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Global bifurcation results on degenerate quasilinear elliptic systems

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Abstract

In this paper we prove certain bifurcation results for the following degenerate quasilinear system

$$\begin{aligned}&-\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v + f(x,\lambda,u,v),\\ &-\nabla(\nu_2(x)|\nabla u|^{p-2}\nabla u) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}\, u + g(x,\lambda,u,v),\\ &x\in\Omega,\quad u|_{\partial\Omega} = v|_{\partial\Omega} = 0,\end{aligned}$$

where Ω is a bounded and connected subset of $\mathbb{R}^N$, with $N \geq 2$. This is achieved by applying topological degree and global bifurcation theory (in the sense of Rabinowitz).

MSC: 35B40; 35B45; 35B65; 35D05; 35D10; 35J70; 35P30

Keywords: Quasilinear elliptic systems; Degenerate equations; Topological degree; Global bifurcation (Rabinowitz)

1. Introduction

In this paper we prove the global bifurcation of a continuum of positive solutions for the following quasilinear elliptic system, defined on Ω,

$$\begin{aligned}&-\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v + f(x,\lambda,u,v),\\ &-\nabla(\nu_2(x)|\nabla u|^{p-2}\nabla u) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}\, u + g(x,\lambda,u,v),\end{aligned} \tag{1.1}$$

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$$x \in \Omega, \qquad u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{1.2}$$

where Ω is bounded and connected, $N \geq 2$. This continuum of solutions is bifurcating from the positive principal eigenvalue of the following unperturbed system,

$$\begin{aligned} &-\nabla(\nu_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}v, \\ &-\nabla(\nu_2(x)|\nabla u|^{p-2}\nabla u) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}u, \end{aligned} \tag{1.3}$$

$$x \in \Omega, \quad u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{1.4}$$

where $\lambda \in \mathbb{R}$. The degeneracy of this system is considered in the sense that the measurable, non-negative diffusion coefficients ν_1, ν_2 are allowed to vanish in Ω (as well as at the boundary $\partial\Omega$) and/or to blow up in $\bar{\Omega}$. The degenerate scalar equation of the system (1.1) was studied in [2]. The system (1.3) and (1.4) under certain conditions on the constants α, β, p, q, N and on the functions a, b and d, forms an eigenvalue problem which has been studied in [9]. Nondegenerate systems of this type where studied for first time in all $\mathbb{R}^N$ using the homogeneous Sobolev space $D^{1,p}(\mathbb{R}^N)$ in the work [4]. Throughout this paper we assume that $N, p, q, \alpha, \beta, a, b, d, f, g$ satisfy the following conditions:

($\mathcal{H}$) $p > 1, q > 1, \alpha \geq 0, \beta \leq 0$ and

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

Let $\nu(x), \mu(x)$ be some nonnegative weighted functions in Ω satisfying the conditions:

($\mathcal{N}_p$)

$$\nu, \mu \in L^1_{\text{loc}}(\Omega), \quad \nu^{-\frac{1}{p-1}}, \mu^{-\frac{1}{q-1}} \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \nu^{-s_p}, \mu^{-s_q} \in L^1(\Omega),$$

for some $p > 1, q > 1, s_p > \max\{\frac{N}{p}, \frac{1}{p-1}\}, s_q > \max\{\frac{N}{q}, \frac{1}{q-1}\}$ satisfying $ps \leq N(s+1), qs \leq N(s+1)$.

($\mathcal{N}$) Let us suppose that ν, μ satisfies condition ($\mathcal{N}_p$). Let $\nu_1(x), \nu_2(x)$ be measurable functions satisfying

$$\frac{\nu(x)}{c_1} \leq \nu_1(x) \leq c_1\nu(x) \quad \text{and} \quad \frac{\mu(x)}{c_2} \leq \nu_2(x) \leq c_2\mu(x), \tag{1.5}$$

for a.e. $x \in \Omega$, with some constants $c_1 > 1$ and $c_2 > 1$.

We also suppose that the coefficient functions satisfy the following conditions:

($\mathcal{A}$) $a \in L^{\frac{p^*}{p^*-p}}(\Omega) \cap C^{0,\zeta}_{\text{loc}}(\Omega)$ for some $\zeta \in (0, 1)$ and either there exists $\Omega_a^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega_a^+| > 0$, such that $a(x) > 0$, for all $x \in \Omega_a^+$, or $a(x) \equiv 0$, in Ω. With $p < p^* < p_s^*$, where

$$p_s^* = \frac{Nps}{N(s+1) - ps}$$

($\mathcal{D}$) $d \in L^{\frac{q^*}{q^*-q}}(\Omega) \cap C^{0,\zeta}_{\text{loc}}(\Omega)$ for some $\zeta \in (0, 1)$ and either there exists $\Omega_d^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega_d^+| > 0$, such that $d(x) > 0$, for all $x \in \Omega_d^+$, or $d(x) \equiv 0$, in Ω. With $q < q^* < q_s^*$, where

$$q_s^* = \frac{Nqs}{N(s+1) - qs}.$$

($\mathcal{B}$) $b(x) \geq 0$, a.e. in Ω, $b \neq 0$ and $b \in L^{\omega}(\Omega) \cap L^{\infty}(\Omega)$, where $\omega = [1 - \frac{a+1}{p^*} - \frac{\beta+1}{q^*}]^{-1}$.

($\mathcal{F}$) The perturbations f and g are of the form:

$$|f(x,\lambda,u,v)| \leq \sigma_1(\lambda)m(x)|u|^{\gamma_1-1}|v|^{\delta_1+1}u + \sigma_2(\lambda)\mu(x)|u|^{\eta-1}u,$$
$$|g(x,\lambda,u,v)| \leq \sigma_3(\lambda)m(x)|u|^{\gamma_2+1}|v|^{\delta_2-1}v + \sigma_4(\lambda)\mu(x)|u|^{\theta-1}v,$$

where the exponents $\gamma_i, \delta_i, i = 1, 2, \eta$ and θ satisfy the following conditions: $\gamma_i + 1 > p$ or $\delta_i + 1 > q$, $\frac{\gamma_i+1}{p^*} + \frac{\delta_i+1}{q^*} < 1, i = 1, 2, p < \eta + 1 < p^*$ and $q < \theta + 1 < q^*$ while $\sigma_i(\lambda), i = 1, 2, 3, 4$, are bounded, $m(x), n(x), \in L^{\omega_{1,i}} \cap L^{\infty}$, where $\omega_{1,i} = [1 - \frac{\gamma_i+1}{p^*} - \frac{\delta_i+1}{q^*}]^{-1}, i = 1, 2$, respectively, $\mu(x) \in L^{\omega_2} \cap L^{\infty}$ where $\omega_2 = [1 - (\eta + 1)]^{-1}$ and $\nu(x) \in L^{\omega_3} \cap L^{\infty}$ where $\omega_3 = [1 - (\theta + 1)]^{-1}$.

