Manuscript submitted to
DCDS
Supplement Volume 2007

Website: www.aimSciences.org

pp. **1–4**

GLOBAL EXISTENCE FOR A WAVE EQUATION ON $\mathbb{R}^N$.

Perikles G. Papadopoulos & Nikolaos M. Stavrakakis

Department of Mathematics
National Technical University
Zografou Campus 157 80
Athens, Hellas

Abstract. We study on the initial-bountary value problem for some degenerate non-linear dissipative wave equations of Kirchhoff type:

$$u_{tt} - \phi(x)||\bigtriangledown u(t)||^{2\gamma}\Delta u + \delta u_t = f(u), \quad x \in \mathbb{R}^N, \quad t \geq 0,$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$, $\delta > 0$, $\gamma \geq 1$, $f(u) = |u|^a u$ with $a > 0$ and $(\phi(x))^{-1} = g(x)$ is a positive function lying in $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If the initial data $\{u_0, u_1\}$ are small and $||\bigtriangledown u_0|| > 0$, then the unique solution exists globally and has certain decay properties.

1. **Introduction-Preliminary Results.** In this work we study the following degenerate nonlocal quasilinear wave equation of Kirchhoff type with a weak dissipative term

$$\begin{aligned} u_{tt} - \phi(x)||\bigtriangledown u(t)||^{2\gamma}\Delta u + \delta u_t &= f(u), \quad x \in \mathbb{R}^N, \quad t \geq 0, & (1)\\ u(x,0) = u_0(x), \qquad u_t(x,0) &= u_1(x), \qquad x \in \mathbb{R}^N, & (2)\end{aligned}$$

with initial conditions u_0, u_1 in appropriate function spaces, $N \geq 3$, $\delta > 0$, $\gamma \geq 1$ and $f(u) = |u|^a u$, $a > 0$. Throughout the paper we assume that the function $\phi : \mathbb{R}^N \to \mathbb{R}$ satisfy the following condition

(G) $\phi(x) > 0$, for all $x \in \mathbb{R}^N$ and $(\phi(x))^{-1} := g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

The space $D^{1,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the energy norm $||u||_{D^{1,2}} =: \int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx$. It is known that

$$D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \ \bigtriangledown u \in (L^2(\mathbb{R}^N))^N \right\}$$

and $D^{1,2}(\mathbb{R}^N)$ is embedded continuously in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, that is, there exists $k > 0$ such that

$$||u||_{\frac{2N}{N-2}} \leq k||u||_{D^{1,2}}. \tag{3}$$

2000 *Mathematics Subject Classification.* 35A07, 35B30, 35B40, 35B45,35L15, 35L70, 35L80.

Key words and phrases. Quasilinear Hyperbolic Equations, Global Solution, Blow-Up, Dissipation, Potential Well, Concavity Method, Unbounded Domains, Kirchhoff Strings, Generalised Sobolev Spaces, Weighted L^p Spaces.

AIMS conference is partially supported by NSF.

It is known that $D^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. The space $L^2_g(\mathbb{R}^N)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product

$$(u,v)_{L^2_g(\mathbb{R}^N)} =: \int_{\mathbb{R}^N} guvdx. \tag{4}$$

It is clear that $L^2_g(\mathbb{R}^N)$ is a separable Hilbert space, too. We shall frequently use the following version of the generalized Poincaré's inequality

$$\int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx \geq \alpha \int_{\mathbb{R}^N} gu^2 dx, \tag{5}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha =: k^{-2}||g||^{-1}_{N/2}$.

To study the properties of the operator $-\phi\Delta$, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \; x \in \mathbb{R}^N, \tag{6}$$

without boundary conditions. Since for every $u, \; v \in C_0^\infty(\mathbb{R}^N)$ we have

$$(-\phi\Delta u, \; v)_{L^2_g} = \int_{\mathbb{R}^N} \bigtriangledown u \bigtriangledown v \; dx, \tag{7}$$

we may consider equation (6) as an operator equation of the form

$$A_0 u = \eta, \;\; A_0 : D(A_0) \subseteq L^2_g(\mathbb{R}^N) \to L^2_g(\mathbb{R}^N), \;\; \eta \in L^2_g(\mathbb{R}^N). \tag{8}$$

Relation (7) implies that the operator $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^N)$, is symmetric. From inequality (5) and equation (7) we have that

$$(A_0 u, \; u)_{L^2_g} \geq \alpha ||u||^2_{L^2_g}, \;\; \text{for all} \;\; u \in D(A_0). \tag{9}$$

