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ESTIMATES ON THE DIMENSION OF A GLOBAL ATTRACTOR FOR A SEMILINEAR DISSIPATIVE WAVE EQUATION ON $\mathbb{R}^N$

Nikos I. Karachalios

Department of Statistics and Actuarial Science, University of the Aegean,
Karlovassi 83200, Samos, GREECE

Nikos M. Stavrakakis

Department of Mathematics, National Technical University,
Zografos Campus 15780, Athens, GREECE

Abstract. We discuss estimates of the Hausdorff and fractal dimension of a global attractor for the semilinear wave equation

$$u_{tt} + \delta u_t - \phi(x)\Delta u + \lambda f(u) = \eta(x), \quad x \in \mathbb{R}^N, \quad t \geq 0,$$

with the initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, where $N \geq 3$, $\delta > 0$ and $(\phi(x))^{-1} := g(x)$ lies in $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The energy space $\mathcal{X}_0 = \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$ is introduced, to overcome the difficulties related with the non-compactness of operators, which arise in unbounded domains. The estimates on the Hausdorff dimension are in terms of given parameters, due to an asymptotic estimate for the eigenvalues μ of the eigenvalue problem $-\phi(x)\Delta u = \mu u, \ x \in \mathbb{R}^N$.

1. **Introduction.** In this paper we are concerned with estimates on the dimension of a global attractor for the semilinear hyperbolic initial value problem

$$u_{tt} + \delta u_t - \phi(x)\Delta u + \lambda f(u) = \eta(x), \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x) \ \text{ and } \ u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with the initial conditions $u_0(x)$, $u_1(x)$ in appropriate function spaces, $N \geq 3$ and $\delta > 0$. Models of this type arise mainly in wave phenomena of various areas in mathematical physics (see [2, 20, 28, 33]). Throughout the paper we assume that the functions ϕ, g, $\eta : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the following conditions:

$(\mathcal{G})$ $\phi(x) > 0$, for all $x \in \mathbb{R}^N$, $(\phi(x))^{-1} := g(x)$ is a smooth function and $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Functions ϕ of this type arise in wave phenomena involving a slowly varying wave speed (e.g. see [33, p. 632]),

$(\mathcal{H})$ $\eta \in L^2_g(\mathbb{R}^N)$,

$(\mathcal{F})$ $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f(0) = 0$. Furthermore, $|f(s)| \leq c^*|s|$ and $|f'(s)| \leq c_2|s|$, where c^*, c_2 are positive constants.

In certain cases we shall impose some extra conditions on f, which are

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$(\mathcal{F}_\infty)$ f' is in $L^\infty(\mathbb{R})$,
$(\mathcal{F}_\delta)$ There exist $\delta_0 \in (0, 2/(N-2))$ such that $|f'(s_1) - f'(s_2)| \leq C|s_1 - s_2|^{\delta_0}$.

For the existence of attractors of evolution equations in the bounded domain case we refer to the monographs of A V Babin and M I Vishik [3], J K Hale [16], O A Ladyzenskaja [21], R Temam [31]. For the unbounded domain case, among other contributions, we refer to the works of F Abergel [1], A V Babin and M I Vishik [4], E Feiresl, Ph Laurençot, F Simondon, H Touré [14], R Rosa [30] for a class of parabolic and Navier-Stokes equations and of J Ball [5], E Feiresl [12, 13] for semilinear damped wave equations. The recent works of Belleri-Pata and Zelik, [6, 34], consider nonautonomous semilinear damped wave equations: Paper [6], studies a strongly damped equation in $\mathbb{R}^3$. In the case of nonautonomous exterior force, the corresponding semigroup possesses an absorbing set, while in the case of the autonomous system, the authors prove existence of global attractor in the usual phase space, via an appropriate cut-off decomposition. The work [34], extends the weighted Sobolev space-setting of [4], in order to show existence of finite dimensional global attractor for the nonautonomous damped wave equation in $\mathbb{R}^N$.