The mathematical modelling of various physical processes, ranging from physics to biology, where spatial heterogeneity has a primary role, is reduced to modelling nonlinear evolution equations with variable diffusion or dispersion. Note also that our problem is closely related (see [2]) to the following system

$$-\nabla(\nu_1(x,u,v)|\nabla u|^{p-2}\nabla u) = f(\lambda, x, u, v, \nabla u, \nabla v),$$
$$-\nabla(\nu_2(x,u,v)|\nabla v|^{q-2}\nabla v) = g(\lambda, x, u, v, \nabla u, \nabla v),$$
$$x \in \Omega, \quad u|_{\partial\Omega} = v|_{\partial\Omega} = 0.$$

Problems of such a type have been successfully applied to the heat propagation in heterogeneous materials, to the study of transport of the electron temperature in a confined plasma, to the propagation of varying amplitude waves in a nonlinear medium, to the study of electromagnetic phenomena in nonhomogeneous superconductors and the dynamics of Josephson junctions, to electrochemistry, to nuclear reaction kinetics, to image segmentation, to the spread of microorganisms, to the growth and control of brain tumors and to population dynamics (see [9] and the references therein).

An example of the physical motivation of the assumptions ($\mathcal{N}$), $(\mathcal{N})_p$ may be found in [1, p. 79]. These assumptions are related to the modelling of reaction–diffusion processes in composite materials occupying a bounded domain Ω, which at some points behave as *perfect insulators*. When at some points the medium is perfectly insulating, it is natural to assume that $\nu_1(x), \nu_2(x)$ vanish in $\bar{\Omega}$. For more information we refer the reader to [5,6].

The rest of the paper is organized in six sections. In Section 2, we introduce the necessary operators and establish their basic characteristics. In Section 3, we prove that the operators generated by the system (1.1) satisfy a condition under which it is possible to define their degree (condition $(S)_+$). In Section 4, the existence of a continuum of nontrivial solutions bifurcating out from the first eigenvalue of the problem (1.3) and (1.4) is achieved. In Section 5, considering the regularity of the solutions we describe the behavior of the continuum of nontrivial solutions for the perturbed problem (1.1) in the product space $D^{1,p}(\Omega) \times D^{1,q}(\Omega)$.

Notation. We *denote* by B_R, $B_R(c)$ the open ball in Ω with center 0 and radius R, c respectively. For *simplicity reasons* sometimes we use the symbols C_0^∞, L^p, $D^{1,p}$ respectively for the spaces $C_0^\infty(\Omega)$, $L^p(\Omega)$, $D^{1,p}(\Omega)$ and $\|\cdot\|_{1,p}$ for the norm $\|\cdot\|_{D^{1,p}(\Omega)}$. Also, sometimes when the domain of integration is not stated, it is assumed to be all of $\mathbb{R}^N$. Equalities introducing definitions are denoted by "=:". The ends of the proofs are marked by "$\lhd$".

2. Space and operator setting

Let $\bar{\nu}(x)$ be a nonnegative weight function in Ω which satisfies condition $(\mathcal{N}_p)$. We consider the weighted Sobolev space $D_0^{1,p}(\Omega, \bar{\nu})$ to be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D_0^{1,p}(\Omega,\bar{\nu})} := \left(\int_\Omega \bar{\nu}(x)|\nabla u|^p\right)^{1/p}.$$

Assuming that $\bar{\nu}(x)$ satisfies $(\mathcal{N}_p)$ then the weighted Sobolev space $D_0^{1,p}(\Omega, \bar{\nu})$ is a reflexive Banach space. For a discussion about the space setting we refer the reader to [2] and the references therein. The following Lemma holds:

Lemma 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ and the weight $\bar{\nu}$ satisfies $(\mathcal{N})_p$. Then the following embeddings hold:*

(i) $\mathcal{D}_0^{1,p}(\Omega, \bar{\nu}) \hookrightarrow L^{p^*}(\Omega)$ *continuously for* $1 < p^* < N$,
(ii) $\mathcal{D}_0^{1,p}(\Omega, \bar{\nu}) \hookrightarrow L^r(\Omega)$ *compactly for any* $r \in [1, p^*)$.

The space setting of our problem is $Z := \mathcal{D}_0^{1,p}(\Omega, \nu_1) \times \mathcal{D}_0^{1,q}(\Omega, \nu_2)$, equipped with the following norm:

$$\|z\|_Z := \|u\|_{\mathcal{D}_0^{1,p}(\Omega,\nu_1)} + \|v\|_{\mathcal{D}_0^{1,q}(\Omega,\nu_2)}, z = (u, v) \in Z.$$

Note also that from condition $(\mathcal{N})$ we can deduce that $\mathcal{D}_0^{1,p}(\Omega, \nu_1) \times \mathcal{D}_0^{1,q}(\Omega, \nu_2)$ and $\mathcal{D}_0^{1,p}(\Omega, \nu) \times \mathcal{D}_0^{1,q}(\Omega, \mu)$ are equivalent.

