Asymptotic Behavior of Solutions for a Semibounded Nonmonotone Evolution Equation*

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{Abstract and Applied Analysis, Vol. 2003, No. 9, (2003), 521-538}

Abstract

We consider a nonlinear parabolic equation involving nonmonotone diffusion. Existence and uniqueness of solutions are obtained, employing methods for semibounded evolution equations. Also shown is the existence of a global attractor for the corresponding dynamical system.

1 Introduction

We consider the following nonlinear parabolic initial-boundary value problem in the open, bounded interval $\Omega \subset \mathbb{R}$

$$
\begin{aligned}
u_t - a(u)u_{xx} &- b(u)u_x^2 - \lambda\sigma(u) = f(x),\ x \in \Omega,\ t > 0, && (1.1)\\
u(x,0) &= u_0(x), && (1.2)\\
u|_{\partial\Omega} &= 0,\ t > 0. && (1.3)
\end{aligned}
$$

This problem extends the well studied porous medium diffusion, since no certain relationship between the coefficients $a(u)$, $b(u)$ is assumed. Let us mention that special cases of this system may typically arise in plasma physics within the context of the fluid treatment of charged particles, and in density-dependent reaction diffusion processes in mathematical biology. Naturally enough, these systems imply only positive values for $u(x,t)$; however, in the following treatment, we *do not* impose such a restriction.

*Keywords and Phrases: Nonlinear PDE of parabolic type, asymptotic behavior of solutions, attractors. AMS Subject Classification: 35K55, 35B40, 35B41.

In order to demonstrate a specific case modelled by the parabolic system, we consider the collisionless evolution equation for the electron pressure $P = nT$, which, if we ignore viscosity, gets the following form in the x-direction (see, R. Balescu [4])

$$\frac{3}{2}P_t = -q_x - \frac{3}{2}uP_x - \frac{5}{2}Pu_x,$$

where u represents the electron velocity and q is the heat flux. Now, applying Darcy's law (see, D. Aronson [3])

$$u = -cP_x, \quad c > 0,$$

to the above equation, we get

$$\frac{3}{2}P_t = -q_x + \frac{3}{2}cP_x^2 + \frac{5}{2}cPP_{xx}.$$

We see that the first term on the right hand side corresponds to porous medium diffusion (not considered here), whereas the other two terms constitute a specific case of (1.1), with $a(P) = \frac{5}{3}cP$ and $b(P) = c$.

Concerning the applications in the dynamics of cell populations, with a spatial distribution of cells, we may associate an energy density $e(u)$, that is an internal energy per unit volume of an evolving spatial pattern, where $u(x,t)$ denotes the cell density (see [6, 14]). In this case, the total energy $E(u)$ in a volume V is given by

$$E(u) = \int_V e(u)dx. \tag{1.4}$$

The change in energy δE, that is the work done in changing states by an amount δu, is given by the variational derivative $\delta E/\delta u$ which defines a potential

$$\mu(u) = \frac{\delta E}{\delta u} = e'(u). \tag{1.5}$$

The gradient of the potential μ produces a flux J, which classically is proportional to this gradient, that is

$$\mathrm{J} = -k\mu'(u). \tag{1.6}$$

By using (1.5) and (1.6), the continuity equation for the density u is

$$\frac{\partial u}{\partial t} = (a(u)u_x)_x, \quad a(u) = ke''(u). \tag{1.7}$$

Writing out the diffusion term in full, we end up with the nonlinear operator that appears in (1.1), in the special case where it holds $a'(u) = b(u)$, i.e. the porous medium case. Also, the nonlinearity $\sigma(u)$ may stand for the possible growth dynamics.

For completeness, let us mention some of the results, concerning the large time behavior of bounded solutions of nonlinear diffusion equations. Most of them are related to porous medium type equations (degenerate, monotone diffusion). In [2], the existence of a global attractor for the one-dimensional porous medium equation, attracting all orbits starting from L^∞-initial data, is demonstrated. Extensive studies in [1], [13] and [15] show that the ω-limit set is contained in the set of stationary solutions. Extensions for the unbounded domain case can be found in [9], [10]. We also mention [5], [7], [16], on the existence of global attractors for degenerate or nondegenerate quasilinear parabolic equations.

The principal assumption that will be used throughout this paper in the study of the problem (1.1)-(1.3) is the following

(A) $a, \quad b, \quad \sigma \in C^2(\mathbb{R}),\ \lambda \in \mathbb{R}$ and there exists $c_* > 0$ such that $a(s) \geq c_*$,

i.e., we consider nondegenerate but *nonmonotone* diffusion. Due to the nonmonotonicity, the standard compactness methods on existence of solutions are not sufficient. To this end, *the diffusion operator is treated as a semibounded operator within the functional setting of an admissible triple* . This procedure allows for the construction of unique solutions in $C_w([0,T], H^2 \cap H_0^1(\Omega))$, the space of weakly continuous functions $u : [0,T] \to H^2 \cap H_0^1(\Omega)$.

The existence of a global attractor in the phase space $\mathrm{H} = H^2 \cap H_0^1(\Omega)$ is proved in Section 3. The result is shown assuming monotonicity for the nonlinearity $b(\cdot)$, considered to be nonincreasing. *Nevertheless, this assumption does not imply monotonicity for the diffusion operator itself.* An important feature is that this assumption is sufficient to prove further regularity with respect to time for the solutions of (1.1)-(1.3) constructed in Section 2. Further, using this result, we may define the semigroup $S(t) : u_0 \in \mathrm{H} \mapsto u(t) \in \mathrm{H}$, corresponding to our problem.

We conclude by recalling some well known results, which will be frequently used (see, [17], [18]).

