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BIFURCATION RESULTS FOR QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. We prove certain bifurcation results for the quasilinear elliptic system

$$\begin{aligned} -\Delta_p u &= \lambda\, a(x)\,|u|^{p-2}u + \lambda\, b(x)\,|u|^{\alpha}\,|v|^{\beta}\,v + f(x,\lambda,u,v),\\ -\Delta_q v &= \lambda\, d(x)\,|v|^{q-2}v + \lambda\, b(x)\,|u|^{\alpha}\,|v|^{\beta}\,u + g(x,\lambda,u,v), \end{aligned}$$

defined on an arbitrary domain (bounded or unbounded) of $\mathbb{R}^N$, where the functions a, d, f and g may change sign. To this end we establish the isolation of the principal eigenvalue of the corresponding unperturbed system and apply topological degree theory.

1. Introduction

In this paper we shall deal with the bifurcation of a continuum of positive solutions for the following quasilinear elliptic system, defined on $\Omega \subseteq \mathbb{R}^N$,

$$\begin{aligned} -\Delta_p u &= \lambda\, a(x)|\,u|^{p-2}u + \lambda\, b(x)\,|u|^{\alpha}\,|v|^{\beta}\,v + f(x,\lambda,u,v),\\ -\Delta_q v &= \lambda\, d(x)\,|v|^{q-2}v + \lambda\, b(x)\,|u|^{\alpha}\,|v|^{\beta}u\; + g(x,\lambda,u,v), \end{aligned} \tag{1.1}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is an arbitrary domain (bounded or unbounded). This continuum is bifurcating from the positive principal eigenvalue of the following unperturbed system

$$\begin{aligned} -\Delta_p u &= \lambda\, a(x)\,|u|^{p-2}u + \lambda\, b(x)\,|u|^{\alpha}\,|v|^{\beta}\,v, \quad x \in \Omega,\\ -\Delta_q v &= \lambda\, d(x)\,|v|^{q-2}\,v + \lambda\, b(x)\,|u|^{\alpha}\,|v|^{\beta}\,u, \quad x \in \Omega, \end{aligned} \tag{1.3}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{1.4}$$

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System (1.3), (1.4) under certain conditions on the constants α, β, p, q, N and on the functions a, b and d, forms an eigenvalue problem. Throughout this work we assume that $N, p, q, \alpha, \beta, a, b, d, f, g$ satisfy the following conditions:

$$(\mathcal{H}) \qquad N > p > 1, \qquad N > q > 1, \qquad \alpha \geq 0, \qquad \beta \geq 0 \qquad \text{and}$$
$$\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1.$$

We also suppose that the coefficient functions satisfy the following conditions:

(Υ_1) a is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, such that $a \in L^{N/p}(\Omega) \cap L^\infty(\Omega)$ and there exists $\Omega^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega^+| > 0$, such that $a(x) > 0$, for all $x \in \Omega^+$.

(Υ_2) d is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, such that $d \in L^{N/q}(\Omega) \cap L^\infty(\Omega)$ and there exists $\Omega^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega^+| > 0$, such that $d(x) > 0$, for all $x \in \Omega^+$.

(Υ_3) the functions a and d satisfy one of the following hypothesis
(G^+) $a(x) \geq 0$, $d(x) \geq 0$, in Ω, or
(G^-) $a(x) < 0$ and $d(x) < 0$, for all $x \in \Omega^-$, on some subset $\Omega^- \subseteq \Omega$ with $|\Omega^-| > 0$.

(Υ_4) b is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, such that $b(x) \geq 0$ in Ω, $b(x) \not\equiv 0$ and $b \in L^{\omega_1}(\Omega) \cap L^\infty(\Omega)$, where $\omega_1 = p^* q^*/[p^* q^* - (\alpha+1)q^* - (\beta+1)p^*]$. With p^* and q^* we denote the critical Sobolev exponents: $p^* = Np/(N-p)$ and $q^* = Nq/(N-q)$.

($\mathcal{F}$) The perturbations f and g are of the form

$$|f(x, \lambda, u, v)| \leq \sigma_1(\lambda)\, m(x)\, |u|^{\gamma_1 - 1}\, |v|^{\delta_1 + 1}\, u + \sigma_2(\lambda)\, \mu(x)\, |u|^{\eta - 1}\, u,$$
$$|g(x, \lambda, u, v)| \leq \sigma_3(\lambda)\, n(x)\, |u|^{\gamma_2 + 1}\, |v|^{\delta_2 - 1}\, v + \sigma_4(\lambda)\, \nu(x)\, |v|^{\theta - 1}\, v,$$

where the exponents γ_i, δ_i, $i = 1, 2$, η and θ satisfy the following conditions: $\gamma_i + 1 > p$ or $\delta_i + 1 > q$, $\frac{\gamma_i + 1}{p^*} + \frac{\delta_i + 1}{q^*} < 1$, $i = 1, 2$, $p < \eta + 1 < p^*$ and $q < \theta + 1 < q^*$ while, the coefficient functions satisfy the following:

(Υ_5) $\sigma_i(\lambda)$, $i = 1, 2, 3, 4$, $m(x)$, $n(x)$, $\mu(x)$ and $\nu(x)$ are smooth functions, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0,1)$, such that σ_i, $i = 1, 2, 3, 4$, are bounded, $m, n \in L^{\omega_{2,i}}(\Omega) \cap L^\infty(\Omega)$, where $\omega_{2,i} = p^* q^*/[p^* q^* - (\gamma_i + 1)q^* - (\delta_i + 1)p^*]$, $i = 1, 2$, respectively, $\mu \in L^{\omega_3}(\Omega) \cap L^\infty(\Omega)$, where $\omega_3 = p^*/[p^* - (\eta + 1)]$ and $\nu \in L^{\omega_4}(\Omega) \cap L^\infty(\Omega)$, where $\omega_4 = q^*/[q^* - (\theta + 1)]$.

Moreover, we assume that there exists a function F such that

$$(\mathcal{P}) \qquad F_u(u,v) = (\alpha+1)\,f(x,u,v) \quad \text{and} \quad F_v(u,v) = (\beta+1)\,g(x,u,v).$$

Systems of the form (1.1) where hypothesis $(\mathcal{P})$ is satisfied are called *potential* systems; This hypothesis enables us to define the corresponding functional on a proper product space and to use variational techniques. As a consequence, a basic property (as it is stated in [13]) of the potential systems is that they behave in a certain sense like scalar equations. This will be clear throughout this paper, as the arguments of [6] for the equation can be carried out to the case of the system. Also, we may note that the isolation result (see Theorem 3.6) completes the basic properties (as in the scalar equation case) of the principal eigenvalue for the system (see [2, 10]).

However, it is expected that the system and the scalar equation will have certain differences. For example, systems of the form (1.1) may have semitrivial solutions, i.e., solutions of the form $(u,0)$ or $(0,v)$ (see Lemma (6.2) below), where bifurcation from semitrivial solutions may occur. Another difference, in the case where $p \neq q$ is that the eigenspaces corresponding to the eigenvalue problem (1.3) are not linear subspaces. Actually, they are not homogeneous in the Cartesian product with components u and v, as it is in the case of the p-Laplacian equation (see [16, Remark 5.4]). Finally, there is a lack of general regularity theory when we treat nonlinear elliptic systems. For more details we refer to [5] and the references therein.

As an application of system (1.1) consider the following problem ($p = q = 2$)

$$\begin{aligned} -\Delta u &= \lambda\, a(x)\, u + \lambda\, b(x)\, v + f(x,\lambda,u,v),\\ -\Delta v &= \lambda\, d(x)\, v + \lambda\, b(x)\, u + g(x,\lambda,u,v), \end{aligned} \tag{1.5}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{1.6}$$

The solutions of the problem (1.5), (1.6) correspond to the steady-state solutions of a competition model arising in popular dynamics. The coefficient functions $a(x), b(x), d(x)$ represent the fact that this competition is taking place in a spatial heterogeneous environment. The case of spatial homogeneous environment, i.e., when a, b, d are positive constants, is studied in the papers [11] and [12]). The boundary conditions (1.6) means that the environment outside Ω is lethal.

Problems where the p - Laplacian operator $-\Delta_p$ is present arise both from pure mathematics (*theory of quasiregular* and *quasiconformal mappings*), as well as from applications, e.g. *Non-Newtonian fluids, reaction-diffusion problems, porous media, astronomy*, etc. In the case of bounded domain, under various boundary conditions, there is quite an extensive literature for the eigenvalue problem under consideration. We refer to [10] and the references therein for the equation and to the works [2, 4] for the system.

Several works dealing with the eigenvalue problem in unbounded domains have recently appeared (see, for example, [2, 6, 7, 10, 15]). Furthermore, in [6, 7] bifurcation technics are used to prove existence results for the p-Laplacian equation in $\mathbb{R}^N$.