Next let us introduce the functionals $I_i, J_i, F_i : Z \to \mathbb{R}$ *with* $i = 1, 2$ in the following way:

$$(I_1(u, v), (\phi, z))_Z := \frac{\alpha+1}{p}\int_\Omega \nu_1(x)|\nabla u|^{p-2}\nabla u \nabla \phi \mathrm{d}x,$$
$$(I_2(u, v), (\phi, z))_Z := \frac{\beta+1}{q}\int_\Omega \nu_2(x)|\nabla v|^{q-2}\nabla v \nabla z \mathrm{d}x,$$
$$(J_1(u, v), (\phi, z))_Z := \frac{\alpha+1}{p}\left\{\int_\Omega a(x)|u|^{p-2}u\mathrm{d}x + \int_\Omega b(x)|u|^\alpha|u|^\beta v\phi\mathrm{d}x\right\},$$
$$(J_2(u, v), (\phi, z))_Z := \frac{\beta+1}{q}\left\{\int_\Omega d(x)|v|^{q-2}\mathrm{d}x + \int_\Omega b(x)|u|^\alpha|u|^\beta uz\mathrm{d}x\right\},$$
$$(F_1(u, v), (\phi, z))_Z := \frac{\alpha+1}{p}\int_\Omega f(x, \lambda, u, v)\phi\mathrm{d}x,$$
$$(F_2(u, v), (\phi, z))_Z := \frac{\beta+1}{p}\int_\Omega g(x, \lambda, u, v)z\mathrm{d}x,$$

Lemma 2.2. *The functionals I_i, J_i are well defined. Moreover, I_i continuous and J_i is compact.*

Proof. The proof follows the standard procedure (see also [3]).

Lemma 2.3. *The functionals F_i, $i = 1, 2$, are well defined, compact and satisfy the relations*

$$\lim_{\|(u,v)\|_Z \to 0} \frac{\|F_i(u, v)\|_{Z^*}}{\|u\|_{1,p}^{p-1} + \|v\|_{1,q}^{q-1}} = 0. \tag{2.1}$$

Proof. From $(\mathcal{F})$ we have

$$\begin{aligned}|\langle F_1(u,v),(\phi,z)\rangle| &\le \sigma_1(\lambda)\left(\int_\Omega m(x)|u|^{\gamma_1}|v|^{\delta_1+1}\phi\right)+\sigma_2(\lambda)\left(\int_\Omega \mu(x)|u|^{\eta}\phi\right),\\ &\le c_1\left(\int_\Omega |m(x)|^{\omega_1}\right)^{1/\omega_1}\left(\int_\Omega |u|^{p^*}\right)^{\gamma_1/p^*}\left(\int_\Omega |v|^{q^*}\right)^{\delta_1+1/q^*}\\ &\quad\times\left(\int_\Omega |\phi|^{p^*}\right)^{1/p^*}+c_2\left(|\mu(x)|^{\Omega_2}\right)^{1/\Omega_2}\left(\int_\Omega |u|^{p^*}\right)^{\eta/p^*}\\ &\quad\times\left(\int_\Omega |\phi|^{p^*}\right)^{1/p^*}<\infty.\end{aligned}$$

Thus F_1 is well defined. Let us prove now the compactness.

The continuity of $F_i, i = 1,2$, follows from the continuity of the Nemytskij operator associated with f and acting from $Z(B_R)$ into $Z(B_R)$. Let $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in Z. For $(\phi, z) \in Z$ it follows that

$$\begin{aligned}\|F_1(u_n,v_n)-F_1(u_0,v_0)\|_{Z^*} \le{}& \sup_{\|\phi\|_Z\le 1}\left|\int_{B_R} f(x,\lambda,u_n,v_n)\phi - f(x,\lambda,u_0,v_0)\phi\right|\\ &+\sup_{\|\phi\|_Z\le 1}\left|\int_{\Omega\setminus B_R} f(x,\lambda,u_n,v_n)\phi - f(x,\lambda,u_0,v_0)\phi\right|.\end{aligned}$$

From the continuity of the Nemytskij operator and by Lemma 2.1 we can deduce that the integral over (B_R) tends to zero as $n\to\infty$. Now let us estimate the integral over $(\Omega\setminus B_R)$.

$$\begin{aligned}&\sup_{\|\phi\|_Z\le 1}\left|\int_{\Omega\setminus B_R} f(x,\lambda,u,v)\phi\right|\\ &\quad\le c_1\left(\int_\Omega |m(x)|^{\omega_1}\right)^{1/\omega_1}\left(\int_\Omega |u|^{p^*}\right)^{\gamma_1/p^*}\left(\int_\Omega |v|^{q^*}\right)^{\delta_1+1/q^*}\left(\int_\Omega |\phi|^{p^*}\right)^{1/p^*}\\ &\quad+c_2\left(|\mu(x)|^{\omega_2}\right)^{1/\omega_2}\left(\int_\Omega |u|^{p^*}\right)^{\eta/p^*}\left(\int_\Omega |\phi|^{p^*}\right)^{1/p^*}\\ &\quad< \|m(x)\|_{\omega_{1,1}}\|u\|_{p^*}^{\gamma_1/p^*}\|v\|_{q^*}^{\delta_1+1/q^*}\|\phi\|_{p^*}+\|\mu(x)\|_{\omega_2}\|u\|_{p^*}^{\eta}\|\phi\|_{p^*}.\end{aligned}$$

Hence for any $\epsilon > 0$ there exists an R large enough such that

$$\sup_{\|\phi\|_Z\le 1}\left|\int_{\Omega\setminus B_R} f(x,\lambda,u,v)\phi\right|<\epsilon$$

Therefore the functional F_1 is compact. The analogue holds for the operator $F_2(u, v)$. Now concerning relation (2.1) we have that

$$\frac{\|F_1(u,v)\|_{Z^*}}{\|u\|_{1,p}^{p-1}+\|v\|_{1,q}^{q-1}} \le \|m(x)\|_{\omega_{1,1}}\|u\|_{p^*}^{\gamma-(p-1)}\|v\|_{q^*}^{\beta+1}+\|\mu(x)\|_{\omega_2}\|u\|_{p^*}^{\eta-(p-1)}\to 0,$$

as $\|z\|\to 0$, and similarly for F_2. $\lhd$

Let us now define $\widetilde{A}_\lambda, A_\lambda : Z \to Z^*$ as:

$$\widetilde{A}_\lambda =: I_1(u,v)-J_1(u,v)+I_2(u,v)-J_2(u,v), \tag{2.2}$$

and

$$A_\lambda =: \widetilde{A}_\lambda - F_1(u, v) - F_2(u, v). \tag{2.3}$$

We say that (u, v) is a weak solution of the system (1.1) and (1.2), if and only if $A_\lambda(u, v) = 0 \in Z^*$ where (u, v) is a critical point of the functional $\Phi : Z \to \mathbb{R}$, defined by

$$\Phi(u, v) =: \lambda \frac{\alpha + 1}{p} \int |\nabla u|^p + \lambda \frac{\beta + 1}{q} \int |\nabla v|^q - \lambda \frac{\alpha + 1}{p} \int a(x)|u|^p$$
$$- \lambda \frac{\beta + 1}{q} \int d(x)|v|^q - \lambda \int b(x)|u|^\alpha |v|^\beta uv - \lambda \int F(u, v).$$

Since $\Phi(|u|, |v|) = \Phi(u, v)$, we may assume that there exists an eigenfunction (u_1, v_1) corresponding to λ_1, such that $u_1 \geq 0$ and $v_1 \geq 0$, a.e., in Ω.

