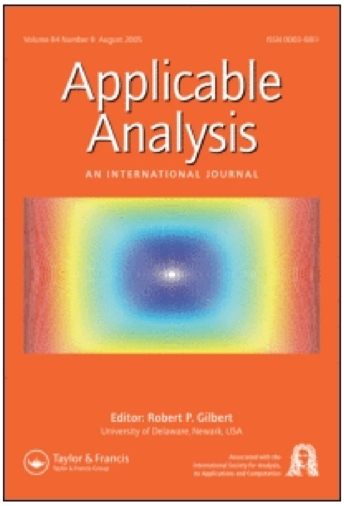

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Compact invariant sets for some quasilinear nonlocal Kirchhoff strings on <i>N</i>

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Compact invariant sets for some quasilinear nonlocal Kirchhoff strings on $\mathbb{R}^N$

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We consider the quasilinear nonlocal dissipative Kirchhoff String problem $u_{tt} - \phi(x)\| \times \nabla u(t)\|^2 \Delta u + \delta u_t + f(u) = 0$, $x \in \mathbb{R}^N$, $t \geq 0$, with the initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$, $\delta \geq 0$, $f(u) = |u|^a u$ for example, and $(\phi(x))^{-1} \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a positive function. The purpose of our work is to study the long-time behaviour of the solution of this equation. The compactness of the embeddings $D(A) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N)$, is widely applied.

Keywords: Quasilinear hyperbolic equations; Kirchhoff strings; Global attractor; Unbounded domains; Generalised sobolev spaces; Weighted L^p spaces

AMS Subject Classifications: 35A07; 35B30; 35B40; 35B45; 35L15; 35L70; 35L80; 47F05; 47H20

1. Introduction

Our aim in this work is to study the following nonlocal quasilinear hyperbolic initial value problem

$$u_{tt} - \phi(x)\| \nabla u(t)\|^2 \Delta u + \delta u_t + f(u) = 0, \quad x \in \mathbb{R}^N, \ t \geq 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with initial conditions u_0, u_1 in appropriate function spaces, $N \geq 3$, and $\delta \geq 0$. Throughout this article, we assume that the functions $\phi, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfy the following condition:

($\mathcal{G}$) $\phi(x) > 0$, for all $x \in \mathbb{R}^N$ and $(\phi(x))^{-1} =: g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

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G. Kirchhoff in 1883 [1] proposed the so-called Kirchhoff string model in the study of oscillations of stretched strings and plates

$$ph\frac{\vartheta^2 u}{\vartheta t^2}+\delta\frac{\vartheta u}{\vartheta t}=\left\{p_0+\frac{Eh}{2L}\int_0^L\left(\frac{\vartheta u}{\vartheta x}\right)^2\mathrm{d}x\right\}\frac{\vartheta^2 u}{\vartheta x^2}+f,\quad 0<x<L,\ t\geq 0, \tag{1.3}$$

where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modules, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modules and f the external force [1]. When $p_0=0$ the equation is considered to be of degenerate type, otherwise it is of nondegenerate type.

In bounded domains, there is a vast literature concerning the attractors of the semilinear wave equation

$$u_{tt}+du_t-\Delta u+f(x,u)=0,\quad x\in\Omega,\ t\geq 0,\ d\geq 0,$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega,$$

provided that the spatial variable x belongs to a bounded domain $\Omega\subset\mathbb{R}^N$ and u satisfies certain boundary conditions. We refer to the works [2,3] and the monographs [4–7]. K. Ono [8–10], for $\delta\geq 0$, has proved global existence, decay estimates, asymptotic stability and blow up results for a (mildly) degenerate nonlinear wave equation of Kirchhoff type with a strong dissipation.

On the other hand, it seems that very few results are achieved for the unbounded domain case. Using weighed Sobolev spaces, A.V. Babin and M.I. Vishik in the pioneering work [11] considered the problem of the existence of attractors for equations of parabolic type on $\mathbb{R}^N$. P. Brenner [12] studied the existence of strong global solutions for nonlinear hyperbolic equations. E. Feiresl [13,14] studied the asymptotic behaviour and compact attractors for semilinear damped wave equations on $\mathbb{R}^N$. Recently, N.I. Karahalios and N.M. Stavrakakis [15–17] proved existence of global attractors and estimated their dimension for a semilinear dissipative wave equation on $\mathbb{R}^N$.

The presentation of this article is as follows: In section 2, we discuss the space setting of the problem and the necessary embeddings for constructing the evolution triple. In section 3, we discuss the existence of a local weak solution of the problem (1.1) and (1.2) with $(u_0,u_1)\in D(A)\times\mathcal{D}^{1,2}(\mathbb{R}^N)$. In section 4, we prove the existence of an absorbing set in the space $D(A)\times\mathcal{D}^{1,2}(\mathbb{R}^N)$. We achieved global results for $N=3$, $a\in[0,\ 2/(N-2)]$ and for the initial energy we have that $E(0)\geq 0$. Then we prove that the semigroup generated by the problem possesses an invariant compact set $\mathcal{A}$. We do not call it an attractor because it is not clear if it attracts all the initial solutions.

Notation: We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for simplicity we use the symbols $C_0^\infty,\mathcal{D}^{1,2},L^p,1\leq p\leq\infty,$ for the spaces $C_0^\infty(\mathbb{R}^N),\mathcal{D}^{1,2}(\mathbb{R}^N),L^p(\mathbb{R}^N)$, respectively; $\|\cdot\|_p$ for the norm $\|.\|_{L^p(\mathbb{R}^N)}$, where in case of $p=2$ we may omit the index. The symbol $=:$ is used for definitions.

2. Space setting: formulation of the problem

As we have already seen in [18] the space setting for the initial conditions and the solutions of our problem is the product space

$$\mathcal{X}_0 =: D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N), \quad N \geq 3.$$

The homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the energy norm $\|u\|_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$. It is known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{(2N/N-2)}(\mathbb{R}^N) \colon \nabla u \in (L^2(\mathbb{R}^N))^N \right\}$$

and $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is embedded continuously in $L^{(2N/N-2)}(\mathbb{R}^N)$, that is, there exists $k > 0$ such that

$$\|u\|_{(2N/N-2)} \leq k \|u\|_{\mathcal{D}^{1,2}}. \tag{2.1}$$

The space $D(A)$ is going to be introduced and studied later in this section. We shall frequently use the following generalised version of Poincaré's inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \alpha \int_{\mathbb{R}^N} g u^2 \, dx, \tag{2.2}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha =: k^{-2} \|g\|_{N/2}^{-1}$ [19, Lemma 2.1]. It is shown that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. The space $L_g^2(\mathbb{R}^N)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product

$$(u, v)_{L_g^2(\mathbb{R}^N)} =: \int_{\mathbb{R}^N} g u v \, dx. \tag{2.3}$$

It is clear that $L_g^2(\mathbb{R}^N)$ is also a separable Hilbert space. Moreover, we have the following compact embedding.