Lemma 1.1 *(Gagliardo-Nirenberg inequality) Let* $1 \leq p, q, r \leq \infty$, *$j$ an integer,* $0 \leq j \leq m$ *and* $j/m \leq \theta \leq 1$. *Then*

$$||D^j u||_p \leq \text{const}||u||_q^{1-\theta}||D^m u||_r^\theta, \ \ u \in L^q \cap W^{m,r}(\Omega), \ \ \Omega \subseteq \mathbb{R}^n$$

where
$$\frac{1}{p}=\frac{j}{n}+\theta\left(\frac{1}{r}-\frac{m}{n}\right)+\frac{1-\theta}{q}.$$
If $m-j-n/r$ is not a nonnegative integer, then the inequality holds for $j/m\leq\theta<1$.

Lemma 1.2 *(Uniform Gronwall) Let g,h,y be three positive locally integrable functions for $t_0\leq t<\infty$ which satisfy*
$$\frac{dy}{dt}\leq gy+h,\ \ \textit{for all}\ \ t\geq t_0$$
and
$$\int_t^{t+r}g(s)ds\leq\alpha_1,\ \ \int_t^{t+r}h(s)ds\leq\alpha_2,\ \ \int_t^{t+r}y(s)ds\leq\alpha_3,$$
for all $t\geq t_0$, where α_1, α_2, α_3 are positive constants. Then
$$y(t+r)\leq(\frac{\alpha_3}{r}+\alpha_2)\exp(\alpha_1),\ \ \textit{for all}\ \ t\geq t_0.$$

We also use the short (equivalent) norms $||u_x||_2$, $||u_{xx}||_2$, $||u_{xxx}||_2$ in $H_0^1(\Omega)$, $H^2\cap H_0^1(\Omega)$ and $H^3\cap H_0^1(\Omega)$, respectively (see Section 3). From the embedding $H^k\cap H_0^1(\Omega)\hookrightarrow C_b^{k-1}(\Omega)$, $k=1,2\dots$, and the Poincaré inequality (see [8], pg. 242) we have
$$||u^{(k-1)}||_\infty\leq\text{const}||u||_{H^k\cap H_0^1}\leq\text{const}||u^{(k)}||_2. \tag{1.8}$$

2 Local Existence

To obtain results on local existence of solutions we intend to write problem (1.1)-(1.3) as a nonlinear evolution equation in an appropriate functional setting. More precisely, we shall consider an admissible triple of Banach spaces, which is defined as follows (See [18, pg. 784]):

Definition 2.1 *An admissible triple $\mathrm{V}\hookrightarrow\mathrm{H}\hookrightarrow\mathrm{W}$ has the following properties: (i) H is a real separable Hilbert space with scalar product $(\cdot|\cdot)_{\mathrm{H}}$, (ii) $\{\mathrm{V},\mathrm{W}\}$ is a dual pair of real separable Banach spaces with the corresponding bilinear form $<\cdot,\cdot>$ (i.e., $<\cdot,\cdot>$ is continuous, $<w,v>=0$, for every $w\in\mathrm{W}$, implies $v=0$ and $<w,v>=0$, for every $v\in\mathrm{V}$, implies $w=0$), (iii) the embeddings $\mathrm{V}\hookrightarrow\mathrm{H}\hookrightarrow\mathrm{W}$ are continuous and dense, (iv) it holds $<h,v>=(h|v)_{\mathrm{H}}$, for all $h\in\mathrm{H}$, $v\in\mathrm{V}$.*

Clearly, an admissible triple generalizes the notion of the evolution triple, in the sense that for an admissible triple it may hold $\mathrm{W} \neq \mathrm{V}^*$. This generalization is necessary in order to tackle the extended version of diffusion in hand. For the problem (1.1)-(1.3), we select the spaces

$$\mathrm{V} = H^4 \cap H_0^1(\Omega), \quad \mathrm{H} = H^2 \cap H_0^1(\Omega), \quad \mathrm{W} = L^2(\Omega). \tag{2.1}$$

Lemma 2.2 *The embedding* $\mathrm{V} \hookrightarrow \mathrm{H} \hookrightarrow \mathrm{W}$ *for the spaces (2.1) defines an admissible triple.*

Sketch of Proof Consider the bilinear form $< \cdot, \cdot >: \mathrm{W} \times \mathrm{V} \mapsto \mathbb{R}$, defined by the integral

$$< w, v >= \int_\Omega vw + wv_{xxxx} dx, \quad \text{for all } \ v \in \mathrm{V}, \ \ w \in \mathrm{W}.$$

Now, it is easy to check that the inner product stemming from the bilinear form $< \cdot, \cdot >$

$$(w|v)_\mathrm{H} = \int_\Omega vw + w_{xx}v_{xx} dx, \quad \text{for every } \ w \in \mathrm{H}, v \in \mathrm{H}$$

induces an equivalent norm in H. We also have that

$$\begin{aligned} |< w, v >| = \left| \int_\Omega vw + wv_{xxxx} dx \right| &\leq ||w||_2||v||_2 + ||w||_2||v_{xxxx}||_2 \\ &\leq c||w||_\mathrm{W}||v||_\mathrm{V}, \end{aligned}$$

hence the bilinear form $< \cdot, \cdot >$ is continuous. Now assume that, for some $w \in \mathrm{W}$, it holds $< w, v >= 0$, for every $v \in \mathrm{V}$. Classical arguments on existence and regularity of solutions for linear elliptic equations (see [12, Chapter II]) imply the existence of solutions for the problem

$$v - v_{xxxx} = w, \quad v \in \mathrm{V}.$$

For this solution v we have that

$$0 =< w, v >= \int_\Omega w^2 dx,$$

which implies that $w = 0$ and the proof is complete. $\diamondsuit$

We introduce the nonlinear operators $\mathbf{A}, \mathbf{B} : \mathrm{V} \mapsto \mathrm{W}$ defined by

$$\mathbf{A}u = -a(u)u_{xx}, \quad \mathbf{B}u = -b(u)u_x^2.$$

The following results outline the basic properties of the operators $\mathbf{A}, \mathbf{B}$.

Proposition 2.3 *The operator* $\mathbf{A}+\mathbf{B}:\mathrm{H}\mapsto\mathrm{W}$ *is bounded on bounded sets of* H.