Hence from [9] we have the following theorem.

Theorem 2.4. *Let Ω be a bounded domain of $\mathbb{R}^N, N \geq 2$. Assume that hypotheses $(\mathcal{H}), (\mathcal{N}), (\mathcal{A}), (\mathcal{B}), (\mathcal{D})$ are satisfied. Then, the system (1.1) and (1.2) admits a positive principal eigenvalue λ_1, satisfying*

$$\lambda_1 = \inf_{\substack{\int_\Omega b(x)|u|^{\alpha+1}|u|^{\beta+1}\mathrm{d}x \\ + \frac{\alpha}{p}\int_\Omega a(x)|u|^p \mathrm{d}x \\ + \frac{\beta+1}{q}\int_\Omega d(x)|v|^q \mathrm{d}x = 1, \\ (u,v) \in Z.}} \left[\frac{\alpha + 1}{p} \int_\Omega \nu_1(x)|\nabla u|^p \mathrm{d}x + \frac{\beta + 1}{q} \int_\Omega \nu_2(x)|\nabla v|^q \mathrm{d}x \right].$$

The associated normalized eigenfunction (u_1, v_1) belongs to Z and each component is nonnegative. In addition,

(i) *the set of all eigenfunctions corresponding to the principal eigenvalue λ_1 forms a one dimensional manifold, $E_1 \subset Z$, which is defined by*

$$E_1 = \{(c_1 u_1, c_1^{p/q} v_1); c_1 \in \mathbb{R}\}.$$

(ii) *λ_1 is the only eigenvalue of (1.2) to which there corresponds a componentwise nonnegative eigenfunction.*

(iii) *λ_1 is isolated in the following sense: there exists $\eta > 0$, such that the interval $(0, \lambda_1 + \eta)$ does not contain any other eigenvalue that λ_1.*

(iv) *(nondegenerate case) If in addition hypothesis $(\mathcal{T}_i)$, for $i = 1, 2$, is satisfied and ν_1, ν_2 are positive and smooth and the coefficient functions α, d and b are smooth functions, at least $C^{0,\zeta}_{\mathrm{loc}}(\Omega)$, for some $\zeta \in (0, 1)$, then u_1 and v_1 belong to $C^{1,\zeta}_{\mathrm{loc}}(\Omega)$, for some $\zeta \in (0, 1)$ and they are both positive in Ω.*

Finally, taking into consideration certain properties, such as the simplicity and isolation of the positive principal eigenvalue of (1.1) and (1.2), we can proceed to the topological degree theorem, Section 3.

3. Topological degree

For the completeness of the presentation in this section we recall some basic facts on the topological degree theory as well as some necessary conditions for the system. The procedure is analogous to the one presented in [8]. First, we define the topological degree for the operators.

Definition 3.1. Let X be a reflexive Banach space, X^* its dual. Then the operator A: $X \to X^*$ satisfies condition (S_+) if for any sequence $u_n \in X$ satisfying $u_n \rightharpoonup u_0$ in X and

$$\lim_{n\to\infty} \sup(A(u_n), u_n - u_0)_X \leq 0,$$

we have that $u_n \to u_0$ (strongly) in X.

If the operator A satisfies the above condition then it is possible to define the degree $\mathrm{Deg}[A, D, 0]$, where $D \subset X$ is a bounded open set such that $A(u) \neq 0$, for any $u \in \partial D$. Note also that if A satisfies (S_+) then $A+K$ also satisfies (S_+) for any compact operator $K : X \to X^*$.

Lemma 3.2. *Let A be a potential operator with $\Phi'(u) = A(u)$, $u \in X$, for some continuously differentiable functional $\Phi : X \to \mathbb{R}$. Let u_0 be a local minimum of Φ and an isolated point for which $A(u_0) = 0$. Then*

$$\mathrm{Ind}(A, u_0) = 1.$$

Lemma 3.3. *Assume that $\langle A(u), u\rangle_X > 0$, for all $u \in X$ with $\|u\|_X = r$. Then*

$$\mathrm{Deg}[A, B_r(0), 0] = 1.$$

Now in order to define the topological degree for our system we need to prove the following lemma.

Lemma 3.4. *The functionals $\widetilde{A}_\lambda$, A_λ satisfy the (S_+) condition, where $\widetilde{A}_\lambda$, A_λ are given by (2.1) and (2.2) respectively.*

Proof. We already know that J_i, F_i are compact. Therefore it suffices to show that the functional $I(u, v) =: I_1 + I_2 : Z \to Z^*$ satisfies condition (S_+). Let us suppose that the sequence (u_n, v_n) converges to (u_0, v_0) weakly in the space Z and

$$\limsup_{n\to\infty} \langle I(u_n, v_n), (u_n - u_0, v_n - v_0)\rangle_Z \leq 0.$$

From the weak convergence we have that

$$\lim_{n\to\infty} \langle I(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle_Z = 0.$$

So

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \langle I(u_n, v_n) - I(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle_Z \\ &= \limsup_{n\to\infty} \left\{ \frac{\alpha+1}{p} \int (\nu_1(x)| \nabla u_n|^{p-2} \nabla u_n - \nu_1(x)| \nabla u_0|^{p-2} \nabla u_0)(\nabla u_n - \nabla u_0) \right. \\ &\quad \left. + \frac{\beta+1}{q} \int (\nu_2(x)| \nabla v_n|^{q-2} \nabla v_n - \nu_2(x)| \nabla v_0|^{q-2} \nabla v_0)(\nabla v_n - \nabla v_0) \right\}. \end{aligned} \tag{3.1}$$