LEMMA 2.1 *Let $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the embedding $\mathcal{D}^{1,2} \subset L_g^2$ is compact.*

Proof For the proof we refer to [16, Lemma 2.1]. ■

The following lemmas will be proved to be useful in the sequel. For the proofs we refer to [16].

LEMMA 2.2 *Let $g \in L^{(2N/2N-pN+2p)}(\mathbb{R}^N)$. Then the following continuous embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^p(\mathbb{R}^N)$ is valid, for all $1 \leq p \leq 2N/(N-2)$.*

Remark 2.3 The assumption of Lemma 2.2 is satisfied under the hypothesis $(\mathcal{G})$, if $p \geq 2$.

LEMMA 2.4 *Let g satisfy condition $(\mathcal{G})$. If $1 \leq q < p < p^* = 2N/(N-2)$, then the following weighted inequality*

$$\|u\|_{L_g^p} \leq C_0 \|u\|_{L_g^q}^{1-\theta} \|u\|_{\mathcal{D}^{1,2}}^{\theta} \tag{2.4}$$

is valid, for all $\theta \in (0,1)$, such that $1/p = (1-\theta)/q + \theta/p^*$ and $C_0 = k^\theta$.

To study the properties of the operator $-\phi\Delta$, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \qquad x \in \mathbb{R}^N, \tag{2.5}$$

without boundary conditions. Since for every $u,\ v \in C_0^\infty(\mathbb{R}^N)$ we have

$$(-\phi\Delta u, v)_{L_g^2} = \int_{\mathbb{R}^N} \nabla u \nabla v \,\mathrm{d}x, \tag{2.6}$$

we may consider equation (2.5) as an operator equation of the form

$$A_0 u = \eta, \quad A_0 : D(A_0) \subseteq L_g^2(\mathbb{R}^N) \to L_g^2(\mathbb{R}^N), \quad \eta \in L_g^2(\mathbb{R}^N). \tag{2.7}$$

Relation (2.6) implies that the operator $A_0 = -\phi\Delta$, with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^N)$, is symmetric. From (2.2) and equation (2.6) we have that

$$(A_0 u, u)_{L_g^2} \geq \alpha \|u\|_{L_g^2}^2, \quad \text{for all } u \in D(A_0). \tag{2.8}$$

So the operator $A_0 = -\phi\Delta$ is a symmetric, strongly monotone operator on $L_g^2(\mathbb{R}^N)$. Hence, Friedrich's extension theorem [20, Theorem 19.C] is applicable. The energy scalar product given by (2.6) is

$$(u, v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v \,\mathrm{d}x$$

and the energy space X_E is the completion of $D(A_0)$ with respect to $(u, v)_E$. It is obvious that the energy space is the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The energy extension $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.9}$$

is defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. We define $D(A)$ to be the set of all solutions of equations (2.5), for arbitrary $\eta \in L_g^2(\mathbb{R}^N)$. Friedrich's extension A of A_0 is the restriction of the energy extension A_E to the set $D(A)$. The operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u, v)_{D(A)} = (u, v)_{L_g^2} + (Au, Av)_{L_g^2}, \quad \text{for all } u, v \in D(A).$$

The norm induced by the scalar product is

$$\|u\|_{D(A)} = \left\{ \int_{\mathbb{R}^N} g|u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \phi|\Delta u|^2 \,\mathrm{d}x \right\}^{1/2},$$

which is equivalent to the norm

$$\|Au\|_{L_g^2} = \left\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2 \,\mathrm{d}x \right\}^{1/2}.$$

So we have established the evolution quartet

$$D(A) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^2(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.10}$$

where all the embeddings are dense and compact. Finally, for later use, it is necessary to remind that the eigenvalue problem

$$-\phi(x)\Delta u = \mu u, \qquad x \in \mathbb{R}^N, \tag{2.11}$$

has a complete system of eigensolutions $\{w_n, \mu_n\}$ satisfying the following properties

$$\begin{cases} -\phi\Delta w_j = \mu_j w_j, & j = 1, 2, \ldots, \quad w_j \in \mathcal{D}^{1,2}(\mathbb{R}^N), \\ 0 < \mu_1 \le \mu_2 \le \cdots, & \mu_j \to \infty, \qquad \text{as } j \to \infty. \end{cases} \tag{2.12}$$

For the positive self-adjoint operator $A = -\varphi\Delta$, we may define the fractional powers in the following way. For every $s > 0$, A^s is an unbounded self-adjoint operator in $L^2_g(\mathbb{R}^N)$ with its domain $D(A^s)$ to be a dense subset in $L^2_g(\mathbb{R}^N)$. The operator A^s is strictly positive and injective. Also $D(A^s)$, endowned with the scalar product

$$(u, v)_{D(A^s)} = (u, v)_{L^2_g} + (A^s u, A^s v)_{L^2_g},$$

becomes a Hilbert space. We write as usual $V_{2s} = D(A^s)$ and we have the following identifications

$$D(A^{-1/2}) = \mathcal{D}^{-1,2}(\mathbb{R}^N), \quad D(A^0) = L^2_g, \quad D(A^{1/2}) = \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Moreover, the mapping

$$A^{s/2} : V_x \longrightarrow V_{x-s}$$

is an isomorphism. Furthermore, as a consequence of the relation (2.10) the injection $D(A^{s_1}) \subset D(A^{s_2})$ is compact and dense, for every $s_1, s_2 \in R$, $s_1 > s_2$. For more on the fractional spaces, we refer to Henry [21]. Finally, we give the definition of weak solutions for the problems, (1.1) and (1.2).

Definition 2.5 A weak solution of the problems (1.1) and (1.2) is a function u such that

(i) $u \in L^2[0, T; D(A)]$, $\quad u_t \in L^2[0, T; \mathcal{D}^{1,2}(\mathbb{R}^N)]$, $\quad u_{tt} \in L^2[0, T; L^2_g(\mathbb{R}^N)]$,
(ii) for all $v \in C_0^\infty([0, T] \times (\mathbb{R}^N))$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g} \,d\tau + \int_0^T \left(\| \nabla u(t) \|^2 \int_{\mathbb{R}^N} \nabla u(\tau) \nabla v(\tau) dx \,d\tau \right)$$
$$+ \delta \int_0^T (u_t(\tau), v(\tau))_{L^2_g} \,d\tau + \int_0^T (f(u(\tau)), v(\tau))_{L^2_g} \,d\tau = 0, \tag{2.13}$$

where $f(s) = |s|^a s$, and
(iii) satisfies the initial conditions

$$u(x, 0) = u_0(x), \quad u_0 \in D(A), \qquad u_t(x, 0) = u_1(x), \quad u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Remark 2.6 Using a density argument, we may see that the generalised formula (2.13) is satisfied for every $v \in L^2[0, T; \ \mathcal{D}^{1,2}(\mathbb{R}^N)]$. By the compactness and density of the embeddings in the evolution quartet (2.10) we may see that, as in [15, Proposition 3.2], the above Definition 2.5 of weak solutions implies that

$$u \in C[0, T; \ \mathcal{D}^{1,2}(\mathbb{R}^N)] \quad \text{and} \quad u_t \in C[0, T; \ L^2_g(\mathbb{R}^N)].$$