So the operator $A_0 = -\phi\Delta$ is a symmetric, strongly monotone operator on $L^2_g(\mathbb{R}^N)$. Hence, Friedrich's extension theorem is applicable. The energy scalar product given by (7) is

$$(u, \; v)_E = \int_{\mathbb{R}^N} \bigtriangledown u \bigtriangledown v dx$$

and the energy space is the completion of $D(A_0)$ with respect to $(u, \; v)_E$. It is obvious that the energetic space X_E is the homogeneous Sobolev space $D^{1,2}(\mathbb{R}^N)$. The energy extension $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta : D^{1,2}(\mathbb{R}^N) \; \to D^{-1,2}(\mathbb{R}^N), \tag{10}$$

is defined to be the duality mapping of $D^{1,2}(\mathbb{R}^N)$. We define $D(A)$ to be the set of all solutions of equations (6), for arbitrary $\eta \in L^2_g(\mathbb{R}^N)$. Friedrich's extension A of A_0 is the restriction of the energetic extension A_E to the set $D(A)$. The operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u, \; v)_{D(A)} = (u, \; v)_{L^2_g} + (Au, \; Av)_{L^2_g}, \;\; \text{for all} \;\; u, \; v \in D(A).$$

The norm induced by the scalar product is

$$||u||_{D(A)} = \left\{ \int_{\mathbb{R}^N} g|u|^2 \; dx + \int_{\mathbb{R}^N} \phi|\Delta u|^2 \; dx \right\}^{\frac{1}{2}},$$

which is equivalent to the norm

$$||Au||_{L^2_g} = \left\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2 \; dx \right\}^{\frac{1}{2}}.$$

So we have established the evolution quartet

$$D(A) \subset D^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N) \subset D^{-1,2}(\mathbb{R}^N), \tag{11}$$

where all the embeddings are dense and compact. Finally, for later use, it is necessary to remind that the eigenvalue problem

$$-\phi(x)\Delta u = \mu u, \quad x \in \mathbb{R}^N, \tag{12}$$

has a complete system of eigensolutions $\{w_n,\ \mu_n\}$ satisfying the following properties

$$\begin{cases} -\phi\Delta w_j = \mu_j w_j, & j = 1,2,..., \quad w_j \in D^{1,2}(\mathbb{R}^N), \\ 0 < \mu_1 \leq \mu_2 \leq ..., & \mu_j \to \infty, \quad \text{as } j \to \infty. \end{cases} \tag{13}$$

In order to clarify the kind of solutions we are going to obtain for the problem (1)-(2), we give the definition of the weak solution for this problem.

Definition 5. A *weak solution* of the problem (1)-(2) is a function u such that
(i) $u \in L^2[0,T;D(A)]$, $u_t \in L^2[0,T;D^{1,2}(\mathbb{R}^N)]$, $u_{tt} \in L^2[0,T;L^2_g(\mathbb{R}^N)]$,
(ii) for all $v \in C_0^\infty([0,T] \times (\mathbb{R}^N))$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g}\, d\tau \quad + \quad \int_0^T \left(|| \bigtriangledown u(t)||^{2\gamma} \int_{\mathbb{R}^N} \bigtriangledown u(\tau) \bigtriangledown v(\tau) dx d\tau \right)$$

$$+\delta \int_0^T (u_t(\tau), v(\tau))_{L^2_g}\, d\tau \quad - \quad \int_0^T (f(u(\tau)),\, v(\tau))_{L^2_g}\, d\tau = 0, \tag{14}$$

where $f(s) = |s|^a s$, and

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x) \in D(A), \qquad u_t(x,0) = u_1(x) \in D^{1,2}(\mathbb{R}^N).$$

2. **Local-Global Existence Results.** First we state the result concerning the local solution of our problem

Theorem 1. *Let* $f(u) = |u|^a u$ *nonlinear* C^1*-function and also let* $0 \leq a \leq 4/(N-2)$, $N \geq 3$, $\delta > 0$, $\gamma \geq 1$. *If* $(u_0,\ u_1) \in D(A) \times D^{1,2}$ *and satisfy the nondegenerate condition*

$$|| \bigtriangledown u_0|| > 0,$$

then there exists $T = T(||u_0||_{D(A)}, || \bigtriangledown u_1||^2) > 0$ *such that problem (1.1)-(1.2) admits a unique local weak solution* u *satisfying*

$$u \in C(0,T;D(A)), \ \ u_t \in C(0,T;D^{1,2}).$$

Moreover, at least one of the following statements holds true, either

(i) $T = +\infty$, *or*

(ii) $\lim e(u(t)) \equiv \lim(||u_t(t)||^2_{D^{1,2}} + ||u(t)||^2_{D(A)}) = \infty$, *as* $t \to T_-$.

