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Central manifold theory for the generalized equation of Kirchhoff strings on $\mathbb{R}^N$

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Abstract

We consider the generalized quasilinear dissipative Kirchhoff's String problem

$$u_{tt} = -\|A^{1/2}u\|_H^2 Au - \delta Au_t + f(u), \quad x \in \mathbb{R}^N, \ \ t \geqslant 0$$

with the initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geqslant 3, \ \ \delta > 0$. The purpose of our work is to study the stability of the initial solution $u = 0$ for this equation using central manifold theory.

MSC: 35B30; 35B40; 35B45; 35L15; 35L70; 35L80

Keywords: Quasilinear wave equations; Unbounded domains; Local and global solutions; Central manifold theory; Strong and weak dissipation; Kirchhoff's equations

1. Introduction

Our aim in this work is to study the following nonlocal quasilinear hyperbolic initial value problem:

$$u_{tt} = -\|A^{1/2}u\|_H^2 Au - \delta Au_t + f(u), \quad x \in \mathbb{R}^N, \ \ t \geqslant 0, \tag{1.1}$$

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$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \quad x\in\mathbb{R}^N \tag{1.2}$$

with initial conditions u_0, u_1 in appropriate function spaces, $N\geqslant 3$, and $\delta>0$.

Kirchhoff in 1883 proposed the so-called Kirchhoff string model in the study of oscillations of stretched strings and plates

$$ph\,\frac{\partial^2 u}{\partial t^2}+\delta\,\frac{\partial u}{\partial t}=\left\{p_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2\mathrm{d}x\right\}\frac{\partial^2 u}{\vartheta x^2}+f,\quad 0<x<L,\ \ t\geqslant 0,$$

where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modules, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modules and f the external force (see [6]). When $p_0=0$ the equation is considered to be of degenerate type and the equation models an unstretched string or its higher dimensional generalization. Otherwise it is of nondegenerate type and the equation models a stretched string or its higher dimensional generalization.

The global existence and the uniqueness of solutions have been established in the energy class (see [27]). Once global existence is known, it is not difficult to show that all solutions decay as $t\to\infty$. Furthermore, in the nondegenerate case a simple calculation of the energy shows that solutions decay at least exponentially.

In the degenerate case, however, estimates of the rate of decay requires far more delicate analysis. Much of the efforts have been focused on estimates from above (see [12,15]). But it is difficult to obtain the estimates from below. In fact, except for some special cases (see [16,17]), a few things are known about the lower estimates. Also Ono in [18], proved global existence, asymptotic stability and blowing up results of solutions for some degenerate nonlinear wave equation with a strong dissipation (see also [14,19,20]). Mizumachi [13] studied the asymptotic behavior of solutions to the Kirchhoff equation with a viscous damping term with no external force.

In our previous work (see [21,24]), we prove global existence blow-up results and existence of global attractor for some equations of Kirchhoff's type in all of $\mathbb{R}^N$. Also, in [23] we prove the existence of compact invariant sets for the same equation. Recently Karachalios and Stavrakakis (see [7–11]) studied global existence, blow-up results and asymptotic behavior of solutions for some semilinear wave equations with weak dissipation in all of $\mathbb{R}^N$.

It should be noted that, unlike most earlier works, our method makes use of the dynamical systems point of view involving the theory of central manifolds. The advantage of such an approach is that we may obtain exact decay estimates by relatively simple computations.

The presentation of this paper is as follows: in Section 2, we discuss the space setting of the problem and the necessary embeddings for constructing the evolution triple. We also give some examples in which our results hold. In Section 3, we make a review of the local invariant manifold theory. In Section 4, we prove the existence and uniqueness of the solution for our problem. In Section 5, we study the stability of the initial solution $u=0$. In order to study the stability, in fact we study the spectrum of the operator $\widehat{A}$. In our problem we have an external force $f(u)$ and the stability of the solution depends on the sign of $f'(0)\neq 0$. In Section 6, we examine the stability of the solution $u=0$, in the case where $f'(0)=0$. For this purpose we use the central manifold theory. Finally, in Section 7, we study the stability of the solution $u=0$, for the equation with weak dissipation. In this case, using Pego's transformation the problem becomes a system of ordinary differential equations.

2. Formulation of the problem

The space $D(\mathscr{A})$ is going to be introduced and studied later in this section. We shall frequently use the following Sobolev–Poincaré inequality:

$$\|v\|_p \leqslant c_* \|A^{m/2} v\| \tag{2.1}$$

for $v \in D(A^{m/2}),\ 1 \leqslant p \leqslant 2N/[N-2m]^+,\ (1 \leqslant p < \infty \text{ if } N = 2m)$.

We also need to make the following remarks: Let $V,\ H$ two Hilbert spaces where

$$V \subset H, \quad \text{and} \quad V \text{ is dense in } H. \tag{2.2}$$

We also have that

$$V \subset\subset H \text{ (compactly)}. \tag{2.3}$$

The scalar product and the norm in H are, respectively, denoted $(.,.)$, $\|.\|_H$. We identify H with its dual H', and H' with a dense subspace of the dual V' of V; thus we get the evolution triple

$$V \subset H \subset V', \tag{2.4}$$

where the injections are continuous and dense.

Let $a(u,v)$ be a bilinear continuous form on V which is symmetric and coercive

$$\exists\ a_0 > 0, \quad a(u,u) \geqslant a_0 \|u\|^2, \quad \forall u \in V. \tag{2.5}$$

With this form we associate the linear operator $\mathscr{A}$ from V into V' defined by

$$(\mathscr{A}u, v) = a(u,v), \quad \forall u, v \in V.$$

Operator $\mathscr{A}$ is an isomorphism from V onto V' and it can also be considered as a selfadjoint unbounded operator in H with domain $D(\mathscr{A}) \subset V$

$$D(\mathscr{A}) = \{v \in V,\ \mathscr{A}v \in H\}.$$

Due to (2.2) there exists an orthonormal basis of H, $\{w_j\}_{j \in N}$ which consists of eigenvectors of $\mathscr{A}$,

$$\begin{cases} \mathscr{A} w_j = \lambda_j w_j, & j = 1, 2, \ldots, \quad w_j \in H, \\ 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots, & \lambda_j \to \infty \quad \text{as } j \to \infty. \end{cases} \tag{2.6}$$

Our results cover the following problems:

Example 1. Let $\mathscr{A} = -\Delta,\ V = H_0^1(\Omega),\ H = L^2(\Omega),\ V^* = H^{-1}(\Omega),\ \Omega \subseteq \mathbb{R}^N$ (Ω bounded or unbounded), for the equation

$$u_{tt} - \|\nabla u\|_{L^2}^2 \Delta u - \delta u_t = 0, \quad x \in \Omega, \quad t > 0$$

with initial conditions $u(x,0) = u_0,\ u_t(x,0) = u_1,\ u|_{\partial\Omega} = 0$ (see [13]).