The rest of the paper is organized in six sections. In Section 2, we introduce the necessary operators and establish the basic characteristics of them. In Section 3, we recall the main results from [10] concerning the existence of positive principal eigenvalue as well as the regularity and the asymptotic behavior of the corresponding eigenfunctions in $D^{1,p}(\Omega) \times D^{1,q}(\Omega)$ for a related to (1.3) and (1.4) problem (see system (3.1), (3.2)). Then we prove that the principal eigenvalue of (1.1) is isolated. *The main idea is to consider system* (1.3) *rather than system* (3.1). Then, it was possible to adapt standard arguments in order to prove this isolation result for system (1.3). In Section 4, we prove that the operators, generated by the system (1.1) satisfy a condition under which it is possible to define their degree (condition $(S)_+$). In Section 5, the existence of a continuum of nontrivial solutions bifurcating out from the first eigenvalue of the problem (1.3)-(1.4) is achieved. In Section 6, considering the regularity of the solutions we describe the behavior of the continuum of nontrivial solutions for the perturbed problem (1.1) in the product space $D^{1,p}(\Omega) \times D^{1,q}(\Omega)$.

This work constitutes the first attempt to apply bifurcation techniques on quasilinear elliptic systems defined in a general domain (bounded or unbounded) with varying coefficients. So, in this direction, the present paper may be considered as a generalization of the work done on the equation (e.g. see [6] and [7]). The procedure we follow is based on the isolation of the principal eigenvalue of the system (1.3) (which is of independent interest). As far as we know, the isolation result is the first one in this direction, both in bounded and unbounded case. Observe, that in general we have $p \neq q$ (compare with the decoupling method used in [9] and the method used in [11, 12]). Our main bifurcation result is based on the classical Rabinowitz Theorem [14], for similar results we refer to the works [6, 7, 8, 9, 11, 12]. Meanwhile, here we extend the results achieved in [16] concerning the exponents and the multiplicity results achieved in [17].

Notation. For simplicity we use the symbol $||.||_p$ for the norm $||.||_{L^p(\Omega)}$ and $\mathcal{D}^{1,p}$ for the space $\mathcal{D}^{1,p}(\Omega)$. B_R and $B_R(c)$ will denote the balls in Ω of radius R and centers zero and c, respectively. Also the Lebesgue measure of a set $\Omega \subset \mathbb{R}^N$ will be denoted by $|\Omega|$. An equality introducing definition is denoted by =:. Integration in all of Ω will be denoted with the integral symbol $\int$ without any indication.

2. Space and operator settings

Consider the product space $Z := \mathcal{D}^{1,p}(\Omega) \times \mathcal{D}^{1,q}(\Omega)$ equipped with the norm $||z||_Z := ||u||_{1,p} + ||v||_{1,q}$, $z = (u, v) \in Z$, where

$$||u||_{1,p} := \Big(\int_\Omega | \bigtriangledown u|^p \, dx \Big)^{1/p}.$$

In the bounded domain case, we notice that $||u||_{1,p}$ is a norm equivalent to the standard Sobolev norm in the space $W_0^{1,p}$, i.e., $W_0^{1,p}(\Omega) = \mathcal{D}^{1,p}(\Omega)$. However, in the case of Ω been an unbounded domain, $||u||_{1,p}$ is the norm of the space $\mathcal{D}^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega) \subsetneqq \mathcal{D}^{1,p}(\Omega)$.

It is known that $\mathcal{D}^{1,p}(\Omega) = \{u \in L^{\frac{Np}{N-p}}(\Omega) : \bigtriangledown u \in (L^p(\Omega))^N\}$ and that there exists $K_0 > 0$, such that for all $u \in \mathcal{D}^{1,p}(\Omega)$, the following inequality holds

$$||u||_{\frac{Np}{N-p}} \quad \leq \quad K_0 \, ||u||_{\mathcal{D}^{1,p}}. \tag{2.1}$$

Clearly, the space $\mathcal{D}^{1,p}$ is a *reflexive Banach space*. For more details we refer to [1]. Our approach is based on the following generalized Poincare's inequality.

Lemma 2.1. *Suppose $g \in L^{N/p}(\Omega)$. Then there exists $\alpha > 0$ such that*

$$\int_\Omega | \bigtriangledown u|^p dx \geq \alpha \int_\Omega |g||u|^p dx, \tag{2.2}$$

for all $u \in \mathcal{D}^{1,p}(\Omega)$.

We introduce the operators $J_1, J_2, D_1, D_2, B_1, B_2, F : Z \to Z^*$ in the following way.

$$\langle J_1(u,v), (\phi, \psi)\rangle =: \int_\Omega | \bigtriangledown u|^{p-2} \bigtriangledown u \bigtriangledown \phi$$

$$\langle J_2(u,v), (\phi, \psi)\rangle =: \int_\Omega | \bigtriangledown v|^{q-2} \bigtriangledown v \bigtriangledown \psi$$

$$\langle D_1(u,v), (\phi, \psi)\rangle =: \int_\Omega a(x)|u|^{p-2} u \, \phi$$

$$\langle D_2(u,v), (\phi, \psi)\rangle =: \int_\Omega d(x)|v|^{q-2} v \, \psi$$

$$\langle B_1(u,v),(\phi,\psi)\rangle =: \int_\Omega b(x)|u|^\alpha|v|^\beta v\,\phi$$
$$\langle B_2(u,v),(\phi,\psi)\rangle =: \int_\Omega b(x)|u|^\alpha|v|^\beta u\,\psi$$
$$\langle F_1(u,v),(\phi,\psi)\rangle =: \int_\Omega f(x,\lambda,u,v)\,\phi$$
$$\langle F_2(u,v),(\phi,\psi)\rangle =: \int_\Omega g(x,\lambda,u,v)\,\psi.$$

Lemma 2.2. *The operators J_i, D_i, B_i, $i = 1,2$, are well defined.*

Proof. The result may be obtained by Hölder's inequality and the properties of the coefficient functions. □

Lemma 2.3. *The operators $J_i, i = 1,2$, are continuous. The operators D_i, $B_i, F_i, i = 1,2$, are compact. Furthermore, the operators F_i, $i = 1,2$, satisfy the relations*

$$\lim_{||(u,v)||_Z\to 0} \frac{||F_i(u,v)||_{Z^*}}{||u||_{1,p}^{p-1} + ||v||_{1,q}^{q-1}} = 0. \tag{2.3}$$

Proof. The proof of the continuity and compactness properties of the operators follows the same lines as in [10] and [16]. Concerning relation (2.3) we have that

$$||F_1(u,v)||_{Z^*} \le \sup_{||\phi||_{1,p}\le 1}\left\{\int m(x)|u|^\gamma|v|^{\delta+1}|\phi|\,dx + \int \mu(x)|u|^\eta|\phi|\,dx\right\}$$
$$\le c \sup_{||\phi||_{1,p}\le 1}\left\{||m(x)||_{\omega_{2,1}}||u||_{p^*}^\gamma ||v||_{q^*}^{\delta+1}||\phi||_{p^*} + ||\mu(x)||_{\omega_3}||u||_{p^*}^\eta||\phi||_{p^*}\right\}.$$

The range of the exponents γ, δ and η implies the following

$$\frac{\|F_1(u,v)\|_{Z^*}}{\|u\|_{1,p}^{p-1} + \|v\|_{1,q}^{q-1}} \le \|m(x)\|_{\omega_{2,1}}\|u\|_{p^*}^{\gamma-(p-1)}\|v\|_{q^*}^{\beta+1} + \|\mu(x)\|_{\omega_3}\|u\|_{p^*}^{\eta-(p-1)} \to 0,$$

or

$$\frac{\|F_1(u,v)\|_{Z^*}}{\|u\|_{1,p}^{p-1} + \|v\|_{1,q}^{q-1}} \le \|m(x)\|_{\omega_{2,1}}\|u\|_{p^*}^{\gamma}\|v\|_{q^*}^{\beta+1-(q-1)} + \|\mu(x)\|_{\omega_3}\|u\|_{p^*}^{\eta-(p-1)} \to 0,$$

as $||z|| \to 0$. The analogous holds for the operator $F_2(u,v)$. □

Next, we introduce the operators $\tilde{A}_\lambda, A_\lambda : Z \to Z^*$ as:

$$\begin{aligned}\tilde{A}_\lambda(u,v) =:& (\alpha+1)\big[J_1(u,v) - \lambda D_1(u,v) - \lambda B_1(u,v)\big]\\ &+ (\beta+1)\big[J_2(u,v) - \lambda D_2(u,v) - \lambda B_2(u,v)\big].\end{aligned} \tag{2.4}$$

$$A_\lambda(u,v) =: \tilde{A}_\lambda(u,v) - F'(u,v). \tag{2.5}$$

Finally, we define the notion of the weak solution for the problem (1.1). We say that (u,v) is a *weak solution* of the system (1.1), if and only if $A_\lambda(u,v) = 0$ in Z^*. This definition follows from the fact that (u,v) must be a critical point of the functional $\Phi : Z \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi(u,v) =: \frac{\alpha+1}{p}\int |\bigtriangledown u|^p + \frac{\beta+1}{q}\int |\bigtriangledown v|^q - \lambda\frac{\alpha+1}{p}\int a(x)|u|^p \\ -\lambda\frac{\beta+1}{q}\int d(x)|v|^q - \lambda\int b(x)|u|^\alpha|v|^\beta uv - \lambda\int F(u,v).\end{aligned}$$

