it follows that $\xi=-\Delta_p u$. Moreover, the above estimates are independent of T, so the solutions can be extended to the whole interval $(0,+\infty)$. □

Define the energy functional of problem (2.1) as

$$E(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \int_\Omega G\big(u(t)\big)dx.$$

Then the next energy decay result is true.

Theorem 2.5. *Assume that $g(s)s - G(s) > 0$, for all $s \in \mathbb{R}$, and the function $f(x,t)$ decays to zero as $t \to \infty$. Then, the solutions of the problem* (2.1) *satisfy the following decay properties:*

(a) *If $p = 2$, then there exist positive constants γ and c, such that*

$$E(t) \leq c \cdot e^{-\gamma t}, \quad \textit{for all} \;\; t > 0, \qquad \textit{and}$$

(b) *if $p \geq 3$, then there exists a positive constant c, such that*

$$E(t) \leq (1+t)^{-\frac{p}{p-2}}, \quad \textit{for all} \;\; t > 0.$$

Proof. The proof follows the ideas developed in [11, Theorem 2]. $\square$

Remark 2.6. It is a direct consequence of Theorem 2.5 that $u \in L^p((0,\infty); W_0^{1,p}(\Omega))$. Indeed, if p=2,

$$\frac{1}{2}\|u(t)\|_2^2 \leq E(t) \leq c \cdot e^{-\gamma t},$$

$$\int_0^t \|u(s)\|_2^2 ds \leq 2c \int_0^t e^{-\gamma s} ds = \frac{2c}{\gamma}\big(1 - e^{-\gamma t}\big) \leq \frac{2c}{\gamma}, \text{ for all } t > 0.$$

Similarly, if $p \geq 3$, then

$$\int_0^t \|u(s)\|_p^p ds \leq \frac{p-2}{2}\Big(1 - (1+t)^{-\frac{2}{p-2}}\Big) \leq \frac{p-2}{2}, \text{ for all } t > 0.$$

It is essential for the rest of the paper to give the following definition.

Definition 2.7. *Let X be a Banach space. A generalized semiflow $\mathcal{G}$ on X is a family of maps $\phi : [0,\infty) \to X$ satisfying the following hypotheses:*

(H1) (Existence) *For each $z \in X$ there exists at least one $\phi \in \mathcal{G}$ with $\phi(0) = z$.*

(H2) (Translates of solutions are solutions) *If $\phi \in \mathcal{G}$ and $\tau \geq 0$, then $\phi(t+\tau) \in \mathcal{G}$, where $t \in [0,\infty)$.*

(H3) (Concatenation)*If $\phi, \psi \in \mathcal{G}$, $t \geq 0$, with $\psi(0) = \phi(t)$, then $\theta \in \mathcal{G}$, where*

$$\theta(\tau) = \begin{cases} \phi(\tau) & \textit{for} \;\; 0 \leq \tau \leq t, \\ \psi(\tau - t) & \textit{for} \;\; t < \tau. \end{cases}$$

(H4) (Upper-semicontinuity with respect to initial data) *Let the sequence* $\{\phi_j\} \subset \mathcal{G}$ *with* $\phi_j(0) \to z$. *Then there exists a subsequence* $\{\phi_m\}$ *of* $\{\phi_j\}$ *and* $\phi \in \mathcal{G}$ *with* $\phi(0) = z$, *such that* $\phi_m(t) \to \phi(t)$, *for each* $t \geq 0$.

The definition of a generalized semiflow is the natural extension of the classical semigroup theory, if uniqueness is dropped. From now on, for consistency reasons, we will represent a generalized semiflow $\mathcal{G}$ with the usual semigroup notation $\{G_t\}_{t\in\mathbb{R}^+}$, but we will continue to assume that the generalized semiflow consists of solutions of the problem under consideration.

Theorem 2.6. *The solution operators* $\{G_t\}_{t\in\mathbb{R}^+}$ *of the problem* (2.1) *are well defined and form a generalized semiflow.*

Proof. Property (H1) is implied by the existence Theorem 2.4. (H2) and (H3) are derived from the definition of a solution. Concerning property (H4), let $\{u_n(t)\}$ be a sequence of solutions for the problem (2.1), satisfying the hypotheses stated in (H4). Then, by Galerkin estimates, $\{u_n(t)\}$ converges strongly to a solution u(t) in $L^2((0,\infty); L^2(\Omega))$. Thus there exists a subsequence that converges almost everywhere. □

2.2. **Existence of a Global Attractor.** The following definitions are necessary for the rest of this work.