Rewriting (3.1) we have

$$\begin{aligned} &\int (\nu_1(x)| \nabla u_n|^p + \nu_1(x)| \nabla u_0|^p - \nu_1(x)| \nabla u_n| \nabla u_n \nabla u_0 - \nu_1(x)| \nabla u_0| \nabla u_0 \nabla u_n) \\ &\quad \geq \int (\nu_1(x)| \nabla u_n|^p + \nu_1(x)| \nabla u_0|^p) - \left(\int \nu_1(x)| \nabla u_n|^p\right)^{1/p'} \left(\nu_1(x) \int | \nabla v|^p\right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&- \left(\int \nu_1(x) |\nabla u_n|^p\right)^{1/p} \left(\nu_1(x) \int |\nabla v|^p\right)^{1/p'} \\
&= \left[\left(\int \nu_1(x) |\nabla u_n|^p\right)^{(p-1)/p} - \left(\int \nu_1(x) |\nabla u_0|^p\right)^{(p-1)/p}\right] \\
&\quad \times \left[\left(\int \nu_1(x) |\nabla u_n|^p\right)^{1/p} - \left(\int \nu_1(x) |\nabla u_0|^p\right)^{1/p}\right] \geq 0.
\end{aligned}
$$

Following the same procedure for (3.1) we obtain

$$
\int \nu_1(x) |\nabla u_n|^p \to \int \nu_1(x) |\nabla u_0|^p \quad \text{and} \quad \int \nu_2(x) |\nabla v_n|^q \to \int \nu_2 |\nabla v_0|^q.
$$

Therefore the proof of the lemma is completed. $\lhd$

4. Bifurcation from λ_1

In this section we shall prove the existence of a bifurcation from the principal eigenvalue λ_1.

Definition 4.1. Let $E = \mathbb{R} \times Z$ be equipped with the norm

$$
\|(\lambda, u, v)\|_E = (|\lambda|^2 + \|(u, v)\|_Z^2)^{1/2}, \quad (\lambda, u, v) \in E. \tag{4.1}
$$

We say that the set

$$
C = \{(\lambda, u, v) \in E : (\lambda, u, v) \text{ solves (1.1)}, (u, v) \neq (0, 0)\}
$$

is a continuum of nontrivial solutions of (1.1), if it is a connected set in E with respect to the topology induced by the norm (4.1). We say $\lambda_0 \in \mathbb{R}$ is a bifurcation point of the system (1.1) (in the sense of Rabinowitz), if there is a continuum of nontrivial solutions C of (1.1) such that $(\lambda_0, 0, 0) \in \bar{C}$ and C is either unbounded in E or there is an eigenvalue $\hat{\lambda} \neq \lambda_0$, such that $(\hat{\lambda}, 0, 0) \in \bar{C}$.

Let us consider now a real nonnegative C^1-function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$
\psi(t) =: \begin{cases} 0, & t \leq K, \\ \dfrac{2\delta}{\lambda_1}(t - 2K), & t \geq 3K, \end{cases}
$$

for $K > 0$ and δ such that the interval $(\lambda_1, \lambda_1 + \delta)$ contains no eigenvalue of (1.3). The function $\psi(t)$ can be chosen positive and strictly convex in $(K, 3K)$. We define the functional

$$
\Psi^\lambda(u, v) =: \langle I(u, v), (u, v)\rangle - \lambda\langle J(u, v), (u, v)\rangle + \psi(\langle I(u, v), (u, v)\rangle).
$$

where

$$
I(u, v) =: I_1(u, v) + I_2(u, v) \quad \text{and} \quad J(u, v) =: J_1(u, v) + J_2(u, v)
$$

Then Ψ^λ is continuously Fréchet differentiable with derivative

$$
\langle(\Psi^\lambda)'(u, v), (w, z)\rangle = \langle \Psi_u^\lambda(u, v), (w, z)\rangle + \langle \Psi_v^\lambda(u, v), (w, z)\rangle,
$$

where

$$\langle \Psi_u^\lambda(u,v),(w,z)\rangle = (\alpha+1)\left\{\left(1+\psi'(\langle I(u,v),(u,v)\rangle)\right)\langle I_1(u,v),(w,z)\rangle - \lambda\langle J_1(u,v),(w,z)\rangle\right\}, \tag{4.2}$$

$$\langle \Psi_v^\lambda(u,v),(w,z)\rangle = (\beta+1)\left\{\left(1+\psi'(\langle I(u,v),(u,v)\rangle)\right)\langle I_2(u,v),(w,z)\rangle - \lambda\langle J_2(u,v),(w,z)\rangle\right\}. \tag{4.3}$$

In addition, the critical points (u_0, v_0) of Ψ^λ occur, if $\Psi_u^\lambda = \Psi_v^\lambda = 0$, and

$$\psi'((I(u_0,v_0),(u_0,v_0))) = \frac{\lambda}{\lambda_1} - 1. \tag{4.4}$$

Hence, we must have $(I(u_0,v_0),(u_0,v_0)) \in (K, 3K)$. In this case either $(u_0,v_0) = (u_1,v_1)$ or $(u_0,v_0) = (-u_1,-v_1)$. So for $\lambda \in (\lambda_1, \lambda_1+\delta)$ we have precisely three isolated critical points 0, (u_1,v_1), $(-u_1,-v_1)$.

Lemma 4.2. *The functional Ψ^λ is* (a) *weakly lower semicontinuous and* (b) *weakly coercive, with $\lambda \in (\lambda_1, \lambda_1+\delta)$.*

Proof. (a) Let $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in Z. Then, we have

$$\liminf_{n\to\infty} \{\langle I(u_n,v_n),(u_n,v_n)\rangle + \psi(\langle I(u_n,v_n),(u_n,v_n)\rangle)\} \geq \langle I(u_0,v_0),(u_0,v_0)\rangle + \psi(\langle I.(u_0,v_0),(u_0,v_0)\rangle), \tag{4.5}$$

since J is compact, $\liminf_{n\to\infty} \|\nabla u_n\|_p \geq \|\nabla u_0\|_p$, $\liminf_{n\to\infty} \|\nabla v_n\|_q \geq \|\nabla v_0\|_q$ and ψ is nondecreasing. Therefore we obtain

$$\liminf_{n\to\infty} \Psi^\lambda(u_n,v_n) \geq \Psi^\lambda(u_0,v_0).$$

(b) The proof follows steps like those for Lemma (5.2) in [8]. Hence the proof of the lemma is completed. $\lhd$