Notation. For the convenience of the representation in the sequel we introduce the operators

$$\begin{aligned}J(u,v) &=: \frac{\alpha+1}{p}J_1(u,v) + \frac{\beta+1}{q}J_2(u,v),\\ D(u,v) &=: \frac{\alpha+1}{p}D_1(u,v) + \frac{\beta+1}{q}D_2(u,v),\\ B(u,v) &=: \frac{\alpha+1}{p}B_1(u,v) + \frac{\beta+1}{q}B_2(u,v),\\ C(u,v) &=: D(u,v) + B(u,v).\end{aligned}$$

3. The unperturbed system

In this section we study the unperturbed problem (1.3), (1.4). To this end, it is convenient to recall from the works [2, 10] some results concerning the following eigenvalue problem

$$\begin{aligned}-\Delta_p u &= \lambda\, a(x)\,|u|^{p-2}u + \lambda\, b(x)\,|u|^{\alpha-1}\,|v|^{\beta+1}\,u, \quad x\in\Omega,\\ -\Delta_q v &= \lambda\, d(x)\,|v|^{q-2}\,v + \lambda\, b(x)\,|u|^{\alpha+1}\,|v|^{\beta-1}\,v, \quad x\in\Omega,\end{aligned} \tag{3.1}$$

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \tag{3.2}$$

We shall restrict our study to the case (G^+). Similar results may be obtained for the case (G^-) by symmetry.

Theorem 3.1. (i) *The system* (3.1), (3.2) *admits a positive principal eigenvalue* λ_1*, given by*

$$\lambda_1 = \inf_{\langle C(|u|,|v|),(|u|,|v|)\rangle=1}\langle J(u,v),(u,v)\rangle. \tag{3.3}$$

The associated normalized eigenfunction (u_1,v_1) *belongs to* $\mathcal{D}^{1,p}\times\mathcal{D}^{1,q}$*, each component is positive and of class* $\mathcal{C}^{1,\zeta}(B_r)$*, for any* $r>0$*, where* $\zeta=\zeta(r)\in(0,1)$*. In addition,* (*i*) *the set of all eigenfunctions corresponding to the principal eigenvalue* λ_1 *forms a one dimensional manifold,* $E_1 \subset Z$*, which is defined by* $E_1 = \{(c_1 u_1, c_1^{q/p} v_1); c_1 \in \mathbb{R}\}$.

(*ii*) λ_1 *is the only eigenvalue of* (3.1), (3.2) *to which corresponds a componentwise positive eigenfunction.*

Remark 3.2. In the case where $a \equiv 0$ and $d \equiv 0$, Theorem 3.1 still hold, see [16, Theorems 5.1 and 5.3].

Concerning now, system (1.3), (1.4) we may obtain the following results.

Theorem 3.3. *The system* (1.3), (1.4) *admits a positive principal eigenvalue* λ_1 *given by* (3.3)*. The associated normalized eigenfunction* (u_1, v_1) *belongs to* $\mathcal{D}^{1,p} \times \mathcal{D}^{1,q}$*, each component is positive and of class* $\mathcal{C}^{1,\zeta}(B_r)$*, for any* $r > 0$*, where* $\zeta = \zeta(r) \in (0,1)$*.*

Proof. Since for positive solutions systems (1.3), (1.4) and (3.1), (3.2) coincide, we deduce from Theorem 3.1 that (λ_1, u_1, v_1) is also an eigenpair for the system (1.3), (1.4). Assume that there exists another nontrivial eigenpair (λ_*, u_*, v_*) of the system (1.3), (1.4), such that $0 < \lambda_* < \lambda_1$. Then the following equality must be satisfied

$$\lambda_* = \frac{\langle J(u_*, v_*), (u_*, v_*)\rangle}{\langle C(u_*, v_*), (u_*, v_*)\rangle}, \tag{3.4}$$

with $\langle C(u_*, v_*), (u_*, v_*)\rangle > 0$. Set

$$u^* =: \frac{|u_*|}{\langle C(|u_*|, |v_*|), (|u_*|, |v_*|)\rangle^{1/p}} \quad \text{and} \quad v^* =: \frac{|v_*|}{\langle C(|u_*|, |v_*|), (|u_*|, |v_*|)\rangle^{1/q}}\,.$$

Using condition $(\mathcal{H})$ we may easily verify that

$$\langle J(u^*, v^*), (u^*, v^*)\rangle = \frac{\langle J(u_*, v_*), (u_*, v_*)\rangle}{\langle C(|u_*|, |v_*|), (|u_*|, |v_*|)\rangle} \tag{3.5}$$

and

$$\langle C(u^*, v^*), (u^*, v^*)\rangle = 1. \tag{3.6}$$

Relations (3.3)–(3.6) and the following inequality

$$\frac{\langle J(u,v), (u,v)\rangle}{\langle C(|u|, |v|), (|u|, |v|)\rangle} \le \frac{\langle J(u,v), (u,v)\rangle}{\langle C(u,v), (u,v)\rangle}, \tag{3.7}$$

which holds for every $(u,v) \in Z$, such that $\langle C(u,v), (u,v)\rangle > 0$, imply that

$$\lambda_1 \;=\; \inf_{\langle C(|u|,|v|),(|u|,|v|)\rangle = 1} \langle J(u,v), (u,v)\rangle$$

$$\begin{aligned} &\leq \quad \langle J(u^*, v^*), (u^*, v^*)\rangle = \frac{\langle J(u_*, v_*), (u_*, v_*)\rangle}{\langle C(|u_*|, |v_*|), (|u_*|, |v_*|)\rangle} \\ &\leq \quad \frac{\langle J(u_*, v_*), (u_*, v_*)\rangle}{\langle C(u_*, v_*), (u_*, v_*)\rangle} = \lambda_*. \end{aligned}$$

The last inequality leads to a contradiction and the proof is completed. □

Theorem 3.4. (*i*) *The set of all eigenfunctions corresponding to the principal eigenvalue* λ_1 *of the system* (1.3), (1.4) *forms a one dimensional manifold,* $E_1 \subset Z$*, which is defined by* $E_1 = \{(c_1\, u_1,\, c_1^{q/p}\, v_1)\, ;\ c_1 \in \mathbb{R}\}$*.*

(*ii*) λ_1 *is the only eigenvalue of* (1.3), (1.4) *to which corresponds a componentwise positive eigenfunction.*

Proof. (i) Inequality (3.7) implies that if $(u, v) \not\equiv (0, 0)$ is an eigenfunction of (1.3), (1.4) corresponding to λ_1, then $(|u|, |v|)$ must be an eigenfunction, too. Since for positive solutions systems (1.3), (1.4) and (3.1), (3.2) coincide, we deduce from the proof of Theorem 3.1 (see [2]), that there exists a positive constant k, so that $|u| = k^p u_1$ and $|v| = k^q v_1$, for every $x \in \Omega$. Suppose, now, that u or v changes sign. Then, since they are smooth functions there exists some $x_0 \in \Omega$, such that $u(x_0) = 0$ or $v(x_0) = 0$. Hence, $u_1(x_0) = 0$ or $v_1(x_0) = 0$, which is a contradiction.

(ii) The result follows from the fact that systems (1.3), (1.4) and (3.1), (3.2) are the same for positive solutions. □

Remark 3.5. We want to emphasize the fact that the solution sets of (1.3), (1.4) and (3.1), (3.2) are not the same. For example, $(u_1, -v_1)$ and $(-u_1, v_1)$ are solutions of (3.1), (3.2), while they are not satisfying the system (1.3), (1.4).

Finally, we shall prove that the eigenvalue λ_1 is isolated. The proof will extend the ideas developed in the proof of the analogous result for the equation [7, Proposition 2.2].