3. Existence and uniqueness of solution

In this section, we give existence and uniqueness results for the problems (1.1) and (1.2) using the space setting established previously. Let $(w, w_t); \in C(0, T;\ D(A) \times \mathcal{D}^{1,2})$ be given. In order to obtain a local existence result for the problems (1.1) and (1.2), we need information concerning the solvability of the corresponding nonhomogeneous linearised (around the function w) problem restricted to the sphere B_R:

$$\begin{aligned} &u_{tt} - \phi(x)\|\nabla w(t)\|^2 \Delta u + \delta u_t + f(w) = 0, \quad (x,t) \in B_R \times (0, T), \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in B_R, \\ &u(x,t) = 0, \quad (x,t) \in \partial B_R \times (0, T). \end{aligned} \tag{3.1}$$

PROPOSITION 3.1 *Assume that* $u_0 \in D(A)$, $u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and* $0 \le a \le 4/(N-2)$, *where* $N \ge 3$. *Then the linear wave equation* (3.1) *has a unique solution such that*

$$u \in C(0, T; D(A)) \quad \text{and} \quad u_t \in C\big(0, T;\ \mathcal{D}^{1,2}\big).$$

Proof The proof follows the lines of [16, Proposition 3.1]. The Galerkin method is used, based on the information taken from the eigenvalue problem (2.11). ∎

THEOREM 3.2 *Consider that* $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and satisfy the nondegenerate condition*

$$\|\nabla u_0\| > 0. \tag{3.2}$$

Then there exists $T = T(\|u_0\|_{D(A)}, \|\nabla u_1\|^2) > 0$ *such that the problems* (1.1) and (1.2) *admits a unique local weak solution u satisfying*

$$u \in C(0, T; D(A)) \ \text{and} \ u_t \in C(0, T; \mathcal{D}^{1,2}).$$

Proof For the proof we refer to [18, Theorem 3.2]. ∎

THEOREM 3.3 *Assume that* $f(u) = |u|^a u$ *is a nonlinear* C^1 *function and* $0 \le a \le 4/(N-2)$, *where* $N \ge 3$. If $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ *and satisfy the nondegenerate condition*

$$\|\nabla u_0\| > 0, \tag{3.3}$$

then there exists $T > 0$ *such that the problems* (1.1) and (1.2) *admits a unique local weak solution u satisfying*

$$u \in C(0, T; \mathcal{D}^{1,2}) \quad \text{and} \quad u_t \in C(0, T; L^2_g).$$

Proof The proof follows the lines of [18, Theorem 3.2]. In this case, because of the compact embedding $\mathcal{X}_0 \subset \mathcal{X}_1 =: \mathcal{D}^{1,2} \times L^2_g$, we obtain for the associated norms that

$$e_1(u(t)) \le e(u(t)),$$

where $e_1(u(t)) =: \|u\|^2_{\mathcal{D}^{1,2}} + \|u'\|^2_{L^2_g}$ and $e(u(t)) =: \|u\|^2_{D(A)} + \|u'\|^2_{\mathcal{D}^{1,2}}$. Following the same steps as in Theorem 3.2 we take the inequality

$$e_1(u(t)) \le e(u(t)) \le R^2,$$

where R is a positive parameter. So, u is a solution such that

$$u \in L^\infty(0, T;\ V), \quad u' \in L^\infty(0, T;\ H).$$

The continuity properties, are also proved with the methods indicated in [7, sections II.3 and II.4]. Finally, the uniqueness of the solution can also be taken from [7, Proposition 4.1, p. 215]. ■

4. Global existence of the solution

In this section, we shall prove that solution of (3.1) exists globally in time. We set $v = u_t + \varepsilon u$ for sufficiently small ε. Then following [7, page 207], for calculation needs we rewrite (3.1) as follows

$$v_t + (\delta - \varepsilon)v + \big(-\phi(x)\|\nabla w\|^2\Delta - \varepsilon(\delta - \varepsilon)\big)u + f(w) = 0. \tag{4.1}$$

To obtain the estimates needed, we shall implement the Faedo–Galerkin approximation based on the eigenfunctions w_j of the eigenvalue problem (2.11). For each n, we look for an approximating solution of the form $u^n(t,x) = \sum_{i=1}^n b_{in}(t)w_i$. We also define the quantity $v^n = u^n_t + \varepsilon u^n$. The Galerkin system for the linear problem of (4.1) is

$$(v^n_t, w_i)_{L^2_g} + (\delta - \varepsilon)(v^n, w_i)_{L^2_g} + \int_{B_R} \|\bigtriangledown w\|^2 \bigtriangledown u^n \bigtriangledown w_i \,\mathrm{d}x - \varepsilon(\delta - \varepsilon)(u^n, w_i)_{L^2_g} + (f(w), w_i)_{L^2_g} = 0, \tag{4.2}$$

$$u^n(x,0) = P_n u_0(x), \quad u^n_t(x,0) = P_n u_1(x), \tag{4.3}$$

around the function $w \in C(0,T; D(A))$, with $w_t \in C(0,T;\mathcal{D}^{1,2})$, where P_n is the continuous orthogonal projector operator of the spaces $D(A)(B_R)$ and $\mathcal{D}^{1,2}(B_R)$ into the span $\{w_i\colon i = 1,2,\ldots,n\}$. For every eigenvalue μ_j and eigenfunctions w_j we have that

$$\mu_j(v, w_j)_{L^2_g} = \int_{\mathbb{R}^N} g v \mu_j w_j = \int_{\mathbb{R}^N} g v(-\varphi\Delta w_j) = ((v, w_j))_{\mathcal{D}^{1,2}}, \tag{4.4}$$

$$\mu_j \int_{\mathbb{R}^N} \bigtriangledown v \bigtriangledown w_j = \int_{\mathbb{R}^N} g(-\varphi\Delta v)\mu_j w_j = \int_{\mathbb{R}^N} g(-\varphi\Delta v)(-\varphi\Delta w_j) = \int_{\mathbb{R}^N} \varphi\Delta v\Delta w_j, \tag{4.5}$$

where $v \in \mathcal{D}^{1,2}$ and $v \in D(A)$, respectively.