Lemma 4.3. *The critical points (u_1,v_1), $(-u_1,-v_1)$ of Ψ^λ are of minimum type, with $\lambda \in (\lambda_1, \lambda_1+\delta)$.*

Proof. Lemma 4.2 implies that Ψ^λ attains a minimum on Z; in addition with (4.4) and the strict convexity of ψ on $(K, 3K)$ we have that

$$\Psi^\lambda(u_1,v_1) = \frac{\lambda-\lambda_1}{\lambda_1}\langle I(u_1,v_1),(u_1,v_1)\rangle + \psi(\langle I(u_1,v_1),(u_1,v_1)\rangle) < 0 = \Psi^\lambda(0,0).$$

Since $\Psi^\lambda(u_1,v_1) = \Psi^\lambda(-u_1,-v_1)$ we obtain the conclusion. $\lhd$

Lemma 4.4. *The quantity $\langle(\Psi^\lambda)'(u,v),(u,v)\rangle$ is strictly positive for any $(u,v) \in Z$ with $\|(u,v)\|_Z > k$, for some large enough positive constant k and $\lambda \in (\lambda_1, \lambda_1+\delta)$.*

Proof. From (4.2) we have

$$\left\langle \frac{1}{p}\Psi_u^\lambda(u,v),(u,v)\right\rangle = \frac{\alpha+1}{p}\langle I_1(u,v) - \lambda J_1(u,v)\rangle + \frac{\alpha+1}{p}\psi'(\langle I(u,v),(u,v)\rangle)\int \nu_1(x)|\nabla u|^p. \tag{4.6}$$

Adding $\Psi_u^\lambda(u, v)$ and $\Psi_v^\lambda(u, v)$ we obtain

$$\left\langle \frac{1}{p}\Psi_u^\lambda(u, v) + \frac{1}{q}\Psi_v^\lambda(u, v), (u, v)\right\rangle = \langle I(u, v), (u, v)\rangle - \lambda\langle J(u, v), (u, v)\rangle$$
$$+ \psi'(\langle I(u, v), (u, v)\rangle)\langle I(u, v), (u, v)\rangle. \tag{4.7}$$

Let $\|(u_n, v_n)\|_Z \to \infty$. Then $\langle J(u_n, v_n), (u_n, v_n)\rangle \to \infty$. Therefore (4.7) becomes

$$\begin{aligned}
&\langle I(u, v) - \lambda J(u, v), (u, v)\rangle + \psi'(\langle I(u, v), (u, v)\rangle)\langle I(u, v), (u, v)\rangle \\
&\quad = \langle I(u, v), (u, v)\rangle - \lambda_1(J(u, v), (u, v)) + \psi'(\langle I(u, v), (u, v)\rangle) \\
&\qquad \times \left[\langle I(u, v), (u, v)\rangle - \frac{\lambda - \lambda_1}{\psi'(\langle I(u, v), (u, v)\rangle)}\langle J(u, v), (u, v)\rangle\right] \\
&\quad \geq \frac{2\delta}{\lambda_1}[\langle I(u, v), (u, v)\rangle - 2K]\left[\langle I(u, v), (u, v)\rangle - \frac{\lambda_1}{2}\langle J(u, v), (u, v)\rangle\right].
\end{aligned}$$

Hence

$$\left\langle \frac{1}{p}\Psi_u^\lambda(u_n, v_n) + \frac{1}{q}\Psi_v^\lambda(u_n, v_n), (u_n, v_n)\right\rangle \to \infty,$$

which means that

$$\left\langle (\Psi^\lambda)'(u_n, v_n), (u_n, v_n)\right\rangle = \left\langle \Psi_u^\lambda(u_n, v_n) + \Psi_v^\lambda(u_n, v_n), (u_n, v_n)\right\rangle \to \infty$$

and the proof of the lemma is completed. ◁

Lemma 4.5. *For the operator $A_\lambda(u, v)$ the following are true*

$$\mathrm{Ind}(A_\lambda, 0) = 1, \quad \lambda \in (0, \lambda_1) \quad \textit{and} \quad \mathrm{Ind}(A_\lambda, 0) = -1, \quad \lambda \in (\lambda_1, \lambda_1 + \delta).$$

Proof. The proof follows steps like those for Lemma 5.5 of [8]. ◁

According to Definition 4.1 we have the following characterization.

Theorem 4.6. *The principal eigenvalue $\lambda_1 > 0$ of the unperturbed problem (1.3) and (1.4) is a bifurcation point (in the sense of Rabinowitz) of the perturbed system (1.1).*

Proof. The index jump result of Lemma 4.5 and the homotopy invariance of the degree imply that $(\lambda_1, 0, 0)$ is a bifurcation point of (1.1). The rest of the proof is similar to that of the Rabinowitz Theorem, see [7]. ◁

Finally, we discuss the sign of the solution branch close to the bifurcation point.

Proposition 4.7. *There exists an $\eta > 0$ small enough, such that for each $(\lambda, u, v) \in C \cap B_\eta(\lambda_1, 0)$, we have $u(x) \geq 0$ and $v(x) \geq 0$, almost everywhere in Ω.*

Proof. Let $(\lambda_n, u_n, v_n) \in C$ be a sequence such that $(\lambda_n, u_n, v_n) \to (\lambda_1, 0, 0)$. We introduce the sequences $\tilde{u}_n$ and $\tilde{v}_n$ in the following way.

$$\tilde{u}_n =: \frac{u_n}{(\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q)^{1/p}} \quad \text{and} \quad \tilde{v}_n =: \frac{v_n}{(\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q)^{1/q}}.$$