Theorem 3.6. *The principal eigenvalue* $\lambda_1 > 0$ *is isolated in the following sense: there exists* $\eta > 0$*, such that the interval* $(0, \lambda_1 + \eta)$ *does not contain any other eigenvalue than* λ_1*.*

Proof. Suppose there exists a sequence of eigenpairs (λ_n, u_n, v_n) of (1.3), (1.4) with $\lambda_n \to \lambda_1$. By the variational characterization of λ_1 we know that $\lambda_n \geq \lambda_1$. So, we may have that $\lambda_n \in (\lambda_1, \lambda_1 + \eta)$, for each $n \in \mathbb{N}$. Furthermore, without loss of generality we may assume that $||u_n|| = ||v_n|| = 1$, for all $n \in \mathbb{N}$. Hence, there exists some $(\tilde{u}, \tilde{v}) \in Z$, such that $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$. This weak convergence and the simplicity of λ_1 imply that $(\tilde{u}, \tilde{v}) = (u_1, v_1)$

or $(\tilde{u},\tilde{v}) = (-u_1,-v_1)$. Let us suppose that $(u_n,v_n) \rightharpoonup (u_1,v_1)$ in Z. For any two pairs of eigenfunctions (u_n,v_n), (u_m,v_m) substituted to the system (1.3), we get

$$\begin{aligned}
-(\Delta_p u_n - \Delta_p u_m) &= \lambda_n a(x)|u_n|^{p-2}u_n - \lambda_m a(x)|u_m|^{p-2}u_m \\
&\quad +\lambda_n b(x)|u_n|^{\alpha}|v_n|^{\beta}v_n - \lambda_m b(x)|u_m|^{\alpha}|v_m|^{\beta}v_m, \\
-(\Delta_q v_n - \Delta_q v_m) &= \lambda_n d(x)|v_n|^{q-2}v_n - \lambda_m d(x)|v_m|^{q-2}v_m \\
&\quad +\lambda_n b(x)|u_n|^{\alpha}|v_n|^{\beta}u_n - \lambda_m b(x)|u_m|^{\alpha}|v_m|^{\beta}u_m.
\end{aligned}$$

Multiplying the first equation by $(u_n - u_m)$, the second by $(v_n - v_m)$, integrating by parts and following the estimates developed in [6] (using the compactness of the operators D and B), we obtain

$$\int |\bigtriangledown u_n|^p \to \int |\bigtriangledown u_1|^p \quad \text{and} \quad \int |\bigtriangledown v_n|^q \to \int |\bigtriangledown v_1|^q,$$

i.e., we get the strong convergence of $u_n \to u_1$ in $D^{1,p}$ and $v_n \to v_1$ in $D^{1,q}$, respectively. This means that $(u_n,v_n) \to (u_1,v_1)$ strongly in the product space Z. Given n fixed, we may get from the system (1.3), (1.4) that

$$\int |\bigtriangledown u_n|^{p-2} \bigtriangledown u_n \bigtriangledown \phi = \lambda_n \int a(x)|u_n|^{p-2}u_n\phi + \lambda_n \int b(x)|u_n|^{\alpha}|v_n|^{\beta}v_n\phi,$$

$$\int |\bigtriangledown v_n|^{q-2} \bigtriangledown v_n \bigtriangledown \psi = \lambda_n \int d(x)|v_n|^{q-2}v_n\psi + \lambda_n \int b(x)|u_n|^{\alpha}|v_n|^{\beta}u_n\psi,$$

for all $(\phi,\psi) \in Z$. Since from Theorem 3.4, λ_1 is the only eigenvalue of (1.3), (1.4), to which corresponds a positive eigenfunction (u_1,v_1), we may introduce the sets $\mathcal{U}_n^- =: \{x \in \Omega : u_n(x) < 0\}$ and $\mathcal{V}_n^- =: \{x \in \Omega : v_n(x) < 0\}$, where we must have $|\Omega_n^-| > 0$, with $\Omega_n^- = \mathcal{U}_n^- \cup \mathcal{V}_n^-$.

Denoting by $u_n^- = \min\{0,u_n\}$ and $v_n^- = \min\{0,v_n\}$ and choosing $\phi \equiv u_n^-$ and $\psi \equiv v_n^-$, it follows that

$$\int_{\mathcal{U}_n^-} |\bigtriangledown u_n^-|^p = \lambda_n \int_{\mathcal{U}_n^-} a(x)|u_n^-|^p + \lambda_n \int_{\mathcal{U}_n^-} b(x)|u_n^-|^{\alpha}|v_n|^{\beta}u_n^- v_n,$$

$$\int_{\mathcal{V}_n^-} |\bigtriangledown v_n^-|^q = \lambda_n \int_{\mathcal{V}_n^-} d(x)|v_n^-|^q + \lambda_n \int_{\mathcal{V}_n^-} b(x)|u_n|^{\alpha}|v_n^-|^{\beta}u_n v_n^-.$$

Since the products $u_n^- v_n^+$ and $u_n^+ v_n^-$ are negative, from the above system of equations we obtain that

$$\int_{\mathcal{U}_n^-} |\bigtriangledown u_n^-|^p \le \lambda_n \int_{\mathcal{U}_n^-} a(x)|u_n^-|^p + \lambda_n \int_{\mathcal{U}_n^-} b(x)|u_n^-|^{\alpha}|v_n^-|^{\beta}u_n^- v_n^-,$$

$$\int_{\mathcal{V}_n^-} |\bigtriangledown v_n^-|^q \le \lambda_n \int_{\mathcal{V}_n^-} d(x)|v_n^-|^q + \lambda_n \int_{\mathcal{V}_n^-} b(x)|u_n^-|^{\alpha}|v_n^-|^{\beta}u_n^- v_n^-.$$

From the definition of the space $D^{1,p}$, Hölder and Young inequalities and the embedding (2.1) we derive that

$$\begin{aligned} ||u_n^-||^p_{D^{1,p}(\Omega_n^-)} \leq\ & c_1\,(\lambda_1+\eta)\Big[||a(x)||_{L^{\frac{N}{p}}(\Omega_n^-)}||u_n^-||^p_{D^{1,p}(\Omega_n^-)} \\ & +||b(x)||_{L^{\omega_1}(\Omega_n^-)}\Big(||u_n^-||^p_{D^{1,p}((\Omega_n^-)}+||v_n^-||^q_{D^{1,q}((\Omega_n^-)}\Big)\Big] \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} ||v_n^-||^q_{D^{1,q}(\Omega_n^-)} \leq\ & c_2\,(\lambda_1+\eta)\Big[||d(x)||_{L^{\frac{N}{q}}(\Omega_n^-)}||v_n^-||^q_{D^{1,q}(\Omega_n^-)} \\ & +||b(x)||_{L^{\omega_1}(\Omega_n^-)}\Big(||u_n^-||^p_{D^{1,p}((\Omega_n^-)}+||v_n^-||^q_{D^{1,q}((\Omega_n^-)}\Big)\Big]. \end{aligned} \tag{3.9}$$

Finally, from the inequalities (3.8) and (3.9), we may obtain

$$1 \leq c_3\Big(\max\Big\{||a(x)||_{L^{\frac{N}{p}}(\Omega_n^-)}, ||d(x)||_{L^{\frac{N}{q}}(\Omega_n^-)}\Big\}+||b(x)||_{L^{\omega_1}(\Omega_n^-)}\Big).$$

So, there exists some constant $c_4>0$ independent from u_n, v_n and λ_n such that

$$|\Omega_n^-|>c_4>0. \tag{3.10}$$

From relation (3.10) we may choose some constant K_0 large enough, such that

$$|\Omega_n^-\cap B_K(0)|>c_5>0, \tag{3.11}$$

for any $K\geq K_0$, where the constant c_5 depends neither on λ_n nor on u_n. From the fact that $u_n\to u_1$ strongly in $D^{1,p}$ and $v_n\to v_1$ strongly in $D^{1,q}$, we have that $u_n\to u_1$ in $L^{p^*}(\Omega)$ and $v_n\to v_1$ in $L^{q^*}(\Omega)$. So $u_n\to u_1$ in $L^{p^*}(B_K(0))$ and $v_n\to v_1$ in $L^{q^*}(B_K(0))$. By Egorov's Theorem we conclude that $u_n(x)$ converges uniformly to $u_1(x)$ on $B_K(0)$ with the exception of a set with arbitrarily small measure and $v_n(x)$ does the same to $v_1(x)$ on $B_K(0)$. But this contradicts (3.11) and the conclusion follows. □

Remark 3.7. We notice that the above result is still valid in the case where $a(x)\equiv 0$ or $d(x)\equiv 0$, see Remark 3.2.

4. Topological degree

For completeness of the presentation in this section we recall some basic facts on the topological degree theory and prove the necessary conditions for the system case. The procedure is analogous to that in [6] for the equation. First, we define the topological degree for operators from a Banach space X to its dual X^*.

Definition 4.1. Let X be a reflexive Banach space, X^* its dual and $A : X \to X^*$ be a demicontinuous operator. We say that the operator A satisfies *the condition* $(S)_+$, if for any sequence $u_n \in X$ satisfying $u_n \rightharpoonup u_0$ (weakly) in X and

$$\limsup_{n\to\infty} (A(u_n), u_n - u_0)_X \le 0,$$

we have that $u_n \to u_0$ (strongly) in X.

For more details about this property we refer to [19, pp. 583]. We recall that if the operator A satisfies property $(S)_+$, then it is possible to define the degree $Deg\,[A, D, 0]$, where $D \subset X$ is a bounded open set such that $A(u) \neq 0$, for any $u \in \partial D$. We also recall that if A satisfies the condition $(S)_+$, then $A+K$ also satisfies the condition $(S)_+$, for any compact operator $K : X \to X^*$. The next two lemmas will be useful in the next section.

Lemma 4.2. *Let A be a potential operator with $\Phi'(u) = A(u)$, $u \in X$, for some continuously differentiable functional $\Phi : X \to \mathbb{R}$. Let u_0 be a local minimum of Φ and an isolated point for which $A(u_0) = 0$. Then*

$$Ind(A, u_0) = 1.$$

Lemma 4.3. *Assume that $\langle A(u), u\rangle_X > 0$, for all $u \in X$ with $||u||_X = r$. Then*

$$Deg[A, B_r(0), 0] = 1.$$

Now, we are going to prove that the operators $\tilde{A}$ and A satisfy the (S_+) condition.