We set $w_j = \mu_i(b'_{in}(t) + \varepsilon b_{in}(t))$ in relation (4.2), summarise for i from 1 to n, integrate over the ball B_R and use relations (4.4), (4.5) to obtain the inequality

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|w\|^2_{\mathcal{D}^{1,2}}\|u^n\|^2_{D(A)} + \|v^n\|^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}\|u^n\|^2_{\mathcal{D}^{1,2}}\right\} + (\delta - \varepsilon)\|v^n\|^2_{\mathcal{D}^{1,2}} + \varepsilon\|w\|^2_{\mathcal{D}^{1,2}}\|u^n\|^2_{D(A)} \\ &\quad + \varepsilon^2(\delta-\varepsilon)\|u^n\|^2_{\mathcal{D}^{1,2}} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\mathcal{D}^{1,2}}\right)\|u^n\|^2_{D(A)} - \int_{B_R} \bigtriangledown(f(w))\bigtriangledown v^n \mathrm{d}x \\ &\quad \le \left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\mathcal{D}^{1,2}}\right)\|u^n\|^2_{D(A)}\right| + \left|\int_{B_R} \bigtriangledown(f(w))\bigtriangledown v^n \,\mathrm{d}x.\right| \end{aligned} \tag{4.6}$$

We observe that

$$\left|\int_{B_R} \bigtriangledown(f(w))\bigtriangledown v^n \,\mathrm{d}x\right| = \left|\int_{B_R} (f(w))' \bigtriangledown w \bigtriangledown v^n \,\mathrm{d}x\right| \le k_2\|w\|^a_{L^{Na}}\|\bigtriangledown w\|_{L^{(2N/N-2)}}\|\bigtriangledown v^n\|, \tag{4.7}$$

where we used Hölder inequality with $p^{-1} = 1/N$, $q^{-1} = (N-2/2N)$, $r^{-1} = 1/2$. Then applying relation (4.7), inequality (4.6) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|w\|^2_{\mathcal{D}^{1,2}}\|u^n\|^2_{D(A)} + \|v^n\|^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}\|u^n\|^2_{\mathcal{D}^{1,2}}\right\}$$
$$+ (\delta-\varepsilon)\|v^n\|^2_{\mathcal{D}^{1,2}} + \varepsilon\|w\|^2_{\mathcal{D}^{1,2}}\|u^n\|^2_{D(A)} + \varepsilon^2(\delta-\varepsilon)\|u^n\|^2_{\mathcal{D}^{1,2}}$$
$$\leq \left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\mathcal{D}^{1,2}}\right)\|u^n\|^2_{D(A)}\right| k_2\|w\|^a_{L^{Na}}\|\nabla w\|_{L^{(2N/N-2)}}\|\nabla v^n\|. \tag{4.8}$$

Next, to define the energy associated with the equation (1.1), we multiply equation (1.1) by $2gu_t$ and integrate over $\mathbb{R}^N$ to get the following relation (for simplicity we set $\delta = 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} + \frac{2}{a+2}\|u(t)\|^{a+2}_{L^{a+2}_g}\right\} + 2\delta\|u_t(t)\|^2_{L^2_g} = 0. \tag{4.9}$$

Then, we define as the energy functional of the problems (1.1) and (1.2) the quantity

$$E(t) =: E(u(t), u_t(t)) =: \|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} + \frac{2}{a+2}\|u(t)\|^{a+2}_{L^{a+2}_g}. \tag{4.10}$$

Then, equation (4.9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + 2\|u_t(t)\|^2_{L^2_g} = 0. \tag{4.11}$$

Concerning the time behaviour of the energy, we have the following remarks. Integrate equation (4.11) over $[0, t]$, to get

$$E(t) + 2\int_0^t \|u_t(t)\|^2_{L^2_g}\,\mathrm{d}x = E(0). \tag{4.12}$$

From equation (4.11) and definition (4.10), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u, u_t) = -2\|u_t(t)\|^2_{L^2_g} \leq 0. \tag{4.13}$$

Therefore, the energy $E(t)$ is a nonincreasing function of t. Hence, we get that

$$E(t) \leq E(0), \quad \text{for all} \quad t \in [0, T). \tag{4.14}$$

For the energy $E(t)$ we have the following estimate (for the proof we refer to [18, Theorem 4.3, p. 102]

$$\frac{\|u_t\|^2_{L^2_g}}{\|\nabla u(t)\|^2} + d_*^{-1}\|\nabla u\|^2 \leq \left\{E(u_0, u_1)^{-1/2} + d_0^{-1}[t-1]^+\right\}^{-2}, \tag{4.15}$$

where d_*, is a positive constant $d_0 \geq 1$, and $[t-1]^+ = \max\ \{0, t-1\}$, for $0 \leq t < T$. Next, we prove the following lemma.

Lemma 4.1 *Assume that $f(u)$ is a C^1-function, $a \geq 0$, $N \geq 3$. If the initial data $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ and satisfy the condition*

$$\|\nabla u_0\| > 0, \tag{4.16}$$

then we have that

$$\|\nabla u(t)\| > 0, \quad \text{for all } t \geq 0. \tag{4.17}$$

Proof Let $u(t)$ be a unique solution of the problems (1.1) and (1.2) in the sense of Theorem 3.2 on $[0, T)$. Multiplying (1.1) by $-2\Delta u_t$ and integrating it over $\mathbb{R}^N$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_t(t)\|^2 + \|\nabla u(t)\|^2 \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{D(A)} + 2\|\nabla u_t(t)\|^2 + 2(f(u(t)), \Delta u_t(t)) = 0 \tag{4.18}$$

Since $\|\nabla u_0\| > 0$ by (4.16), we see that $\|\nabla u(t)\| > 0$ near $t = 0$. Let

$$T =: \sup\{t \in [0, +\infty) : \|\nabla u(s)\| > 0 \quad \text{for } 0 \leq s < t\},$$

then $T > 0$ and $\|\nabla u(t)\| > 0$ for $0 \leq t < T$. If $T < +\infty$, we have

$$\lim_{t \to T_-} \|\nabla u(t)\| = 0. \tag{4.19}$$

So, using (4.19) we see from (4.15) that $\lim_{t\to T_-}\|u_t(t)\|_{L^2_g}$ must be zero. We perform the change of variable $t/T - t$. Then $\widetilde{u}(t) = u(T - t)$ satisfies

$$\begin{aligned} &\widetilde{u}''(t) - \phi(x)\|\nabla\widetilde{u}(t)\|^2\Delta\widetilde{u}(t) - \widetilde{u}_t(t) + f(\widetilde{u}(t)) = 0, \quad x \in \mathbb{R}^N, \ t \geq 0, \\ &\widetilde{u}(0) = 0, \quad \widetilde{u}'(0) = 0, \quad x \in \mathbb{R}^N. \end{aligned} \tag{4.20}$$

We note that $\widetilde{u} \in C^0([0, T]; D(A)) \cap C^1([0, T]; \mathcal{D}^{1,2})$. Multiplying (4.20) by $2g\widetilde{u}'$ and integrating it over $\mathbb{R}^N$, we have an equation similar to (4.11), that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\widetilde{u}(t), \widetilde{u}'(t)) = 2\|\widetilde{u}'(t)\|^2_{L^2_g} \leq 2E(\widetilde{u}(t), \widetilde{u}'(t)).$$

Integrating it over $[0, t]$, we have

$$E(\widetilde{u}(t), \widetilde{u}'(t)) \leq 2\int_0^t E(\widetilde{u}(s), \ \widetilde{u}'(s))\mathrm{d}s,$$

for $0 \leq t \leq T$. Noting $E(\widetilde{u}(0), \widetilde{u}'(0)) = 0$ and applying the Gronwall inequality we see that

$$E(\widetilde{u}(t), \widetilde{u}'(t)) = 0, \quad \text{on } [0, T],$$

that is, $\|\nabla u(T - t)\| = 0$ on $[0, T]$ which contradicts $\|\nabla u_0\| > 0$, and hence, we obtain $T = +\infty$ and

$$\|\nabla u(t)\| > 0, \quad \text{for all } t \geq 0. \tag{4.21}$$

■

Now, we need the following result.