Example 2. Let $V = \mathscr{D}^{1,2}(\mathbb{R}^N),\ \ H = L^2_g(\mathbb{R}^N),\ \ V^* = \mathscr{D}^{-1,2}(\mathbb{R}^N),\ \ \mathscr{A} = -\phi\Delta$, for the Kirchhoff's equation in all of $\mathbb{R}^N$

$$u_{tt} - \phi(x)\Delta u\|\nabla u\|^2_{L^2_g} - \delta\phi(x)\Delta u_t = f(u), \quad x \in \mathbb{R}^N,\ \ t > 0,$$

where $u(x, 0) = u_0,\ \ u_t(x, 0) = u_1$ and $\phi(x) \to \infty$ as $|x| \to \infty$ (see [18,21]).

Example 3. Let $V = H^2_0(\Omega),\ \ H = L^2(\Omega),\ \ D(\mathscr{A}) = H^2_0(\Omega) \cap H^4(\Omega),\ \ \mathscr{A} = \Delta^2,\ \ \Omega = [0, L]$, for the equation

$$\frac{\partial^2 u}{\partial t^2} + \delta u_{xxxxt} + \|u_{xx}\|^2_2 u_{xxxx} = f(u), \quad x \in \Omega,\ \ t > 0$$

with $u(x, 0) = u_0,\ \ u_t(x, 0) = u_1,\ \ u|_{\partial\Omega} = u_{xx}|_{\partial\Omega} = 0$. In higher dimension the equation may be written in the form

$$u_{tt} + \delta\Delta^2 u_t + \|\Delta u\|^2_2 \Delta^2 u = f(u), \quad x \in \Omega, \quad \text{where } \Omega \subset \mathbb{R}^N,\ \ t > 0,\ \ N \geqslant 2.$$

For more information about the problem we refer to [26, p. 223]. Also, our results hold for equations with strong dissipation (see [26, p. 221]).

Finally, we give the notion of the *weak solution* for problem (1.1)–(1.2).

Definition 2.1. A weak solution of problem (1.1)–(1.2) is a function $u(x, t)$ such that

(i) $u \in L^2[0, T; D(\mathscr{A})],\ \ u_t \in L^2[0, T; V],\ \ u_{tt} \in L^2[0, T; H]$,
(ii) for all $v \in C^\infty_0([0, T] \times (\mathbb{R}^N))$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_H \,\mathrm{d}\tau + \int_0^T \left(\|\mathscr{A}^{1/2} u(t)\|^2_H \int_{\mathbb{R}^N} \mathscr{A}^{1/2} u(\tau) \mathscr{A}^{1/2} v(\tau) \,\mathrm{d}x \,\mathrm{d}\tau \right)$$
$$+ \delta \int_0^T (\mathscr{A} u_t(\tau), v(\tau))_H \,\mathrm{d}\tau - \int_0^T (f(u(\tau)), v(\tau))_H \,\mathrm{d}\tau = 0$$

and
(iii) satisfies the initial conditions

$$u(x, 0) = u_0(x), \quad u_0 \in D(\mathscr{A}), \quad u_t(x, 0) = u_1(x), \quad u_1 \in V.$$

3. Invariant manifolds

In this section we make a brief review of the local invariant manifold theory.
Let X be a Banach space and consider the semilinear evolution equation on X:

$$z_t + \mathscr{A} z = g(z). \tag{3.1}$$

We assume that $\mathscr{A}$ and g satisfy the following hypotheses:

(A_1) $\mathscr{A}$ is a sectorial operator, or $-\mathscr{A}$ generates an analytic semigroup.

(A_2) The spectrum of $\mathscr{A}$, $\sigma(\mathscr{A})$, can be written as $\sigma(\mathscr{A}) = \sigma_0 \cup \sigma_+$, where σ_0 and σ_+ are spectral sets such that

$$\sigma_0 = \{\lambda \in \sigma(\mathscr{A}) | \mathrm{Re}\lambda = 0\},$$
$$\sigma_+ = \{\lambda \in \sigma(\mathscr{A}) | \mathrm{Re}\lambda > 0\}.$$

(A_3) There exists a real number a, $0 \leqslant a < 1$, and an integer $k \geqslant 1$ such that g is a C^k-mapping from X^a to X. Furthermore, $g(0) = 0$ and $Dg(0) = 0$.

(A_4) X has C^k-norm, that is, $F(z) = \|z\|_X$ $(z \neq 0)$ is a C^k-function.

Here we define the space X^a as follows: Let $\gamma > 0$ be a sufficiently large constant such that $\inf_{n \in \sigma(\mathscr{A}+\gamma I)} \mathrm{Re}\, n > 0$. Then the fractional powers of $(\mathscr{A} + \gamma I)$ can be defined for all $a \in \mathbb{R}$. We denote by X^a the domain of the operator $(\mathscr{A} + \gamma I)^a$, which is a Banach space equipped with the norm $\|w\|_{X^a} =: \|(\mathscr{A} + \gamma I)^a w\|_X$.

Under hypothesis (A_2), there exist projections P_0 and P_+ associated with σ_0 and σ_+ such that the sets $X_0 =: P_0 X$ and $X_+ =: P_+ X$ are invariant subspaces, $X = X_0 \oplus X_+$, $\sigma(A|_{X_0}) = \sigma_0$ and $\sigma(A|_{X_+}) = \sigma_+$, (for the proof see [4]).

Let $X_0^a = X^a \cap X_0$ and $X_+^a = X^a \cap X_+$. Since σ_0 is a bounded spectral set, we easily see that $X_0^a = X_0$.

Let $\{T(t)\}_{t \geqslant 0}$ be a semiflow defined by (3.1). In other words, by $T(t)z_0$, we mean the solution to (3.1) that equals z_0 at $t = 0$.

Definition 3.1. The set of all $\lambda \in \sigma(\mathscr{A})$, for which the operator $\lambda I - \mathscr{A}$ is one to one and $(\lambda I - \mathscr{A})\mathscr{X}$ is dense in $\mathscr{X}$, but such that $(\lambda I - \mathscr{A})\mathscr{X} \neq \mathscr{X}$, where $\mathscr{X}$ is a Banach space, is called the continuous spectrum of $\mathscr{A}$ and denoted by $\sigma_c(\mathscr{A})$ (see [4, p. 580]).

Definition 3.2. For a given neighborhood U of 0 in X^a, a local center manifold $W^c_{\mathrm{loc}}(0)$ is a C^1-submanifold in U satisfying the following:

(1) $W^c_{\mathrm{loc}}(0) = \{\xi + \eta \,|\, \xi = h^c(\eta),\ \xi \in X_+^a \cap U,\ \eta \in X_0 \cap U\}$, where we have that $h^c : X_0 \cap U \to X_+^a$ is a C^1-mapping with $h^c(0) = 0$ and $Dh^c(0) = 0$.

(2) For each $z \in W^c_{\mathrm{loc}}(0)$, $\{T(t)z \,|\, t_1 \leqslant t \leqslant t_2\} \subset U$ implies that $\{T(t)z \,|\, t_1 \leqslant t \leqslant t_2\} \subset W^c_{\mathrm{loc}}(0)$, for any t_1, t_2 with $t_1 < 0 < t_2$.