Lemma 4.4. *The operators $\tilde{A}_\lambda, A_\lambda$ satisfy the (S_+) condition, where $\tilde{A}_\lambda$ and A_λ are given by* (2.4) *and* (2.5), *respectively.*

Proof. We note that since the operators D_i, B_i and F' are compact, it suffices to prove that the operator $J(u, v)$ satisfies the (S_+) condition. Let us suppose that the sequence (u_n, v_n) converges to (u_0, v_0) weakly in the space Z and

$$\limsup_{n\to\infty}\ \langle J(u_n, v_n), (u_n - u_0, v_n - v_0)\rangle_Z \le 0.$$

From the weak convergence we have that

$$\lim_{n\to\infty}\ \langle J(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle_Z = 0.$$

So

$$0 \ge \limsup_{n\to\infty}\ \langle J(u_n, v_n) - J(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle_Z$$

$$= \limsup_{n\to\infty}\Big\{ \frac{\alpha+1}{p}\int(|\bigtriangledown u_n|^{p-2}\bigtriangledown u_n - |\bigtriangledown u_0|^{p-2}\bigtriangledown u_0)(\bigtriangledown u_n - \bigtriangledown u_0)$$
$$+\frac{\beta+1}{q}\int(|\bigtriangledown v_n|^{q-2}\bigtriangledown v_n - |\bigtriangledown v_0|^{q-2}\bigtriangledown v_0)(\bigtriangledown v_n - \bigtriangledown v_0)\Big\}. \tag{4.1}$$

It is also valid, that, for any $s, t \in L^p(\Omega)$ (see [6, relation 2.12])

$$\int(|s|^{p-2}s - |t|^{p-2}t)(s-t) \geq (||s||_p^{p-1} - ||t||_p^{p-1})(||s||_p - ||t||_p) \geq 0. \tag{4.2}$$

From (4.1) and (4.2) we obtain that

$$||\bigtriangledown u_n||_p \to ||\bigtriangledown u_0||_p \quad \text{and} \quad ||\bigtriangledown v_n||_p \to ||\bigtriangledown v_0||_p.$$

Then, the conclusion of the lemma is obvious. □

5. Bifurcation from λ_1

In this section we shall prove the existence of a bifurcation from the principal eigenvalue λ_1, by using the topological degree as it is defined in the last section for the operators $\tilde{A}_\lambda$, A_λ.

Definition 5.1. Let $E = \mathbb{R} \times Z$ be equipped with the norm

$$||(\lambda, u, v)||_E = (|\lambda|^2 + ||(u,v)||_Z^2)^{1/2}, \ (\lambda, u, v) \in E. \tag{5.1}$$

We say that the set

$$C = \{(\lambda, u, v) \in E : (\lambda, u, v) \ \text{ solves } \ (1.1), \ (u,v) \neq (0,0)\}$$

is a continuum of nontrivial solutions of (1.1), if it is a connected set in E with respect to the topology induced by the norm (5.1). We say $\lambda_0 \in \mathbb{R}$ is a bifurcation point of the system (1.1) (in the sense of Rabinowitz), if there is a continuum of nontrivial solutions C of (1.1) such that $(\lambda_0, 0, 0) \in \bar{C}$ and C is either unbounded in E or there is an eigenvalue $\hat{\lambda} \neq \lambda_0$, such that $(\hat{\lambda}, 0, 0) \in \bar{C}$.

To prove the main result of the present work we follow the ideas developed for the equation in [6]. For this we need the following construction. Fix $K > 0$ and define a function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) =: \begin{cases} 0, & t \leq K, \\ \frac{2\delta}{\lambda_1}(t-2K), & t \geq 3K, \end{cases}$$

where δ is such that the interval $(\lambda_1, \lambda_1+\delta)$ contains none eigenvalue of (1.3). The function $\psi(t)$ can be chosen positive and strictly convex in $(K, 3K)$. We define the functional

$$\Psi^\lambda(u,v) =: \langle J(u,v),(u,v)\rangle - \lambda\langle C(u,v),(u,v)\rangle + \psi(\langle J(u,v),(u,v)\rangle).$$

Then Ψ^λ is continuously Fréchet differentiable with derivative

$$\langle (\Psi^\lambda)'(u,v),(w,z)\rangle = \langle \Psi^\lambda_u(u,v),(w,z)\rangle + \langle \Psi^\lambda_v(u,v),(w,z)\rangle,$$

where

$$\begin{aligned}\langle \Psi^\lambda_u(u,v),(w,z)\rangle = (\alpha+1)\Big\{\Big(1+\psi'(\langle J(u,v),(u,v)\rangle)\Big)\langle J_1(u,v),(w,z)\rangle \\ -\lambda\langle D_1(u,v),(w,z)\rangle - \lambda\langle B_1(u,v),(w,z)\rangle\Big\},\end{aligned} \tag{5.2}$$

$$\begin{aligned}\langle \Psi^\lambda_v(u,v),(w,z)\rangle = (\beta+1)\Big\{\Big(1+\psi'(\langle J(u,v),(u,v)\rangle)\Big)\langle J_2(u,v),(w,z)\rangle \\ -\lambda\langle D_2(u,v),(w,z)\rangle - \lambda\langle B_2(u,v),(w,z)\rangle\Big\}.\end{aligned} \tag{5.3}$$

In addition, the critical points (u_0,v_0) of Ψ^λ occur, if $\Psi^\lambda_u = \Psi^\lambda_v = 0$, i.e.,

$$\int |\bigtriangledown u_0|^{p-2}\bigtriangledown u_0 \bigtriangledown w - \frac{\lambda}{L}\int a(x)\,|u_0|^{p-2}\,u_0\,w - \frac{\lambda}{L}\int b(x)|u_0|^\alpha|v_0|^\beta v_0 w = 0$$

and

$$\int |\bigtriangledown v_0|^{q-2}\bigtriangledown v_0 \bigtriangledown z + \frac{\lambda}{L}\int d(x)\,|v_0|^{q-2}\,v_0\,z - \frac{\lambda}{L}\int b(x)|u_0|^\alpha|v_0|^\beta u_0 z = 0,$$

where $L =: 1+\psi'(\langle J(u_0,v_0),(u_0,v_0)\rangle)$. However, since $\lambda\in(\lambda_1,\lambda_1+\delta)$, the only nontrivial critical points of Ψ_λ occur if

$$\psi'((J(u_0,v_0),(u_0,v_0))) = \frac{\lambda}{\lambda_1} - 1. \tag{5.4}$$

Hence, we must have $(J(u_0,v_0),(u_0,v_0)) \in (K,3K)$. In this case either $(u_0,v_0) = (u_1,v_1)$ or $(u_0,v_0) = (-u_1,-v_1)$. So for $\lambda\in(\lambda_1,\lambda_1+\delta)$ we have precisely three isolated critical points 0, (u_1,v_1), $(-u_1,-v_1)$. The next lemmas describe the main characteristics of the functional Ψ^λ.

Lemma 5.2. *The functional Ψ^λ is a) weakly lower semicontinuous and b) weakly coercive, with $\lambda\in(\lambda_1,\lambda_1+\delta)$.*

Proof. a) Let $(u_n,v_n) \rightharpoonup (u_0,v_0)$ weakly in Z. Then, from the compactness of C we get that

$$\langle C(u_n,v_n),(u_n,v_n)\rangle \to \langle C(u_0,v_0),(u_0,v_0)\rangle. \tag{5.5}$$

Since $\liminf_{n\to\infty}||\bigtriangledown u_n||_p \geq ||\bigtriangledown u_0||_p$, $\liminf_{n\to\infty}||\bigtriangledown v_n||_q \geq ||\bigtriangledown v_0||_q$ and ψ is nondecreasing, we have that

$$\begin{aligned}\liminf_{n\to\infty}\quad &\{\langle J(u_n,v_n),(u_n,v_n)\rangle + \psi(\langle J(u_n,v_n),(u_n,v_n)\rangle)\}\\ &\geq \langle J(u_0,v_0),(u_0,v_0)\rangle + \psi(\langle J(u_0,v_0),(u_0,v_0)\rangle).\end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we obtain

$$\liminf_{n\to\infty} \Psi^\lambda(u_n, v_n) \geq \Psi^\lambda(u_0, v_0).$$

b) Assume that $||(u_n, v_n)||_Z \to \infty$. Actually, $\langle J(u_n, v_n), (u_n, v_n)\rangle \to \infty$. For $\langle C(u_n, v_n), (u_n, v_n)\rangle \leq 0$, holds that $\Psi^\lambda(u_n, v_n) \geq J(u_n, v_n), (u_n, v_n)\rangle \to \infty$. While, for $\langle C(u_n, v_n), (u_n, v_n)\rangle > 0$, it follows that

$$\begin{aligned}
\Psi^\lambda(u_n, v_n) &= \langle J(u_n, v_n), (u_n, v_n)\rangle - \lambda\langle C(u_n, v_n), (u_n, v_n)\rangle \\
&+ \psi(\langle J(u_n, v_n), (u_n, v_n)\rangle) \\
&= \langle J(u_n, v_n), (u_n, v_n)\rangle - \lambda_1\langle C(u_n, v_n), (u_n, v_n)\rangle \\
&+ (\lambda_1 - \lambda)\langle C(u_n, v_n), (u_n, v_n)\rangle + \psi(\langle J(u_n, v_n), (u_n, v_n)\rangle) \\
&\geq (\lambda_1 - \lambda)\langle C(u_n, v_n), (u_n, v_n)\rangle + \psi(\langle J(u_n, v_n), (u_n, v_n)\rangle) \\
&\geq -\delta\langle C(u_n, v_n), (u_n, v_n)\rangle + \psi(\langle J(u_n, v_n), (u_n, v_n)\rangle) \\
&\geq -\frac{\delta}{\lambda_1}\langle J(u_n, v_n), (u_n, v_n)\rangle + \frac{2\delta}{\lambda_1}[\langle J(u_n, v_n), (u_n, v_n)\rangle - 2K],
\end{aligned}$$

where we used the variational characterization (3.3) for λ_1 and the definition of ψ. So $\Psi^\lambda(u_n, v_n) \to \infty$ and the proof of the lemma is completed. $\square$