Definition 2.8. *A generalized semiflow* $\{G_t\}_{t\in\mathbb{R}^+}$ *in* X *is called asymptotically compact, if for any sequence* $\{\phi_j\} \subset G$ *with* $\{\phi_j(0)\}$ *bounded and for any sequence* $\{t_j\}_{j\in\mathbb{N}}$, *where* $\lim_{j\to+\infty} t_j = +\infty$, *the sequence* $\{\phi_j(t_j)\}$ *has a convergent subsequence.*

Definition 2.9. *A generalized semiflow* $\{G_t\}_{t\in\mathbb{R}^+}$ *in* X *is called eventually bounded, if for any bounded* $B \subset X$, *there exists* $\tau \geq 0$, *such that* $\gamma_+^\tau(B)$ *is bounded, where* $\gamma_+^\tau(B) = \bigcup_{x\in B} \gamma_+^\tau(x)$ *and* $\gamma_+^\tau(x)$ *is the positive orbit starting from* x *at time* τ.

Definition 2.10. *A generalized semiflow* $\{G_t\}_{t\in\mathbb{R}^+}$ *in* X *is called point dissipative, if there exists a bounded set* $B_0 \subset X$, *such that for any solution* $u \in G$, $u(t) \in B_0$ *for all sufficiently large* t.

Definition 2.11. *A semigroup* $\{G_t\}_{t\in\mathbb{R}^+}$ *belongs in the class AK if it possesses the following property: for every bounded subset* B *in* X *such that* $\gamma_+^0(B)$ *is bounded, any sequence of real positive numbers* $\{t_k\}_{k\in\mathbb{N}}$, *with* $\lim_{k\to+\infty} t_k = +\infty$ *and any sequence* $\{x_k\}$ *in* B, *the sequence of the form* $\{G_{t_k}(x_k)\}_{k\in\mathbb{N}}$ *is precompact.*

In the generalized semiflow cases the following statements are necessary for the method to be used for the proof of the existence of a global attractor.

Theorem 2.12. *Suppose the semigroup $\{G_t\}_{t\in\mathbb{R}^+}$ is defined on a Banach space X with a norm $\|\cdot\|_X$. Suppose also that S_t can be decomposed in the sum W_t+V_t with the following properties:*

(a) $\{V_t\}_{t\in\mathbb{R}^+}$ is a family of operators such that for any bounded set $B\subset X$ we have $\|V_t(B)\|_X\le m_1(t)m_2(\|B\|_X)$, for $m_1,m_2:\mathbb{R}^+\to\mathbb{R}^+$ continuous and $\lim_{t\to\infty}m_1(t)=0$, $\|B\|_X=sup_{x\in B}\|x\|_X$;

(b) for any bounded subset B of X, $W_t(B)$ is precompact. Then $\{G_t\}_{t\in\mathbb{R}^+}$ is of class AK.

Proof. For the proof we refer to [9]. $\square$

Proposition 2.13. *A generalized semiflow is asymptotically compact, if and only if it is of class AK and eventually bounded.*

Proof. For the proof we refer to [2]. $\square$

Theorem 2.14. *A generalized semiflow has a global attractor $\mathcal{A}$, if and only if it is point dissipative and asymptotically compact. The global attractor $\mathcal{A}$ is the unique maximal compact invariant subset of X and it is given by*

$$\mathcal{A}=\omega(X)=\bigcup\{\omega(B):B\subset X,\ B\ bounded\}$$

Proof. For the proof we refer to [1]. $\square$

Let us now deal with the decomposition of the semiflow. To that end one needs the following result.