Proof For $T > 0$ and $R > 0$, we define the two parameter space of solutions

$$X_{T,R} \quad =: \quad \left\{ v \in C(0,T;D(A)) : v_t \in C(0,T;D^{1,2}),\ v(0) = u_0, \right.$$
$$\left. v_t(0) = u_1,\ e(v(t)) \leq R^2, \text{ for all } t \in [0,T] \right\}.$$

It is easy to see that $X_{T,R}$ can be organized as a complete metric space with the distance

$$d(u,v) =: \sup_{0 \leq t \leq T} e_1(u(t) - v(t)), \ \ \text{where } e_1(v) =: ||v_t||^2_{L^2_g} + ||v||^2_{D^{1,2}}. \tag{15}$$

We define the non-linear mapping S in the following way. For every $v \in X_{T,R}$, $u = Sv$ is the unique solution of the linear wave equation (**??**). Using the fact that $|| \bigtriangledown u_0|| \equiv M_0 > 0$, we prove that there exist $T > 0, R > 0$ such that S maps $X_{T,R}$ into itself and S is a contraction mapping with respect to the metric $d(.,.)$. By applying the Banach contraction mapping theorem, we obtain a unique solution u belonging to $X_{T,R}$. Therefore it follows from the continuity argument for wave equations that this solution u belongs to our space. For more details of the proof we refer to [3] and [4]. $\diamondsuit$

Next, we shall consider the global existence and decay properties of the nontrivial solutions for the degenerate nonlinear wave equations (1)-(2), where $\gamma \geq 1$ and $\delta = 1$ for simplicity. We note that the problem (1)-(2) has the trivial solution $u \equiv 0$. We define *energy and potential functionals* associated with the equation (1)-(2) by

$$\begin{aligned} E(u, u_t) &\equiv ||u_t||^2_{L^2_g} + J(u), &(16)\\ J(u) &\equiv \frac{1}{\gamma+1}|| \bigtriangledown u||^{2(\gamma+1)} - \frac{2}{a+2}||u(t)||^{a+2}_{L^{a+2}_g}, &(17)\end{aligned}$$

respectively, where we denote $E(t) = E(u(t), u_t(t))$ for simplicity. Then it is easy to see that $E(t) \leq E(0)$, and hence, we see that

$$|| \bigtriangledown (u(t)|| \leq \{(\gamma+1)E(t)\}^{1/(2(\gamma+1))} \leq \{(\gamma+1)E(0)\}^{1/(2(\gamma+1))}. \tag{18}$$

Also the K-positive set associated with problem (1)- (2) is

$$W_* \equiv \left\{ u \in D(A) : K(u) \equiv || \bigtriangledown u||^{2(\gamma+1)} - ||u||^{a+2}_{L^{a+2}_g} > 0 \right\} \cup \{0\}. \tag{19}$$

Then we observe the following

Proposition 2. *(i) Let* $2\gamma < a \leq \frac{4}{N-2}$, *then* W_* *is a neighborhood of* 0 *in* $D^{1,2}$.
(ii) If $u \in W_*$ *and* $a > 2\gamma$, *then*

$$0 \leq d_*^{-1}||u||^{2(\gamma+1)}_{D^{1,2}} \leq J(u) \leq E(u, u_t), \quad \textit{where } d_* = \frac{(a+2)(\gamma+1)}{(a-2\gamma)}. \tag{20}$$

Proof (i) Indeed, using the generalized Poincare's inequality, we have that

$$\begin{aligned} ||u||^{a+2}_{L^{a+2}_g} &\leq C_0||u||^{(1-\theta)(a+2)}_{L^2_g}||u||^{\theta(a+2)}_{D^{1,2}} \leq C_0||u||^{(1-\theta)(a+2)}_{L^2_g}||u||^{\theta(a+2)-2(\gamma+1)}_{D^{1,2}}||u||^{2(\gamma+1)}_{D^{1,2}}\\ &\leq \frac{C_0}{\alpha}||u||^{a-2\gamma}_{D^{1,2}}||u||^{2(\gamma+1)}_{D^{1,2}}. &(21)\end{aligned}$$

Hence, by (21) we get

$$K = ||u||^{2(\gamma+1)}_{D^{1,2}} - ||u||^{a+2}_{L^{a+2}_g} \geq (1 - \frac{C_0}{\alpha}||u||^{a-2\gamma}_{D^{1,2}})||u||^{2(\gamma+1)}_{D^{1,2}}.$$

Therefore, if $||u||^{a-2\gamma}_{D^{1,2}} \leq (k^{-\theta-2}||g||^{-1}_{N/2})^{1/(a-2\gamma)}$, then $K(u) \geq 0$ and $0 \in W_*$.
(ii) Since $a > 2\gamma$, we have for $u \in W_*$ that,

$$\begin{aligned} J(u) &= \frac{1}{\gamma+1}||u(t)||^{2(\gamma+1)}_{D^{1,2}} - \frac{2}{a+2}||u(t)||^{a+2}_{L^{a+2}_g}\\ &\geq \frac{1}{\gamma+1}||u(t)||^{2(\gamma+1)}_{D^{1,2}} - \frac{2}{a+2}||u(t)||^{2(\gamma+1)}_{D^{1,2}} = \frac{a-2\gamma}{(\gamma+1)(a+2)}||u||^{2(\gamma+1)}_{D^{1,2}}. \quad \diamondsuit\end{aligned}$$

In order to derive the decay estimate of the energy $E(t)$, we have the following proposition

