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GLOBAL ATTRACTOR FOR A KLEIN-GORDON-SCHRÖDINGER TYPE SYSTEM

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Abstract. In this paper we prove the existence and uniqueness of solutions for the following evolution system of Klein-Gordon-Schrödinger type

$$\begin{aligned}
i\psi_t + \kappa\psi_{xx} + i\alpha\psi &= \phi\psi + f(x),\\
\phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t &= -Re\psi_x + g(x),\\
\psi(x,0)=\psi_0(x),\ \phi(x,0) &= \phi_0(x),\ \phi_t(x,0)=\phi_1(x),\\
\psi(x,t)=\phi(x,t)=0, &\quad x\in\partial\Omega,\ \ t>0,
\end{aligned}$$

where $x \in \Omega$, $t > 0$, $\kappa > 0$, $\alpha > 0$, $\lambda > 0$, $f(x)$ and $g(x)$ are the driving terms and Ω (bounded) $\subset \mathbb{R}$. Also we prove the continuous dependence of solutions of the system on the initial data as well as the existence of a global attractor.

1. **Introduction.** The aim of this paper is to prove the existence of a global attractor for the following Klein-Gordon-Schrödinger type system defined in a bounded interval $\Omega \subset \mathbb{R}$

$$i\psi_t + \kappa\psi_{xx} + i\alpha\psi = \phi\psi + f(x), \quad x\in\Omega,\ t>0, \tag{1}$$

$$\phi_{tt} - \phi_{xx} + \phi + \lambda\phi_t = -Re\psi_x + g(x), \quad x\in\Omega,\ t>0, \tag{2}$$

$$\psi(x,0)=\psi_0(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0)=\phi_1(x), \quad x\in\Omega,\ t>0, \tag{3}$$

$$\psi(x,t)=\phi(x,t)=0, \quad x\in\partial\Omega,\ \ t>0,\ \ \kappa>0,\ \ \alpha>0,\ \lambda>0, \tag{4}$$

where $f(x), g(x)$ are the driving terms. Systems of Klein-Gordon-Schrödinger type have been studied for many years. To our knowledge, it seems that the first problems of this type is the so called *Yukawa System* [11], which goes back to 1935 (see, [2] and the references therein). An other model which is of the same type is the so called *Zakharov System*, which is formed by V. E. Zakharov [13] in early seventies (see, [3] and [4] and the references therein).
Here we consider a *Klein-Gordon-Schrödinger system of a third type*, which is the problem (1) - (4). This problem is the outcome of a modeling process, described in all details in a work by N. Karachalios, N. Stavrakakis and P. Xanthopoulos [6]. Problem (1) - (4) models the *Upper Hybrid Heating* (UHH) scheme for plasmas in fusion devices. (UHH) is the dominant branch of the general *Electron Cyclotron Resonance Heating* (ECRH) scheme, which, for Tokamaks and Stellarators, constitutes a basic method of plasma build-up and heating. The celebrated Zakharov system, is highly successful in a multitude of applications. However, regarding the study of (UHH) Zakharov system cannot not be implemented for certain reasons.

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The variable ψ stands for the dimensionless low frequency electron field, whereas the (real) variable ϕ denotes the dimensionless low frequency density. For more details on the physical interpretation and the modeling process of the system the reader may refer to [6], [12] and the references therein.
In this paper we prove the existence of a global attractor in the space $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ which attracts all bounded sets of $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ in the norm topology. This paper is divided in four Sections. In the Second Section, we derive some useful estimates on the solutions of the system (1) - (4) in $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$. The Third Section, is based on an energy method first introduced by J. Ball [1]. We are going to use the energy equations of the problem to prove the continuity of solutions on the initial data in the space $(H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$. In Section 4, we show the asymptotic compactness of the dynamical system and the existence of a global attractor. In a recent joint work we study the finite dimensionality of the global attractor (see, [10]).
***Notation*:** Denote by H^s both the standard real and complex Sobolev spaces. For *simplicity reasons* sometimes we use $H^s,\ L^s$ for $H^s(\Omega),\ L^s(\Omega)$ and $||.||$, $(.,.)$ for the norm and the inner product of $L^2(\Omega)$, respectively. $\int dx$ denotes the integration over Ω. Finally, C is a general symbol for any positive constant.

2. **Global Existence.** In this section we derive a priori estimates for the solutions of the Klein-Gordon-Schrödinger system. Let us introduce the transformation $\theta = \phi_t + \delta\phi$, where θ is real, with δ a small positive constant to be specified later. Then, system (1) - (4) takes the form

$$\begin{aligned} i\psi_t + \kappa\psi_{xx} + i\alpha\psi &= \phi\psi + f,\ \ x \in \Omega,\ t > 0, && (5)\\ \phi_t + \delta\phi &= \theta,\ \ x \in \Omega,\ t > 0, && (6)\\ \theta_t + (\lambda - \delta)\theta - \phi_{xx} + (1 - \delta(\lambda - \delta))\phi &= -Re\psi_x + g,\ \ x \in \Omega,\ t > 0, && (7) \end{aligned}$$

Also the initial and boundary conditions take the form

$$\begin{aligned} \psi(x,0) = \psi_0(x),\ \phi(x,0) &= \phi_0(x),\ \theta(x,0) = \theta_0(x),\quad x \in \Omega && (8)\\ \psi(x,t) = \phi(x,t) = 0, &\qquad x \in \partial\Omega,\ \ t > 0. && (9) \end{aligned}$$

Lemma 1. *Let $||\psi_0(t)|| \le R$, for some $R > 0$ and $f \in L^2(\Omega)$. Every solution of (5)-(9) satisfies*

$$||\psi(t)|| \le R^*, \quad t \ge t_1,$$

where R^* is a constant depending on α, $||f||$; t_1 depending on α, $||f||$ and R.