Lemma 2.15. *Let $n<p<+\infty$ and the function $g:\mathbb{R}\to\mathbb{R}$ satisfy the growth condition (2.2). Suppose $u\in L^p((0,\infty);W_0^{1,p}(\Omega))$. Then $g(u(t))\in L^2((0,\infty);L^2(\Omega))$.*

Proof. First, using the Orlicz inequality we obtain

$$\begin{aligned}
\int_0^t\|g(u(s))\|_2^2ds&=\int_0^t\int_\Omega|g(u(x,s))|^2dxds\le C_b\int_0^t\int_\Omega exp(2b|u(x,s)|^{\frac{p}{p-1}})dxds\\
&\le C_bc\int_0^t\exp\Big(2b\alpha\int_\Omega|u(x,s)|^{\frac{pn}{p-1}}dx+2b\beta\int_\Omega|\nabla|u(x,s)|^{\frac{p}{p-1}}|^ndx\Big)ds\\
&\le C_bcc'\cdot\exp\Big[\alpha'\int_0^t\Big\{2b\alpha\int_\Omega|u(x,s)|^{\frac{pn}{p-1}}dx+2b\beta\int_\Omega\Big|\nabla|u(x,s)|^{\frac{p}{p-1}}\Big|^n\Big\}ds\\
&\quad+\beta'\int_0^t\Big\{\Big(2b\alpha\int_\Omega|u(x,s)|^{\frac{pn}{p-1}}dx+2b\beta\int_\Omega|\nabla|u(x,s)|^{\frac{p}{p-1}}|^ndx\Big)'\Big\}ds\Big]
\end{aligned}$$

$$= C_b c c' \cdot \exp\Big[\alpha'\Big\{2b\alpha\int_0^t\int_\Omega |u(x,s)|^{\frac{pn}{p-1}}dxds + 2b\beta\int_0^t\int_\Omega \Big|\nabla|u(x,s)|^{\frac{p}{p-1}}\Big|^n dxds\Big\}$$

$$+\beta'\Big\{\Big(2b\alpha\int_0^t\int_\Omega |u(x,s)|^{\frac{pn}{p-1}}dxds + 2b\beta\int_0^t\int_\Omega |\nabla|u(x,s)|^{\frac{p}{p-1}}|^n dxds\Big)'\Big\}\Big], \quad (2.5)$$

where $\alpha,\ \beta,\ c,\ \alpha',\ \beta',\ c'$ are the Orlicz constants. Since $p > n$ and Ω is bounded Morrey's embedding theorem and Calderon's lemma imply that $u(s)$ belongs to $L^{\frac{pn}{p-1}}(\Omega)$. Hence,

$$\int_0^t\Big(\int_\Omega |u(x,s)|^{\frac{pn}{p-1}}dx\Big)ds \le c_1\int_0^t \|u(s)\|_p^p ds. \quad (2.6)$$

Similarly, using Calderon's lemma, Morrey's and Sobolev's embedding theorems and Hölder's inequality we obtain

$$\begin{aligned}
&\int_0^t\Big(\int_\Omega |\nabla|u(x,s)|^{\frac{p}{p-1}}|^n dx\Big)ds \\
&= \int_0^t\Big(\int_\Omega \Big||u(x,s)|^{\frac{1}{p-1}}\frac{u(x,s)}{|u(s)|}\nabla u(x,s)\Big|^n dx\Big)ds \\
&= \int_0^t\Big(\int_\Omega |u(x,s)|^{\frac{n}{p-1}}|\nabla u(x,s)|^n dx\Big)ds \\
&\le \int_0^t\Big(\text{ess sup}_{x\in\Omega}\Big\{|u(x,s)|^{\frac{n}{p-1}}\Big\}\int_\Omega |\nabla u(x,s)|^n dx\Big)ds \\
&\le \Big(c_2\text{ess sup}_{t\in\mathbb{R}}\|u(t)\|_p\Big)\cdot\Big(c_3\int_0^t \|u(s)\|^p_{W_0^{1,p}}ds\Big),
\end{aligned} \quad (2.7)$$

where $c_1,\ c_2,\ c_3$ are embedding constants. Now, use estimates (2.6) and (2.7) in the inequality (2.5) to get

$$\begin{aligned}
&\int_0^t \|g(u(s))\|_2^2 ds \le C_b c c'\cdot\exp\Big[2\alpha' a b c_1\int_0^t \|u(s)\|_p^p ds \\
&+ 2\alpha' b\beta\Big(c_2\text{ess sup}_{t\in\mathbb{R}}\|u(t)\|_p\Big)\Big(c_3\int_0^t \|u(s)\|^p_{W_0^{1,p}}ds\Big) \\
&+ 2b\beta'\alpha c_1\Big(\|u(t)\|_p^p - \|u(0)\|_p^p\Big) \\
&+ 2b\beta\beta' c_2 c_3\Big(\|u(t)\|_p^p\|u(t)\|^p_{W_0^{1,p}} - \|u(0)\|_p^p\|u(0)\|^p_{W_0^{1,p}}\Big)\Big] < \infty,\ t\in\mathbb{R}.
\end{aligned} \quad (2.8)$$

To complete the proof of the lemma, we take the limit of (2.8), as $t \to +\infty$. $\square$

Now, for the decomposition of the semiflow one has the following.