LEMMA 4.2 *Assume that* $0 \leq a < 2/(N-2)$, $N \geq 3$, $\|\nabla u_0\| > 0$ *and*

$$\rho_1 > 4\alpha^{-1/2}R^2c_3^2. \tag{4.22}$$

Then the unique local solution defined by Theorem 3.2 *exists globally in time.*

Proof Given the constants $T > 0$, $R > 0$, we introduce the two parameter space of solutions

$$X_{T,R} =: \{w \in C(0,T; D(A)) : w_t \in C(0,T; \mathcal{D}^{1,2}), w(0) = u_0,\\ w_t(0) = u_1,\ e(w) \le R^2,\ t \in [0,T]\},$$

where $e(w) =: \|w_t\|^2_{\mathcal{D}^{1,2}} + \|w\|^2_{D(A)}$. Also u_0 satisfies the nondegenerate condition (3.2). It is easy to see that the set $X_{T,R}$ is a complete metric space under the distance $d(u,v) =: \sup_{0 \le t \le T} e(u(t) - v(t))$. We may introduce the notation

$$M_0 =: \frac{1}{2}\|\nabla u_0\|^2, T_0 =: \sup\{t \in [0,\infty) : \|\nabla w(s)\|^2 > M_0,\ 0 \le s \le t\}.$$

By condition (3.2) and the relation (4.21), we may see that $M_0 > 0$, $T_0 > 0$ and $\|\nabla w(t)\|^2 > M_0 > 0$, for all $t \in [0, T_0]$. Multiplying equation (3.1) by

$$gAv = g(-\varphi\Delta)v = -\Delta v = -\Delta(u_t + \varepsilon u),$$

where $v = u_t + \varepsilon u$ and integrating over $\mathbb{R}^N$, we obtain an inequality analog to (4.26) on all of $\mathbb{R}^N$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|w\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \|v\|^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}\|u\|^2_{\mathcal{D}^{1,2}}\right\} + (\delta-\varepsilon)\|v\|^2_{\mathcal{D}^{1,2}} + \varepsilon\|w\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)}\\ + \varepsilon^2(\delta-\varepsilon)\|u\|^2_{\mathcal{D}^{1,2}}\\ \le \left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\mathcal{D}^{1,2}}\right)\|u\|^2_{D(A)}\right| + k_2\|w\|^a_{L^{Na}}\|\nabla w\|_{L^{(2N/N-2)}}\|\nabla v\|. \tag{4.23}$$

We observe that

$$\theta(t) =: \|w\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \|v\|^2_{\mathcal{D}^{1,2}} + \frac{\varepsilon(\delta-\varepsilon)}{2}\|u\|^2_{\mathcal{D}^{1,2}}\\ \ge \|w\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \|v\|^2_{\mathcal{D}^{1,2}}\\ \ge M_0\|u\|^2_{D(A)} + \|u_t\|^2_{\mathcal{D}^{1,2}} \ge c_3^{-2}e(u), \tag{4.24}$$

with $c_3 =: \left(\max\{1, M_0^{-1}\}\right)^{1/2}$. We also have that

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\mathcal{D}^{1,2}}\right)\|u\|^2_{D(A)}\right| = \left|\left(2\int_{\mathbb{R}^N}\Delta w w_t \varphi g\,\mathrm{d}x\right)\|u\|^2_{D(A)}\right|\\ \le 2\left(\|w\|^2_{D(A)}\right)^{1/2}\left(\|w_t\|^2_{L^2_g}\right)^{1/2}\|u\|^2_{D(A)}\\ \le 2\alpha^{-1/2}R\|w_t\|_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)}\\ \le 2\alpha^{-1/2}R^2 e(u) \le 2\alpha^{-1/2}R^2c_3^2\ \theta(t). \tag{4.25}$$

By relations (4.24) and (4.25) the inequality (4.23) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) + (\delta-\varepsilon)\|v\|^2_{\mathcal{D}^{1,2}} + \varepsilon\|w\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \frac{2\varepsilon\varepsilon(\delta-\varepsilon)}{2}\|u\|^2_{D(A)}\\ \le 2\alpha^{-1/2}R^2c_3^2\ \theta(t) + k_2\|w\|^a_{L^{Na}}\|\nabla w\|_{L^{(2N/N-2)}}\|\nabla v\|. \tag{4.26}$$

We also have that

$$\|w\|^a_{L^{Na}} \leq R^a \quad \text{and} \quad \| \nabla w\|_{L^{(2N/N-2)}} \leq \|w\|_{D(A)} \leq R. \tag{4.27}$$

Applying Young's inequality for $\varepsilon = \delta/2$, in the last term of (4.26) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) + \frac{\rho_1}{2}\theta(t) - 2\alpha^{-1/2}R^2 c_3^2\, \theta(t) \leq \frac{C(R)}{\delta}, \tag{4.28}$$

where $\rho_1 = \min((\delta/2) - \varepsilon, \varepsilon, 2\varepsilon)$ and $C(R) = k_2 R^{2(a+1)}$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) + C_*\theta(t) \leq \frac{C(R)}{\delta}, \tag{4.29}$$

where $C_* = (1/2)\big(\rho_1 - 4\alpha^{-1/2}R^2c_3^2\big) > 0$. Applying Gronwall's Lemma in (4.29) we get

$$\theta(t) \leq \theta(0)\, e^{-C_* t} + \frac{1 - e^{-C_* t}}{C_*}\frac{C(R)}{\delta}. \tag{4.30}$$

By using the nondegenerate condition $\|\nabla u_0\| > 0$ and relation (4.21), we may assume that $\|\nabla w(s)\| > M_0 > 0$, $0 \leq s \leq t$, $t \in [0, +\infty)$. Letting $t \to \infty$, in relation (4.30) we conclude that

$$\lim_{t\to\infty} \sup \theta(t) \leq \frac{C(R)}{\delta C_*} =: R_*^2. \tag{4.31}$$

From inequality (4.31) and following the arguments of Theorem 3.2 [18], we conclude that the solution of (3.1) exists globally in time. ■

Remark 4.3 (Global solutions) From the last Lemma 4.2 we may observe that solutions of the problems (1.1) and (1.2), (given by Theorem 3.2), belong to the space $C_b(\mathbb{R}_+, \chi_0)$, the space of bounded continuous functions from $\mathbb{R}_+$ to χ_0, i.e., we have achieved global solutions for the given problem. We achieved global results for $\alpha \in [0, 2/(N{-}2))$, for $N = 3$ and for the initial energy $E(0) \geq 0$.