Proof Let $B=B_{\mathrm{H}}(R)$ a closed ball in H. We shall show that there exist constants $K_1(R),K_2(R)$ such that

$$||\mathbf{A}u||_2\le K_1(R)||u||_{\mathrm{H}},\ \ ||\mathbf{B}u||_2\le K_2(R)||u||_{\mathrm{H}},\ \text{ for all }\ u\in B.$$

Since $a,b,\sigma\in C^2(\mathbb{R})$ and the embedding $\mathrm{H}\hookrightarrow C^1_b(\Omega)$ is continuous, it follows that there exist constants $C_{1,m}(R),C_{2,m}(R)$, $m=0,1,2$, such that

$$\sup_{x\in\Omega}|a^{(m)}(u(x))| \ \le\ C_{1,m}(R),\ \ m=0,1,2, \tag{2.2}$$

$$\sup_{x\in\Omega}|b^{(m)}(u(x))| \ \le\ C_{2,m}(R),\ \ m=0,1,2. \tag{2.3}$$

Using (2.2), (2.3) and the fact that $H^1_0(\Omega)$ is a generalized Banach algebra, we may obtain the inequalities

$$\begin{aligned}||\mathbf{A}u||_2\le\sup_{x\in\Omega}|a(u(x))|\,||u_{xx}||_2 &\le K_1(R)||u||_{\mathrm{H}},\\ ||\mathbf{B}u||_2\le\sup_{x\in\Omega}|b(u(x))|\,||u_x^2||_2 &\le \text{const}\sup_{x\in\Omega}|b(u(x))|\,||u||^2_{\mathrm{H}}\\ &\le K_2(R)||u||_{\mathrm{H}}.\end{aligned}$$

Finally, we conclude that

$$||(\mathbf{A}+\mathbf{B})u||_2\le K(R)||u||_{\mathrm{H}}, \tag{2.4}$$

where $K(R)=\max\{K_1(R),K_2(R)\}$. $\diamondsuit$

Proposition 2.4 *The operator* $\mathbf{A}+\mathbf{B}:\mathrm{H}\mapsto\mathrm{W}$ *is locally Lipschitz continuous.*

Proof Let $u,v\in B=B_{\mathrm{H}}(R)$ a closed ball in H. We have that

$$||\mathbf{A}u-\mathbf{A}v||_2\le||(a(u)-a(v))v_{xx}||_2+||a(u)(u_{xx}-v_{xx})||_2.$$

From the Mean Value Theorem and (2.2), we get

$$|a(u(x))-a(v(x))|\le C_{1,1}(R)|u(x)-v(x)|, \tag{2.5}$$

$$|a'(u(x))-a'(v(x))|\le C_{1,2}(R)|u(x)-v(x)|. \tag{2.6}$$

Therefore,

$$\begin{aligned} ||(a(u)-a(v))v_{xx}||_2^2 &\le C_{1,1}(R)^2||u-v||_\infty^2||v_{xx}||_2^2 \le C(R)||u-v||_{\mathrm{H}}^2, \\ ||a(u)(u_{xx}-v_{xx})||_2^2 &\le C_{1,0}^2(R)||u_{xx}-v_{xx}||_2^2 \le C(R)||u-v||_{\mathrm{H}}^2, \end{aligned}$$

where $C(R)$ is a common symbol for the constants. Similar inequalities hold for the operator $\mathbf{B}$. So finally it holds that

$$||(\mathbf{A}+\mathbf{B})u-(\mathbf{A}+\mathbf{B})v||_2 \le C(R)||u-v||_{\mathrm{H}}. \quad \diamondsuit$$

Proposition 2.5 *The operator* $\mathbf{A}+\mathbf{B}:\mathrm{H}\mapsto\mathrm{W}$ *is semibounded.*

Proof By definition, it must be proved that there exists a monotone increasing function $d_1 \in C^1(\mathbb{R})$ such that

$$<(\mathbf{A}+\mathbf{B})u,u> \ \ge -d_1(||u||_{\mathrm{H}}^2), \quad \text{for every } \ u \in V. \tag{2.7}$$

Let $u \in C_0^\infty(\Omega)\cap C(\overline{\Omega})$. For the operator $\mathbf{A}$, it holds

$$<\mathbf{A}u,u>=\int_\Omega \mathbf{A}uu\,dx+\int_\Omega \mathbf{A}uu_{xxxx}dx. \tag{2.8}$$

Integration by parts in the second integral on the right-hand-side of (2.8) gives

$$\begin{aligned} -\int_\Omega a(u)u_{xx}u_{xxxx}dx &= -\frac{1}{2}\int_\Omega a''(u)u_x^2u_{xx}^2dx-\frac{1}{2}\int_\Omega a'(u)u_{xx}^3dx \\ &+ \int_\Omega a(u)u_{xxx}^2dx. \end{aligned} \tag{2.9}$$

Using Lemma 1.1 we obtain the inequality

$$||u_{xx}||_4 \le \mathrm{const}||u||_2^{1/4}||u_{xxx}||_2^{3/4},$$

which, with the aid of (2.2) and Young's inequality, gives the following estimate

$$\begin{aligned} -\frac{1}{2}\int_\Omega a''(u)u_x^2u_{xx}^2dx &- \frac{1}{2}\int_\Omega a'(u)u_{xx}^3dx \\ &\ge -C_{1,2}||u_x||_\infty^2||u_{xx}||_2^2-C_{1,1}||u_{xx}||_2||u_{xx}||_4^2 \\ &\ge -\hat{C}_1||u||_{\mathrm{H}}^4-\hat{C}_2||u||_{\mathrm{H}}||u||_2^{\frac{1}{2}}||u_{xxx}||_2^{\frac{3}{2}} \\ &\ge -\hat{C}_1||u||_{\mathrm{H}}^4-\hat{C}_3||u||_{\mathrm{H}}^{\frac{3}{2}}||u_{xxx}||_2^{\frac{3}{2}} \\ &\ge -\hat{C}_1||u||_{\mathrm{H}}^4-\hat{C}_4||u||_{\mathrm{H}}^6-\frac{c_*}{2}||u_{xxx}||_2^2. \end{aligned} \tag{2.10}$$