Proposition 2.16. *Let $p > n$ and hypotheses* (2.2) *and* (2.3) *be satisfied. Then the generalized semiflow $\{U_t\}_{t\in\mathbb{R}^+}$ of problem* (2.1) *is of class AK.*

Proof. (a) Consider the linearization of problem (2.1)

$$v''(t) - \Delta v'(t) = f - g + \Delta_p u(t),\ v(0) = v'(0) = 0, \tag{2.9}$$

where $u(t)$ is a solution of (2.1). Now, multiply equation (2.9) by v', integrate over Ω and use the Cauchy-Schwartz and Young inequalities to obtain

$$\frac{d}{dt}\|v'(t)\|_2^2 + \frac{2}{\lambda_1}\|v'(t)\|_2^2 \leq 2(\|f-g\|_2 + \|\Delta_p u(t)\|_{W^{-1,\,q}})\|v'(t)\|_2,$$

$$\frac{d}{dt}\|v'(t)\|_2^2 + \frac{2}{\lambda_1}\|v'(t)\|_2^2 \leq \epsilon(\|f-g\|_2 + \|\Delta_p u(t)\|_{W^{-1,\,q}})^2 + \frac{1}{\epsilon}\|v'(t)\|_2^2,$$

$$\frac{d}{dt}\|v'(t)\|_2^2 \leq \epsilon(\|f-g\|_2 + \|\Delta_p u(t)\|_{W^{-1,\,q}})^2 + \Big(\frac{1}{\epsilon} - \frac{2}{\lambda_1}\Big)\|v'(t)\|_2^2,$$

$$\|v'(t)\|_2^2 \leq \epsilon\int_0^t \Big(\|f(x,s) - g(u(s))\|_2 + \|\Delta_p u(s)\|_{W^{-1,\,q}}\Big)^2 ds + \Big(\frac{1}{\epsilon} - \frac{2}{\lambda_1}\Big)\int_0^t \|v'(s)\|_2^2 ds.$$

The use of Gronwall's inequality implies that

$$\|v'(t)\|_2^2 \leq \epsilon\Big[\int_0^t \Big(\|f(x,s) - g(u(s))\|_2 + \|\Delta_p u(s)\|_{W^{-1,\,q}}\Big)^2 ds\Big] e^{(\frac{1}{\epsilon} - \frac{2}{\lambda_1})t}. \tag{2.10}$$

Furthermore, since $u''(t) \in L^2((0,\infty); H^{-r}(\Omega))$ (see [11]), one has

$$\begin{aligned}
&\int_0^t \Big(\|f(x,s) - g(u(s))\|_2 + \|\Delta_p u(s)\|_{W^{-1,\,q}}\Big)^2 ds \\
&\leq 2\int_0^t \Big(\|f(x,s) - g(u(s))\|_2^2 + \|\Delta_p u(s)\|_{W^{-1,\,q}}^2\Big) ds \\
&= 2\int_0^t \|f(x,s) - g(u(s))\|_2^2 ds + 2\int_0^t \|\Delta_p u(s)\|_{W^{-1,\,q}}^2 ds \\
&= 2\int_0^t \|f(x,s) - g(u(s))\|_2^2 ds \\
&\qquad + 2\int_0^t \|u''(s) - \Delta u(s) + g(u(s)) - f(x,s)\|_{W^{-1,\,q}}^2 ds \\
&\leq 8\int_0^t \|f(x,s)\|_2^2 ds + 8\int_0^t \|g(u(s))\|_2^2 ds + 6\int_0^t \|u''(s)\|_{-r}^2 ds
\end{aligned}$$

$$+ 6\int_0^t \|\Delta u'(s)\|^2_{W^{-1,\,q}} ds \leq M \tag{2.11}$$

by Lemma 2.15 and the boundedness of the Laplace operator. Thus, inequalities (2.10) and (2.11) imply that

$$\Big\| \int_0^t v'(s) ds \Big\|_2 \leq \int_0^t \|v'(s)\|_2 ds \leq K_\epsilon(t), \tag{2.12}$$

where

$$K_\epsilon(t) = \frac{2\epsilon\lambda_1\sqrt{\epsilon M}}{2\epsilon - \lambda_1}\Big(1 - \sqrt{\exp\Big[\Big(\frac{1}{\epsilon} - \frac{2}{\lambda_1}\Big)t\Big]}\Big).$$