Finally Lemma 4.1 has an immediate consequence:

Remark 4.4 A nonlinear semigroup $S(t) : \chi_0 \to \chi_0$, $t \geq 0$, may be associate to the problems (1.1) and (1.1) such that for $\psi = \{u_0, u_1\} \in \chi_0$, $S(t)\psi = \{u(t), u_t(t)\}$ is the weak solution of the problems (1.1) and (1.2). Moreover the ball $B_0 =: B_{\chi_0}(0, \bar{R}_*)$ for any $\bar{R}_* > R_*$, where R_* defined by (4.31), is an absorbing set for the semigroup $S(t)$ in the energy space $\chi_0 \subset \chi_1$, compactly.

In the rest of this article we show that the ω-limit set of the absorbing set is a compact invariant set. To this end, we need to decompose the semigroup $S(t)$, in the form $S(t) = S_1(t) + S_2(t)$, where for a suitable bounded set $\mathcal{B} \subset \chi_0$ the semigroups $S_1(t)$, $S_2(t)$ satisfy the following properties

($\mathcal{S}_1$) $S_1(t)$ is uniformly compact for t large, i.e., $\cup_{t\geq t_0} S_1(t)\mathcal{B}$ is relatively compact in χ_1.
($\mathcal{S}_2$) $\sup_{k\in\mathcal{B}} \|S_2(t)k\|_{\chi 1} \to 0$, as $t \to \infty$.

The next lemma implies that the semigroup associated with the problem (4.38), satisfy the property ($\mathcal{S}_2$).

LEMMA 4.5 *For the initial value problem*

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t = 0, \quad x \in \mathbb{R}^N, \quad t \in [0, T],$$
$$u(.,0) = u_0 \in D(A), \quad u_t(.,0) = u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{4.32}$$

there exists a unique solution such that

$$u \in C_b\big(\mathbb{R}_+, \mathcal{D}^{1,2}\big), \quad u_t \in C_b\Big(\mathbb{R}_+, L^2_g\Big).$$

Moreover, this solution decays exponentially, as $t \to \infty$.

Proof We proceed as in [15, Proposition 3.2] and the Lemma 4.2 to obtain the estimate

$$\begin{aligned}&\|w\|^2_{\mathcal{D}^{1,2}}\|u\|^2_{D(A)} + \|u_t + \varepsilon u\|^2_{\mathcal{D}^{1,2}} + \rho_2\|u\|^2_{\mathcal{D}^{1,2}} \\ &\quad \le \Big\{\|w_0\|^2_{\mathcal{D}^{1,2}}\|u_0\|^2_{D(A)} + \|u_1 + \varepsilon u_0\|^2_{\mathcal{D}^{1,2}} + \rho_2\|u_0\|^2_{\mathcal{D}^{1,2}}\Big\} e^{-C_* t},\end{aligned}$$

where $\rho_2 = \varepsilon(\delta - \varepsilon)/2$ and $C_* > 0$. The last estimate apart of giving the existence and uniqueness results for the problem (4.32), implies also the exponential decay of solutions, by letting $t \to \infty$. ■

Concerning property $(\mathcal{S}_1)$, we need to prove the following three lemmas. By $C_b(X, Y)$ we denote the set of bounded continuous functions from X to Y.

LEMMA 4.6 *Consider the initial value problem*

$$\bar{u}_{tt} - \phi(x)\|\nabla \bar{u}(t)\|^2 \Delta \bar{u} + \delta \bar{u}_t + f(u) = 0, \quad x \in \mathbb{R}^N, \quad t \in [0, T],$$
$$\bar{u}(.,0) = \bar{u}_0 \in D(A), \quad \bar{u}_t(.,0) = \bar{u}_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{4.33}$$

where u denotes the solution of the original problem given by Theorem 3.2. *Then problem* (4.33) *posses a unique solution such that*

$$\bar{u} \in C_b\big(\mathbb{R}_+, \mathcal{D}^{1,2}\big), \quad \bar{u}_t \in C_b\Big(\mathbb{R}_+, L^2_g\Big).$$

Proof Working as in Lemmas 4.2 and 4.5, we obtain the inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|w\|^2_{\mathcal{D}^{1,2}}\|\bar{u}\|^2_{D(A)} + \|\bar{v}\|^2_{\mathcal{D}^{1,2}} + \rho_2\|\bar{u}\|^2_{\mathcal{D}^{1,2}}\Big\} \\ &\quad + (\delta - \varepsilon)\|\bar{v}\|^2_{\mathcal{D}^{1,2}} + \varepsilon\|w\|^2_{\mathcal{D}^{1,2}}\|\bar{u}\|^2_{D(A)} + \varepsilon^2(\delta - \varepsilon)\|\bar{u}\|^2_{\mathcal{D}^{1,2}} \\ &\quad \le \left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{\mathcal{D}^{1,2}}\right)\|\bar{u}\|^2_{D(A)}\right| + \left|\int_{\mathbb{R}^N} \nabla(f(w))\nabla \bar{v}\,\mathrm{d}x.\right|\end{aligned}$$

Standard procedure gives the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|w\|^2_{\mathcal{D}^{1,2}}\|\bar{u}\|^2_{D(A)} + \|\bar{v}\|^2_{\mathcal{D}^{1,2}} + \rho_2\|\bar{u}\|^2_{\mathcal{D}^{1,2}}\Big\} + C_*\Big\{\|w\|^2_{\mathcal{D}^{1,2}}\|\bar{u}\|^2_{D(A)} + \|\bar{v}\|^2_{\mathcal{D}^{1,2}} + \rho_2\|\bar{u}\|^2_{\mathcal{D}^{1,2}}\Big\} \le \tilde{C},$$

where $\tilde{C} = C(R)/\delta$ and $C(R)$, C_*, are defined in Lemma 4.2. Using Gronwall's lemma we have

$$\begin{aligned}&\|w\|^2_{\mathcal{D}^{1,2}}\|\bar{u}\|^2_{D(A)} + \|\bar{u}_t + \varepsilon\bar{u}\|^2_{\mathcal{D}^{1,2}} + \rho_2\|\bar{u}\|^2_{\mathcal{D}^{1,2}}\\ &\quad\leq \left\{\|w_0\|^2_{\mathcal{D}^{1,2}}\|\bar{u}_0\|^2_{D(A)} + \|\bar{u}_1 + \varepsilon\bar{u}_0\|^2_{\mathcal{D}^{1,2}} + \rho_2\|\bar{u}_0\|^2_{\mathcal{D}^{1,2}}\right\}e^{-C_* t} + \tilde{C}\big(1 - e^{-C_* t}\big).\end{aligned}$$

Finally, letting $t \to \infty$, we obtain the result. ■

This lemma gives the existence of the semigroup S_1. By $\mathcal{L}(X, Y)$ we denote the set of linear continuous mappings from X to Y. To prove uniform compactness for t large, i.e., property $(\mathcal{S}_1)$, we need the next technical lemma.