Finally, we have the following useful result for the existence of a local center manifold. (For the proof we refer to [3,2,5]).

Proposition 3.3. *Under hypotheses* (A_1)–(A_4), *there exists a neighborhood U of* 0 *in X^a satisfying the following*:

$W^c_{\mathrm{loc}}(0)$ *is a local C^k-submanifold of X^a that is tangent to X_0 at* 0, *that is, there exists a C^k-function $h^c : X_0 \cap U \to X_+^a$ satisfying $h^c(0) = Dh^c(0) = 0$ and*

$$W^c_{\mathrm{loc}}(0) = \{\xi + \eta \,|\, \xi = h^c(\eta),\ \xi \in X_+^a \cap U,\ \eta \in X_0 \cap U\}.$$

4. Global existence

In this section we prove existence of a global solution for the problem (1.1)–(1.2), under small initial data.

Theorem 4.1 (*Local existence*). *Let* $f(u)$ *a* C^1*-function such that* $|f(u)|\leqslant k_1|u|^{a+1}$, $|f'(u)|\leqslant k_2|u|^a$, $0\leqslant a\leqslant 4/(N-2)$ *and* $N\geqslant 3$. *Consider that* $(u_0,\ u_1)\in D(A)\times V$ *and satisfy the* nondegenerate condition

$$\|A^{1/2}u_0\|^2>0. \tag{4.1}$$

Then there exists $T_*=T(\|Au_0\|,\ \|A^{1/2}u_1\|)>0$ *such that the problem* (1.1)–(1.2) *admits a unique local weak solution* u *satisfying*

$$u\in C(0,T_*;D(A)) \quad \textit{and} \quad u_t\in C(0,T_*;V).$$

Proof. We take $\delta=1$. Given the constants $T_*>0,\ \ R>0$, we introduce the two parameter space of solutions

$$X_{T_*,R}=:\{v\in C^0_w(0,T_*;D(A)):v_t\in C^0_w(0,T_*;V),v(0)=u_0,\ v_t(0)=u_1,\\ e(v)\leqslant R^2,\ t\in[0,T_*]\},$$

where $e(v(t))=:\|Av(t)\|_H^2+\|A^{1/2}v'(t)\|_H^2$, the norm in the space $\mathscr{X}_0=: D(A)\times V$. Also u_0 satisfies the nondegenerate condition (4.1). It is easy to see that the set $X_{T_*,R}$ is a complete metric space under the distance $d(u,\ v)=:\sup_{0\leqslant t\leqslant T}e_1(u(t)-v(t))$, where

$$e_1(v)=\|v'(t)\|_H^2+\|A^{1/2}v\|_H^2,$$

the norm in the space $\mathscr{X}_1=: V\times H$. We have that $\mathscr{X}_0\subset\mathscr{X}_1$ compactly, that is, $e_1(u(t))\leqslant e(u(t))$.

Next, we introduce the nonlinear mapping S in the following way. Given $v\in X_{T_*,R}$, we define $u=Sv$ to be the unique solution of the linear wave equation with strong dissipation

$$u''(t)+\|A^{1/2}v(t)\|^2Au(t)+Au'(t)=f(v), \tag{4.2}$$

$$u(0)=u_0,\ u'(0)=u_1. \tag{4.3}$$

In the sequel we shall show that there exists $T_*>0$ and $R>0$ such that the following two conditions are valid:

$$\text{(i) } S \text{ maps } X_{T_*,R} \text{ into itself}, \tag{4.4}$$

$$\text{(ii) } S \text{ is a contraction with respect to the metric } d(.,.). \tag{4.5}$$

Set $2M_0=:\|A^{1/2}u_0\|_H^2>0$ and

$$T_0=:\sup\{t\in[0,+\infty);\|A^{1/2}v(s)\|_H^2>M_0,\ \text{for } 0\leqslant s\leqslant t\}.$$

Then

$$T_0>0 \quad \text{and} \quad \|A^{1/2}v(t)\|_H^2\geqslant M_0, \quad \text{on } [0,T_0]. \tag{4.6}$$

(i) To check property (4.4), we multiply Eq. (4.2) by $2Au_t$ and integrate over $\mathbb{R}^N$ to get

$$2\int_{\mathbb{R}^N} Au_t u_{tt}\,\mathrm{d}x + 2\int_{\mathbb{R}^N} \|A^{1/2}v(t)\|_H^2 Au_t Au\,\mathrm{d}x + 2\int_{\mathbb{R}^N} Au_t Au_t\,\mathrm{d}x = 2\int_{R^n} f(v)Au_t\,\mathrm{d}x. \tag{4.7}$$

So, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|A^{1/2}u_t(t)\|_H^2 + \|A^{1/2}v(t)\|_H^2\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\|Au(t)\|_H^2) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\|A^{1/2}v(t)\|_H^2\right)\|Au(t)\|^2 + 2(f(v),\ Au_t(t)).$$

Finally, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,e_2^*(u(t)) + 2\|Au_t(t)\|_H^2 = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\|A^{1/2}v(t)\|_H^2\right)\|Au(t)\|_H^2 + 2(f(v),\,Au_t(t)), \tag{4.8}$$

where we set

$$e_2^*(u(t)) =:\ \|A^{1/2}u_t(t)\|_H^2 + \|A^{1/2}v(t)\|_H^2\|Au(t)\|_H^2.$$

Note that

$$e_2^*(u) \geqslant \|A^{1/2}u_t\|^2 + M_0\|Au(t)\|_H^2 \geqslant c_1^{-2}e(u(t)) \geqslant c_1^{-2}e_1(u(t)) \tag{4.9}$$

with $c_1 = (\max\{1,\, M_0^{-1}\})^{1/2}$. To proceed further, we observe that

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\|A^{1/2}v\|_H^2\right)\|Au(t)\|_H^2 &= \left(2\int_{\mathbb{R}^N} Avv_t\,\mathrm{d}x\right)\|Au(t)\|_H^2\\ &\leqslant 2(\|Av\|_H^2)^{1/2}(\|v_t\|_H^2)^{1/2}\,\|Au(t)\|_H^2\\ &\leqslant 2Rk(\|v_t\|_H^2)^{1/2}\ e(u(t))\\ &\leqslant 2RkRc_1^2e_2^*(u(t)) \leqslant c_2R^2e_2^*(u(t)))\end{aligned} \tag{4.10}$$

with $c_2 = 2kc_1^2$, where k is the constant of the embedding $V \subset H$. We also have that

$$\begin{aligned}2(f(v),\,Au_t) &= 2\int_{\mathbb{R}^N} f'(v)A^{1/2}vA^{1/2}u_t\,\mathrm{d}x\\ &\leqslant 2k_2\|v\|_{Na}^a\|A^{1/2}v\|_{2N/N-2}\|A^{1/2}u_t\|,\end{aligned}$$

where we used Hölder inequality with $p^{-1} = 1/N,\ q^{-1} = N-2/2N,\ r^{-1} = \frac{1}{2}$. From the embeddings $D(A) \subset V \subset H$ and using Sobolev–Poincaré inequality we get

$$\|v\|_{Na}^a \leqslant c_*^a\|Av\|_H^a \leqslant c_*^aR^a,\ \ \|A^{1/2}v\|_{2N/N-2} \leqslant c_*\|Av\|_H \leqslant c_*R.$$