For the first integral of the right-hand side of (2.8), we have

$$-\int_{\Omega} a(u)u_{xx}u\,dx \geq -C_{1,0}||u||_{\infty}||u_{xx}||_1 \geq -\hat{C}_0||u||^2_{\mathrm{H}}. \tag{2.11}$$

Using assumption (A), (2.8)-(2.11) and density arguments, we obtain that

$$< \mathbf{A}u, u > \geq -\hat{C}_0||u||^2_{\mathrm{H}} - \hat{C}_1||u||^4_{\mathrm{H}} - \hat{C}_4||u||^6_{\mathrm{H}} := -d_{1,1}(||u||^2_{\mathrm{H}}). \tag{2.12}$$

A similar procedure may be followed for the operator $\mathbf{B}$, to derive the relation

$$< \mathbf{B}u, u > \geq -d_{1,2}(||u||^2_{\mathrm{H}}). \tag{2.13}$$

Finally, from the estimates (2.12) and (2.13) we get that there exists a monotone increasing C^1- function $d_1 : \mathbb{R} \to \mathbb{R}$ satisfying relation (2.7). $\diamondsuit$

The previous propositions enable us to show local existence of solutions. The result is stated as follows:

Theorem 2.6 *Let* $u_0,\ f \in \mathrm{H}$. *Assume that condition* (A) *is satisfied. Then there exists* $T > 0$ *such that the problem (1.1)-(1.3) has a unique solution*

$$u \in C_w([0,T],\mathrm{H}) \ \ \textit{and} \ \ u_t \in C_w([0,T],\mathrm{W}).$$

Moreover, the solution $u : [0,T] \to W$ *is Lipschitz continuous.*

Proof A. *Existence*: The first step is to show existence of at least one solution in a finite dimensional subspace $\mathrm{V}_n = \mathrm{span}\{e_1, ..., e_n\}$ of V, where $\{e_i\}_{i\geq 1}$ is an orthonormal basis of $\mathrm{V_n}$ with respect to $(\cdot|\cdot)_{\mathrm{H}}$. It holds that $\overline{\bigcup_n \mathrm{V}_n} = \mathrm{V} \hookrightarrow \mathrm{H}$.
We define the linear and continuous operator $\tilde{\mathrm{P}}_n : \mathrm{W} \mapsto \mathrm{V}$ as $\tilde{\mathrm{P}}_n w = \sum_i^n < w, e_i > e_i,\ \ w \in \mathrm{W}$. Now, the Galerkin equation for the problem (1.1)-(1.3) on $\mathrm{V}_n \hookrightarrow \mathrm{V} \hookrightarrow \mathrm{H}$ reads

$$u_n'(t) + \tilde{\mathrm{P}}_n(\mathbf{A}+\mathbf{B})u_n(t) = \tilde{\mathrm{P}}_n\mathbf{C}u_n(t),\ \ t \in [0,T],\ \ u_n(0) = \tilde{\mathrm{P}}_n u_0, \tag{2.14}$$

where

$$\mathbf{C}u_n(t) = \lambda\sigma(u_n(t)) + f.$$

Using Propositions 2.3 and 2.4, Peano's Theorem justifies the existence of a C^1 solution for (2.14), $u_n : [0,T_0] \to \mathrm{V}_n$, for some $T_0 > 0$ which depends on n.
The next step is to obtain an a priori estimate for u_n in H. Note that $\tilde{\mathrm{P}}_n : \mathrm{H} \mapsto \mathrm{V_n}$ is an orthogonal projection onto the space $\mathrm{V_n}$, since it holds

$\tilde{\mathrm{P}}_n u = \sum_i^n (u|e_i)_{\mathrm{H}} e_i, \quad u \in \mathrm{H}$. Since u_n is continuous on $[0,\ T_0]$, equation (2.14) implies that

$$\begin{aligned}(u_n'|u_n)_{\mathrm{H}} &= -(\tilde{\mathrm{P}}_n(\mathbf{A}+\mathbf{B})u_n|u_n)_{\mathrm{H}} + (\tilde{\mathrm{P}}_n\mathbf{C}u_n|u_n)_{\mathrm{H}} \\ &= - < (\mathbf{A}+\mathbf{B})u_n, u_n > + < \mathbf{C}u_n, u_n > . \end{aligned} \tag{2.15}$$

Now, it is not hard to verify that there exists a monotone increasing function $d_2 \in C^1(\mathbb{R})$ such that

$$| < \mathbf{C}u, u) > | \leq d_2(||u||_{\mathrm{H}}^2), \text{ for all } u \in \mathrm{V}. \tag{2.16}$$

Hence, from (2.7), (2.15) and (2.16) we obtain the differential inequality

$$\frac{d}{dt}||u_n(t)||_{\mathrm{H}}^2 \leq 2d\left(||u_n(t)||_{\mathrm{H}}^2\right), \quad t \in [0, T_0], \tag{2.17}$$

where $||u_n(0)||_{\mathrm{H}} = ||\mathrm{P}_n u_0||_{\mathrm{H}} \leq ||u_0||_{\mathrm{H}}$. Since the function $d(\cdot)$ is Lipschitz continuous as a C^1 function, we may apply the Theorem of Picard-Lindelöf to conclude that there exists a $T > 0$, this time independent of n, such that

$$||u_n(t)||_H^2 \leq \max_{t\in[0,T]} g(t) \leq R, \quad t \in [0, T]. \tag{2.18}$$

Finally, using standard arguments, we can extend the solution u_n to the interval $[0, T]$.