Proposition 3. *Let* u *be a solution of (1)-(2) on* $[0,T]$. *If* $u \in \overline{W}_*$, $a > 2\gamma$, *and*

$$K(u) \geq \frac{1}{2}|| \bigtriangledown u||^{2(\gamma+1)}, \tag{22}$$

then we have

$$E(t) \leq \left\{E(0)^{-\gamma/(\gamma+1)} + d_0^{-1}[t-1]^+\right\}^{-(\gamma+1)/\gamma}, \tag{23}$$

for $0 \leq t \leq T$, *where* d_0 *is some positive constant if* $E(0) \leq 1$, *that is,* $|| \bigtriangledown u||^{2(\gamma+1)} \leq C_*(1+t)^{-1/\gamma}$, *where* $C_* = C_*\left(||u_0||^{2(\gamma+1)}_{D^{1,2}},\ ||u_1||_{L^2_g}\right)$.

Proof Multiplying equation (1) by $2u_t g$ and integrating over $\mathbb{R}^N$, we have

$$\frac{d}{dt}\left\{||u_t(t)||^2_{L^2_g} + \frac{1}{\gamma+1}||u(t)||^{2(\gamma+1)}_{D^{1,2}} - \frac{2}{a+2}||u(t)||^{a+2}_{L^{a+2}_g}\right\} + 2||u_t(t)||^2_{L^2_g} = 0.$$

By the definition of the energy $E(t)$, we get that

$$\frac{d}{dt}E(t) + 2||u_t(t)||^2_{L^2_g} = 0, \tag{24}$$

and hence, $E(t)$ is a non-increasing function. Moreover, from Proposition 4.1 (ii), $E(t)$ is non-negative. Integrating (24) over $[0,T]$, we have that

$$E(t) + 2\int_0^t ||u_t(s)||^2_{L^2_g} ds = E(0). \tag{25}$$

For a moment, we assume that $T > 1$. Integrating (24) over $[t, t+1]$, $0 < t < T-1$, we have

$$2\int_t^{t+1} ||u_t(s)||^2_{L^2_g} ds = E(t) - E(t-1)\ (\equiv D(t)^2). \tag{26}$$

Then there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$||u_t(t_i)||^2_{L^2_g} \leq 2D(t), \quad \text{for } i = 1, 2. \tag{27}$$

Multiplying (1)-(2) by ug and integrating over $\mathbb{R}^N$, we have

$$K(u(t)) = ||u_t(t)||^2_{L^2_g} - \frac{d}{dt}(u(t), u_t(t))_{L^2_g} - (u(t), u_t(t))_{L^2_g}. \tag{28}$$

Integrating (28) over $[t_1, t_2]$, we observe from (22) that

$$\begin{aligned} 1/2 \quad & \int_{t_1}^{t_2} || \bigtriangledown u(s)||^{2(\gamma+1)} ds \leq \int_{t_1}^{t_2} K(u(s)) ds \\ \leq \quad & \int_t^{t+1} ||u_t(s)||^2_{L^2_g} ds + \left\{\left(\int_t^{t+1} ||u_t(s)||^2_{L^2_g} ds\right)^{1/2} + \sum_{i=1}^{2} ||u_t(t_i)||_{L^2_g}\right\} \times \\ & \times \sup_{t \leq s \leq t+1} ||u(s)||_{L^2_g} \leq D(t)^2 + 5D(t)\alpha^{-1}(d_* E(t))^{1/(2(\gamma+1))}, \end{aligned} \tag{29}$$

where we used (5), (26), (27) at the last inequality. Integrating (24) over $[t, t_2]$, we have from (26) and (29) the following estimation for the energy

$$\begin{aligned} E(t) &= E(t_2) + 2\int_t^{t_2} \|u_t(s)\|^2_{L^2_g} ds \\ &\leq 2\int_{t_1}^{t_2} E(s)ds + 2\int_t^{t+1} \|u_t(s)\|^2_{L^2_g} ds \\ &\leq 4\int_t^{t+1} \|u_t(s)\|^2_{L^2_g} ds + \int_{t_1}^{t_2} \|\bigtriangledown u(s)\|^{2(\gamma+1)} ds \\ &\leq 6D(t)^2 + 10\alpha^{-1}D(t)(d_* E(t))^{1/(2(\gamma+1))} \\ &\leq 6D(t)^2 + d_*^2(10\alpha^{-1}D(t))^{2(\gamma+1)/(2\gamma+1)} + \frac{1}{2}E(t). \end{aligned}$$

Noting the fact that $2D(t)^2 \leq E(t) \leq E(0)$ (see (25), (26)), we have

$$E(t) \leq C_* D(t)^{\frac{2(\gamma+1)}{(2\gamma+1)}},$$

with

$$C_* = 2\left\{6\left(\frac{E(0)}{2}\right)^{\gamma/(2\gamma+1)} + d_*^2(10\alpha^{-1})^{2(\gamma+1)/(2\gamma+1)}\right\}.$$

Hence from relation (26) we see that

$$E(t)^{1+\gamma/(\gamma+1)} \leq 2^{-1}C_*^{2(\gamma+1)/(2\gamma+1)}\left\{E(t) - E(t+1)\right\}.$$