Proof The proof is analogue to the one of Lemma 2.1 in [9]. $\lhd$

Lemma 2. *Assume that $f, g \in L^2(\Omega)$ and $||(\psi_0, \phi_0, \theta_0)||_{H_0^1 \times H_0^1 \times L^2} \le R$, where $R > 0$. Then, there exists a constant δ_1 such that when $\delta \le \delta_1$, every solution (ψ, ϕ, θ) of problem (5)-(9) satisfies*

$$||\psi(t)||_{H_0^1} + ||\phi(t)||_{H_0^1} + ||\theta(t)|| \le M_1, \quad t \ge t_2,$$

where M_1 depends on $\alpha, \kappa, \lambda, \delta_1, ||f||, ||g||$ and t_2 on $\alpha, \kappa, \lambda, \delta_1, ||f||, ||g||$ and R.

Proof. Multiplying equation (5) by $-\psi_t$, integrating and taking the real part gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\kappa||\psi_x||^2 + \int \phi|\psi|^2 + 2Re\int f\bar{\psi}\right) + \kappa\alpha||\psi_x||^2 + \left(\alpha + \frac{\delta}{2}\right)\int \phi|\psi|^2 \\ = \frac{1}{2}\int \theta|\psi|^2 + \alpha Re\int f\bar{\psi}. \end{aligned} \tag{10}$$

Next, multiplying equation (7) by θ and substituting θ from equation (6) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(||\theta||^2 + ||\phi_x||^2 + (1-\delta(\lambda-\delta))||\phi||^2) + (\lambda-\delta)||\theta||^2 + \delta||\phi_x||^2 \\ + \delta(1-\delta(\lambda-\delta))||\phi||^2 = -Re\int \theta\psi_x + \int g\theta.\end{aligned} \tag{11}$$

Adding relations $2\times$ (10) and $2\times$ (11) gives

$$F_1^{'}(t) + \delta F_1(t) = G_1(t), \tag{12}$$

where for simplification reasons the following quantities are introduced

$$\begin{aligned}F_1 :=& \kappa||\psi_x||^2 + \int \phi|\psi|^2 dx + ||\theta||^2 + ||\phi_x||^2 + (1-\delta(\lambda-\delta))||\phi||^2 + 2Re\int f\bar{\psi},\\ G_1 :=& (\delta-2\kappa\alpha)||\psi_x||^2 - 2\alpha\int \phi|\psi|^2 + (3\delta-2\lambda)||\theta||^2 - \delta(1-\delta(\lambda-\delta))||\phi||^2\\ &- \delta||\phi_x||^2 + \int \theta|\psi|^2 + 2(\delta-\alpha)Re\int f\bar{\psi} - 2Re\int \theta\psi_x + 2\int g\theta.\end{aligned}$$

Taking δ small enough such that $\delta - 2\kappa\alpha < 0,\ 3\delta - 2\lambda < 0,\ 1-\delta(\lambda-\delta) > 0$, one can render several terms of G_1 negative. Let us proceed by majorizing the integrals of G_1 as follows

$$\left|\int \theta|\psi|^2\right| \le ||\theta||\ ||\psi||_4^2 \le ||\theta||\ ||\psi||^{1/2}||\psi||^{3/2} \le \frac{\epsilon_1}{2}||\theta||^2 + \frac{\epsilon_2}{2}||\psi_x||^2 + C,$$

$$\left|2\alpha\int \phi|\psi|^2\right| \le \epsilon_3||\phi||^2 + \frac{\epsilon_2}{2}||\psi_x||^2 + C,$$

$$\text{and } \left|\int \theta\psi_x\right| \le ||\psi_x||\ ||\theta|| \le \frac{\epsilon}{2}||\psi_x||^2 + \frac{1}{2\epsilon}||\theta||^2,$$

$$\left|2(\delta-\alpha)\int f\bar{\psi}\right| \le C||f||\ ||\psi|| \le C, \text{ and } \left|2\int g\theta\right| \le 2||g||\ ||\theta|| \le \frac{\epsilon_1}{2}||\theta||^2 + C.$$

The next step is to estimate the arbitrary positive constants $\epsilon_1,\ \epsilon_2,\ \epsilon$, such that the following two inequalities hold simultaneously true $\epsilon_1 + \dfrac{1}{2\epsilon} \le -(3\delta - 2\lambda),\ \epsilon_2 + \dfrac{\epsilon}{2} \le -(\delta - 2\kappa\alpha)$. Let $\nu > 0,\ \nu \neq \frac{1}{2},\ \bar{\alpha} = -(3\delta - 2\lambda)$ and $\bar{\beta} = -(\delta - 2\kappa\alpha)$. Setting $\epsilon_1 = \dfrac{\bar{\alpha}}{2\nu},\ \epsilon_2 = \dfrac{\bar{\beta}}{2\nu}$ we have the following necessary condition: $\bar{\alpha}\bar{\beta} \ge \dfrac{\nu^2}{(2\nu-1)^2}$. Since $\bar{\alpha},\ \bar{\beta} > 0$ the inequality is always true for sufficiently small ν. Finally, taking ϵ_3 small enough, so that $\epsilon_3 < -\delta(1-\delta(\lambda-\delta))$ implies