If we choose $\epsilon = \epsilon(t) = (1 + e^{-t})\frac{\lambda_1}{2}$, then

$$\lim_{t\to+\infty}\Big[\Big(\frac{1}{\epsilon(t)} - \frac{2}{\lambda_1}\Big)t\Big] = 0.$$

Thus

$$\lim_{t\to+\infty} K_\epsilon(t) = 0, \tag{2.13}$$

which implies decay for the solutions of the linearized system (2.9).

(b) Now consider the following equation:

$$w''(t) - \Delta w'(t) = 0,\ w(0) = u_0,\ w'(0) = u_1. \tag{2.14}$$

Multiplying equation (2.14) by $w'(t)$, integrating over Ω and using the integration by parts rule, we obtain

$$\frac{d}{dt}\|w'(t)\|_2^2 + \frac{2}{\lambda_1}\|w'(t)\|_2^2 \leq 0, \quad \text{and} \quad \|w'(t)\|_2^2 \leq \|u_1\|_2^2 \exp(-\tfrac{2}{\lambda_1}t). \tag{2.15}$$

Consider (2.14) as a first-order equation with respect to t; i.e., substitute $w'(t) = z(t)$. This problem is known to possess a global attractor, which implies that the semigroup $\{Z_t\}_{t\in\mathbb{R}^+}$, associated with solutions $z(t)$, is asymptotically compact. Suppose that $u_1^k,\ k = 1, 2, 3, ...$ is a bounded sequence of initial conditions and $t_k \to \infty$. Then the sequence of functions $\{Z_{t_k}(u_1^k)\}_{k\in\mathbb{N}}$ has a convergent subsequence. Moreover, relation (2.15) implies that the convergence is uniform. Thus, the integral over (0, t) of the subsequence converges, which, in turn, implies asymptotic compactness for the semigroup $\{W_t\}_{t\in\mathbb{R}^+}$ associated with the solutions $w(t)$ of problem (2.14). Equation (2.1) is the sum of (2.9) and (2.14). So, the application of Theorem 2.12 completes the proof. □

We can now state the main result about the attractor.

Theorem 2.17. *Let hypotheses* (2.2)–(2.3) *be satisfied. Then, problem* (2.1) *possesses a unique global attractor in the space* $W_0^{1,p}(\Omega) \times L^2(\Omega)$.

Proof. Since $u \in L^\infty((0,\infty); W_0^{1,p}(\Omega))$, the generalized semiflow $\{U_t\}_{t\in\mathbb{R}^+}$ is point dissipative and eventually bounded. Thus, following Proposition 2.13 and Theorem 2.14, we conclude that problem (2.1) possesses a global attractor. □

Remark 2.18. In the case of $p = n$, the necessary L^∞ embedding in Lemma 2.15 fails. To avoid such a situation, one has to impose a stronger assumption on the function g.

Theorem 2.19. *Let $p = n$ and the continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfy the following property: there exists a positive number S, such that for $s < S$ g is bounded and, for $s \geq S$, one has that $|g(s)| \leq \alpha|s|^{\frac{p}{2}}$, for some $\alpha > 0$. Then problem* (2.1) *possesses a global attractor.*

Proof. We have to prove as in Lemma 2.15 that $g \in L^2((0,\infty); L^2(\Omega))$. Indeed, we have

$$\int_0^\infty \|g(u(s))\|_2^2 ds \leq \int_0^T \|g(u(s))\|_2^2 ds + \alpha \int_T^\infty \Big\| |u(s)|^{\frac{p}{2}} \Big\|_2^2 ds < \infty.$$

The rest of the proof follows the same steps as in the case $p > n$. ◁

3. The Anisotropic Case

In this section we deal with the $p(x)$-Laplacian case, which is more complicated than the classical one, due to a number of reasons. First, the $p(x)$-Laplacian operator is inhomogeneous and second, it may not possess a first eigenvalue; i.e., there are situations where the sequence λ_n of eigenvalues converges to zero (see [8]). The natural spaces where the solutions of an equation with the $p(x)$-Laplacian operator lie are the anisotropic Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. Next, some general notation and results on these spaces will be presented together with a compactness lemma in order to establish the necessary embeddings needed to deploy the standard Galerkin method.