Noting (25) and applying [2, Lemma 2.2], we obtain estimate (23). $\diamondsuit$

For later use, we introduce a function $H(t)$ as follows

$$H(t) =: \frac{\|\bigtriangledown u_t(t)\|^2}{\|\bigtriangledown u(t)\|^{2\gamma}} + \|u(t)\|^2_{D(A)}, \quad t \geq 0. \tag{30}$$

We have the following theorem

Theorem 4. *Let* $N = 3$, $a > \max\{4\gamma - 2, 2\gamma + [a-4]^+/2\}$, *and* $\{u_0, u_1\} \in W_* \times D^{1,2}$, *with*

$$\|\bigtriangledown u_0\| > 0.$$

Also, we have that the initial conditions are suitably small, that is, we have the following inequality:

$$0 < \left\{\left(d_1 E(0)^{(\gamma-1)(2(\gamma+1))}\right)^{a-2} + d_2 E(0)^{\frac{(a+2-4\gamma)}{(2(\gamma+1))}}\right\}^{\frac{2}{a-2}} H(0) < 1, \tag{31}$$

where d_1, d_2 *are certain positive constants and* $H(0) =: \frac{\|\bigtriangledown u_1\|^2}{\|\bigtriangledown u_0\|^{2\gamma}} + \|u_0\|_{D(A)}$. *Then (1)-(2) has a unique global solution* $u \in W_*$ *such that*

$$u \in C^0([0,\infty); D(A)) \cap C^1\left([0,\infty); D^{1,2}\right) \cap C^2\left([0,\infty); L^2_g\right).$$

Moreover, we obtain the following estimates

$$\|u_t(t)\|^2_{L^2_g} + \|u_{tt}(t)\|^2_{L^2_g} \leq C(1+t)^{-1-1/\gamma}, \tag{32}$$

$$\|\bigtriangledown u_t(t)\|^2 \leq C(1+t)^{-1}, \tag{33}$$

where C *is a positive constant.*

Proof Since $u_0 \in W_*$ and W_* is an open set, putting

$$T_1 = \sup\{t \in [0,\infty); u(s) \in W_* \text{ for } 0 \le s < t\},$$

we see that $T_1 > 0$ and $u(t) \in W_*$ for $0 \le t < T_1$. If $T_1 < \infty$, then one of the following two cases happens:

$$K(u(T_1)) = 0, \quad \text{and} \quad u(T_1) \neq 0, \tag{34}$$

or

$$K(u(t)) \to \infty, \quad \text{as} \quad t \to T_1^-. \tag{35}$$

Using relation (11) if $(a-2\gamma)-(a+2)\theta > 0$ and $\theta = (3a-4)^+/(2(a+2))$, we obtain the following estimate

$$||u(t)||^{a+2}_{L^{a+2}_g} \le C_0^{a+2} || \bigtriangledown u(t)||^{(a-2\gamma)-(a+2)\theta} ||u(t)||^{(a+2)\theta}_{D(A)} || \bigtriangledown u(t)||^{2(\gamma+1)}. \tag{36}$$

Now, we see from (20), (25), (30) and (36) that

$$||u(t)||^{a+2}_{L^{a+2}_g} \le \frac{1}{2} G(t) || \bigtriangledown u(t)||^{2(\gamma+1)}, \tag{37}$$

for $0 \le t < T_1$, where we have

$$G(t) \equiv d_1 E(0)^{((a-2\gamma)-(a+2)\theta)/(2(\gamma+1))} H(t)^{(a+2)\theta/2}, \tag{38}$$

with $d_1 = 2C_0^{a+2} d_*^{((a-2\gamma)-(a+2)\theta)/(2(\gamma+1))}$. Since we have that $G(0) < 1$ for small initial data, putting

$$T_2 \equiv \sup\{t \in [0,\infty) : G(s) < 1 \quad \text{for} \quad 0 \le s < t\},$$

we see that $T_2 > 0$ and $G(t) < 1$, for $0 \le t < T_2$. If $T_2 < T_1$ $(< \infty)$, then we get that

$$G(T_2) = 1. \tag{39}$$

Relations (19) and (37) imply that

$$\begin{aligned} K(u(t)) &\ge || \bigtriangledown u(t)||^{2(\gamma+1)} - (1/2) G(t) || \bigtriangledown u(t)||^{2(\gamma+1)} \\ &\ge (1/2) || \bigtriangledown u(t)||^{2(\gamma+1)}, \end{aligned} \tag{40}$$

for $0 \le t \le T_2$. Since we have that $|| \bigtriangledown u_0|| > 0$, setting

$$T_3 \equiv \sup\{t \in [0,\infty) : || \bigtriangledown u(s)|| > 0, \quad \text{for} \quad 0 \le s < t\}, \tag{41}$$

we get that $T_3 > 0$ and $|| \bigtriangledown u(t)|| > 0$ for $0 \le t < T_3$. If we have that $T_3 < T_2$, then we obtain

$$|| \bigtriangledown u(T_3)|| = 0. \tag{42}$$

Multiplying equation (1) by $-2\Delta u_t$ and integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned} \frac{d}{dt} || \bigtriangledown u_t(t)||^2 &+ || \bigtriangledown u(t)||^{2\gamma} \frac{d}{dt} ||u(t)||^2_{D(A)} + 2 || \bigtriangledown u_t(t)||^2 \\ &= 2(\bigtriangledown f(u(t)), \ \bigtriangledown u_t(t)). \end{aligned}$$