LEMMA 4.7 *Let $f(s) = \|s\|^a s$, where $0 \leq a \leq N/(2(N-2))$. Then there exists $\delta \in (0,1)$, such that for every $\varphi \in D(A)$ the functional $f'(\varphi) \in \mathcal{L}(\mathcal{D}^{1,2}, V_{-\delta})$ and for every $R > 0$*

$$\sup_{\|u\|_{D(A)} \leq R} \left|f'(\varphi)\right|_{\mathcal{L}\left(\mathcal{D}^{1,2},\, V_{-\delta}\right)} < \infty.$$

Proof For every $\delta \in (0,1)$ we have the following compact embedding $V_\delta \subset V_0 \equiv L^2_g(\mathbb{R}^N)$. Let $\psi \in V_\delta$ and $z \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. We apply the Holder inequality with exponents $p = 2$, $q = 4$, $r = 4$ and we get the next estimation

$$\begin{aligned}\langle f'(\varphi)z, \psi\rangle_{(V_\delta, V_{-\delta})} &= \left|\int_{\mathbb{R}^N} g f'(\varphi)\psi z \,\mathrm{d}x\right|\\ &\leq c_1\left|\int_{\mathbb{R}^N} |\varphi|^a g^{1/2} g^{1/4} g^{1/4}\psi z\,\mathrm{d}x\right|, (c_1 = 1 + a)\\ &\leq c_1\|\varphi\|^a_{L^{4a}_g}\|z\|_{L^4_g}\|\psi\|_{L^2_g}\\ &\leq c_1 c_2\|\varphi\|^a_{\mathcal{D}^{1,2}}\|z\|_{\mathcal{D}^{1,2}}\|\psi\|_{L^2_g}\\ &\leq c^* c(R)\|z\|_{\mathcal{D}^{1,2}}\|\psi\|_{V_\delta},\end{aligned} \tag{4.34}$$

where we have applied the embedding $L^{4a}_g(\mathbb{R}^N) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$, which is valid for $0 \leq a \leq N/2(N-2)$ (Lemma 2.2). Relation (4.34) shows that $f'(\varphi)z$, is in the dual $V_{-\delta}$ of V_δ and that its norm in $V_{-\delta}$ is bounded by $c^* c(R)\|z\|_{\mathcal{D}^{1,2}}$. So the proof is completed. ■

Remark 4.8 Let $f \in L^\infty(R)$. Then we have that $f'(u) \in L^\infty(\mathbb{R}^N)$, for every $u \in D(A)$. Since for any $\delta \in (0,1)$ the embedding $L^2_g \equiv V_0 \subset V_{-\delta}$, is compact we get that

$$\begin{aligned}\|f'(\varphi)z\|_{V_{-\delta}} &\leq \|f'(\varphi)z\|_{L^2_g}\\ &\leq \|f'(\varphi)\|_{L^\infty}\|z\|_{L^2_g}\\ &\leq \|f'(\varphi)\|_{L^\infty}\|z\|_{\mathcal{D}^{1,2}}\end{aligned} \tag{4.35}$$

and we get the same result as in Lemma 4.7.

LEMMA 4.9 *The semigroup $S_1(t)$ satisfies the property $(\mathcal{S}_1)$.*

Proof We decompose the solution of the problems (1.1) and (1.2) as $u = \bar{w} + \bar{u}$, where $\bar{w}$ is the solution of the problem (4.32) and $\bar{u} = u - \bar{w}$ is the solution of the problem (4.33), with initial conditions $\bar{u}(x,0) = 0$ and $\bar{u}_t(x,0) = 0$. The semigroup $S_2(t)$

associated with solution $\bar{w}$ has the property $(\mathcal{S}_2)$. We shall show that $S_1(t) = S(t) - S_2(t)$ is uniformly compact. Let $\{u_0, u_1\}$ be in a bounded set $\mathcal{B}$ of $\mathcal{X}_0$ with $\|\nabla u_0\| > 0$. We have that $\mathcal{B}$ is also bounded in $\mathcal{X}_1$, since the embedding $\mathcal{X}_0 \subset \mathcal{X}_1$ is compact. According to Lemma 4.2, we have that for all $t \geq t_0$, $\{u, u_t\}$ is in $\mathcal{B}_0$ and

$$\|u(t)\|^2_{\mathcal{D}^{1,2}} \|u\|^2_{D(A)} + \|u_t\|^2_{\mathcal{D}^{1,2}} \leq R^2_*, \quad \text{for all } \ t \geq t_0. \tag{4.36}$$

We differentiate equation (3.1) with respect to the time. Then $V = \bar{u}_t$ is the solution of the problem

$$\begin{aligned} &V_{tt} - \phi(x)\| \nabla w\|^2 \Delta V + \delta V_t + f'(w) w_t = -2\left(\int_{\mathbb{R}^N} \nabla w_t \nabla w \, \mathrm{d}x\right)(-\phi \Delta \bar{u}), \\ &\quad V(x, 0) = 0, \quad V_t(x, 0) = f(u_0(x)). \end{aligned} \tag{4.37}$$

For the rest of the proof we follow ideas developed in [2]. By Theorem 3.2 and Lemma 4.6, $V \in C_b(R_+, V_1)$ and $V_t \in C_b(R_+, V_0)$. Also by Lemma 4.7, $f'(w) w_t \in C_b(R_+, V_{-\delta})$. So applying the operator $A^{-\delta/2}$ to the equation (4.37) and setting $\psi =: A^{-\delta/2} V$, $\xi =: A^{-\delta/2}(f'(w) w_t)$ and $\xi^* =: A^{-\delta/2}(-2 \int_{\mathbb{R}^N} \nabla w_t \nabla w \, \mathrm{d}x(-\phi \Delta \bar{u}))$, we have that

$$\psi_{tt} - \phi(x)\| \nabla w\|^2 \Delta \psi + \delta \psi_t = -\xi + \xi^*. \tag{4.38}$$

From the properties of the operator A^s and relation

$$A^{s/2} \colon V_x \to V_{x-s}, \tag{4.39}$$

we have that the following mappings

$$\begin{aligned} A^{-\delta/2} &: V_{-\delta} \longrightarrow V_0, \\ A^{-\delta/2} &: V_0 \longrightarrow V_\delta, \\ A^{-\delta/2} &: V_1 \longrightarrow V_{1+\delta}, \end{aligned}$$

are isomorphisms. Therefore $\{\psi, \psi_t\} \in C_b(R_+, V_{1+\delta} \times V_\delta)$. Since we have that ξ, $\xi^* \in C_b(R_+, V_0)$ by Lemma 4.6 we obtain that $\{\psi, \psi_t\} \in C_b(R_+, \ V_1 \times V_0)$ [7, p. 182] and [2]). Furthermore, the isomorphisms

$$\begin{aligned} A^{\delta/2} &: V_1 \longrightarrow V_{-\delta+1}, \\ A^{\delta/2} &: V_0 \longrightarrow V_{-\delta}, \end{aligned}$$

imply that the following relation is true

$$\{\bar{u}_t, \bar{u}_{tt}\} = \{V, V_t\} = A^{\delta/2}\{\psi, \psi_t\} \in C_b(R_+, V_{-\delta+1} \times V_{-\delta}). \tag{4.40}$$