$$F_1^{'}(t) + \delta F_1(t) \le C.$$

The application of Gronwall's Inequality completes the proof. □

Lemma 3. *Assume that $f, g \in H_0^1(\Omega)$ and $||(\psi_0, \phi_0, \theta_0)||_{(H_0^1\cap H^2)^2\times H_0^1} \le R$, where $R > 0$. Then, there exists a constant δ_1 such that when $\delta \le \delta_1$, every solution (ψ, ϕ, θ) of the problem (5)-(9) satisfies*

$$||\psi(t)||_{H_0^1\cap H^2} + ||\phi(t)||_{H_0^1\cap H^2} + ||\theta(t)||_{H_0^1} \le M_2, \quad t \ge t_3,$$

where M depends on $\alpha, \kappa, \lambda, \delta_1, ||f||_{H_0^1}, ||g||_{H_0^1}$; and t_3 on $\alpha, \kappa, \lambda, \delta_1, |f||_{H_0^1}, ||g||_{H_0^1}, R$.

Proof. Multiplying relation (5) by $\psi_{xx,t}+\alpha\psi_{xx}$ and taking the real part, produces

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\kappa||\psi_{xx}||^2-2Re\int\phi\psi\bar{\psi}_{xx}+2Re\int f_x\bar{\psi}_x\right)+\kappa\alpha||\psi_{xx}||^2\\&-(2\alpha+\delta)Re\int\phi\psi\bar{\psi}_{xx}=-Re\int\theta\psi\bar{\psi}_{xx}dx-Im\int\phi^2\psi\bar{\psi}_{xx}-\alpha Re\int f_x\bar{\psi}_x.\end{aligned}\tag{13}$$

Next, multiplication of equation (7) by $-\theta_{xx}$ and integration gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(||\theta_x||^2+||\phi_{xx}||^2+(1-\delta(\lambda-\delta))||\phi_x||^2\right)+(\lambda-\delta)||\theta_x||^2\\&\quad+\delta||\phi_{xx}||^2+\delta(1-\delta(\lambda-\delta))||\phi_x||^2=-Re\int\theta_x\psi_{xx}+\int g_x\theta_x.\end{aligned}\tag{14}$$

Furthermore, the addition of equations $2\times$ (13) and $2\times$ (14) implies,

$$F_2^{'}(t)+\delta F_2(t)=G_2(t),\quad t\geq t_3,$$

where for simplification reasons the following notation is introduced

$$\begin{aligned}F_2:=\ &\kappa||\psi_{xx}||^2-2Re\int\phi\psi\bar{\psi}_{xx}+2Re\int f_x\bar{\psi}_x+||\theta_x||^2+||\phi_{xx}||^2\\&+(1-\delta(\lambda-\delta))||\phi_x||^2,\\G_2:=\ &(\delta-2\kappa\alpha)||\psi_{xx}||^2+(3\delta-2\lambda)||\theta_x||^2-\delta||\phi_{xx}||^2-\delta(1-\delta(\lambda-\delta))||\phi_x||^2\\&+4\alpha Re\int\phi\psi\bar{\psi}_{xx}-2Re\int\theta_x\psi_{xx}-2Re\int\theta\psi\bar{\psi}_{xx}dx-2Im\int\phi^2\psi\bar{\psi}_{xx}\\&+2(\delta-\alpha)\int f_x\bar{\psi}_x+2\int g_x\theta_x.\end{aligned}$$

Taking δ small enough one can render several terms of G_2 negative. Following the same procedure as the one used for the first estimate we can deduce that,

$$F_2^{'}(t)+\delta F_2(t)\leq C,\quad t\geq t_3,$$

therefore by applying Gronwall's Inequality the proof is completed. $\lhd$

Repeating a similar procedure to the one used in the preceding Lemmas we obtain the following results on a finite time interval.

Lemma 4. *Assume that* $f,g\in L^2(\Omega)$. *Let* $||\psi_0,\phi_0,\theta_0||_{H_0^1\times H_0^1\times L^2}\leq R$, *where* $R>0$. *Then, there exists a constant* δ_1 *such that when* $\delta\leq\delta_1$, *every solution* (ψ,ϕ,θ) *of the problem (5)-(9) satisfies*

$$||\psi(t)||_{H_0^1}+||\phi(t)||_{H_0^1}+||\theta||\leq L_1,\quad 0\leq t\leq T,$$

where L_1 *depends on* $\alpha,\kappa,\lambda,\delta,||f||,||g||$ *and* T.

Lemma 5. *Assume that* $f,\ g\in H_0^1(\Omega)$. *Let* $||\psi_0,\phi_0,\theta_0||_{(H_0^1\cap H^2)^2\times H_0^1}\leq R$, *where* $R>0$. *Then, there exists a constant* δ_1 *such that when* $\delta\leq\delta_1$, *every solution* (ψ,ϕ,θ) *of the problem (5)-(9) satisfies*

$$||\psi(t)||_{H_0^1\cap H^2}+||\phi(t)||_{H_0^1\cap H^2}+||\theta||_{H_0^1}\leq L_2,\quad 0\leq t\leq T,$$

where L_2 *depends on* $\alpha,\kappa,\lambda,\delta,\ ||f||_{H_0^1},\ ||g||_{H_0^1}$ *and* T.

Therefore we are ready to state the main result of this section.