Thus, we obtain

$$2(f(v),\ Au_t(t)) \leqslant 2k_2c_*^aR^ac_*Re(u(t))^{1/2}. \tag{4.11}$$

Using estimates (4.10)–(4.11), we get from (4.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\, e_2^*(u(t)) \leqslant c_2 R^2 e_2^*(u(t)) + c_3 R^{a+1} e_2^*(u(t))^{1/2},$$

where $c_3 = 2k_2 c_*^{a+1} c_1$. Hence, from Gronwall's inequality we get

$$e_2^*(u(t)) \leqslant \{e_2^*(u(0))^{1/2} + c_3 R^{a+1} T_*\}^2 e^{c_2 R^2 T_*}.$$

Thus from estimation (4.9) we obtain

$$\begin{aligned} e_1(u) \leqslant e(u(t)) &\leqslant c_1^2((\|A^{1/2}u_1\|_H^2 + \|A^{1/2}u_0\|_H^2\ \|Au_0\|_H^2)^{1/2} + c_3 R^{a+1} T_*)^2 e^{c_2 R^2 T_*} \\ &=: C_1(T_*, R) \end{aligned} \tag{4.12}$$

for any $t \in [0, T_*]$, with $T_* \leqslant T_0$. Since we have that function $u \in L^\infty(0, T_*;\ D(A)) \cap W^{1,\infty}(0, T_*;\ V)$ and $u(t)$ satisfies Eq. (4.2), it follows that $u'' \in L^\infty(0, T_*;\ H)$ and hence, $u \in C_w^0([0, T_*];\ D(A)) \cap C_w^1([0, T_*];\ V)$. Thus, for the map S to verify condition (4.4) it will be enough that the parameters $T_*,\ R$ satisfy

$$C_1(T_*, R) < R^2 \tag{4.13}$$

which is true for T_* and the norms small enough.

(ii) We take $v_1, v_2 \in X_{T_*,R}$ and denote by $u_1 = Sv_1, u_2 = Sv_2$. Hereafter, we suppose that (4.13) is valid, i.e. $u_1, u_2 \in X_{T_*,R}$. We set $w = u_1 - u_2$. The function w satisfies the following problem:

$$\begin{aligned} w_{tt} + \|A^{1/2}v_1\|_H^2 Aw + Aw_t = &- \{\|A^{1/2}v_1\|_H^2 - \|A^{1/2}v_2\|_H^2\}Au_2 \\ &+ f(v_1) - f(v_2), \end{aligned} \tag{4.14}$$

$$w(0) = 0,\ \ w_t(0) = 0. \tag{4.15}$$

Multiplying Eq. (4.14) by $2w_t$ and integrating over $\mathbb{R}^N$ we have the equation

$$\begin{aligned} &2\int_{\mathbb{R}^N} w_t w_{tt}\,\mathrm{d}x + 2\int_{\mathbb{R}^N} \|A^{1/2}v_1\|_H^2 Aww_t\,\mathrm{d}x + 2\int_{\mathbb{R}^N} Aw_t^2\,\mathrm{d}x \\ &\quad = -2\{\|A^{1/2}v_1\|_H^2 - \|A^{1/2}v_2\|_H^2\}\int_{\mathbb{R}^N} Au_2 w_t\,\mathrm{d}x + 2\int_{\mathbb{R}^N}(f(v_1) - f(v_2))w_t\,\mathrm{d}x \end{aligned} \tag{4.16}$$

which may also be written as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\, e_{v_1}^*(w) + 2\|A^{1/2}w_t\|_H^2 = &+ \frac{\mathrm{d}}{\mathrm{d}t}\,\|A^{1/2}v_1\|_H^2\|A^{1/2}w\|_H^2 \\ &- 2\{\|A^{1/2}v_1\|_H^2 - \|A^{1/2}v_2\|_H^2\}(Au_2, w_t) \\ &+ 2(f(v_1) - f(v_2), w_t) \\ \equiv &+ I_1(t) + I_2(t) + I_3(t), \end{aligned} \tag{4.17}$$

where we set $e_{v_1}^*(w(t)) =: \|w_t(t)\|^2 + \|A^{1/2}v_1(t)\|_H^2\|A^{1/2}w(t)\|_H^2$. Note that the following estimations are valid:

$$e_{v_1}^*(w) \geqslant \|w_t\|^2 + M_0\|A^{1/2}w\|_H^2 \geqslant c_1^{-2} e_1(w). \tag{4.18}$$

As in (4.10), we observe that

$$I_1(t) \leqslant c_2 R^2 e^*_{v_1}(w) \tag{4.19}$$

and

$$\begin{aligned} I_2(t) &\leqslant 2(\|A^{1/2}v_1\|_H + \|A^{1/2}v_2\|_H)\|A^{1/2}(v_1 - v_2)\|_H \|Au_2\|\|w_t(t)\| \\ &\leqslant 2(R+R)e_1(v_1 - v_2)^{1/2} R e_1(w(t))^{1/2} \\ &\leqslant 4R^2 e_1(v_1 - v_2)^{1/2} c_1 e^*_{v_1}(w) = c_4 R^2 e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w), \end{aligned} \tag{4.20}$$

where $c_4 = 4c_1$. Next, applying the Sobolev–Poincaré inequality (2.1) and the embeddings $D(A) \subset V \subset H$, we obtain

$$\begin{aligned} I_3(t) &\leqslant 2k_1 \alpha^{-1}(\|A^{1/2}v_1\|^a_H + \|A^{1/2}v_2\|^a_H)\|A^{1/2}(v_1 - v_2)\|_H \|w_t\| \\ &\leqslant c_6 R^a e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w(t))^{1/2}, \end{aligned} \tag{4.21}$$

where $c_6 = 4k_1\alpha^{-1}c_1$. From estimates (4.19)–(4.21), we obtain the following estimate for the relation (4.17):

$$\frac{\mathrm{d}}{\mathrm{d}t} e^*_{v_1}(w) \leqslant c_2 R^2 e^*_{v_1}(w) + (c_4 R^2 + c_6 R^a) e_1(v_1 - v_2)^{1/2} e^*_{v_1}(w)^{1/2}.$$

Gronwall's inequality and the fact that $e^*_{v_1}(w(0)) = 0$ imply

$$e^*_{v_1}(w) \leqslant (c_4 R^2 + c_6 R^a)^2 T_*^2 e^{c_2 R^2 T_*} \sup_{0 \leqslant t \leqslant T_*} e_1(v_1(t) - v_2(t)). \tag{4.22}$$