Now, from (2.18) we have that there exists a subsequence, denoted again by $\{u_n\}$, such that

$$u_n(t) \rightharpoonup u(t), \text{ in H}, \text{ as } n \to \infty, \tag{2.19}$$

at least in a dense countable subset of $[0, T]$. Let $v \in \mathrm{V}_k \hookrightarrow \mathrm{H}$, $k \leq n$. Since $\tilde{\mathrm{P}}_n v = v$, for every $k \leq n$, it follows that

$$\begin{aligned}(u_n'(t)|v)_{\mathrm{H}} &= -(\tilde{\mathrm{P}}_n(\mathbf{A}+\mathbf{B}-\mathbf{C})u_n(t)|v)_{\mathrm{H}} \\ &= - < (\mathbf{A}+\mathbf{B}-\mathbf{C})u_n(t), v > . \end{aligned} \tag{2.20}$$

Using Proposition 2.3 and estimate (2.18), we conclude that $\{u_n\}$ is equicontinuous on $[0, T]$, which implies that (2.19) holds in the whole interval $[0, T]$. Finally, from (2.20) and density in H, we obtain that $u \in C_w([0, T], \mathrm{H})$, $u_t \in C_w([0, T], \mathrm{W})$ and, as a consequence, that $u : [0, T] \to \mathrm{W}$ is Lipschitz continuous.

B. *Uniqueness*: The difference of solutions $w = u - v$ of the problem (1.1)-(1.3) satisfies the following initial value problem

$$w_t - a(u)w_{xx} - \mathrm{A}(u, v)v_{xx} - \mathrm{B}(u, v) - \lambda\Sigma(u, v) = 0, \quad w(0) = 0, \tag{2.21}$$

where $\mathrm{A}(u,v) = a(u) - a(v)$, $\mathrm{B}(u,v) = (b(u) - b(v))v_x^2 + b(u)(u_x^2 - v_x^2)$ and $\Sigma(u,v) = \sigma(u) - \sigma(v)$. Multiplying (2.21) by u and integrating over Ω, we obtain the equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||w||_2^2 &+ \int_\Omega a'(u)v_x w w_x\,dx + \int_\Omega (a(u)-a(v))w_x v_x\,dx \\ &+ \int_\Omega a'(u)u_x w w_x dx + \int_\Omega (a'(u)-a'(v))v_x^2 w\,dx \\ &- \int_\Omega (b(u)-b(v))v_x^2 w dx - \int_\Omega b(u)(u_x^2 - v_x^2)w dx \\ &+ \int_\Omega a(u)w_x^2 dx - \lambda\int_\Omega (\sigma(u)-\sigma(v))w dx = 0. \qquad (2.22)\end{aligned}$$

Using the estimate (2.18) and the relations (2.2), (2.3), (2.6) the following estimates are derived

$$\begin{aligned}\left|\int_\Omega (a'(u)-a'(v))v_x^2 w\,dx\right| &\le C_{1,2}||v_x||_\infty^2||w||_2^2 \le C(R)||w||_2^2, \\ \left|\int_\Omega a'(u)v_x w w_x\,dx\right| &\le C_{1,1}||v_x||_\infty||w||_2||w_x||_2 \\ &\le \epsilon_0||w_x||_2^2 + C(R)||w||_2^2.\end{aligned}$$

The rest of the integrals in the equation (2.22) can be estimated in a similar way. Hence, for sufficiently small ϵ_0, we get the inequality

$$\frac{1}{2}\frac{d}{dt}||w(t)||_{\mathrm{W}}^2 + \frac{c_*}{2}||w_x||_2^2 \le C||w(t)||_{\mathrm{W}}^2.$$

Application of the standard Gronwall's Lemma implies uniqueness. $\diamondsuit$

3 Existence of a Global Attractor in H

In this section we discuss the asymptotic behavior of solutions of the nonlinear parabolic problem (1.1)-(1.3). To this end, in addition to the principal hypothesis (A), we assume that the nonlinear functions b, σ satisfy the following hypotheses

(B) $b'(s) \le 0$ and there exist $c_m > 0$, such that $|\sigma^{(m)}(s)| \le c_m|s|$, for all $m = 0, 1, 2$.

First, we prove that under the extra hypothesis (B), the unique local solution $u(x,t)$ of the problem (1.1)-(1.3), obtained in Theorem 2.6, exists globally in time. We denote by λ_*, the positive constant induced by Poincaré's inequality.

Lemma 3.1 *Let hypotheses* (A), (B) *be fulfilled and* $u_0,\ f \in \mathrm{H}$. *Assume also that*

$$\lambda < \frac{c_* \lambda_*}{2c_0}. \tag{3.1}$$

Then there exists a constant ρ_2 *independent of* t, *such that,*

$$\limsup_{t\to\infty} ||u_x(t)||_2 \le \rho_2. \tag{3.2}$$

Proof We multiply equation (1.1) by $-u_{xx}$ and integrate over Ω to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u_x||_2^2 &+ \int_\Omega a(u)u_{xx}^2 dx + \int_\Omega b(u)u_x^2 u_{xx} dx \\ &+ \lambda \int_\Omega \sigma(u) u_{xx}\, dx = \int_\Omega f u_{xx}\, dx. \end{aligned} \tag{3.3}$$

Using hypothesis (A), we observe that

$$\int_\Omega a(u)u_{xx}^2 dx \ge c_* ||u_{xx}||_2^2, \tag{3.4}$$

whereas from hypothesis (B) we have

$$\int_\Omega b(u)u_x^2 u_{xx} dx = -\frac{1}{3}\int_\Omega b'(u)u_x^4 dx \ge 0. \tag{3.5}$$

Furthermore, hypothesis (B) together with Poincaré's inequality

$$||u||_2 \le \lambda_*^{-1/2} ||u_x||_2, \tag{3.6}$$

imply that

$$\lambda \left| \int_\Omega \sigma(u) u_{xx}\, dx \right| \le \lambda c_0 ||u||_2 ||u_{xx}||_2 \le \lambda \lambda_*^{-1} c_0 ||u_{xx}||_2^2. \tag{3.7}$$