But $f(w) \in C_b(R_+, V_{-\delta})$. So by (4.40) we obtain that $-\phi(x)\| \nabla w\|^2 \Delta \bar{u} = -\bar{u}_{tt} - \delta \bar{u}_t - f(w) \in V_{-\delta}$. Using again (4.39) we have the isomorphism

$$(-\phi \Delta)^{-1} = A^{-2/2} : V_{-\delta} \longrightarrow V_{-\delta+2}.$$

Therefore

$$\{\bar{u}, \bar{u}_t\} = \left\{A^{-1} \bar{u}, \bar{u}_t\right\} \in C_b(R_+, V_{-\delta+2} \times V_{-\delta+1}),$$

that is, $\cup_{t \geq t_0} S_1(t)\mathcal{B}$ is in a bounded set of $V_{-\delta+2} \times V_{-\delta+1}$. Hence, the compact embeddings $V_{-\delta+2} \subset V_1$ and $V_{-\delta+1} \subset V_0$ imply that the set $\cup_{t \geq t_0} S_1(t)\mathcal{B}$ is relatively compact in $\mathcal{X}_1$. ■

As a consequence of the above lemmas we have the following result

THEOREM 4.10 *Let ϕ satisfying $(\mathcal{G})$. Then the semigroup $S(t)$ associated with problems* (1.1) and (1.2) *possesses a functional invariant set $\mathcal{A} = \omega(B_0)$, which is compact in the weak topology of $\mathcal{X}_1$.*

Remark 4.11 The set $\overline{\cup_{t \geq t_0} S_1(t)\mathcal{B}}$ is compact with respect to the strong topology in $\mathcal{X}_1$. For the functional invariant compact set $\mathcal{A} = \omega(B_0)$, we observe that $(u_0, u_1) \in \mathcal{A}$, if $\|\nabla u_0\| > 0$. So, this set is attractor like.

Remark 4.12 The above set $\mathcal{A} = \omega(B_0)$, is a positively invariant set in the space $\mathcal{X}_0$, because we have that $S(t)\mathcal{A} \subset \mathcal{A}$, from the definition of the absorbing set. This set is not invariant in the space $\mathcal{X}_0$ because the semigroup $S(t)$ is weakly continuous in $\mathcal{X}_0$, see Lemma 4.13, but it is not continuous in $\mathcal{X}_0$.

Finally, we prove the following lemma.

LEMMA 4.13 *For every $t \in R$, the mapping $S(t)$ is weakly continuous from $\mathcal{X}_0$ into $\mathcal{X}_0$.*

Proof Let $\{u^n\}$ be a weakly convergent sequence in $\mathcal{X}_0$ and u its (weak) limit. We fix $t \in R$; we have that the sequence $\{S(t)u^n\}$ is bounded in $\mathcal{X}_0$. We extract a subsequence $\{S(t)u^{n'}\}$ that converges weakly to $v \in \mathcal{X}_0$. On the other hand, the compactness of the injection of $\mathcal{X}_0$ into $\mathcal{X}_1$ insures that $\{u^n\}$ converges strongly to u in $\mathcal{X}_1$. Hence, $\{S(t)u^n\}$ converges strongly to $S(t)u$ in $\mathcal{X}_1$ and then $v = S(t)u$. Therefore, the whole sequence $\{S(t)u^n\}$ weakly converges to $S(t)u$ in $\mathcal{X}_0$ and the lemma is proved. ■

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References

[1] Kirchhoff, G., 1883, *Vorlesungen Über Mechanik* (Leipzig: Teubner).
[2] Ghidaglia, JM. and Temam, R., 1987, Attractors for damped nonlinear wave equations. *Journal de Mathématiques Pures et Appliquées*, **66**, 273–319.
[3] Ladyzenskaja, OA., 1987, On the determination of minimal global attractors for the Navier-Stokes equations and other partial differential equations. *Uspekhi Matematicheskikh Nauk*, **42**, 25–60. [English Translation: *Russian Mathematical Surveys*, **42**, 27–73.]
[4] Babin, A.V. and Vishik, M.I., 1990, Attractors for partial differential evolution eguations in an unbounded domain. *Proceedings of the Royal Society of Edinburgh*, **116A**, 221–243.
[5] Hale, JK., 1988, Asymptotic behaviour of dissipative systems. *Mathematical Surveys and Monographs 25* (AMS: Providence, RI).
[6] Haraux, A., 1991, Systemes dynamiques dissipatifs et applications. *RMA 17* (Paris: Masson).
[7] Temam, R., 1997, Infinite-dimensional dynamical systems in mechanics and physics. *Applications in Mathematical Science*, 2nd Edn, Vol. 68 (New York: Springer-Verlag).

[8] Ono, K., 1997, On global existence, asymptotic stability and blowing – up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation. *Mathematical Methods in Applied Science*, **20**, 151–177.

[9] Ono, K., 1997, Global existence and decay properties of solutions for some mildly degenerate nonlinear dissipative Kirchhoff strings. *Funkcialaj Ekvacioj*, **40**, 255–270.

[10] Ono, K., 1997, Global existence, decay, and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings. *Journal of Differential Equations*, **137**, 273–301.

[11] Babin, A.V. and Vishik, M.I., 1992, *Attractors for evolution equations* (Amsterdam: North-Holland).

[12] Brenner, P., 1989, On space-time means and strong global solutions on nonlinear hyperbolic equations. *Mathematische Zeitschrift*, **201**, 45–55.

[13] Feiresl, E., 1994, Attractors for semilinear damped wave equations on R^3. *Nonlinear Analysis TMA*, **23**(2), 187–195.

[14] Feiresl, E., 1995, Asymptotic behaviour and attractors for a semilinear damped wave equations with supercritical exponent. *Proceedings of the Royal Society Edinburgh*, **125A**, 1051–1062.

[15] Karachalios, NI. and Stavrakakis, NM., 1999, Existence of global attractors for semilinear dissipative wave equations on $\mathbb{R}^N$. *Journal of Differential Equations*, **157**, 183–205.

[16] Karachalios, NI. and Stavrakakis, NM., 2001, Global existence and blow-up results for some nonlinear wave equations on $\mathbb{R}^N$. *Advances in Differential Equations*, **6**, 155–174.

[17] Karachalios, NI., 2002, Estimates on the dimension of a global attractor for a semilinear dissipative wave equation on $\mathbb{R}^N$. *Discrete and Continuous Dynamical Systems*, **8**, 939–951.

[18] Papadopoulos, PG. and Stavrakakis, NM., 2001, Global existence and blow-up results for an equation of Kirchhoff type on $\mathbb{R}^N$. *Topological Methods in Nonlinear Analysis*, **17**, 91–109.

[19] Brown, K.J. and Stavrakakis, NM., 1996, Global bifurcation results for a semilinear elliptic equation on all of $\mathbb{R}^N$. *Duke Mathematical Journal*, **85**, 77–94.

[20] Zeidler, E., 1990, *Nonlinear Functional Analysis and its Applications*. Monotone Operators, Vol. II (New York: Springer-Verlag).

[21] Henry, D., 1981, *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics (Berlin: Springer-Verlag).