Again, consider Ω an open, bounded subset of $\mathbb{R}^n$. For $u : \Omega \times (0,T) \to \mathbb{R}$ and $p : \Omega \to \mathbb{R}$, measurable, with $p(x) \geq 1$ for all $x \in \bar{\Omega}$, define

$$-\Delta_{p(x)} u(t) = -\mathrm{div}\Big(|\nabla u(t)|^{p(x)-2} \nabla u(t) \Big).$$

We now move on to discuss the anisotropic Lebesgue and Sobolev spaces.

Definition 3.1. *Suppose that the function $p : \Omega \to \mathbb{R}$ is measurable and let $p^- = \text{ess inf}_{x\in\Omega} p(x)$ and $p^+ = \text{ess sup}_{x\in\Omega} p(x)$, such that $1 \leq p^- \leq p^+ < \infty$. Define*

$$\begin{aligned} L^{p(x)}(\Omega) &= \Big\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\} \qquad \text{and} \\ W^{1,p(x)}(\Omega) &= \Big\{ u \in L^{p(x)}(\Omega) \mid \nabla u \in \big(L^{p(x)}(\Omega)\big)^n \Big\}. \end{aligned}$$

As usual, we define $W_0^{1,p(x)}(\Omega)$ to be the closure of the space $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The above spaces are Banach spaces endowed with the Luxemburg norm

$$\|u\|_{p(x)} = \inf_{\lambda>0} \Big\{ \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} dx \leq 1 \Big\} \text{ and } \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

Moreover, the norm $\|\nabla u\|_{p(x)}$ is equivalent to $\|u\|_{1,p(x)}$ in $W_0^{1,p(x)}(\Omega)$.

These spaces have geometric and topological properties similar to the classical Lebesgue - Sobolev spaces. However, there are some difficulties arising in density theorems, which depend on the exponent function $p(x)$ and, thus, require a finer approach, but these issues will not be addressed here. For more detailed information on the definition and properties of generalized Lebesgue - Sobolev spaces see [7] and the references therein. The following theorems summarize the basic embedding and density properties of these spaces.

Theorem 3.2. *The following statements are true.*

(a) The spaces $\big(L^{p(x)}(\Omega), \|\cdot\|_{p(x)}\big)$ and $\big(W^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)}\big)$ are separable;

(b) if Ω is open, then $C_0^\infty(\Omega)$ is dense in $\big(L^{p(x)}(\Omega), \|\cdot\|_{p(x)}\big)$;

(c) if $p^- > 1$, then $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are uniformly convex and reflexive;

(d) for all $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$, one has

$$\Big| \int_\Omega u(x)v(x)dx \Big| \leq 2\|u\|_{p(x)}\|v\|_{q(x)}; \quad \text{and}$$

(e) the dual of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $1/p(x)+1/q(x) = 1$ and the Riesz representation theorem is valid.

Theorem 3.3. *The following statements are true.*

(a) Let $p_1(x) \leq p_2(x)$ almost everywhere in Ω. Then the embeddings $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and $W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega)$ are continuous;

(b) let the exponent functions $p(x)$, $q(x)$ satisfy also that $p,\ q \in C(\bar{\Omega})$ and that

$$p(x) < n, \quad q(x) < \frac{np(x)}{n - p(x)}, \text{ for all } x \in \bar{\Omega};$$

then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact;

(c) assume that there exists $L > 0$, such that for all $x, y \in \bar{\Omega}$ one has that

$$-\,|p(x) - p(y)| \cdot \ln|x - y| \leq L; \tag{3.1}$$

then, $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ are dense in $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ respectively.

Let us now formulate the analogous problem

$$\begin{aligned} u''(t) - \Delta_{p(x)}u(t) - \Delta u'(t) + g(u) \quad &= \quad f(x,t),\ x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u'(x,0) \quad &= \quad u_1(x),\ x \in \Omega, \\ u(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \quad \text{where} & \quad (u_0, u_1) \in W_0^{1,p(x)}(\Omega) \times L^2(\Omega), \end{aligned} \tag{3.2}$$

under the assumptions (2.2)-(2.3). The main result of this section is the following.