Therefore, from (4.12) and (4.22) we get

$$d(u_1, u_2) \leqslant C_2(T_*, R) d(v_1, v_2), \tag{4.23}$$

where the constant $C_2(T_*, R)$ depending on T_* and R is

$$C_2(T_*, R) \equiv c_1^2 \{c_4 R^2 + c_6 R^a\}^2 T_*^2 e^{c_2 R^2 T_*}.$$

For small enough $T_* > 0$, we have $C_2(T_*, R) < 1$. From the above argument, by applying the Banach contraction mapping principle we know that the problem (1.1)–(1.2) admits a unique solution $u(t)$ in the class

$$C^0_w([0, T_*];\ D(A)) \cap C^1_w([0, T_*];\ V).$$

Moreover, we see that $u \in L^\infty(0, T_*;\ D(A)) \cap W^{1,\infty}(0, T_*;\ V)$ and $f(u) \in L^\infty(0, T_*;\ V)$. Therefore, it follows from the continuity argument for wave equations (see [26]) that this solution u belongs to

$$C^0([0, T_*];\ D(A)) \cap C^1([0, T_*];\ V).$$

The proof of theorem is now completed. □

Remark 4.2. In the above theorem if we assume that $u_0 \in D(A)$, $u_1 \in V$, f is a nonlinear C^1 function, then it is easy to see following the same steps, that the solution u belongs to

$$C^0([0, T];\ V) \cap C^1([0, T];\ H). \tag{4.24}$$

In that case, because of the inequalities

$$e_1(u(t)) \leqslant e(u(t)) \leqslant R^2,$$

we find that u is a solution such that

$$u \in L^\infty(0, T;\ V),\ u' \in L^\infty(0, T;\ H).$$

The continuity properties (4.24), are proved with the methods indicated in [26, Sections II.3 and II.4].

Corollary 4.3 (*Global existence*). *We assume that* $(u_0,\ u_1)\ \in D(A) \times V$. *Then, there exists a unique solution of problem* (1.1)–(1.2), *such that*

$$u \in C([0,\ +\infty)\ ;\ V),\ u_t \in C([0,\ +\infty)\ ;\ H).$$

Proof. The proof follows the main ideas developed in the work [21, Theorem 4.3]. We also refer to [18]. □

5. The linearized system

In this section we study the stability of the initial solution $u = 0$ in the case of $f'(0) \neq 0$. The linearized equation of the system around solution u is

$$v_{tt} = -\|A^{1/2}u\|_H Av - \delta Av_t + f'(u)v. \tag{5.1}$$

In the case where $u = 0$, Eq. (5.1) becomes

$$v_{tt} + \delta Av_t - f'(0)v = 0. \tag{5.2}$$

Setting $v_t = w$, Eq. (5.2) gets the following form:

$$\begin{bmatrix} w \\ v \end{bmatrix}_t + \begin{bmatrix} \delta A & -f'(0) \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\bar{u_t} + \hat{A}\bar{u} = 0, \tag{5.3}$$

where

$$\bar{u_t} = (w,\ v)^{\mathrm{T}} \quad \text{and} \quad \hat{A} = \begin{bmatrix} \delta A & -f'(0) \\ -1 & 0 \end{bmatrix}. \tag{5.4}$$

So, in order to study the stability of the solution, we study the spectrum of the operator $\hat{A}$. For later use we state the following theorems.

Theorem 5.1. *Let* $\mathscr{A}$ *be a sectorial linear operator in a Banach space* X, *and let* $f : U \to X$, *where* U *is a cylindrical neighborhood in* $\mathbb{R}^N \times X^\delta$, (*for some* $\delta < 1$). *Also let* x_0 *be an equilibrium point. Suppose*

$$f(t,\ x_0 + z) = f(t,\ x_0) + Bz + g(t,\ z),$$

where B is a bounded linear map from X^δ to X and $\|g(t,\ z)\| = O(\|z\|_\delta)$ as $\|z\|_\delta \to 0$, uniformly in t, and $f(t,x)$ is locally Hölder continuous in t, locally Lipschitzian in x, on U.

If the spectrum of $L = \mathscr{A} - B$ lies in $\{\mathrm{Re}\,\lambda > \beta\}$, for some $\beta > 0$, i.e., $\sigma(L) \subset \{\mathrm{Re}\,\lambda > \beta\}$, or equivalently if the linearization

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \mathscr{A}z = Bz$$

is uniformly asymptotically stable, then the original equation has the solution x_0 uniformly asymptotically stable in X^δ.

Proof. See [5, Theorem 5.1.1, p. 98]. □

Theorem 5.2. *Let $\mathscr{A},\ f$ be as in Theorem 5.1. Assume also $\mathscr{A}x_0 = f(t,\ x_0)$ for $t \geqslant t_0$, where*

$$\begin{aligned}
&f(t,\ x_0 + z) = f(t,\ x_0) + Bz + g(t,z), \quad g(t,0) = 0,\\
&\|g(t,z_1) - g(t,z_2)\| \leqslant k(\rho)\|z_1 - z_2\|_\delta,\\
&\|z_1\|_\delta \leqslant \rho,\ \|z_2\|_\delta \leqslant \rho, \quad \text{and} \quad k(\rho) \to 0,\ \ \rho \to 0^+.
\end{aligned}$$

If $L = \mathscr{A} - B$ assume $\sigma(L) \cap \{\mathrm{Re}\,\lambda < 0\}$ is a nonempty spectral set. Then the equilibrium solution x_0 is unstable.

Proof. See [5, Theorem 5.1.3, p. 102]. □

Next, we will compute the eigenvalues of $\hat{A}$. Let $\bar{x_j} = [\phi_j,\ \psi_j] \in D(A)$. Eigenvalues of $\hat{A}$ satisfy the following relation:

$$\hat{A}\bar{x_j} = \mu_j \bar{x_j}$$

or

$$\begin{bmatrix} \delta A & -f'(0) \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} = \mu_j \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix}.$$

Therefore, we have the following system:

$$\begin{cases} \delta A\phi_j - f'(0)\psi_j = \mu_j\phi_j, \\ -\phi_j = \mu_j\psi_j. \end{cases} \tag{5.5}$$

But, we have that $\{\phi_j\}_{j\in N}$ are eigenfunctions of A, i.e.

$$\begin{cases} A\phi_j = \lambda_j\phi_j, & j = 1, 2, \ldots, \quad \phi_j \in D(A), \\ 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots, & \lambda_j \to \infty, \quad \text{as } j \to \infty. \end{cases} \tag{5.6}$$

So, system (5.5) becomes

$$\begin{cases} \delta\lambda_j\phi_j - f'(0)\psi_j = \mu_j\phi_j, \\ -\phi_j = \mu_j\psi_j, \end{cases}$$

or

$$\begin{bmatrix} \delta\lambda_j & -f'(0) \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} = \mu_j \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix}. \tag{5.7}$$