It is easy to prove that the sequences $\tilde{u}_n$ and $\tilde{v}_n$ are bounded. We also have

$$\frac{|u_n|^\alpha |v_n|^\beta u_n v_n}{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q} = |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta \tilde{u}_n \tilde{v}_n, \tag{4.8}$$

for every $n \in \mathbb{N}$. Let us take any pair of eigenfunctions $(\tilde{u}_n, \tilde{v}_n)$, $(\tilde{u}_m, \tilde{v}_m)$ and substitute into the system (1.1). Multiplying the first equation by $(\tilde{u}_n - \tilde{u}_m)$, and integrating by parts we obtain

$$\begin{aligned}
&\int_\Omega \nu_1(x)(|\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n - |\tilde{u}_m|^{p-2} \nabla \tilde{u}_m)(\nabla\tilde{u}_n - \nabla\tilde{u}_m)\mathrm{d}x \\
&= \lambda_n \int_\Omega a(x) \left(|\tilde{u}_n|^{p-2}\tilde{u}_n - |\tilde{u}_m|^{p-2}\tilde{u}_m\right)(\tilde{u}_n - \tilde{u}_m)\mathrm{d}x \\
&\quad + \lambda_n \int_\Omega b(x) \left(|\tilde{u}_n|^{\alpha}|\tilde{v}_n|^{\beta}\tilde{v}_n - |\tilde{u}_m|^{\alpha}|\tilde{v}_m|^{\beta}\tilde{v}_m\right)(\tilde{u}_n - \tilde{u}_m)\mathrm{d}x \\
&\quad + (\lambda_n - \lambda_m)\left[\int_\Omega a(x)|\tilde{u}_m|^{p-2}\tilde{u}_m(\tilde{u}_n - \tilde{u}_m)\mathrm{d}x + \int_\Omega b(x)|\tilde{u}_m|^{\alpha}|\tilde{v}_m|^{\beta}\tilde{v}_m\right].
\end{aligned}$$

Similarly for the second equation. Using relations (2.1) and (4.8) we have that

$$\int |\nu_1(x) \nabla \tilde{u}_n|^p = \lambda_n \int a(x)|\tilde{u}_n|^p + \lambda_n \int b(x)|\tilde{u}_n|^{\alpha}|\tilde{v}_n|^{\beta}\tilde{u}_n\tilde{v}_n + O(\|(u_n, v_n)\|_Z),$$

$$\int |\nu_2(x) \nabla \tilde{v}_n|^q = \lambda_n \int d(x)|\tilde{v}_n|^q + \lambda_n \int b(x)|\tilde{u}_n|^{\alpha}|\tilde{v}_n|^{\beta}\tilde{u}_n\tilde{v}_n + O(\|(u_n, v_n)\|_Z).$$

From the compactness of J_1, J_2 and the monotonicity of the degenerate p-Laplacian we derive that for some positive constant k, $\tilde{u}_n \to k^p u_1$ and $\tilde{v}_n \to k^q v_1$ (strongly) in the spaces $D^{1,p}$ and $D^{1,q}$, respectively. Assume that the sets $\mathcal{U}_n^- = \{x \in \Omega : \tilde{u}_n(x) < 0\}$ and $\mathcal{V}_n^- = \{x \in \Omega : \tilde{v}_n(x) < 0\}$ are non-empty. Using (2.1) we obtain that

$$1 \le c_0 \left(\max\left\{\|a(x)\|_{L^{\frac{p^*}{p^*-p}}(\Omega_n^-)}, \|d(x)\|_{L^{\frac{q^*}{q^*-q}}(\Omega_n^-)}\right\} + \|b(x)\|_{L^{\omega}(\Omega_n^-)}\right),$$

where $\Omega_n^- = \mathcal{U}_n^- \cup \mathcal{V}_n^-$. Since $\|(u_n, v_n\|_Z \to 0$, $a \in L^{p^*/(p^*-p)}(\Omega)$, $d \in L^{q^*/(q^*-q)}(\Omega)$, $b \in L^{\omega}(\Omega)$ and c_0 does not depend on u_n or v_n, we derive that for some $K_0 > 0$ large enough

$$|\Omega_n^- \cap B_K(0)| \ge c_1,$$

for any $K > K_0$, where $c_1 > 0$ depends neither on λ_n nor on u_n or v_n. Now, by Egorov's Theorem we deduce that $\tilde{u}_n$ and $\tilde{v}_n$ (and hence u_n and v_n) are nonnegative in Ω, for n large enough. Then, it follows that $u_n \ge 0$ and $v_n \ge 0$, for any $(\lambda, u_n, v_n) \in C \cap B_\eta(\lambda_1, 0)$, with $\eta > 0$ small enough. $\triangleleft$

5. Properties of the continuum C

In order to prove some additional properties of the system (1.1) we have to make some further restrictive assumptions:

$(\mathcal{T}_1)$ There exists a positive number t, such that

$$p\left(1 - \frac{\delta_i + 1}{q^*}\right)^{-1} < t < p^*,$$

$\mu \in L^{\frac{t}{t-p}}(\Omega)$ and $m \in L^{\chi_3}(\Omega)$, where $\chi_3 := \max\{[1 - \frac{\gamma_i+1}{p^*} - \frac{\delta_i+1}{q^*}]^{-1}, [1 - \frac{p}{t} - \frac{\delta_i+1}{q^*}]^{-1}\}$.

$(\mathcal{T}_2)$ There exists a positive number t, such that

$$q\left(1 - \frac{\gamma_i + 1}{p^*}\right)^{-1} < t < q^*,$$

$v \in L^{\frac{t}{t-q}}(\Omega)$ and $n \in L^{\chi_4}(\Omega)$, where $\chi_4 := \max\{[1 - \frac{\gamma_i+1}{p^*} - \frac{\delta_i+1}{q^*}]^{-1}, [1 - \frac{q}{t} - \frac{\gamma_i+1}{p^*}]^{-1}\}$.

Proposition 5.1. *Let $p > 1, q > 1$, and assume that $(\mathcal{H})$, $(\mathcal{N})$, $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{D})$ and $(\mathcal{T}_i)$, for $i = 1, 2$, hold. Then for any weak solution of (1.1) with $\lambda \geq 0$ we have that $u \in L^r(\mathbb{R}^N)$ (or $v \in L^r$), with $p^* \leq r \leq \infty$ and $u(x)$ (or $v(x)$ respectively) decaying uniformly as $|x| \to \infty$. Moreover, $u \in C^{1,\zeta}(B_K(0))$ (or $v(x)$ respectively) for any $K > 0$ with some $\zeta \in (0, 1)$. However, if both hypotheses hold then both u and v are uniformly bounded a.e. in Ω.*