Theorem 1. *Assume that* $f,\, g \in H_0^1(\Omega)$. *Let* $||\psi_0, \phi_0, \theta_0||_{(H_0^1 \cap H^2)^2 \times H_0^1} \leq R,$ *where* $R > 0$. *Then, there exists a constant* δ_1 *such that when* $\delta \leq \delta_1$, *the system (5) - (9) admits a unique solution satisfying*

$$\begin{aligned}
&\psi \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)),\ \psi_t \ \in L^\infty(0,\infty; L^2(\Omega)),\\
&\phi \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)),\ \phi_t \ \in L^\infty(0,\infty; H_0^1(\Omega)),\ \phi_{tt} \in L^\infty(0,\infty; L^2(\Omega)),\\
&\psi(x,0) = \psi_0(x),\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x),\ x \in \Omega.
\end{aligned}$$

Proof. The proof follows the lines of Theorem 3.1 in [6]. □

3. **The Solution Semigroup.** Problem (5)-(9) defines a **semigroup** $S(t)$,

$$S(t) : (H_0^1 \cap H^2)^2 \times H_0^1 \to (H_0^1 \cap H^2)^2 \times H_0^1 \text{ (Theorem 1)}$$

Let B_1, B_2 denote the following balls of center zero and radius M_1, M_2 respectively

$$\begin{aligned}
&B_1 = \{(\psi,\phi,\theta) \in H_0^1 \times H_0^1 \times L^2 : ||\psi||_{H_0^1} + ||\phi||_{H_0^1} + ||\theta|| \leq M_1\},\\
&B_2 = \{(\psi,\phi,\theta) \in (H_0^1 \cap H^2)^2 \times H_0^1 : ||\psi||_{H_0^1 \cap H^2} + ||\phi||_{H_0^1 \cap H^2} + ||\theta||_{H_0^1} \leq M_2\},
\end{aligned} \tag{15}$$

where $M_1,\ M_2$ are the constants introduced in Lemmas 2, 3. Therefore, B_1, B_2 are bounded absorbing sets for (5)-(9). Since B_1 is bounded, we see that there exists a constant $T(B_1)$ depending on B_1 such that $S(t)B_1 \subset B_1$, for all $t \geq T(B_1)$.

Lemma 6. *If* $(\psi_n, \phi_n, \theta_n) \rightharpoonup (\psi, \phi, \theta)$ *weakly in* $H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, *then for every* $T > 0$, *we have*

$$S(\cdot)(\psi_n, \phi_n, \theta_n) \rightharpoonup S(\cdot)(\psi,\phi,\theta), \quad \textit{weakly in} \quad L^2(0,T; H_0^1 \times H_0^1 \times L^2), \tag{16}$$

$$S(t)(\psi_n, \phi_n, \theta_n) \rightharpoonup S(t)(\psi,\phi,\theta), \quad \textit{weakly in} \quad H_0^1 \times H_0^1 \times L^2,\ \ 0 \leq t \leq T. \tag{17}$$

Proof. From the weak convergence and Lemma 4 it follows that $\{S(t)(\psi_n, \phi_n, \theta_n)\}$ is bounded in $L^\infty(0,T; H_0^1 \times H_0^1 \times L^2)$ with

$$\left\{\frac{\partial}{\partial t} S(t)\psi_n\right\}, \left\{\frac{\partial}{\partial t} S(t)\theta_n\right\} \text{ bounded in } \quad L^\infty(0,T; H^{-1}),$$

$$\left\{\frac{\partial}{\partial t} S(t)\phi_n\right\} \text{ bounded in } \quad L^\infty(0,T; L^2).$$

Therefore, exist a subsequence $\{(\psi_{n_j}, \phi_{n_j}, \theta_{n_j})\}$ of $\{(\psi_n, \phi_n, \theta_n)\}$ and $(\psi_\infty, \phi_\infty, \theta_\infty) \in L^\infty(0,T; H_0^1 \times H_0^1 \times L^2)$ such that

$$S(t)(\psi_{n_j}, \phi_{n_j}, \theta_{n_j}) \rightharpoonup (\psi_\infty, \phi_\infty, \theta_\infty), \text{ weakly in } L^2(0,T; H_0^1 \times H_0^1 \times L^2), \tag{18}$$

$$\frac{\partial}{\partial t} S(t)\psi_{n_j} \rightharpoonup \frac{\partial}{\partial t}\phi_\infty \quad \text{and} \quad \frac{\partial}{\partial t} S(t)\theta_{n_j} \rightharpoonup \frac{\partial}{\partial t}\theta_\infty, \text{ weakly in } L^2(0,T; H^{-1}), \tag{19}$$

$$\frac{\partial}{\partial t} S(t)\phi_{n_j} \rightharpoonup \frac{\partial}{\partial t}\phi_\infty, \quad \text{weakly in } L^2(0,T; L^2). \tag{20}$$

Then, it can be shown that $(\psi_\infty, \phi_\infty, \theta_\infty)$ is a solution of the system (5)-(9) with initial conditions $(\psi_\infty(0), \phi_\infty(0), \theta_\infty(0)) = (\psi, \phi, \theta)$. But in Theorem 1 we established the uniqueness of solutions, that implies $(\psi_\infty, \phi_\infty, \theta_\infty) = S(t)(\psi, \phi, \theta)$ which together with (18) concludes the proof of (16).