Therefore, in order to find the eigenvalues of $\hat{A}$, we compute the characteristic polynomial of $\hat{A}$, i.e.

$$\begin{vmatrix} -\delta\lambda_j + \mu_j & f'(0) \\ 1 & \mu_j \end{vmatrix} = 0$$

or equivalently

$$\mu_j^2 - \delta\lambda_j\mu_j - f'(0) = 0.$$

Let, $\Delta = \delta^2\lambda_j^2 + 4f'(0)$. Then according to the sign of $f'(0)$, we have the following cases:

Case I: Let $f'(0) > 0$. Then the operator $\hat{A}$ admits the following two real eigenvalues of different sign:

$$\mu_j^{\pm} = \frac{\delta\lambda_j \pm (\delta^2\lambda_j^2 + 4f'(0))^{1/2}}{2}. \tag{5.8}$$

Also, since $f'(0) > 0$ we may easily see that the continuous spectrum of $\hat{A}$, $\sigma_c(\hat{A}) = \emptyset$. So, by Theorem 5.2 we have that min $\mathrm{Re}_{n\in\sigma(\hat{A})}\, n < 0$, which implies that 0 is unstable for the initial Kirchhoff's system.

Case II: Let $f'(0) < 0$. Then we have the following cases:

Case a: Let $\Delta = \delta^2\lambda_1^2 + 4f'(0) \geqslant 0$. Then we have

$$\frac{\delta^2\lambda_1^2}{4} \geqslant -f'(0). \tag{5.9}$$

Then the operator $\hat{A}$ admits two real eigenvalues which are both positive. Indeed, we have for the smallest eigenvalue μ_1^-

$$\mu_1^- = \frac{\delta\lambda_1 - (\delta^2\lambda_1^2 + 4f'(0))^{1/2}}{2} > 0, \quad \text{that is} \quad 0 > 4f'(0)$$

which holds. In order to find the continuous spectrum of $\hat{A}$, we examine when $\Delta = 0$. This is the case when

$$\lambda_1 = \frac{2\sqrt{-f'(0)}}{\delta}. \tag{5.10}$$

From (5.8) and (5.10) we obtain that

$$\mu_1 = \frac{\delta\lambda_1}{2} = \sqrt{-f'(0)} > 0. \tag{5.11}$$

So, using the fact that $f'(0) < 0$, we have an isolated eigenvalue $\sqrt{-f'(0)}$, being an accumulation point of the sequence of eigenvalues $\{\mu_j^{\pm}\}_{j=1}^{\infty}$. Thus, we have that

$\sigma_c(\hat{A}) = \{\sqrt{-f'(0)}\} \neq \emptyset$. We assume that

$$0 < \mu_1^- = \frac{\delta\lambda_1 - (\delta^2\lambda_1^2 + 4f'(0))^{1/2}}{2} < \sqrt{-f'(0)} \quad \text{or}$$
$$8(-f'(0)) < 4\delta\lambda_1\sqrt{-f'(0)}.$$

Finally, we get

$$\sqrt{-f'(0)} < \frac{\delta\lambda_1}{2}$$

which holds because of (5.9). Thus, applying Theorem 5.1, we have that the solution $u = 0$ is asymptotically stable for the initial Kirchhoff's system.

Case b: Let $\Delta < 0$. In that case we have that the eigenvalues $\mu_1^{\pm}$ are complex and

$$\min \mathrm{Re}_{\mu_1 \in \sigma(\hat{A})} \mu_1^{\pm} = \frac{\delta\lambda_1}{2} > 0.$$

Therefore, using Theorem 5.1, we have that the solution $u = 0$ is asymptotically stable for the initial Kirchhoff's system.

6. Central manifold

In this section we study the stability of the initial solution $u = 0$, in the case where $f'(0) = 0$ by means of central manifold theory. Making use of a change of variables similar to what is introduced by Pego (see [25]), namely

$$\begin{cases} p(x,t) = A^{-1/2}u_t, \\ q(x,t) = -\delta A^{1/2}u - p, \end{cases} \tag{6.1}$$

we can rewrite (1.1)–(1.2) in the form of a *reaction–diffusion system*

$$\begin{cases} p_t(x,t) = -\delta Ap + \left(\frac{1}{\delta^3}\|p+q\|_H^2\right)(p+q) + A^{-1/2}f(u), \\ q_t(x,t) = -\left(\frac{1}{\delta^3}\|p+q\|_H^2\right)(p+q) - A^{-1/2}f(u), \\ p(x,t) = 0, \quad t > 0, \\ p(x,0) = p_0(x), \quad q(x,0) = q_0(x), \end{cases} \tag{6.2}$$

where $p + q = -\delta A^{1/2}u$. Indeed, we have

$$\begin{aligned} p_t &= -\delta Ap + \left(\frac{1}{\delta^3}\|p+q\|_H^2\right)(p+q) + A^{-1/2}f(u) \\ &= -\delta AA^{-1/2}u_t + \left(\frac{1}{\delta^3}\|-\delta A^{1/2}u\|_H^2\right)(-\delta A^{1/2}u) + A^{-1/2}f(u) \\ &= -\delta A^{1/2}u_t - \|A^{1/2}u\|_H^2 A^{1/2}u + A^{-1/2}f(u). \end{aligned}$$

But we have that $p_t = A^{-1/2}u_{tt}$. Thus, we get

$$A^{-1/2}u_{tt} = -\delta A^{1/2}u_t - \|A^{1/2}u\|_H^2 A^{1/2}u + A^{-1/2}f(u). \tag{6.3}$$

Using operator $A^{1/2}$ in (6.3) we have

$$u_{tt} = -\delta A u_t - \|A^{1/2}u\|_H^2 Au + f(u) \text{ (Generalized Kirchhoff's Equation)}.$$

For another example we can examine how Pego's transformation can be used in Example 2. The associated transformation is

$$\begin{cases} p = (-\phi\Delta)^{-1/2}u_t, \\ q = -\delta(-\phi\Delta)^{1/2}u - p \quad \text{and} \quad p + q = -\delta(-\phi\Delta)^{1/2}u. \end{cases}$$

The reaction–diffusion system is

$$\begin{cases} p_t = -\delta(-\phi\Delta)p + \left(\dfrac{1}{\delta^3}\|p+q\|_{L_g^2}^2\right)(p+q) + (-\phi\Delta)^{-1/2}f(u), \\ q_t = -\left(\dfrac{1}{\delta^3}\|p+q\|_{L_g^2}^2\right)(p+q) - (-\phi\Delta)^{-1/2}f(u). \end{cases}$$

Indeed,

$$\begin{aligned} p_t = &- \delta(-\phi\Delta)p + \left(\frac{1}{\delta^3}\|p+q\|_{L_g^2}^2\right)(p+q) + (-\phi\Delta)^{-1/2}f(u) \\ = &- \delta(-\phi\Delta)(-\phi\Delta)^{-1/2}u_t + \left(\frac{1}{\delta^3}\|-\delta(-\phi\Delta)^{1/2}u\|_{L_g^2}^2\right)\{-\delta(-\phi\Delta)^{1/2}u\} \\ &+ (-\phi\Delta)^{-1/2}f(u) \\ = &- \delta(-\phi\Delta)^{1/2}u_t - \|\nabla u\|^2(-\phi\Delta)^{1/2}u + (-\phi\Delta)^{-1/2}f(u). \end{aligned}$$