Questions concerning global existence, and blow-up of solutions for nonlinear wave equations, in bounded or unbounded domains, treated in the recent works of H A Levine and J Serrin [23], H A Levine, S R Park, J Serrin [24], P Pucci and J Serrin [27], G Todorova [32] and to [18], [19], for a problem similar to (1.1)-(1.2), involving blow-up type nonlinearities.

For the wave equation (1.1)-(1.2), it is unclear a priori, which is the appropriate phase space. In [17], homogeneous Sobolev spaces were introduced, for the study of the asymptotic behavior of solutions, of (1.1)-(1.2). This space setting, proved to be the natural one, for the treatment of the unbounded diffusion coefficient, and the unbounded domain. Although weighted L^p-spaces are involved, it is not restrictive, for the initial data and the exterior forces. The main result in [17], is the existence of a global attractor. In the present work, which could be considered as a continuation of [17], we derive an estimate of the Hausdorff and fractal dimension, of the global attractor. At this point we would like to note, that for dissipative evolution equations considered in unbounded domains, the question of finite dimensionality of the global attractor is nontrivial, see [34]: Even in the autonomous case, the global attractor can be infinite dimensional. This is in contrast with the bounded domain case, where infinite dimensional attractors appear only in the nonautonomous systems.

The estimates of the Hausdorff dimension of the global attractor, are in terms of the given parameters and are based on the fundamental results of Constantin-Foias-Temam, on the dimension of functional invariant sets, and the asymptotic distribution of eigenvalues of the differential operator $-\phi\Delta$ in $\mathbb{R}^N$, acting in an appropriate weighted L^2-space (see [8], [22],[26] and references therein, for the asymptotic distribution of eigenvalues of various differential operators).

Notation: We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for simplicity reasons we use the symbols $D(A)$, L^p, $1 \leq p \leq \infty$, $\mathcal{D}^{1,2}$, respectively for the spaces $D(A)(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, $\mathcal{D}^{1,2}(\mathbb{R}^N)$, respectively; $||.||_p$ for the norm $||.||_{L^p(\mathbb{R}^N)}$. By $(\cdot,\cdot)_{L^2_g}$, $(\cdot,\cdot)_{\mathcal{D}^{1,2}}$ we denote the scalar products of the corresponding spaces. The constants C or c are considered in a generic sense. The end of the proofs is marked by "$\diamond$".

2. **Preliminary Results.** For later use, in this section we copy some basic results from [17]. We also give the asymptotic estimate for the eigenvalues of the differential operator, that appears in the equation. The phase space of the problem (1.1)-(1.2), is the product space $\mathcal{X}_0 = \mathcal{D}^{1,2} \times L^2_g$. By $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we define the closure of the $C_0^\infty(\mathbb{R}^N)$ functions with respect to the "energy norm" $||u||_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} | \bigtriangledown u|^2 dx$. It is well known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \bigtriangledown u \in (L^2(\mathbb{R}^N))^N \right\},$$

and that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ can be embedded continuously in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, i.e., there exists $C_E > 0$ such that

$$||u||_{\frac{2N}{N-2}} \leq C_E ||u||_{\mathcal{D}^{1,2}}. \tag{2.1}$$

The best possible embedding constant for (2.1), is ([11, p. 533], and references therein)

$$C_E = \frac{1}{4} N(N-2) \left\{ N\omega_N \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{N}{2}\right) \Gamma\left(\frac{N+1}{2}\right) \right\}^{\frac{2}{N}},$$

where $\omega_N = \pi^{\frac{N}{2}} / \Gamma\left(1 + \frac{N}{2}\right)$, is the volume of the unit ball in $\mathbb{R}^N$. We shall frequently use, the following generalized Poincaré inequality (see [7, Lemma 2.1])

$$\int_{\mathbb{R}^N} | \bigtriangledown u|^2 dx \geq \alpha \int_{\mathbb{R}^N} g u^2 dx, \tag{2.2}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. It is found that $\alpha = C_E^{-2} ||g||^{-1}_{N/2}$.