Now for equation (17), let us choose $t \in [0, T]$ fixed. From the weak convergence and Lemma 4 $\{S(t)(\psi_n, \phi_n, \theta_n)\}$ is bounded in $H_0^1 \times H_0^1 \times L^2$. So there exists a subsequence $\{(\psi_{n_{j,t}}, \phi_{n_{j,t}}, \theta_{n_{j,t}})\}$ of $\{(\psi_n, \phi_n, \theta_n)\}$ such that $S(t)(\psi_{n_{j,t}}, \phi_{n_{j,t}}, \theta_{n_{j,t}}) \rightharpoonup (\psi_t, \phi_t, \theta_t)$ weakly in $H_0^1 \times H_0^1 \times L^2$, with $(\psi_t, \phi_t, \theta_t) \in H_0^1 \times H_0^1 \times L^2$, where $(\psi_{n_{j,t}}, \phi_{n_{j,t}}, \theta_{n_{j,t}})$ and $(\psi_t, \phi_t, \theta_t)$ both depend on t. By (18)-(20) we can see that

$(\psi_\infty, \phi_\infty, \theta_\infty)$ is a solution of the system with $(\psi_\infty(0), \phi_\infty(0), \theta_\infty(0)) = (\psi, \phi, \theta)$ as well as $(\psi_\infty(t), \phi_\infty(t), \theta_\infty(t)) = (\psi_t, \phi_t, \theta_t)$. Therefore, by the uniqueness of the solution we have $(\psi_t, \phi_t, \theta_t) = (\psi_\infty(t), \phi_\infty(t), \theta_\infty(t)) = S(t)(\psi, \phi, \theta)$. Hence we have proved that any sequence $S(t)(\psi_n, \phi_n, \theta_n)$ has a weakly convergent subsequence that will converge to the same limit $S(t)(\psi, \phi, \theta)$. Therefore by contradiction arguments we conclude that relation (17) holds. $\lhd$

Theorem 2. *Assume that $f, g \in L^2(\Omega)$. The solutions $(\psi, \phi, \theta) \in C(\mathbb{R}^+, H_0^1 \times H_0^1 \times L^2)$ of the problem (5)-(9) depend continuously on the initial data in $H_0^1 \times H_0^1 \times L^2$.*

Proof. Assume that $(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \to (\psi_0, \phi_0, \theta_0)$ in $H_0^1 \times H_0^1 \times L^2$, therefore we need to prove that $S(t)(\psi_n, \phi_n, \theta_n) \to S(t)(\psi, \phi, \theta), \forall t > 0$, as $n \to \infty$. Given $t > 0$, we choose $T > t$. From the statement above we know that $\{(\psi_{0,n}, \phi_{0,n}, \theta_{0,n})\}$ is bounded and therefore by Lemma 4 there exists a solution such that

$$||\psi_n(\tau)||_{H_0^1} + ||\phi_n(\tau)||_{H_0^1} + ||\theta_n(\tau)|| \leq C, \quad 0 \leq \tau \leq T, \tag{21}$$

will hold where $(\psi_n(\tau), \phi_n(\tau), \theta_n(\tau)) = S(\tau)(\psi_{0,n}, \phi_{0,n}, \theta_{0,n})$. But from the system (5)-(9) we can see that $\left\| \frac{\partial}{\partial \tau} \psi_n \right\|_{L^2(0,T;H_0^{-1})} + \left\| \frac{\partial}{\partial \tau} \phi_n \right\|_{L^2(0,T;L^2)} \leq C$. Hence, there exists a $(\psi(\tau), \phi(\tau), \theta(\tau)) \in L^\infty(0, T; H_0^1 \times H_0^1 \times L^2)$, such that

$$\begin{aligned} &(\psi_n(\tau), \phi_n(\tau), \theta_n(\tau)) \rightharpoonup (\psi(\tau), \phi(\tau), \theta(\tau)), \text{ weakly in } L^2(0, T; H_0^1 \times H_0^1 \times L^2), \\ &\frac{\partial}{\partial \tau}\psi_n \rightharpoonup \frac{\partial}{\partial \tau}\psi, \text{ weakly in } L^2(0, T; H_0^{-1}), \ \frac{\partial}{\partial \tau}\phi_n \rightharpoonup \frac{\partial}{\partial \tau}\phi, \text{ weakly in } L^2(0, T; L^2). \end{aligned} \tag{22}$$

Thereby, using the relations above and standard compactness results we have

$$(\psi_n, \phi_n) \to (\psi, \phi) \text{ strongly in } L^2(0, T; L^2 \times L^2). \tag{23}$$

Using similar arguments as the ones above and equation (21) we can deduce with the help of Lemma 2 that for a fixed t there exists a $(\psi(t), \phi(t), \theta(t))$ such that

$$(\psi_n(t), \phi_n(t), \theta_n(t)) \rightharpoonup (\psi(t), \phi(t), \theta(t)) \text{ weakly in } H_0^1 \times H_0^1 \times L^2,$$

where $(\psi(t), \phi(t), \theta(t))$ is the solution of the problem (5)-(9) with initial conditions $(\psi_0, \phi_0, \theta_0)$. Taking into consideration Lemma 6, the arguments above imply that

$$S(t)(\psi_n, \phi_n, \theta_n) \rightharpoonup S(t)(\psi, \phi, \theta) \text{ weakly in } H_0^1 \times H_0^1 \times L^2. \tag{24}$$