But, $p_t = (-\phi\Delta)^{-1/2}u_{tt}$. Thus, we get that

$$(-\phi\Delta)^{-1/2}u_{tt} = -\delta(-\phi\Delta)^{1/2}u_t - \|\nabla u\|^2(-\phi\Delta)^{1/2}u + (-\phi\Delta)^{-1/2}f(u).$$

Using operator $(-\phi\Delta)^{1/2}$ in the previous relation we have

$$u_{tt} - \phi(x)\Delta u\|\nabla u\|^2 - \delta\phi(x)\Delta u_t = f(u).$$

As we have already seen, the linearized equation of the system around solution $u = 0$ is

$$v_{tt} = -\delta A v_t + f'(0)v. \tag{6.4}$$

Now, we have that $f'(0) = 0$. So, relation (6.4) becomes

$$v_{tt} + \delta A v_t = 0$$

or

$$\begin{bmatrix} w \\ v \end{bmatrix}_t = -\begin{bmatrix} \delta A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}. \tag{6.5}$$

Finally, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} w \\ v \end{bmatrix} = -\hat{A}\begin{bmatrix} w \\ v \end{bmatrix}, \tag{6.6}$$

where $\hat{A} = [\begin{smallmatrix} \delta A & 0 \\ 0 & 0 \end{smallmatrix}]$ and $w = v_t,\ w_t = -\delta A w$.

Lemma 6.1. *Let $\hat{A}$ be considered in the space $X =: H \times H$. Then the spectrum of $\hat{A}$ consists of eigenvalues* $\{0\} \cup \{\delta\lambda_j\}_{j=1}^{\infty}$.

Proof. Let $\bar{x_j} = [\phi_j, \ \psi_j] \in D(A)$. Eigenvalues of $\hat{A}$ satisfy the following relation:

$$\hat{A}\bar{x_j} = \mu_j \bar{x_j}$$

or

$$\begin{bmatrix} \delta A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} = \mu_j \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix}.$$

Therefore, we have the following system:

$$\begin{cases} \delta A \phi_j + 0\psi_j = \mu_j \phi_j, \\ 0\phi_j + 0\psi_j = \mu_j \psi_j. \end{cases} \tag{6.7}$$

But, we have that $\{\phi_j\}_{j \in N}$ are eigenfunctions of $\hat{A}$, i.e. relation (5.6) holds. So, (6.7) becomes

$$\begin{cases} \delta\lambda_j \phi_j + 0\psi_j = \mu_j \phi_j, \\ 0\phi_j + 0\psi_j = \mu_j \psi_j \end{cases}$$

or

$$\begin{bmatrix} \delta\lambda_j & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} = \mu_j \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix}. \tag{6.8}$$

Therefore, in order to find the eigenvalues of $\hat{A}$, we compute the characteristic polynomial of $\hat{A}$, i.e.

$$\begin{vmatrix} \delta\lambda_j - \mu_j & 0 \\ 0 & -\mu_j \end{vmatrix} = 0 \quad \text{or} \quad \mu_j^2 - \delta\lambda_j \mu_j = 0.$$

So, we have that $\mu_j = 0$ or $\delta\lambda_j$. Thus, we get that

$$\sigma(\hat{A}) = \{0\} \cup \{\delta\lambda_j\}_{j=1}^{\infty}. \qquad \square$$

Let, P_0 and P_1 projections associated with spectral sets $\{0\}$ and $\{\delta\lambda_j\}_{j=1}^{\infty}$, respectively. Put $X_0 = P_0 X$ and $X_+ = P_+ X$. Then

$$X_0 = span\left\{\begin{pmatrix} 0 \\ \phi_j \end{pmatrix}\right\}_{j=1}^{\infty}, \quad X_+ = span\left\{\begin{pmatrix} \phi_j \\ 0 \end{pmatrix}\right\}_{j=1}^{\infty}.$$

Remark 6.2. Using Pego's transformation, the linearized system (6.5) may be written as a reaction–diffusion system, that is,

$$\begin{cases} p_t(x,t) = -\delta A p, \\ q_t(x,t) = 0. \end{cases} \tag{6.9}$$

Setting $z = \binom{p}{q}$, $\hat{A} = \begin{pmatrix} \delta A & 0 \\ 0 & 0 \end{pmatrix}$, $g(z) = \binom{0}{0}$, relation (6.9) becomes

$$z_t + \hat{A} z = g(z). \tag{6.10}$$

Remark 6.3. We assume that $f(0) = f'(0) = 0$. Then system (6.2) may also be written as Eq. (6.10), where

$$z = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \delta A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and}$$
$$g(z) = \frac{1}{\delta^3} \|p + q\|_H^2 (p + q) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + A^{-1/2} f(u) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus, using Remarks 6.2, 6.3 and Proposition 3.3, we have that systems (6.2) and (6.9) have a local center manifold.

Proposition 6.4. *For some neighborhood U of 0 in $X^{1/2} =: V \times H$, systems* (6.2)–(6.9) *have a local center manifold satisfying the following*

$$W^c_{\mathrm{loc}}(0) = \{\xi + \eta \mid \xi = h^c(\eta), \ \xi \in X_+^{1/2} \cap U, \ \eta \in X_0 \cap U\},$$

where we have that $h^c(0) = Dh^c(0) = 0$.

In this final part, we use center manifold theory in order to extract results for the stability of the solution $(p, \ q) = (0, 0)$, for systems (6.2) and (6.9). For this we have the following cases:

Case A: Substituting $p(t) = h^c(q(t))$ into (6.9) and eliminating q_t, we obtain

$$p_t(t) = Dh^c(q(t)) q_t(t) = 0. \tag{6.11}$$

But, we have that $p_t(t) = -\delta A h^c(q(t)) q_t(t)$. Thus, from (6.11) we get

$$0 = -\delta A h^c(q(t)) q_t(t). \tag{6.12}$$

Then, $h^c(q(t))$ is a center manifold if it satisfies (6.12) together with conditions $h^c(0) = Dh^c(0) = 0$. From relation (6.12) we get that $h^c(q(t)) = 0$. So, the solution $(p, \ q) = (0, 0)$ is stable for system (6.9).