It can be shown (see [7, Lemma 2.2]), that $\mathcal{D}^{1,2}$ is a separable Hilbert space. The weighted Lebesque space $L^2_g(\mathbb{R}^N)$, is defined to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions, with respect to the inner product

$$(u, v)_{L^2_g} =: \int_{\mathbb{R}^N} g\, u\, v\, dx.$$

Clearly, $L^2_g(\mathbb{R}^N)$ is a separable Hilbert space. The following lemma is crucial for the analysis of the problem.

Lemma 2.1. *Suppose that $g \in L^{N/2} \bigcap L^\infty$. Then $\mathcal{D}^{1,2}$ is compactly embedded in L^2_g.*

Proof Let $\{u_n\}$ be a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then there exists a constant $c > 0$, such that for all positive integers m, n and any $R > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} g(u_n - u_m)^2 dx &\leq ||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(\mathbb{R}^N)} ||(u_n - u_m)||_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \\ &\leq c\{||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(\mathbb{R}^N \setminus B_R)} + ||g(u_n - u_m||_{L^{\frac{2N}{N+2}}(B_R)}\}. \end{aligned}$$

Since $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $\{u_n\}$ is bounded in $H^1(B_R)$. By a diagonalization procedure, we can find a subsequence (denoted again by $\{u_n\}$), which converges in $L^{\frac{2N}{N+2}}(B_R)$, for all $R > 0$. On the other hand,

$$||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(\mathbb{R}^N \setminus B_R)} \leq ||g||_{L^{\frac{N}{2}}(\mathbb{R}^N \setminus B_R)} ||u_n - u_m||_{L^{\frac{2N}{N-2}}(\mathbb{R}^N \setminus B_R)}.$$

Since $g \in L^{\frac{N}{2}} \cap L^\infty$, we can make $||g||_{L^{\frac{N}{2}}(\mathbb{R}^N \setminus B_R)}$ as small as we please, by choosing R sufficiently large. Hence there exist $\epsilon > 0$

$$\int_{\mathbb{R}^N} g(u_n - u_m)^2 dx \leq \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

for m and n sufficiently large. Therefore $\{u_n\}$ is a Cauchy sequence in $L_g^2(\mathbb{R}^N)$. $\diamond$

To analyze the properties of the operator $-\phi\Delta$ in the space setting described above, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x),\ x \in \mathbb{R}^N \tag{2.3}$$

without a boundary condition, as an operator equation

$$A_0 u = \eta,\ \ A_0 : D(A_0) \subseteq L_g^2(\mathbb{R}^N) \to L_g^2(\mathbb{R}^N), \tag{2.4}$$

where $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^N)$ and $\eta \in L_g^2(\mathbb{R}^N)$. In [17], it is shown by using inequality (2.2), that the assumptions of the *Friedrichs' extension theorem* (see [35, Theorem 19.C]) are satisfied. The *energetic scalar product* is

$$(u,\ v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v dx,$$

and the *energetic space* X_E is defined as the completion of $D(A_0)$ with respect to $(u,\ v)_E$, i.e., the energetic space coincides with the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The *energetic extension* $A_E = -\phi\Delta$ of A_0, is defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The *Friedrichs' extension* A of A_0, is defined as the restriction of the energetic extension A_E, to the Hilbert space $D(A)$, endowed with the graph scalar product

$$(u,\ v)_{D(A)} = (u,\ v)_{L_g^2} + (Au,\ Av)_{L_g^2},\ \text{ for all }\ u,\ v \in D(A).$$

The norm induced by the scalar product $(u,\ v)_{D(A)}$ is

$$||u||_{D(A)} = \left\{ \int_{\mathbb{R}^N} g|u|^2\, dx + \int_{\mathbb{R}^N} \phi|\Delta u|^2\, dx \right\}^{\frac{1}{2}},$$

which is equivalent to the norm

$$||Au||_{L_g^2} = \left\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2\, dx \right\}^{\frac{1}{2}}.$$

Moreover, as a consequence of Lemma 2.1, we obtain the following embedding relations