To prove the strong convergence for the above sequence the energy equation (12) is used. Every solution of the system (5)-(9) verifies the energy equation, hence

$$F_1(S(t)(\psi_0, \phi_0, \theta_0)) = e^{-\delta t} F_1(\psi_0, \phi_0, \theta_0) + \int_0^t e^{-\delta(t-\tau)} G_1(S(t)(\psi_0, \phi_0, \theta_0)) d\tau, \tag{25}$$

where $(\psi(t), \phi(t), \theta(t)) = S(t)(\psi_0, \phi_0, \theta_0)$. The same will also hold for the solution $S(t)(\psi_{0,n}, \phi_{0,n}, \theta_{0,n})$, i.e.,

$$\begin{aligned} F_1(S(t)(\psi_{0,n}, \phi_{0,n}, \theta_{0,n})) =& e^{-\delta t} F_1(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \\ &+ \int_0^t e^{-\delta(t-\tau)} G_1(S(t)(\psi_{0,n}, \phi_{0,n}, \theta_{0,n})) d\tau. \end{aligned} \tag{26}$$

From our assumption that $(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \to (\psi_0, \phi_0, \theta_0)$ strongly in $H_0^1 \times H_0^1 \times L^2$, and the definition of F_1 we can deduce that $F_1(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \to F_1(\psi_0, \phi_0, \theta_0)$, as $n \to \infty$. Now rewriting the last term of (26) as

$$\int_0^t e^{-\delta(t-\tau)} G_1(S(t)(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}))d\tau$$
$$= \int_0^t e^{-\delta(t-\tau)}\Big((\delta - 2k\alpha)||S(t)\psi_{0x,n}||^2 - 2\alpha \int S(t)\phi_{0,n}|S(t)\psi_{0,n}|^2\Big)$$
$$+ \int_0^t e^{-\delta(t-\tau)}\Big((3\delta - 2\lambda)||S(t)\theta_{0,n}||^2 - \delta(1-\delta(\lambda-\delta))||S(t)\phi_{0,n}||^2 - \delta||S(t)\phi_{0x,n}||^2\Big)$$
$$+ \int_0^t e^{-\delta(t-\tau)}\Big(\int S(t)\theta_{0,n}|S(t)\psi_{0,n}|^2 - 2Re\int S(t)\theta_{0,n}S(t)\psi_{0x,n}\Big)$$
$$+ \int_0^t e^{-\delta(t-\tau)}\Big(2(\delta-\alpha)Re\int f\overline{S(t)\psi_{0,n}} + 2\int gS(t)\theta_{0,n}\Big). \tag{27}$$

By the weak convergence of relation (22) we have that

$$\liminf_{n\to\infty} \quad ||e^{-\delta(t-\tau)}S(\tau)\psi_{0,n}||_{L^2(0,t;H_0^1)} \geq ||e^{-\delta(t-\tau)}S(t)\psi_0||_{L^2(0,t;H_0^1)}, \tag{28}$$
$$\liminf_{n\to\infty} \quad ||e^{-\delta(t-\tau)}S(\tau)\phi_{0,n}||_{L^2(0,t;H_0^1)} \geq ||e^{-\delta(t-\tau)}S(t)\phi_0||_{L^2(0,t;H_0^1)}, \tag{29}$$
$$\liminf_{n\to\infty} \quad ||e^{-\delta(t-\tau)}S(\tau)\theta_{0,n}||_{L^2(0,t;L^2)} \geq ||e^{-\delta(t-\tau)}S(t)\theta_0||_{L^2(0,t;L^2)}. \tag{30}$$

Then by choosing δ small enough so that $(2k\alpha - \delta) > 0$, $(2\lambda - 3\delta) > 0$, $\delta(1-\delta(\lambda - \delta)) > 0$, the first terms of relation (27) may be rewritten as

$$\begin{aligned}\limsup_{n\to\infty} -\int_0^t e^{-\delta(t-\tau)}\Big(&(2k\alpha-\delta)||S(t)\psi_{0x,n}||^2 + (2\lambda - 3\delta)||S(t)\theta_{0,n}||^2 \\ &+\delta(1-\delta(\lambda-\delta))||S(t)\phi_{0,n}||^2 - \delta||S(t)\phi_{0x,n}||^2\Big) \\ \leq \quad -\int_0^t e^{-\delta(t-\tau)}\Big(&(2k\alpha-\delta)||S(t)\psi_{0x}||^2 + (2\lambda-3\delta)||S(t)\theta_0||^2 \\ &+\delta(1-\delta(\lambda-\delta))||S(t)\phi_0||^2 - \delta||S(t)\phi_{0x}||^2\Big).\end{aligned} \tag{31}$$

Next,

$$\Big|\int_0^t e^{-\delta(t-\tau)}\int S(t)\phi_{0,n}|S(t)\psi_{0,n}|^2 - \int_0^t e^{-\delta(t-\tau)}\int S(t)\phi_0|S(t)\psi_0|^2\Big|$$
$$\leq \int_0^t e^{-\delta(t-\tau)}||S(t)\phi_{0,n} - S(t)\phi_0||\ ||S(t)\psi_n||_4^2$$
$$+ \int_0^t e^{-\delta(t-\tau)}||S(t)\phi_0||_3\ ||S(t)\psi_{0,n} - S(t)\psi_0||\ (||S(t)\psi_{0,n}||_6 + ||S(t)\psi_0||_6).$$

Using the Sobolev Embedding Theorem and Lemma 4, produces

$$\begin{aligned}&||S(t)\psi_{0,n}||^2_{L^\infty(0,t;H_0^1)}\int_0^t ||S(t)\phi_{0,n} - S(t)\phi_0|| + ||S(t)\phi_0||_{L^\infty(0,t;H_0^1)}\times \\ &\times \int_0^t ||S(t)\psi_{0,n} - S(t)\psi_0||\ (||S(t)\psi_{0,n}||_{H_0^1} + ||S(t)\psi_0||_{H_0^1}) \to 0.\end{aligned} \tag{32}$$