Theorem 3.4. *Let the function p satisfy the assumptions of Definition 3.1 and relation (3.1), with $n/2 < p(x) < n$, almost everywhere in Ω. Then, there exists a function $u : [0, +\infty) \times \Omega \to \mathbb{R}$, with $u \in L^\infty\big((0,\infty); W_0^{1,p(x)}(\Omega)\big)$ and $u' \in L^\infty\big((0,\infty); L^2(\Omega)\big) \cap L^2\big((0,\infty); W_0^{1,2}(\Omega)\big)$, satisfying problem (3.2).*

In order to prove the above theorem, we need the following lemmas.

Lemma 3.5. *Let p satisfy the assumptions of Theorem 3.4 and $u \in W_0^{1,p(x)}(\Omega)$. Then, the following inequality is true:*

$$\int_\Omega \exp\left(a|u|^{\frac{p(x)}{p(x)-1}}\right)dx < \infty.$$

Proof. Since $u \in W_0^{1,p(x)}(\Omega)$, from the variable exponent Sobolev embedding theorem (see [7]), one has that $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\frac{np(x)}{p(x)-1}}(\Omega)$, continuously and compactly. Thus, $|u|^{\frac{p(x)}{p(x)-1}} \in L^n(\Omega)$ and using Orlicz' inequality one gets

$$\begin{aligned} &\int_\Omega \exp\left(a|u|^{\frac{p(x)}{p(x)-1}}\right)dx \leq c\exp\left(a\int_\Omega |u|^{\frac{p(x)n}{p(x)-1}}dx + b\int_\Omega \left|\nabla|u|^{\frac{p(x)}{p(x)-1}}\right|^n dx\right) \\ &\leq c\exp\left(a\int_\Omega |u|^{\frac{p(x)n}{p(x)-1}}dx + b\int_\Omega \Big(\frac{p(x)}{(p(x)-1)}\Big)^n \Big||u|^{\frac{1}{p(x)-1}}\frac{u}{|u|}\nabla u\Big|^n dx\right) \end{aligned}$$

$$\leq c\exp\Big(a\int_\Omega |u|^{\frac{p(x)n}{p(x)-1}}dx + b\cdot \text{meas}(\Omega)\cdot \text{ess sup}_{x\in\Omega}\Big\{\Big(\frac{p(x)}{(p(x)-1)}\Big)^n\Big\}$$
$$\times \int_\Omega |u|^{\frac{n}{p(x)-1}}|\nabla u|^n dx\Big)$$
$$\leq c\exp\Big(a\int_\Omega |u|^{\frac{p(x)n}{p(x)-1}}dx$$
$$+\, 2b\cdot \text{meas}(\Omega)\cdot \text{ess sup}_{x\in\Omega}\Big\{\Big(\frac{p(x)}{(p(x)-1)}\Big)^n\Big\}\|u\|_{\frac{np(x)}{p(x)-1}}\|\nabla u\|_{\frac{np(x)}{p(x)-1}}\Big) < \infty,$$

by the use of the anisotropic Cauchy-Schwartz inequality, since $p(x)$ and $\frac{p(x)}{p(x)-1}$ are conjugate exponents. □

The following compactness result is simple but essential.

Lemma 3.6. *Let p satisfy the assumptions of Theorem 3.4. Then*

$$L^\infty\big((0,T);W_0^{1,p(x)}(\Omega)\big) \hookrightarrow L^2\big((0,T);L^2(\Omega)\big), \text{ continuously and compactly.}$$

Proof. The embedding theorem for anisotropic Sobolev Spaces implies that $W_0^{1,p(x)}(\Omega)$ is compactly embedded in $L^2(\Omega)$. This means that, if $u_n(t)$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$, considering t as a parameter, there exists a $u(t) = lim_{n\to\infty}u_n(t) \in L^2(\Omega)$, such that $u_n(t) \to u(t)$, in $L^2(\Omega)$ (possibly through a subsequence). Equivalently, the sequence of positive numbers $\|u_n(t) - u(t)\|_2^2 \to 0$. From Lebesgue's dominated convergence theorem and the fact that ess $\sup_{t\in[0,T]}\|u_n(t)\|_2 < \infty$, uniformly for all n, one obtains

$$\int_0^T \|u_n(t) - u(t)\|_2^2 dt \to 0,$$

which completes the proof. □

The next proposition is quite useful for calculations concerning the $p(x)$-Laplacian operator. It generalizes the well-known properties of the ordinary $p-$Laplacian and illustrates the fact that the operator is inhomogeneous in the anisotropic case.