Moreover, multiplying the previous equality by $|| \bigtriangledown u(t)||^{-2\gamma}$, for $0 \le t < T_3$, we have

$$\begin{aligned} H'(t) + 2\frac{|| \bigtriangledown u'(t)||^2}{|| \bigtriangledown u(t)||^{2\gamma}} &= 2\gamma \frac{(\bigtriangledown u(t), \bigtriangledown u_t(t)) \ || \bigtriangledown u_t(t)||^2}{|| \bigtriangledown u(t)||^{2(\gamma+1)}} + 2\frac{(\bigtriangledown f(u(t)), \bigtriangledown u_t(t))}{|| \bigtriangledown u(t)||^{2\gamma}} \\ &\equiv I_1(t) + I_2(t), \end{aligned} \tag{43}$$

where, $H(t)$ is given by (30). To estimate $I_1(t)$ we observe from (20) and (25) that

$$\begin{aligned} I_1(t) &\leq 2\gamma || \bigtriangledown u(t)||^{\gamma-1} \frac{|| \bigtriangledown u_t(t)||^3}{|| \bigtriangledown u(t)||^{3\gamma}} \\ &\leq 2\gamma (d_* E(0))^{(\gamma-1)/(2(\gamma+1))} H(t)^{1/2} \frac{|| \bigtriangledown u_t(t)||^2}{|| \bigtriangledown u(t)||^{2\gamma}}. \end{aligned}$$

Using Hölder inequality with $p^{-1} = \frac{1}{3},\ q^{-1} = \frac{1}{6},\ r^{-1} = \frac{1}{2},$ relation (11) we get that

$$\begin{aligned} |(\bigtriangledown f(u),\ \bigtriangledown u_t(t))| &\leq (a+1)||u||^a_{L^{3a}} || \bigtriangledown u||_{L^6} || \bigtriangledown u_t|| \\ &\leq C_0^{a+1}(a+1)|| \bigtriangledown u||^{a/2+1}||u||^{a/2}_{D(A)} || \bigtriangledown u_t||, \end{aligned}$$

and hence

$$I_2(t) \leq \left\{ C_0^{a+1}(a+1)|| \bigtriangledown u(t)||^{a/2+1-\gamma} \right\}^2 H(t)^{a/2} + \frac{|| \bigtriangledown u_t(t)||^2}{|| \bigtriangledown u(t)||^{2\gamma}}.$$

So we get from (43) that

$$\begin{aligned} H'(t) \quad + &\quad [1 - F(t)] \frac{|| \bigtriangledown u_t(t)||^2}{|| \bigtriangledown u(t)||^{2\gamma}} \\ &\leq \left\{ C_0^{a+1}(a+1)|| \bigtriangledown u(t)||^{a/2+1-\gamma} \right\}^2 H(t)^{a/2}, \end{aligned} \tag{44}$$

where we set

$$F(t) \equiv d_2 E(0)^{(\gamma-1)/(2(\gamma+1))} H(t)^{1/2}, \quad \text{with} \quad d_2 = 2\gamma d_*. \tag{45}$$

Since we have that $F(0) < 1$ for small initial data, putting

$$T_4 \equiv \sup \{ t \in [0, \infty) : F(s) < 1, \quad \text{for} \quad 0 \leq s < t \},$$

we observe that $T_4 > 0$ and $F(t) < 1$ for $0 \leq t < T_4$. If $T_4 < T_3$, then we see that

$$F(T_4) = 1. \tag{46}$$

Moreover, we get from (44) that

$$H(t) \leq \left\{ H(0)^{-(a-2)/2} - c_2 \int_0^t || \bigtriangledown u(s)||^{a+2-2\gamma} ds \right\}^{-2/(a-2)}, \tag{47}$$

with $c_2 = C_0^{2(a+1)}(a/2-1)(a+1)^2$. We have from (23) and (36) that

$$\begin{aligned} \int_0^t || \bigtriangledown u(s)||^{\omega} &\quad ds \\ &\leq \int_0^t (d_* E(s))^{\omega/(2(\gamma+1))} ds \\ &\leq \int_0^t [d_* \left\{ E(0)^{-\gamma/(\gamma+1)} + d_0^{-1}[s-1]^+ \right\}^{-(\gamma+1)/\gamma}]^{\omega/(2(\gamma+1))} ds \\ &\leq 2 d_*^{\omega/(2(\gamma+1))} d_0 E(0)^{(\omega-2\gamma)/(2(\gamma+1))}, \end{aligned} \tag{48}$$

under $\omega > 2\gamma$ and $E(0) \geq 1$. Denoting by $d_4 = 2c_2 d_*^{(a+2-2\gamma)/(2(\gamma+1))} d_0$ if $a > 4\gamma - 2$ then condition (31) implies

$$d_2 E(0)^{(\gamma-1)/(2(\gamma+1))} \left\{ H(0)^{-(a-2)/2} - d_4 E(0)^{(a+2-4\gamma)/(2(\gamma+1))} \right\}^{-1/(a-2)} < 1.$$