Following the same procedure for the proceeding integral implies

$$\int_0^t e^{-\delta(t-\tau)}\int S(t)\theta_{0,n}|S(t)\psi_{0,n}|^2 \to \int_0^t e^{-\delta(t-\tau)}\int S(t)\theta_0|S(t)\psi_0|^2.$$

Furthermore,

$$\begin{aligned}&\left|\int_0^t e^{-\delta(t-\tau)}\int S(t)\theta_{0,n}S(t)\psi_{0x,n}-\int_0^t e^{-\delta(t-\tau)}\int S(t)\theta_0 S(t)\psi_{0x}\right|\\&\quad\leq ||S(t)\psi_{0x,n}||_{L^\infty(0,t;L^2)}\int_0^t ||S(t)\theta_{0,n}-S(t)\theta_0||\\&\quad+||S(t)\theta_0||_{L^\infty(0,t;L^2)}\int_0^t ||S(t)\psi_{0x}-S(t)\psi_{0x,n}||\to 0.\end{aligned}\tag{33}$$

Therefore by relations (25) and (28)-(33) we obtain

$$\limsup_{n\to\infty} F_1(S(t)(\psi_{0,n},\phi_{0,n},\theta_{0,n}))\leq F_1(S(t)(\psi_0,\phi_0,\theta_0)).\tag{34}$$

But by relation (24) and the compact embedding $H^1\hookrightarrow L^2$ we find that the following is true $S(t)(\psi_{0,n},\phi_{0,n})\to S(t)(\psi_0,\phi_0)$, strongly in L^2; therefore due to the above relation we have

$$\int\phi_{0,n}|\psi_{0,n}|^2\to\int\phi_0|\psi_0|^2,\ \text{and}\ \int f\overline{S(t)\psi_{0,n}}\to\int f\overline{S(t)\psi_0}.$$

Hence from the definition of F_1

$$\begin{aligned}&\limsup_{n\to\infty}(\kappa||S(t)\psi_{0,n}||_{H_0^1}+||S(t)\theta_{0,n}||^2+||S(t)\phi_{0,n}||^2_{H_0^1}\\&+(1-\delta(\lambda-\delta))||S(t)\phi_{0,n}||^2)\\&\leq(\kappa||S(t)\psi_0||_{H_0^1}+||S(t)\theta_0||^2+||S(t)\phi_0||^2_{H_0^1}+(1-\delta(\lambda-\delta))||S(t)\phi_{0,n}||^2).\end{aligned}$$

But we may consider the right hand side of the relation above as the norm of $H_0^1\times H_0^1\times L^2$ and therefore without loss of generality

$$\limsup_{n\to\infty}||S(t)(\psi_{0,n},\psi_{0,n},\theta_{0,n})||_{H_0^1\times H_0^1\times L^2}\leq||S(t)(\psi_0,\psi_0,\theta_0)||_{H_0^1\times H_0^1\times L^2}.$$

Finally due to the weak convergence, we have completed the proof. □

Theorem 3. *Let $f,g\in H_0^1(\Omega)$. The solutions $(\psi,\phi,\theta)\in C(\mathbb{R}^+,(H_0^1\cap H^2)^2\times H_0^1)$ of the problem (5)-(9) depend continuously on initial data in $(H_0^1\cap H^2)^2\times H_0^1$.*

Proof. The proof is analogue to the proof of Theorem 3.3 in [8]. □

4. **Existence of a Global Attractor.** The aim of this section is to prove the existence of a global attractor for the dynamical system $S(t)$ in the space $(H_0^1\cap H^2)^2\times H_0^1$. Some of the ideas found here where earlier developed in the work by Karachalios and Stavrakakis [7]. To apply Proposision 1 below, it is necessary to prove the asymptotic compactness of the solutions in $(H_0^1\cap H^2)^2\times H_0^1$.

Theorem 4. *Let $f,g\in L^2(\Omega)$. Then the dynamical system $S(t)$ is asymptotically compact in $H_0^1\times H_0^1\times L^2$, that is if $\{(\psi_n,\phi_n,\theta_n)\}$ is bounded in $H_0^1\times H_0^1\times L^2$ and $t_n\to\infty$, then $\{S(t)(\psi_n,\phi_n,\theta_n)\}$ is precompact.*

Proof. Ideas developed in [1] will be used. Since sequence (ψ_n,ϕ_n,θ_n) is bounded, there exists R such that $||(\psi_n,\phi_n,\theta_n)||_{H_0^1\times H_0^1\times L^2}\leq R$. Lemma 2 implies the existence of a constant $T(R)$ that

$$S(t)(\psi_n,\phi_n,\theta_n)\in B_1,\ \text{ for all } t\geq T(R),\tag{35}$$

where B_1 is the absorbing set given by relation (15). Since $t_n \to \infty$, there exists $N_1(R)$ such that if $n \geq N_1$, then $t_n \geq T(R)$, and thereby

$$S(t_n)(\psi_n, \phi_n, \theta_n) \in B_1, \quad \text{for all } n \geq N_1. \tag{36}$$

Hence there exists $(\psi, \phi, \theta) \in B_1$ such that

$$S(t_n)(\psi_n, \phi_n, \theta_n) \rightharpoonup (\psi, \phi, \theta), \text{ weakly in } H_0^1 \times H_0^1 \times L^2. \tag{37}$$