Lemma 5.3. *The critical points (u_1, v_1), $(-u_1, -v_1)$ of Ψ^λ are of minimum type, with $\lambda \in (\lambda_1, \lambda_1 + \delta)$.*

Proof. Lemma 5.2 implies that (see [19, Theorem 25.D]) Ψ^λ attains a minimum on Z. On the other hand, from relation (5.4) and the strict convexity of ψ on $(K, 3K)$ we have that

$$\begin{aligned}
\Psi^\lambda(u_1, v_1) &= \frac{\lambda - \lambda_1}{\lambda_1}\langle J(u_1, v_1), (u_1, v_1)\rangle + \psi(\langle J(u_1, v_1), (u_1, v_1)\rangle) \\
&< 0 = \Psi^\lambda(0, 0).
\end{aligned}$$

Since $\Psi^\lambda(u_1, v_1) = \Psi^\lambda(-u_1, -v_1)$ we obtain the conclusion. $\square$

Lemma 5.4. *The quantity $\langle(\Psi^\lambda)'(u, v), (u, v)\rangle$ is strictly positive for any $(u, v) \in Z$ with $||(u, v)||_Z > k$, for some large enough positive constant k and $\lambda \in (\lambda_1, \lambda_1 + \delta)$.*

Proof. From (5.2), we have

$$\begin{aligned}
\langle\frac{1}{p}\Psi_u^\lambda(u, v), (u, v)\rangle &= \frac{\alpha + 1}{p}\,\langle J_1(u, v) - \lambda D_1(u, v) - \lambda B_1(u, v), (u, v)\rangle \\
&+ \frac{\alpha + 1}{p}\psi'(\langle J(u, v), (u, v)\rangle)\int |\bigtriangledown u|^p. \qquad (5.7)
\end{aligned}$$

Similarly, from (5.7) we have

$$\langle\frac{1}{q}\Psi_v^\lambda(u,v),(u,v)\rangle = \frac{\beta+1}{q}\,\langle J_2(u,v)-\lambda D_2(u,v)-\lambda B_2(u,v),(u,v)\rangle + \frac{\beta+1}{q}\psi^{'}(\langle J(u,v),(u,v)\rangle)\int|\bigtriangledown v|^q. \tag{5.8}$$

Adding (5.7), (5.8) and using condition $(\mathcal{H})$ we obtain

$$\langle\frac{1}{p}\Psi_u^\lambda(u,v)+\frac{1}{q}\Psi_v^\lambda(u,v),(u,v)\rangle = \langle J(u,v),(u,v)\rangle-\lambda\langle C(u,v),(u,v)\rangle + \psi^{'}(\langle J(u,v),(u,v)\rangle)\langle J(u,v),(u,v)\rangle.$$

Assume now that $||(u_n,v_n)||_Z\to\infty$. Then $\langle J(u_n,v_n),(u_n,v_n)\rangle\to\infty$. From the variational characterization (3.3) for λ_1 and the definition of ψ we derive

$$\begin{aligned}\langle J(u,v)-\lambda C(u,v),(u,v)\rangle &+\psi^{'}(\langle J(u,v),(u,v)\rangle)\langle J(u,v),(u,v)\rangle\\ &=\langle J(u,v),(u,v)\rangle-\lambda_1(C(u,v),(u,v))+\psi^{'}(\langle J(u,v),(u,v)\rangle)\times\\ &\quad\times\left[\langle J(u,v),(u,v)\rangle-\frac{\lambda-\lambda_1}{\psi'(\langle J(u,v),(u,v)\rangle)}\langle C(u,v),(u,v)\rangle\right]\\ \geq\frac{2\delta}{\lambda_1}\Big[\langle J(u,v),(u,v)\rangle-2K\Big]&\times\Big[\langle J(u,v),(u,v)\rangle-\frac{\lambda_1}{2}\langle C(u,v),(u,v)\rangle\Big].\end{aligned}$$

Since

$$\langle\frac{1}{p}\Psi_u^\lambda(u_n,v_n)+\frac{1}{q}\Psi_v^\lambda(u_n,v_n),(u_n,v_n)\rangle\to\infty,$$

we have that

$$\langle(\Psi^\lambda)^{'}(u_n,v_n),(u_n,v_n)\rangle=\langle\Psi_u^\lambda(u_n,v_n)+\Psi_v^\lambda(u_n,v_n),(u_n,v_n)\rangle\to\infty$$

and the conclusion of the lemma follows. □

Lemma 5.5. *For the operator $A_\lambda(u,v)$ the following are true*

$$Ind(A_\lambda,0)=1,\ \lambda\in(0,\lambda_1)\quad and\quad Ind(A_\lambda,0)=-1,\ \lambda\in(\lambda_1,\lambda_1+\delta).$$

Proof. From the variational characterization (3.3) of λ_1 it follows that for any $\lambda\in(0,\lambda_1)$, we have

$$\langle\tilde{A}_\lambda(u,v),(u.v)\rangle>0,\ \text{for all }(u,v)\in Z.$$

Then the degree

$$Deg[\tilde{A}_\lambda,B_r(0),0], \tag{5.9}$$

is well defined, for any $\lambda\in(0,\lambda_1)$ and any ball $B_r(0)\subset Z$. Applying Lemma 4.3 we get

$$Deg[\tilde{A}_\lambda,B_r(0),0]=1,\quad\text{for all }\ \lambda\in(0,\lambda_1). \tag{5.10}$$

According to Theorem 3.6 there exists a $\delta > 0$ such that the interval $(\lambda_1, \lambda_1 + \delta)$ does not contain any eigenvalue of the problem (1.3), (1.4). Hence, the degree (5.9) is also well defined for all $\lambda \in (\lambda_1, \lambda_1 + \delta)$. From Lemmas 4.2 and 5.3, we have

$$Ind\big((\Psi^\lambda)', (u_1, v_1)\big) = Ind\big((\Psi^\lambda)', (-u_1, -v_1)\big) = 1. \tag{5.11}$$

In addition, Lemmas 4.3 and 5.4 imply that

$$Deg[(\Psi^\lambda)', B_k, 0] = 1. \tag{5.12}$$

We choose k so large that $(\pm u_1, \pm v_1) \in B_k(0)$. Now, by the additivity property of the degree and (5.11), (5.12) we have

$$Ind((\Psi^\lambda)', 0) = -1. \tag{5.13}$$

Furthermore, by the definition of ψ we obtain

$$Deg[(\Psi^\lambda)', B_r, 0] = Ind((\tilde{A}_\lambda, 0), \tag{5.14}$$

for $r > 0$ small enough. Then we conclude from (5.10), (5.13) and (5.14) that

$$\begin{aligned} Ind(\tilde{A}_\lambda, 0) = &\quad 1, \quad \text{for all} \quad \lambda \in (0, \lambda_1), \\ Ind(\tilde{A}_\lambda, 0) = &\quad -1, \quad \text{for all} \quad \lambda \in (\lambda_1, \lambda_1 + \delta). \end{aligned} \tag{5.15}$$

It follows from relation (2.3) and the homotopy invariance of the degree under compact perturbations that for $r > 0$ small enough

$$Deg[A_\lambda, B_r(0), 0] = Deg[\tilde{A}_\lambda, B_r(0), 0],$$

for $\lambda \in (0, \lambda_1 + \delta) \backslash \lambda_1$. Finally, from (5.15) we obtain

$$\begin{aligned} Ind(A_\lambda, 0) = &\quad 1, \quad \text{for all} \quad \lambda \in (0, \lambda_1), \\ Ind(A_\lambda, 0) = &\quad -1, \quad \text{for all} \quad \lambda \in (\lambda_1, \lambda_1 + \delta) \end{aligned}$$

and the lemma is proved. □

According to Definition 5.1 we have the following characterization concerning the existence and the geometry of some part of the solution set of the system (1.1).

Theorem 5.6. *The principal eigenvalue $\lambda_1 > 0$ of the unperturbed problem* (1.3), (1.4) *is a bifurcation point* (*in the sense of Rabinowitz*) *of the perturbed system* (1.1).