Proof. Let $u_M(x) := \min\{u(x), M\}$. Choose $\phi = u_M^{kp+1} (k \geq 0)$, as a test function in the first equation of (1.1)

$$\int_\Omega \nu_1(x)|\nabla u|^{p-2}\nabla u \cdot \nabla(u_M^{kp+1})\mathrm{d}x \leq \int_\Omega a(x)|u|^{(k+1)p}\mathrm{d}x + \int_\Omega b(x)|v|^\beta|u|^{kp+\alpha}\mathrm{d}x + \int_\Omega f(x, \lambda, u, v)u_M^{kp+1}\mathrm{d}x. \tag{5.1}$$

Taking into consideration Proposition 5.1 of [9] we only need to estimate the last integral of (5.1). Therefore we obtain

$$\sigma_1(\lambda)\left(\int_\Omega m(x)^{\chi_3'}\right)^{\frac{1}{\chi_3'}}\left(\int_\Omega |v|^{q^*}\right)^{\frac{\delta_1+1}{q^*}}\left(\int_\Omega |u|^{(k+1)t}\right)^{\frac{p}{t}}$$

where we have chosen $\chi_3' = [1 - \frac{p}{t} - \frac{\delta_i+1}{q^*}]^{-1}$. Therefore we conclude that

$$\|u_M\|_{(k+1)p^*} \leq C^{\frac{1}{k+1}}\left[\frac{k+1}{(kp+1)^{1/p}}\right]^{\frac{1}{k+1}}\|u\|_{(k+1)t}.$$

The rest of the proof is similar to that in [9]. Thereby the proof of the proposition is completed. ◁

As a consequence of the previous proposition we obtain the following corollary.

Corollary 5.2. *Assume that the hypotheses of Proposition 5.1 are satisfied and (u, v) is a nonnegative eigenfunction corresponding to λ_1. Then u and v are strictly positive in Ω.*

Lemma 5.2. (i) (Local Bifurcation) *The only possible points of the form $(\lambda, 0, 0)$, which the closure of the continuum $\bar{C}$ may contain, are the points $(\lambda_{p,a}, 0, 0)$ and $(\lambda_{q,d}, 0, 0)$.*
(ii) (Bifurcation from semitrivial solutions) *The only possible points of the form $(\lambda, u, 0)$, $u \not\equiv 0$ (or $(\lambda, 0, v)$, $v \not\equiv 0$), which $\bar{C}$ may contain, are the points $(\lambda_{p,a}, c\phi_{p,a}, 0)$ (or $(\lambda_{q,d}, 0, c\phi_{q,d})$, respectively), for some real constant $c \neq 0$.*
(iii) *if $\bar{C}$ contains no point of the form $(\lambda, 0, 0)$, $(\lambda, u, 0)$, $u \not\equiv 0$ and $(\lambda, 0, v)$, $v \not\equiv 0$, then every solution (u, v) in C is strictly positive (componentwise).*

Proof. (i) The proof follows the same steps as in Proposition 4.7.
(ii) The proof follows the same steps as for (i).
(iii) Suppose that $\bar{C}$ contains no point of the form $(\lambda, 0, 0)$ and for some solution $(\lambda, u, v) \in C$ there exists a point $x_0 \in \Omega$, such that $u(x_0) < 0$. By Proposition 4.7 we may observe that $u(x) > 0$, $x \in \Omega$, for all solutions $(\lambda, u, v) \in C$ close to $(\lambda_1, 0, 0)$. Since the continuum C is connected, the $C^{1,a}_{\mathrm{loc}}$- regularity of the solutions implies that there exists $(\lambda_0, u_0, v_0) \in C$, such that $u_0(x) \geq 0$, for all $x \in \Omega$, except some point $x_0 \in \Omega$, such that $u_0(x_0) = 0$ and in

any neighborhood of (λ_0, u_0, v_0) we can find a point $(\hat{\lambda}, \hat{u}, \hat{v}) \in C$, with $\hat{u}(x) < 0$, for some $x \in \Omega$. Let B denote any open ball containing x_0. Then it follows that $u_0 \equiv 0$ on B. Hence $u_0 \equiv 0$ on Ω. Thus, we may construct a sequence $\{(\lambda_n, u_n, v_n)\} \subseteq C$, such that $u_n(x) > 0$ and $v_n(x) > 0$, for all $n \in \mathcal{N}$ and $x \in \Omega$, $u_n \to 0$ in $D^{1,p}$, $v_n \to v_0$ in $D^{1,q}$ and $\lambda_n \to \lambda_0$. Then, the continuum C contains a point of the form $(\lambda_0, 0, v_0)$, which is a contradiction. Similar results may be obtained for v. $\lhd$

Applying the previous results and the fact that the solutions of (1.1) are uniformly bounded we can state the main result of this section in the general case.

Theorem 5.3 (*Global Bifurcation*). *Let $p > 1, q > 1$, and assume that $(\mathcal{H})$, $(\mathcal{N})$, $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{D})$ and $(\mathcal{T}_i)$, for $i = 1, 2, 3, 4$, hold. Then, there exists a continuum $C \subseteq \mathbb{R} \times Z$ of uniformly bounded solutions of the problem (1.1) bifurcating from the zero solution at $(\lambda_1, 0, 0)$, such that one of the following alternatives holds.*

(i) *The continuum $\bar{C}$ (in closure) contains one of the points $(\lambda_{p,a}, 0, 0)$ and $(\lambda_{q,d}, 0, 0)$, and in particular problem (1.1) has a nontrivial positive (componentwise) solution $(u, v) \in Z$, whenever λ is between λ_1 and $\lambda_{p,a}$ or $\lambda_{q,d}$.*
(ii) *The continuum C is unbounded and every (u, v) in C is strictly positive (componentwise).*

Remark 5.1. For the following variation of the system (1.1) and (1.2)

$$\begin{aligned} &-\nabla(\nu_1(x,u,v)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha}|v|^{\beta}u + f(x,\lambda,u,v),\\ &-\nabla(\nu_2(x,u,v)|\nabla u|^{p-2}\nabla u) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta}v + g(x,\lambda,u,v), \end{aligned} \tag{5.2}$$

$$x \in \Omega, \quad u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{5.3}$$

where $\nu_1(x, s, t)$, $\nu_2(x, s, t)$ satisfy similar conditions to $(\mathcal{N})$ and $(\mathcal{N}_p)$ (see [2]) we may apply the same procedure as in this paper to show simplicity and isolation of the positive principal eigenvalue as well as global bifurcation of a continuum of positive solutions from the principal eigenvalue.

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