Since $t_n \to \infty$, for every $T > 0$, there exists $N_2(R, T)$ such that when $n \geq N_2$ one has $t_n - T \geq T(R)$. Therefore by relation (35)

$$S(t_n - T)(\psi_n, \phi_n, \theta_n) \in B_1 \text{ for all } n \geq N_2, \tag{38}$$

$$S(t_n - T)(\psi_n, \phi_n, \theta_n) \rightharpoonup (\psi_T, \phi_T, \theta_T), \text{ weakly in } H_0^1 \times H_0^1 \times L^2, \tag{39}$$

where $(\psi_T, \phi_T, \theta_T) \in B_1$. By relation (17) it follows that

$$S(T)(S(t_n - T)(\psi_n, \phi_n, \theta_n)) \rightharpoonup S(T)(\psi_T, \phi_T, \theta_T), \text{ weakly in } H_0^1 \times H_0^1 \times L^2, \tag{40}$$

and from the uniqueness of the solution we get that

$$(\psi, \phi, \theta) = S(T)(\psi_T, \phi_T, \theta_T). \tag{41}$$

Now from relation (37)

$$\liminf_{n \to \infty} ||S(t_n)(\psi_n, \phi_n, \theta_n)||_{H_0^1 \times H_0^1 \times L^2} \geq ||(\psi, \phi, \theta)||_{H_0^1 \times H_0^1 \times L^2}. \tag{42}$$

Let $S(T)(S(t_n - T)(\psi_n, \phi_n, \theta_n)) \in H_0^1 \times H_0^1 \times L^2$ be a solution of the system (5)-(9). Every solution satisfies (12). Hence relations (37) and (40) will give

$$\begin{aligned} F_1(S(t_n)(\psi_n, \phi_n, \theta_n)) &= e^{-\delta T} F_1(S(t_n - T)(\psi_n, \phi_n, \theta_n)) \\ &\quad + \int_0^T e^{-\delta(T-\tau)} G_1(S(\tau)(S(t_n - T)(\psi_n, \phi_n, \theta_n)))d\tau. \end{aligned} \tag{43}$$

Now since relation (38) holds, $S(t_n - T)(\psi_n, \phi_n, \theta_n)$ is bounded and therefore $e^{-\delta t} F_1(S(t_n - T)(\psi_n, \phi_n, \theta_n)) \leq Ce^{\delta T}$. Estimating the second term of (43) as in Theorem 2 implies that

$$\begin{aligned} \limsup_{n \to \infty} &\int_0^T e^{-\delta(T-\tau)} G_1(S(\tau)(S(t_n - T)(\psi_n, \phi_n, \theta_n)))d\tau \\ &\leq \int_0^T e^{-\delta(T-\tau)} G_1(S(\tau)(\psi_T, \phi_T, \theta_T))d\tau. \end{aligned} \tag{44}$$

Therefore

$$\begin{aligned} &\limsup_{n \to \infty} F_1(S(T)(S(t_n - T)(\psi_n, \phi_n, \theta_n)) \\ &\leq Ce^{\delta T} + \int_0^T e^{\delta(T-\tau)} G_1(S(T)(\psi_T, \phi_T, \theta_T))d\tau. \end{aligned} \tag{45}$$

Furthermore, for the solution $(\psi, \phi, \theta) = S(T)(\psi_T, \phi_T, \theta_T)$ we have

$$F_1(\psi, \phi, \theta) = e^{-\delta T} F_1(\psi_T, \phi_T, \theta_T) + \int_0^T e^{-\delta(T-\tau)} G_1(S(\tau)(\psi_\tau, \phi_\tau, \theta_\tau))d\tau. \tag{46}$$

Substituting relation (46) into (45) implies

$$\limsup_{n \to \infty} F_1(S(t_n)(\psi_n, \phi_n, \theta_n)) \leq C_1 e^{\delta T} + F_1(\psi, \phi, \theta), \tag{47}$$

where $e^{\delta T} F_1(\psi_T, \phi_T, \theta_T) \leq C_1 e^{\delta T}$. Let $T \to +\infty$, then

$$\limsup_{n \to \infty} F_1(S(t_n)(\psi_n, \phi_n, \theta_n)) \leq F_1(\psi, \phi, \theta).$$

The rest of the proof follows the same steps as in Theorem 2. □

Theorem 5. *Assume that* $f, g \in H_0^1(\Omega)$*. Then the dynamical system* $S(t)$ *is asymptotically compact in* $(H_0^1 \cap H^2)^2 \times H_0^1$*.*

Proof. The proof is omitted as it follows similar steps to Theorem 4. □

Proposition 1. *Assume that* X *is a metric space and* $\{S(t)\}_{t\geq 0}$ *is a semigroup of continuous operators in* X*. If* $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set and is asymptotically compact, then* $\{S(t)\}_{t\geq 0}$ *possesses a global attractor which is a compact invariant set and attracts every bounded set in* X*.*

Theorem 6. *Assume that* $f, g \in H_0^1(\Omega)$*. Then the problem (5)-(9) possesses a strong compact global attractor in* $(H_0^1 \cap H^2)^2 \times H_0^1$*, which is a compact invariant subset and attracts every bounded set in the norm topology of* $(H_0^1 \cap H^2)^2 \times H_0^1$*.*

Proof. Taking into consideration the asymptotic compactness of $S(t)$ in $H_0^1 \times H_0^1 \times L^2$ (Theorem 4), the asymptotic compactness of $S(t)$ in $(H_0^1 \cap H^2)^2 \times H_0^1$ (Theorem 5) and Proposition 1 the proof is straight forward. □

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