Relations (3.3), (3.4) and (3.7) imply that

$$\frac{d}{dt}||u_x(t)||_2^2 + \alpha ||u_{xx}(t)||_2^2 \le \frac{1}{c_*}||f||_2^2, \tag{3.8}$$

where $\alpha = c_* - 2c_0 \lambda \lambda_*^{-1}$. Applying again Poincaré's inequality (3.6) to the above estimate (3.8) we get

$$\frac{d}{dt}||u_x(t)||_2^2 + \alpha \lambda_* ||u_x(t)||_2^2 \le \frac{1}{c_*}||f||_2^2. \tag{3.9}$$

If the assumption (3.1) is satisfied, i.e., $\alpha > 0$ Gronwall's Lemma leads to the following estimate

$$||u_x(t)||_2^2 \leq ||u_x(0)||_2^2 \exp(-\alpha\lambda_* t) + \frac{1}{\alpha c_* \lambda_*}||f||_2^2(1-\exp(-\alpha\lambda_* t)). \quad (3.10)$$

Letting $t \to \infty$, from estimate (3.10) we obtain that

$$\limsup_{t\to\infty} ||u_x(t)||_2^2 \leq \rho_2^2,$$

where $\rho_2^2 = \frac{1}{\alpha c_* \lambda_*}||f||_2^2$ and the proof is completed. $\diamondsuit$

Let $\mathcal{B}$ a bounded set of H, included in a ball $\mathrm{B_H}(0, M)$ of H, centered at 0 of radius M. Assuming that $u_0 \in \mathcal{B}$, we infer from Lemma 3.1 that for $\rho_2' > \rho_2$, there exists $t_0(\mathcal{B}, \rho_2') > 0$ such that for $t \geq t_0(\mathcal{B}, \rho_2')$,

$$||u_x(t)||_2 \leq \rho_2', \quad ||u(t)||_2 \leq \rho_1 = \lambda_*^{-1/2}\rho_2'. \quad (3.11)$$

Integrating (3.8) with respect to t, it follows that for every $r > 0$

$$\alpha \int_t^{t+r} ||u_{xx}(s)||_2^2\, ds \leq \frac{r}{c_*}||f||_2^2 + ||u_x(t)||_2^2.$$

Once again, letting $t \to \infty$, we obtain from inequality (3.11) that

$$\limsup_{t\to\infty} \int_t^{t+r} ||u_{xx}(s)||_2^2\, ds \leq \frac{r}{\alpha c_*}||f||_2^2 + \frac{\rho_2^2}{\alpha}, \quad \text{for every } \ r > 0.$$

and for $t \geq t_0(\mathcal{B}, {\rho'}_2)$

$$\int_t^{t+r} ||u_{xx}(s)||_2^2\, ds \leq \frac{r}{\alpha c_*}||f||_2^2 + \frac{{\rho'}_2^2}{\alpha}, \quad \text{for every } \ r > 0. \quad (3.12)$$

Lemma 3.2 *Let hypotheses* (A), (B) *be fulfilled,* $u_0 \in \mathcal{B}$ *and* $f \in$ H. *Assume also that (3.1) is satisfied. Then there exists a constant* ρ_3 *independent of* t*, and* $t_1 > 0$ *such that*

$$||u_{xx}(t)||_2 \leq \rho_3, \quad \textit{for} \ \ t \geq t_1. \quad (3.13)$$

Proof Multiply equation (1.1) by u_{xxxx} and integrate over Ω to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u_{xx}||_2^2 \quad &+ \quad \int_\Omega a'(u)u_x u_{xx} u_{xxx} dx + \int_\Omega a(u) u_{xxx}^2 dx \\ &+ \quad 2\int_\Omega b(u) u_x u_{xx} u_{xxx} dx + \lambda \int_\Omega \sigma'(u) u_x u_{xxx}\, dx \\ &+ \quad \int_\Omega b'(u) u_x^3 u_{xxx} dx = -\int_\Omega f_x u_{xxx}\, dx. \quad (3.14)\end{aligned}$$

Using inequalities (1.8), (3.11) and hypothesis (A), we obtain that inequalities (2.2), (2.3) hold, for all $t \geq t_0(\mathcal{B}, \rho'_2)$, with R replaced by ρ'_2. It follows that

$$\begin{aligned}\left|\int_\Omega a'(u) u_x u_{xx} u_{xxx} dx\right| &\leq C_{1,1} \|u_x\|_\infty \|u_{xx}\|_2 \|u_{xxx}\|_2 \\ &\leq C_{1,1} \text{const} \|u_{xx}\|_2^2 \|u_{xxx}\|_2, \\ &\leq C_1 \|u_{xx}\|_2^4 + \epsilon_1 \|u_{xxx}\|_2^2. \end{aligned} \tag{3.15}$$

Applying Lemma 1.1 we obtain the inequality

$$\|u_x\|_6 \leq \text{const}\|u\|_2^{1/3} \|u_{xx}\|_2^{2/3},$$

which can be used to get the estimate

$$\begin{aligned}\left|\int_\Omega b'(u) u_x^3 u_{xxx} dx\right| &\leq C_{2,1} \|u_x\|_6^3 \|u_{xxx}\|_2 \\ &\leq C_{2,1} \text{const} \|u\|_2 \|u_{xx}\|_2^2 \|u_{xxx}\|_2 \\ &\leq C_2 \|u_{xx}\|^4 + \epsilon_1 \|u_{xxx}\|_2^2. \end{aligned} \tag{3.16}$$

We also have that the estimate

$$\begin{aligned}\lambda\left|\int_\Omega \sigma'(u) u_x u_{xxx} dx\right| &\leq \lambda c_1 \|u\|_\infty \|u_x\|_2 \|u_{xxx}\|_2 \\ &\leq \lambda c_1 \text{const} \|u_{xx}\|_2^2 \|u_{xxx}\|_2 \\ &\leq C_3 \|u_{xx}\|_2^4 + \epsilon_1 \|u_{xxx}\|_2^2. \end{aligned}$$