Thus, we easily see that

$$F(t) < 1,$$

for all $0 \leq t \leq T_4$, which contradicts relation (46). Hence, we have $T_4 \geq T_3$. Relations (31), (47) and (48), imply that

$$H(t) \equiv \frac{||\bigtriangledown u_t(t)||^2}{||\bigtriangledown u(t)||^{2\gamma}} + ||u(t)||^2_{D(A)} \leq (d_2 E(0)^{(\gamma-1)/(2(\gamma+1))})^{-2} \leq 1, \qquad (49)$$

for $0 \leq t < T_3$. Next, we shall show that $||\bigtriangledown u(t)|| > 0$, for $t \geq 0$. Since we have that $||\bigtriangledown u(T_3)|| = 0$, by relation (42), we see from (49) and the continuity that $||\bigtriangledown u_t(T_3)|| = 0$. We perform the change of variable $t \to T_3 - t$, then we have that $\widetilde{u}(t) = u(T_3 - t)$ satisfies the following

$$\begin{aligned} \widetilde{u}_{tt}(t) &- \phi(x)||\bigtriangledown \widetilde{u}(t)||^{2\gamma}\Delta\widetilde{u}(t) = \widetilde{u}_t(t) + f(\widetilde{u}(t)), \quad x \in \mathbb{R}^N, t \geq 0, \\ \widetilde{u}(0) &= \widetilde{u}_t(0) = 0, \quad x \in \mathbb{R}^N. \end{aligned}$$

Multiplying the above equation by $2g\widetilde{u}_t$ as in (24) and integrating it over $\mathbb{R}^N$, we have from (16) and (20) that

$$\frac{d}{dt}E(\widetilde{u}(t), \widetilde{u}_t(t)) = 2||\widetilde{u}_t(t)||^2_{L^2_g} \leq 2\left\{||\widetilde{u}_t(t)||^2_{L^2_g} + J(\widetilde{u}(t))\right\} = 2E(\widetilde{u}(t), \widetilde{u}_t),$$

i.e.,

$$E(\widetilde{u}(t), \widetilde{u}_t(t)) \leq 2\int_0^t E(\widetilde{u}(s), \widetilde{u}_t(s))ds, \quad \text{for all} \quad 0 \leq t \leq T_3.$$

Noting that $E(\widetilde{u}(0), \widetilde{u}_t(0)) = 0$ and applying the Gronwall inequality, we obtain that

$$d_*^{-1}||\bigtriangledown \widetilde{u}(t)||^{2(\gamma+1)} \leq E(\widetilde{u}(t), \widetilde{u}_t(t)) = 0,$$

that is, $||\bigtriangledown u(T_3 - t)|| = 0$, for all $0 \leq t \leq T_3$, which contradicts the condition $||\bigtriangledown u_0|| > 0$. Therefore, we get $T_3 \geq T_2$. Similarly, we get that $T_2 \geq T_1$, for small initial data. Then, since $||u(t)||_{D(A)} \leq C < \infty$ for $0 \leq t < T_1$, we have

$$K(u(t)) \leq ||\bigtriangledown u(t)||^{2(\gamma+1)} \leq C < \infty, \quad \text{for all} \quad 0 \leq t < T_1.$$

Therefore the case (35) does not happen. On the other hand, if (34) holds, the inequalities (37) and (40) are valid for $0 \leq t \leq T_1$. So, from case (34) and estimation (40) we obtain that

$$0 = K(u(T_1)) \geq (1/2)||\bigtriangledown u(T_1)||^{2(\gamma+1)} > 0,$$

which is a contradiction. Hence, we get that $T_1 = \infty$. Thus, we obtain $||\bigtriangledown u(t)|| > 0$, for all $t \geq 0$. Moreover, (23) and (49) hold for $t \geq 0$ and the local solution in the sense of theorem 2.1 can be continued globally in time.