Proposition 3.7. *Let u be a function in $W_0^{1,p(x)}(\Omega)$. Then the following properties hold:*

$$(i) \qquad \langle -\Delta_p u(t), u'(t)\rangle = \frac{d}{dt}\int_\Omega \frac{|\nabla u(t,x)|^{p(x)}}{p(x)}dx$$

$$(ii) \qquad \langle -\Delta_p u(t), u(t)\rangle = \int_\Omega |\nabla u(t,x)|^{p(x)}dx.$$

Proof. The second property is a direct consequence of Green's identity. For the first one, we obtain

$$\begin{aligned}
&\int_\Omega \Big(-\Delta_{p(x)} u(t,x)\Big) u'(t,x)dx \\
&= \int_\Omega \|\nabla u(t,x)\|_{\mathbb{R}^n}^{p(x)-2} (\nabla u(t,x), \nabla u'(t,x))_{\mathbb{R}^n} dx \\
&= \frac{1}{2}\int_\Omega \|\nabla u(t,x)\|_{\mathbb{R}^n}^{p(x)-2} \cdot \frac{d}{dt}\|\nabla u(t,x)\|_{\mathbb{R}^n}^2 dx \\
&= \frac{1}{2}\int_\Omega \Big(\|\nabla u(t,x)\|_{\mathbb{R}^n}^2\Big)^{\frac{p(x)-2}{2}} \cdot \frac{d}{dt}\|\nabla u(t,x)\|_{\mathbb{R}^n}^2 dx \\
&= \frac{1}{2}\int_\Omega \frac{2}{p(x)}\frac{d}{dt}\Big((\|\nabla u(t,x)\|_{\mathbb{R}^n}^2)^{\frac{p(x)}{2}}\Big) dx,
\end{aligned}$$

which concludes the proof. □

Proof of Theorem 3.4. The proof will follow the same steps as in Theorem 2.4. We will only show how to obtain the estimates and the necessary embeddings. The projection and the monotonicity arguments remain the same. We take, in the approximate equation in the finite-dimensional Galerkin spaces, the product with $u_n'(t)$, and integrate over Ω and $(0,t)$, to obtain, after the application of the Cauchy-Schwartz inequality, the uniform Gronwall lemma and Lemma 3.5 that

$$\frac{1}{2}\|u_n'(t)\|_2^2 + \int_\Omega \frac{\|\nabla u_n(t)\|_{\mathbb{R}^n}^{p(x)}}{p(x)} dx + \int_0^t \|\nabla u_n'(s)\|_2^2 ds + \frac{1}{2}\int_\Omega G(u_n(t))dx \le C.$$

Thus, we extend the approximate solutions $u_n(t)$ to the whole interval $[0,T]$ and we get that

$$\begin{gathered}
u_n \text{ is bounded in } L^\infty\big((0,T); W_0^{1,p(x)}(\Omega)\big), \\
u_n' \text{ is bounded in } L^\infty\big((0,T); L^2(\Omega)\big) \text{ and} \\
u_n' \text{ is bounded in } L^2\big((0,T); W_0^{1,2}(\Omega)\big).
\end{gathered}$$

Now, applying Lemma 3.6, going to a subsequence if necessary, one obtains that $u_n \to u$ strongly in $L^2((0,T);L^2(\Omega))$, and $u_n' \to u'$ strongly in $L^2((0,T);L^2(\Omega))$, which implies that $g(u_n) \rightharpoonup g(u)$ in $L^2((0,T);L^2(\Omega))$ (see [10, Lemma 1.3]). Note that the hemicontinuity and monotonicity of the $p-$Laplacian operator are still valid in the anisotropic case. □

Remark 3.8. Setting $p(x) = p$ and imposing the same assumptions on p, we can prove the existence of solutions for the classic $p-$Laplacian, when

$n/2 < p < n$ and thus have a general result on the existence of solutions for problem (2.1).

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