The rest of the integral terms in (3.14) can be bounded similarly. Thus, for sufficiently small ϵ_1, we get the inequalities

$$\frac{d}{dt}\|u_{xx}(t)\|_2^2 + c_* \|u_{xxx}(t)\|_2^2 \leq M_1 + M_2 \|u_{xx}(t)\|_2^4, \tag{3.17}$$

$$\frac{d}{dt}\|u_{xx}(t)\|_2^2 \leq M_1 + M_2 \|u_{xx}(t)\|_2^4, \tag{3.18}$$

where M_1, M_2 are independent of t. We set $y(t) = \|u_{xx}(t)\|_2^2$, $h(t) = M_1$ and $g(t) = M_2 \|u_{xx}(t)\|_2^2$. For fixed $r > 0$, we use (3.12) to deduce that

$$\int_t^{t+r} g(s) ds \leq \alpha_1, \quad \int_t^{t+r} h(s) ds \leq \alpha_2, \quad \int_t^{t+r} y(s) ds \leq \alpha_3,$$

for all $t \geq t_0(\mathcal{B}, \rho'_2)$, where $\alpha_1 = M_2 \alpha_3$, $\alpha_2 = M_1 r$, $\alpha_3 = \frac{r}{\alpha c_*}\|f\|_2^2 + \frac{{\rho'}_2^2}{\alpha}$. Applying uniform Gronwall's Lemma 1.2 to the differential inequality (3.18), we conclude that

$$\|u_{xx}(t)\|_2^2 \leq \left(\frac{\alpha_3}{r} + \alpha_2\right) \exp(\alpha_1) := \rho_3^2, \text{ for all } t \geq t_0(\mathcal{B}, \rho'_2) + r \tag{3.19}$$

and the proof is complete. $\diamondsuit$

Lemma 3.3 *Let hypotheses* (A), (B) *be fulfilled* $u_0 \in \mathcal{B}$ *and* $f \in \mathrm{H}$. *Assume also that (3.1) is satisfied. Then, there exists a constant* ρ_4 *independent of* t *and* $t_2 > 0$, *such that*

$$||u_{xxx}(t)||_2 \leq \rho_4, \ \ for \ \ t \geq t_2. \tag{3.20}$$

Proof We multiply equation (1.1) by $-u^{(6)}$ and integrate over Ω to get the equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u_{xxx}||_2^2 \ &+ \ \int_\Omega a(u)u_{xxxx}^2 dx + \int_\Omega \mathbf{A_1}(u)u_x^2 u_{xx}u_{xxxx}dx \\ &+ \ 2\int_\Omega \mathbf{A_2}(u)u_x u_{xxx}u_{xxxx}dx + \int_\Omega \mathbf{A_3}(u)u_{xx}^2 u_{xxxx}dx \\ &+ \ \lambda\int_\Omega \Big(\sigma''(u)u_x^2 + \sigma'(u)u_{xx}\Big)\, u_{xxxx}\, dx \\ &+ \ \int_\Omega b''(u)u_x^4 u_{xxxx}dx = -\int_\Omega f_{xx}u_{xxxx}\, dx, \end{aligned} \tag{3.21}$$

where $\mathbf{A_1}(u) = a''(u) + 5b'(u)$, $\mathbf{A_2}(u) = a'(u) + b(u)$ and $\mathbf{A_3}(u) = a'(u) + 2b(u)$. Similarly to Lemma 3.2, we arrive at the inequality

$$\frac{d}{dt}||u_{xxx}(t)||_2^2 + c_*||u_{xxxx}(t)||_2^2 \leq M_3 + M_4||u_{xxx}(t)||_2^4, \tag{3.22}$$

where $M_3(\rho_1, \rho_2', \rho_3)$, $M_4(\rho_1, \rho_2', \rho_3)$ are independent of t. Moreover, from inequality (3.17) we obtain that for fixed $r' > 0$

$$\int_t^{t+r'} ||u_{xxx}(s)||^2 ds \ \ \leq \ \ \frac{M_1 r'}{c_*} + \frac{\rho_3^2}{c_*}(M_2\rho_3^2 r' + 1). \tag{3.23}$$

Setting $y(t) = ||u_{xxx}(t)||_2^2$, $h(t) = M_3$, $g(t) = M_4||u_{xxx}(t)||_2^2$ inequality (3.23) implies the following estimates

$$\int_t^{t+r'} g(s)ds \leq \beta_1, \ \ \int_t^{t+r'} h(s)ds \leq \beta_2, \ \ \int_t^{t+r'} y(s)ds \leq \beta_3,$$

where

$$\beta_1 = M_4\beta_3, \ \ \beta_2 = M_3 r', \ \ \beta_3 = \frac{M_1 r'}{c_*} + \frac{\rho_3^2}{c_*}(M_2\rho_3^2 r' + 1).$$

Applying Lemma 1.2 to the differential inequality (3.22), we conclude that

$$||u_{xxx}(t)||_2^2 \leq \left(\frac{\beta_3}{r'} + \beta_2\right)\exp(\beta_1) := \rho_4^2, \ \ \text{for} \ \ t \geq t_1 + r'$$

to complete the proof. $\diamondsuit$

Next we discuss certain regularity questions of the solution and the solution operator for the problem (1.1)-(1.3).