Proof. The index jump result of Lemma 5.5 and the homotopy invariance of the degree imply that $(\lambda_1, 0, 0)$ is a bifurcation point of (1.1). The rest of the proof is similar to that of Rabinowitz Theorem, see [14]. □

Finally, we discuss the sign of the solution branch close to the bifurcation point.

Proposition 5.7. *There exists $\eta > 0$ small enough, such that for each $(\lambda, u, v) \in C \cap B_\eta(\lambda_1, 0)$, we have $u(x) \geq 0$ and $v(x) \geq 0$, almost everywhere in Ω.*

Proof. Let $(\lambda_n, u_n, v_n) \in C$ be a sequence such that $(\lambda_n, u_n, v_n) \to (\lambda_1, 0, 0)$. We introduce the sequences $\tilde{u}_n$ and $\tilde{v}_n$ in the following way.

$$\tilde{u}_n =: \frac{u_n}{(||u_n||_{1,p}^p + ||v_n||_{1,q}^q)^{1/p}} \quad \text{and} \quad \tilde{v}_n =: \frac{v_n}{(||u_n||_{1,p}^p + ||v_n||_{1,q}^q)^{1/q}}.$$

The sequences $\tilde{u}_n$ and $\tilde{v}_n$ are bounded, since

$$||\tilde{u}_n||_{1,p}^p + ||\tilde{v}_n||_{1,q}^q = 1, \quad \text{for every} \ \ n \in \mathbb{N}.$$

Condition $(\mathcal{H})$ implies also that

$$\frac{|u_n|^\alpha \, |v_n|^\beta \, u_n \, v_n}{||u_n||_{1,p}^p + ||v_n||_{1,q}^q} = |\tilde{u}_n|^\alpha \, |\tilde{v}_n|^\beta \, \tilde{u}_n \, \tilde{v}_n, \tag{5.16}$$

for every $n \in \mathbb{N}$. Now, using relations (2.3) and (5.16) we have that

$$\int | \bigtriangledown \tilde{u}_n|^p = \lambda_n \int a(x)|\tilde{u}_n|^p + \lambda_n \int b(x)|\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta\tilde{u}_n\tilde{v}_n + O(||(u_n, v_n)||_Z),$$

$$\int | \bigtriangledown \tilde{v}_n|^q = \lambda_n \int d(x)|\tilde{v}_n|^q + \lambda_n \int b(x)|\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta\tilde{u}_n\tilde{v}_n + O(||(u_n, v_n)||_Z).$$

Similarly, as in the proof of Theorem 3.6 we derive that for some positive constant k, $\tilde{u}_n \to k^p \, u_1$ and $\tilde{v}_n \to k^q \, v_1$ (strongly) in the spaces $D^{1,p}$ and $D^{1,q}$, respectively. We claim that for n large enough, $u_n \geq 0$ and $v_n \geq 0$. Assume that the sets $\mathcal{U}_n^- = \{x \in \Omega : \tilde{u}_n(x) < 0\}$ and $\mathcal{V}_n^- = \{x \in \Omega : \tilde{v}_n(x) < 0\}$ are non empty. Using (2.3) we obtain that

$$1 \leq c_0\big(\max\big\{||a(x)||_{L^{\frac{N}{p}}(\Omega_n^-)}, ||d(x)||_{L^{\frac{N}{q}}(\Omega_n^-)}\big\} + ||b(x)||_{L^{\omega_1}(\Omega_n^-)}\big),$$

where $\Omega_n^- = \mathcal{U}_n^- \cup \mathcal{V}_n^-$. Since $||(u_n, v_n||_Z \to 0$, $a \in L^{N/p}(\Omega)$, $d \in L^{N/q}(\Omega)$, $b \in L^\omega(\Omega)$ and c_0 does not depend on u_n or v_n, we derive that for some $K_0 > 0$ large enough

$$|\Omega_n^- \cap B_K(0)| \geq c_1,$$

for any $K > K_0$, where $c_1 > 0$ depends neither on λ_n nor on u_n or v_n. Now, using the same argument as in the proof of Lemma 3.6 based on the Egorov's Theorem we deduce that $\tilde{u}_n$ and $\tilde{v}_n$ (and hence u_n and v_n) are nonnegative in Ω, for n large enough. Then, it follows that $u_n \geq 0$ and $v_n \geq 0$, for any $(\lambda, u_n, v_n) \in C \cap B_\eta(\lambda_1, 0)$, with $\eta > 0$ small enough. $\square$

6. Properties of the continuum C

All the above arguments were obtained by using no kind of regularity for the solutions of the system (1.1). In the sequel we consider the regularity of these solutions. Assume the following condition.

$$(\Upsilon_6)\qquad |f(\lambda,x,u,v)| \le C\,(|u|^{p-1}+|u|^{p^*-1}+|v|^{q/p'}+|v|^{rq/(rp)'}),$$
$$|g(\lambda,x,u,v)| \le C\,(|v|^{q-1}+|v|^{rq-1}+|u|^{p/q'}+|u|^{rp/(rq)'}),$$

where $m'=\frac{m}{m-1}$ denotes the conjugate of m, $r=\frac{N}{N-p}$ and C is a positive constant.

Lemma 6.1. *Assume that the condition* (Υ_6) *is satisfied and* (u,v) *be a solution of* (1.1). *Then u and v are of class $C^{1,\zeta}(B_r)$, for any $r>0$ and $\zeta=\zeta(r)\in(0,1)$. Moreover, in the unbounded domain case, both u and v decay uniformly to zero as $|x|\to\infty$.*

Proof. Observe that the following inequalities

$$|u|^{\alpha}|v|^{\beta+1}\le c(|u|^{p^*-1}+|v|^{\chi}),\;\; |u|^{\alpha+1}|v|^{\beta}\le c(|u|^{rq-1}+|v|^{\psi}),$$

hold for some $\chi<rq/(rp)'$ and $\psi<rp/(rq)'$, respectively. The rest of the proof follows from [3, Theorem 2.1]. □

As it will be clear later, it is convenient to recall the following eigenvalue problem.

$$-\Delta_p u \;= \lambda\, g(x)|u|^{p-2}u,\quad x\in\Omega, \tag{6.1}$$

$$0<u(x),\quad x\in\Omega, \tag{6.2}$$

where $g(x)$ satisfy certain conditions. It is known, that to the equation (6.1) corresponds a positive principal eigenvalue $\lambda_{p,g}$. This eigenvalue is simple and is the only one to which corresponds a positive eigenfunction $\phi_{p,g}$. For details we refer to the works [2, 10].

Lemma 6.2. (i) (Local Bifurcation) *The only possible points of the form* $(\lambda,0,0)$*, which the closure of the continuum $\bar{C}$ may contain, are the points* $(\lambda_{p,a},0,0)$ *or* $(\lambda_{q,d},0,0)$.

(ii) (Bifurcation from semitrivial solutions) *The only possible points of the form* $(\lambda,u,0)$, $u\not\equiv 0$ (*or* $(\lambda,0,v)$, $v\not\equiv 0$)*, which $\bar{C}$ may contain, are the points* $(\lambda_{p,a},c\,\phi_{p,a},0)$ (*or* $(\lambda_{q,d},0,c\,\phi_{q,d})$*, respectively*)*, for some real constant* $c\neq 0$.

(iii) *If $\bar{C}$ contains no point of the form* $(\lambda,0,0)$, $(\lambda,u,0), u\not\equiv 0$ *and* $(\lambda,0,v)$, $v\not\equiv 0$*, then every solution* (u,v) *in C is strictly positive* (*componentwise*).

Proof. (i) From Lemma 5.7 we may suppose that there exists a sequence $\{(\lambda_n, u_n, v_n)\} \subseteq C$, such that $u_n(x) > 0$, $v_n(x) > 0$, for all $n \in \mathcal{N}$ and $x \in \Omega$, $u_n \to 0$ in $D^{1,p}$, $v_n \to 0$ in $D^{1,q}$ and $\lambda_n \to \lambda_0$. Then, we construct the sequences

$$\tilde{u}_n = \frac{u_n}{(||u_n||^p_{1,p} + ||v_n||^q_{1,q})^{1/p}} \quad \text{and} \quad \tilde{v}_n = \frac{v_n}{(||u_n||^p_{1,p} + ||v_n||^q_{1,q})^{1/q}}.$$

Repeating the same argument as in Lemma 5.7, we may obtain that $(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}_0, \tilde{v}_0)$ (strongly) in Z, for some $(\tilde{u}_0, \tilde{v}_0) \in Z$, such that $\tilde{u}_0 \geq 0$ and $\tilde{v}_0 \geq 0$, satisfying, also, the following equations.

$$\int | \bigtriangledown \tilde{u}_0|^p = \lambda_0 \int a(x)|\tilde{u}_0|^p + \lambda_0 \int b(x)|\tilde{u}_0|^\alpha |\tilde{v}_0|^\beta \tilde{u}_0 \tilde{v}_0,$$

$$\int | \bigtriangledown \tilde{v}_0|^q = \lambda_0 \int d(x)|\tilde{v}_0|^q + \lambda_0 \int b(x)|\tilde{u}_0|^\alpha |\tilde{v}_0|^\beta \tilde{u}_0 \tilde{v}_0.$$

Finally, one of the following must occur.