Case B: Substituting now, $p(t) = h^c(q(t))$ into (6.2) and eliminating $q(t)$, we get

$$p_t(t) = Dh^c(q) q_t = Dh^c(q) \left(-\frac{1}{\delta^3} \|h^c(q) + q\|_H^2 \right) (h^c(q) + q) - A^{-1/2} f(u). \tag{6.13}$$

We also have that

$$p_t(t) = -\delta A (h^c(q)) + \left(\frac{1}{\delta^3} \|h^c(q) + q\|_H^2 \right) (h^c(q) + q) + A^{-1/2} f(u). \tag{6.14}$$

Therefore, from relations (6.13) and (6.14) we obtain

$$\frac{1}{\delta^3} Dh^c(q)\|h^c(q)+q\|_H^2(h^c(q)+q)-\delta A(h^c(q)) + \frac{1}{\delta^3}\|h^c(q)+q\|_H^2(h^c(q)+q)+2A^{-1/2}f(u)=0, \tag{6.15}$$

where Dh_q^c stands for the Frechet derivative of h^c. Thus, $h^c(q(t))$ is a center manifold if it satisfies (6.15) together with conditions $h^c(0)=Dh^c(0)=0$. Now, we see from (6.15) that the center manifold is approximated in the following form:

$$h^c(q)=\frac{1}{\delta^4}\|q\|_H^2A^{-1}q+\frac{2A^{-3/2}f(u)}{\delta}+O(\|q\|_H^4). \tag{6.16}$$

Solutions on the center manifold satisfy

$$\begin{aligned} p(t)&=h^c(q(t)),\\ q_t(t)&=-\frac{1}{\delta^3}\|h^c(q)+q\|_H^2(h^c(q)+q). \end{aligned} \tag{6.17}$$

From system (6.17) we obtain that the stability of the solution $u=0$ depends on f. Thus we have the following cases:

(i) if $f(u_0)<0$, then we get that $(p,q)=(0,0)$ is unstable, so $u=0$ is also unstable for the initial Kirchhoff's system,
(ii) if $f(u_0)>0$, then $(p,q)=(0,0)$ is asymptotically stable, so $u=0$ is also asymptotically stable for the initial system,
(iii) if $f(u_0)=0$, we have that solutions on the center manifold satisfy the following system:

$$\begin{aligned} p(t)&=h^c(q(t)),\\ q_t(t)&=-\frac{1}{\delta^3}\|q\|_H^2q+O(\|q\|_H^5). \end{aligned}$$

So, we obtain that $(p,q)=(0,0)$ is stable, that is, $u=0$ is stable for the initial Kirchhoff's system.

7. Kirchhoff's equation with weak dissipation

In this section we study the stability of the solution $u=0$, for the generalized Kirchhoff's equation with weak dissipation

$$u_{tt}=-\delta u_t+\|A^{1/2}u\|_H^2Au+f(u), \quad f'(0)\neq 0. \tag{7.1}$$

Pego's transformation for Eq. (7.1), is

$$\begin{cases} p(x,t)=A^{-1/2}u_t,\\ q(x,t)=-\delta A^{1/2}u-p. \end{cases} \tag{7.2}$$

The associated reaction–diffusion system is

$$\begin{cases} p_t(x,t) = -\delta p + \left(\dfrac{1}{\delta^3}\,\|p+q\|^2\right)(p+q) + A^{-1/2} f(u), \\ q_t(x,t) = \left(-\dfrac{1}{\delta^3}\|p+q\|^2\right)(p+q) - A^{-1/2} f(u). \end{cases} \tag{7.3}$$

We also have that the linearized system of Eq. (7.1), for $u = 0$ is

$$v_{tt} + \delta v_t = f'(0)v. \tag{7.4}$$

Now, we are able to prove the following lemma.

Lemma 7.1. *Using Pego's transformation* (7.2), *system* (7.4) *can be written as*

$$\begin{cases} p_t = -\delta p + A^{-1/2} f'(0)v, \\ q_t = 0. \end{cases} \tag{7.5}$$

Proof. Indeed, we have that $p_t = -\delta A^{-1/2} v_t + A^{-1/2} f'(0)v$. But, $p_t = A^{-1/2} v_{tt}$. Thus, we get the following relation:

$$A^{-1/2} v_{tt} = -\delta A^{-1/2} v_t + A^{-1/2} f'(0)v. \tag{7.6}$$

Using operator $A^{1/2}$ in (7.6), we have

$$v_{tt} = -\delta v_t + f'(0)v. \qquad \square$$

In order to obtain stability results for the solution $u = 0$ we set

$$\begin{aligned} v_t &= w, \\ w_t &= -\delta w + f'(0)v. \end{aligned}$$

So we get the following system:

$$\begin{cases} v_t = w, \\ w_t = f'(0)v - \delta w \end{cases}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f'(0) & -\delta \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}. \tag{7.7}$$

We study the spectrum of operator $\hat{A}$, where

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ f'(0) & -\delta \end{bmatrix}.$$

We compute the characteristic polynomial of $\hat{A}$, i.e.

$$\begin{vmatrix} \lambda & -1 \\ -f'(0) & \lambda + \delta \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + \lambda\delta - f'(0) = 0.$$

We have to study a system of ordinary differential equations with $\Delta = \delta^2 + 4f'(0)$. So, we obtain the following cases, according to the sign of $f'(0)$:

Case I: Let $f'(0) < 0$.

Case a: $\Delta = \delta^2 + 4f'(0) > 0$. Then the eigenvalues of $\hat{A}$ are

$$\lambda_{1,2} = \frac{-\delta \pm (\delta^2 + 4f'(0))^{1/2}}{2}. \tag{7.8}$$

We observe that the smallest eigenvalue is negative. Indeed, we have that

$$\lambda_2 = -\delta - (\delta^2 + 4f'(0))^{1/2} < 0$$

which holds. Thus $\lambda_2 < \lambda_1 < 0$, which implies that $u = 0$ is asymptotically stable for the initial Kirchhoff's system.

Case b: Let $\Delta = \delta^2 + 4f'(0) = 0$, or $\delta^2 = -4f'(0)$. In this case, we have that $\lambda_{1,2} = -\delta/2 < 0$. Therefore, we have that $u = 0$ is also asymptotically stable for our system.

Case c: Let $\Delta < 0$, or $\delta^2 < -4f'(0)$. In this case we have complex solutions

$$(\lambda_1,\ \lambda_2) = -\frac{\delta \pm i\sqrt{\delta + 4f'(0)}}{2}.$$

Since, $\mathrm{Re}\,\lambda_1 < 0,\ \mathrm{Re}\,\lambda_2 < 0$, we get that 0 is asymptotically stable for the initial Kirchhoff's system.

Case II: Finally, let $f'(0) > 0$.

Case a: If we assume that $\Delta > 0$, or $\delta^2 > -4f'(0)$, we have for the eigenvalues that

$$\lambda_2 = \frac{-\delta - (\delta^2 + 4f'(0))^{1/2}}{2} < 0 \quad \text{and} \quad \lambda_1 = \frac{-\delta + (\delta^2 + 4f'(0))^{1/2}}{2} > 0.$$

Thus we get that solution $u = 0$ is unstable for the initial Kirchhoff's system.

Case b: $\Delta \leqslant 0$, or $\delta^2 \leqslant -4f'(0)$, which does not holds because we have that $f'(0) > 0$.

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