Proposition 3.4 *Let hypotheses* (A), (B), *be fulfilled and* $u_0, f \in H$. *Then, for the unique solution of (1.1)-(1.3), it holds that* $u \in C(0,T;H)$, *for every* $T>0$. *Moreover, the mapping* $S(t): u_0 \in \mathrm{H} \mapsto u(t) \in \mathrm{H}$ *is continuous.*

Proof We shall divide the proof in two parts.
A. *Continuity of Solutions*: Consider the dense imbeddings

$$\mathrm{V} \hookrightarrow \mathrm{H} \hookrightarrow \mathrm{V}^*. \tag{3.24}$$

A consequence of relation (3.22) is that $u \in L^2(0,T;,V)$, for every $T>0$. Also, it can be easily proved that $u_t \in L^2(0,T;W)$. Taking into account the continuous embedding $L^2(0,T;W) \hookrightarrow L^2(0,T;V^*)$, it follows that

$$u \in \mathcal{W} \equiv \left\{ u \in L^2(0,T;V),\ u_t \in L^2(0,T;V^*) \right\} \hookrightarrow C(0,T;H).$$

B. *Continuity of the Solution Mapping*: Multiply equation (2.21) by w_{xxxx} and integrate over Ω to get the following relation

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}||w_{xx}||_2^2 + \int_\Omega a(u) w_{xxx}^2\, dx + \int_\Omega a'(u) u_x w_{xx} w_{xxx}\, dx \\
+ \quad & \int_\Omega (a'(u) - a'(v)) v_x v_{xx} w_{xxx}\, dx + \int_\Omega a'(u) w_x v_{xx} w_{xxx}\, dx \\
+ \quad & \int_\Omega (a(u) - a(v)) v_{xxx} w_{xxx}\, dx + \int_\Omega (b'(u) - b'(v)) v_x^3 w_{xxx}\, dx \\
+ \quad & \int_\Omega b'(u) v_x^2 w_x w_{xxx}\, dx + \int_\Omega b'(u)(u_x + v_x) u_x w_x w_{xxx}\, dx \\
+ \quad & \int_\Omega b(u)(u_x + v_x) w_{xx} w_{xxx}\, dx + \int_\Omega b(u)(u_{xx} + v_{xx}) w_x w_{xxx}\, dx \\
+ \quad & 2\int_\Omega (b(u) - b(v)) v_x v_{xx} w_{xxx}\, dx + \lambda \int_\Omega (\sigma(u) - \sigma(v)) w_{xxxx}\, dx = 0.
\end{aligned}$$

The integral terms in the equation above, may be estimated as follows

$$\begin{aligned}
\left| \int_\Omega (a(u) - a(v)) v_{xxx} w_{xxx}\, dx \right| \quad & \leq \quad C_{1,1} ||w||_\infty ||v_{xxx}||_2 ||w_{xxx}||_2 \\
& \leq \quad K_1 ||w_{xx}||_2 ||v_{xxx}||_2 ||w_{xxx}||_2 \\
& \leq \quad K_2 ||v_{xxx}||_2^2 ||w_{xx}||_2^2 + \epsilon_2 ||w_{xxx}||_2^2,
\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega b(u)(u_{xx}+v_{xx})w_x w_{xxx}\,dx\right| &\leq C_{2,0}||w_x||_\infty||u_{xx}+v_{xx}||_2||w_{xxx}||_2\\ &\leq K_3||w_{xx}||_2||u_{xx}+v_{xx}||_2||w_{xxx}||_2\\ &\leq K_4||w_{xx}||_2^2+\epsilon_2||w_{xxx}||_2^2.\end{aligned}$$

The inequality obtained by this procedure, for sufficiently small ϵ_2, is

$$\begin{aligned}&\frac{d}{dt}||w_{xx}(t)||_2^2+c_*||w_{xxx}(t)||_2^2\leq M_0(t)||w_{xx}(t)||_2^2, \qquad (3.25)\\ &M_0(t)=C_1+C_2||v_{xxx}(t)||_2^2.\end{aligned}$$

Since the solution $v\in L^2(0,T;H^3\cap H_0^1(\Omega))$ (e.g. see Lemma 3.3), the function $M_0(t)$ is integrable on the interval $[0,T]$. Therefore, the standard Gronwall Lemma is applicable to the inequality (3.25) to obtain

$$||w_{xx}(t)||_2^2\leq C_3||w_{xx}(0)||_2^2,\;\; C_3=\exp\left\{\max_{t\in[0,T]}M_0(t)\right\}. \qquad (3.26)$$

Inequality (3.26) implies the continuity of the mapping $S(t):u_0\in \mathrm{H}\mapsto u(t)\in\mathrm{H}$. $\diamondsuit$

Now, we are allowed to define a dynamical system in H as the mapping

$$S(t):u_0\in\mathrm{H}\mapsto u(t)\in\mathrm{H}$$

associated to the problem (1.1)-(1.3). We conclude with the following result:

Theorem 3.5 *If $f\in H$, then the semigroup $S(t)$ possesses global attractor $\mathcal{A}$ in H.*

Proof Restating the result of Lemma 3.2 and taking into account inequality (3.19) for some fixed $r>0$, we have that the closed ball in H,

$$\mathcal{B}_1=\{\phi\in H:\;||\phi||_H\leq\rho_3\},$$

is a bounded absorbing set for the semigroup $S(t)$, i.e., for every bounded set $\mathcal{B}$ in H, there exists $t_1(\mathrm{B})>0$, such that $S(t)\mathcal{B}\subset\mathcal{B}_1$, for every $t\geq t_1(\mathcal{B})$. On the other hand, Lemma 3.3 implies that there exists $t_2(\mathcal{B})>0$ such that $S(t)\mathcal{B}\subset \mathrm{B}_2$ for $t\geq t_2(\mathcal{B})$, where

$$\mathcal{B}_2=\{\phi\in X:\;||\phi||_X\leq\rho_4\},$$

is a closed ball in $X\equiv H^3\cap H_0^1(\Omega)$. The set $\mathcal{B}_2$ is bounded in X and relatively compact in H and the semigroup $S(t)$ is uniformly compact. Hence, the set $\mathcal{A} = \omega(\mathcal{B})$ is a compact attractor for the semigroup $S(t)$. $\diamondsuit$

Acknowledgement. This work was partially sponsored by a grant from the Research Committee of NTUA, Athens, and by a grant from THALES Project of Basic Research Training, NTUA, Athens.

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