Finally, we shall derive decay estimates of $||\bigtriangledown u_t(t)||$ and $||u_{tt}(t)||_{L^2_g}$. It follows from (20), (23) and (49) that estimate (33) is valid. Indeed, we have

$$||\bigtriangledown u_t(t)||^2 \leq (d_2 E(0)^{(\gamma-1)/(2(\gamma+1))})^{-2}||\bigtriangledown u(t)||^{2\gamma} \leq C(1+t)^{-1}.$$

Since u is a solution of equation (1)-(2), we get that

$$\begin{aligned} ||u_{tt}(t)||^2_{L^2_g} &\leq (||\bigtriangledown u(t)||^{2\gamma}||u(t)||_{D(A)} + ||u_t(t)||_{L^2_g} \\ &\quad + C_0^{a+1}||\bigtriangledown u(t)||^{(a+4)/2}||u(t)||^{(a-2)/2}_{D(A)})^2 \\ &\leq C(1+t)^{-\omega}, \end{aligned}$$

where $\omega = \min\{2, 1+1/\gamma, (a+4)/(2\gamma)\} = 1 + 1/\gamma$, and estimate (32) is proved. The proof of theorem is now completed. $\diamondsuit$

Remark 5. (More general nonlinearity). Let the non-linear function $f(u)$ satisfies the following relations

$$\begin{aligned}\int_{\mathbb{R}^N} f(u)udx &\geq k_0^{-1}\int_{\mathbb{R}^N} F(u)dx \geq 0,\ F(u) = 2\int_0^u f(\eta)d\eta,\\ |f(u)| &\leq k_1|u|^{a+1},\ |f'(u)| \leq k_2|u|^a,\end{aligned}$$

with certain positive constants $k_0, k_1, k_2 \geq 1$ for $a > 0$. The conclusion of the above theorem holds also. In that case we need to redefine $J(u)$ and $K(u)$ as

$$\begin{aligned}J(u) &\equiv \frac{1}{\gamma+1}||\bigtriangledown u||^{2(\gamma+1)} - 2\int_{\mathbb{R}^N} F(u)dx,\\ K(u) &\equiv ||\bigtriangledown u||^{2(\gamma+1)} - k_1||u||^{a+2}_{L_g^{a+2}}. \diamondsuit\end{aligned}$$

Remark 6. (Blow-Up problem for both degenerate and non-degenerate cases). For the blow-up problem for the following degenerate or non-degenerate wave equations with the blow-up term $f(u) = |u|^a u$,

$$\begin{aligned}u_{tt} - \phi(x)(\alpha &+ b||\bigtriangledown u||^{2\gamma})\Delta u + \delta u_t = |u|^a u, \qquad (50)\\ u(0) &= u_0, \quad u_t(0) = u_1,\end{aligned}$$

where $\alpha \geq 0,\ b \geq 0,\ \alpha + b > 0,\ \gamma > 0,\ \delta > 0,\ a > 0,$ we define the associated *energy functional*:

$$E(u, u_t) \equiv ||u_t||^2_{L_g^2} + (\alpha + \frac{b}{\gamma+1}||\bigtriangledown u||^{2\gamma})||\bigtriangledown u||^2 - \frac{2}{a+2}||u||^{a+2}_{L_g^{a+2}}. \qquad (51)$$

Then we see that

$$E(t) + 2\delta\int_0^t ||u_t(s)||^2_{L_g^2}ds = E(0), \qquad (52)$$

for $t \geq 0$, where we assume that $E(0) \leq 0$. To show the blow-up properties of the solutions, we implement the so-called *concavity method*. For $\gamma = 1$ we refer to the paper [3]. $\diamondsuit$

Acknowledgments. This work was partially financially supported by a grant from the Department of Mathematics at NTU, Athens and by a grant from the `Pythagoras` Basic Research Program No. 68/831 of the Ministry of Education of the Hellenic Republic.

REFERENCES

[1] G Kirchhoff , *Vorlesungen Über Mechanik*, Teubner, Leipzig, 1883.
[2] K Ono, *Global Existence and Decay Properties of Solutions for some Mildly Degenerate Nonlinear Dissipative Kirchhoff Strings*, Funkcialaj Ekvacioj, 40, (1997), 255-270.
[3] P G Papadopoulos and N M Stavrakakis, *Global Existence and Blow-Up Results for an Equation of Kirchhoff Type on* $\mathbb{R}^N$, Topological Methods in Nonlinear Analysis, 17, (2001), 91-109.
[4] P G Papadopoulos and N M Stavrakakis, *Central Manifold Theory for the Generalized Equation of Kirchhoff Strings on* $\mathbb{R}^N$, Nonlinear Analysis TMA, 61, (2005), 1343-1362.
[5] P G Papadopoulos and N M Stavrakakis, *Strong global attractor for a quasilinear nonlocal wave equation on* $\mathbb{R}^N$, Elec. Jour. Diff. Equat. (EJDE), Vol. 2006(2006), No. 77, pp. 1-10.
[6] P G Papadopoulos and N M Stavrakakis, *Compact Invariant Sets for Some Quasilinear Nonlocal Kirchhoff Strings on* $\mathbb{R}^N$, submitted.
[7] P G Papadopoulos and N M Stavrakakis,*Blow-up for a non-Degenerate non-Local Quasilinear Wave Equation on* $\mathbb{R}^N$, submitted.

E-mail address: `perispap@math.ntua.gr` & `nikolas@central.ntua.gr`