(a) $\tilde{u}_0 \equiv 0$ and $\tilde{v}_0 \equiv \phi_{q,d}$, so that the closure of the continuum C contains the point $(\lambda_{q,d}, 0, 0)$, or
(b) $\tilde{u}_0 \equiv \phi_{p,a}$ and $\tilde{v}_0 \equiv 0$, so that $\bar{C}$ contains the point $(\lambda_{p,a}, 0, 0)$, or
(c) $\tilde{u}_0 \equiv k^{1/p} u_1$ and $\tilde{v}_0 \equiv k^{1/q} v_1$, so that $\bar{C}$ contains no point of the form $(\lambda, 0, 0)$.

(ii) The proof follows the same steps of (i).

(iii) Suppose that $\bar{C}$ contains no point of the form $(\lambda, 0, 0)$ and for some solution $(\lambda, u, v) \in C$ there exists a point $x_0 \in \Omega$, such that $u(x_0) < 0$ (the same will apply, if we assume that $v(x_0) < 0$). By Lemma 5.7, we may observe that $u(x) > 0$, $x \in \Omega$, for all solutions $(\lambda, u, v) \in C$ close to $(\lambda_1, 0, 0)$. Since the continuum C is connected, the $C^{1,a}_{loc}$- regularity of the solutions implies that there exists $(\lambda_0, u_0, v_0) \in C$, such that $u_0(x) \geq 0$, for all $x \in \Omega$, except some point $x_0 \in \Omega$, such that $u_0(x_0) = 0$ and in any neighborhood of (λ_0, u_0, v_0) we can find a point $(\hat{\lambda}, \hat{u}, \hat{v}) \in C$, with $\hat{u}(x) < 0$, for some $x \in \Omega$. Let B denote any open ball containing x_0. Then from Vazquez' Maximum principle (see [18]), it follows that $u_0 \equiv 0$ on B. Hence, $u_0 \equiv 0$ on Ω. Thus, we may construct a sequence $\{(\lambda_n, u_n, v_n)\} \subseteq C$, such that $u_n(x) > 0$ and $v_n(x) > 0$, for all $n \in \mathcal{N}$ and $x \in \Omega$, $u_n \to 0$ in $D^{1,p}$, $v_n \to v_0$ in $D^{1,q}$ and $\lambda_n \to \lambda_0$. Then, the continuum C contains a point of the form $(\lambda_0, 0, v_0)$, which is a contradiction. Similar results may be obtained for v. $\square$

Applying the previous results we are ready to state the main result of this section in the general case.

Theorem 6.3. *Suppose that the hypothesis* $(\mathcal{H})$, $(\mathcal{F})$, $(\mathcal{P})$ *and* $(\Upsilon_1) - (\Upsilon_6)$ *hold. Then, there exists a continuum* $C \subseteq \mathbb{R} \times Z$ *of uniformly bounded solutions of the problem* (1.1) *bifurcating from the zero solution at* $(\lambda_1, 0, 0)$, *such that one of the following alternatives hold.*

(*i*) *The continuum* $\bar{C}$ *(in closure) contains one of the points* $(\lambda_{p,a}, 0, 0)$ *or* $(\lambda_{q,d}, 0, 0)$, *and in particular problem* (1.1) *has a nontrivial positive (componentwise) solution* $(u, v) \in Z$, *whenever* λ *is between* λ_1 *and* $\lambda_{p,a}$ *or* $\lambda_{q,d}$.

(*ii*) *The continuum* C *is unbounded and every* (u, v) *in* C *is strictly positive* (*componentwise*).

In the remaining part of this section we study a special case of the system (1.1), for which the first alternative of Theorem 6.3 is excluded. We assume the following hypothesis.

(Υ_7) $a(x) \equiv 0$ and $d(x) \equiv 0$. (see Remark 3.7).

For this kind of systems we have the following bifurcation theorem.

Theorem 6.4. *Suppose that the hypothesis* $(\mathcal{H})$, $(\mathcal{F})$, $(\mathcal{P})$ *and* $(\Upsilon_4) - (\Upsilon_7)$ *hold. Then, there exists a continuum* $C \subseteq \mathbb{R} \times Z$ *of solutions of the problem* (1.1) *bifurcating from the zero solution at* $(\lambda_1, 0, 0)$, *such that* C *is unbounded and every* (u, v) *in* C *is strictly positive* (*componentwise*).

Proof. Following the same lines as in Lemma 6.2, we may prove that $\bar{C}$ contains no points of the form $(\lambda, 0, 0)$, $(\lambda, u, 0)$, $u \not\equiv 0$ and $(\lambda, 0, v)$, $v \not\equiv 0$. Hence, C must be unbounded, so that every solution (u, v) is strictly positive (componentwise). □

Remark 6.5. Suppose that $\bar{C}$ contains no points of the form $(\lambda, 0, 0)$, e.g., the assumptions of the Theorem 6.4 are happening. Suppose also, that there exists some $\lambda_0 > \lambda_1$, such that $\sigma_i(\lambda_0)$, $i = 1, 2, 3, 4$. Since the system

$$\begin{aligned} -\Delta_p u &= \lambda_0\, a(x)\, |u|^{p-2} u + \lambda_0\, b(x)\, |u|^{\alpha}\, |v|^{\beta}\, v, \quad x \in \Omega, \\ -\Delta_q v &= \lambda_0\, d(x)\, |v|^{q-2}\, v + \lambda_0\, b(x)\, |u|^{\alpha}\, |v|^{\beta}\, u, \quad x \in \Omega, \end{aligned}$$

cannot have a positive solution in Ω, corresponding to λ_0, we conclude that C must "blow up" in $||(u, v)||_Z$. This result is the analogous one of the equation, see [6, Remark 4.7].

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References

[1] R.A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.

[2] W. Allegretto, and Y.X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Analysis TMA, 32 (1998), 819-830.

[3] A. Bechah, *Local and global estimates for solutions of systems involving the p-Laplacian in unbounded domains*, Electr. J. Dif. Eq., 19 (2001), 1-14.

[4] P.A. Binding and Y.X. Huang, *Linked eigenvalue problems for the p-Laplacian*, Proc. R. Soc. Edinburgh, 124A (1994), 1023-1036.

[5] J. Björn, *Boundness and differentiability for nonlinear elliptic systems*, Trans. Amer. Math. Soc., 353 (2001), 4545-4565.

[6] P.Drábek and Y.X. Huang *Bifurcation problems for the p-Laplacian in* $\mathbb{R}^N$, Trans. Amer. Math. Soc., 349 (1997), 171-188.

[7] P.Drábek and N.M. Stavrakakis, *Bifurcation from the first eigenvalue of some nonlinear elliptic operators in Banach spaces*, Nonlinear Analysis TMA, 42 (2000), 561-572.

[8] Y. Du, *Bifurcation from semitrivial solutions bundles and applications to certain equation systems*, Nonlinear Analysis TMA, 27 (1996), 1407-1435.

[9] J. Fleckinger, R. Manasevich, and F. de Thelin, *Global bifurcation from the first eigenvalue for a system of p-Laplacians*, Math. Nach., 182 (1996), 217-242.

[10] J. Fleckinger, R. Manasevich, N.M. Stavrakakis, and F. de Thelin, *Principal eigenvalues for some quasilinear elliptic equations on* $\mathbb{R}^N$, Adv. Diff Equ., 2 (1997), 981-1003.

[11] J. Fleckinger and R. Pardo, *Bifurcation for an elliptic system coupled in the linear part*, Nonlinear Analysis TMA, 37 (1999), 13-30.

[12] J. Fleckinger, R. Pardo, and F. de Thelin, *Four parameter bifurcation for a p-Laplacian system*, Electr. J. Dif. Eq., 6 (2001), 1-15.

[13] J. Hulshof, E. Mitidieri, and R. Vandervorst, *Strongly Indefinite Systems with Critical Sobolev Exponents*, Trans. Amer. Math. Soc., 350 (1998), 2349-2365.

[14] P. Rabinovitz, *Some global results for nonlinear eigenvalue problems*, J. Funct Anal., 7 (1971), 487-513.

[15] N M Stavrakakis and F. de Thelin, *Principal eigenvalues for some quasilinear elliptic equations on* $\mathbb{R}^N$, Math. Nach., 212 (2000), 155-171.

[16] N.M. Stavrakakis and N.B. Zographopoulos, *Existence results for some quasilinear elliptic systems in* $\mathbb{R}^N$, Electr. J. Dif. Eq., 39 (1999), 1-15.

[17] N.M. Stavrakakis and N.B. Zographopoulos, *Multiplicity and regularity results for some quasilinear elliptic systems on* $\mathbb{R}^N$, Nonlinear Analysis TMA, 50 (2002), 55-69.

[18] J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12 (1984), 191-202.

[19] E. Zeidler, "Nonlinear Functional Analysis and its Applications," Vol. II/B: Nonlinear Monotone Operators, Springer-Verlag, Berlin, 1990.