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^2(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.5}$$

which are compact and dense. It follows from the compactness of the embeddings in (2.5), that for the eigenvalue problem

$$-\phi(x)\Delta u = \mu u,\ \ x \in \mathbb{R}^N, \tag{2.6}$$

there exists a complete system of eigensolutions $\{w_n,\ \mu_n\}$, satisfying the following relations

$$\begin{cases} -\phi\Delta w_j = \mu_j w_j, & j = 1,2,..., \quad w_j \in D(A), \\ 0 < \mu_1 \le \mu_2 \le ..., & \mu_j \to \infty, \quad \text{as } j \to \infty. \end{cases}$$

The eigenfunctions $w_j,\ j = 1,2,...,$ belong to the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, and are also eigenfunctions of the *weak eigenvalue problem*

$$\int_{\mathbb{R}^N} \nabla u \nabla v\, dx = \mu \int_{\mathbb{R}^N} guv\, dx,\ \ v \in D^{1,2}(\mathbb{R}^N),\ \ \forall\ u \in C_0^\infty(\mathbb{R}^N). \tag{2.7}$$

Note that the eigenfunctions $w_j,\ j = 1,2,...,$ of problem (2.6) (or (2.7)), can be arranged to form a complete orthonormal system in $L_g^2(\mathbb{R}^N)$. Concerning the asymptotic behavior of the eigenfunctions of problem (2.6), every solution u of (2.6) is such that

$$|u(x)| \to 0, \text{ as } |x| \to \infty, \tag{2.8}$$

(see [7, Theorem 3.2]). For the eigenvalues of the problem (2.6) the following result holds:

Proposition 2.2. *The eigenvalues of (2.6) satisfy the asymptotic estimate*

$$\mu_j \geq K\, j^{\frac{2}{N}}, \quad K = \frac{1}{\mathrm{e}\, C_E ||g||_{N/2}} \tag{2.9}$$

Proof The result follows by adaptation of [11, p. 531-533] in the case where $\Omega = \mathbb{R}^N$. We consider the function

$$\mathrm{H}(x,y,t) = \sum_{i=1}^{\infty} \mathrm{e}^{-\mu_i t} w_i(x) w_i(y), \quad x,\, y \in \mathbb{R}^N, \; t > 0, \tag{2.10}$$

Taking into account (2.8), we observe that $\mathrm{H}(x,y,t)$ satisfies in the weak sense, the properties

$$\left(\phi(y)\Delta_y - \frac{\partial}{\partial t}\right) \mathrm{H}(x,y,t) \;=\; 0, \tag{2.11}$$

$$\mathrm{H}(x,y,t) \;>\; 0, \text{ in } \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty), \tag{2.12}$$

$$\lim_{|x|^2+|y|^2 \to \infty} \mathrm{H}(x,y,t) \;=\; 0, \; t \in (0,\infty). \tag{2.13}$$

Since $w_i, \; i = 1,2,...,$ can be arranged to form a complete orthonormal system in $L^2_g(\mathbb{R}^N)$, the function $\mathrm{h}(t) = \sum_{i=1}^{\infty} \mathrm{e}^{-2\mu_i t}$, can be written as

$$\mathrm{h}(t) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathrm{H}^2(x,y,t) g(x) g(y) dx dy.$$

By using (2.11)-(2.13) we get

$$\frac{\partial \mathrm{h}}{\partial t} = -2 \int_{\mathbb{R}^N} g(x) \int_{\mathbb{R}^N} |\bigtriangledown_y \mathrm{H}(x,y,t)|^2 dy dx, \tag{2.14}$$

and by Hölder's inequality

$$\begin{aligned} \mathrm{h}(t) \;\leq\; & \left[\int_{\mathbb{R}^N} g(x) \left(\int_{\mathbb{R}^N} \mathrm{H}^{\frac{2N}{N-2}}(x,y,t) dy\right)^{\frac{N-2}{N}} dx\right]^{\frac{N}{N+2}} \\ & \times \left[\int_{\mathbb{R}^N} g(x) \left(\int_{\mathbb{R}^N} \mathrm{H}(x,y,t) g^{\frac{N+2}{4}}(y) dy\right)^2 dx\right]^{\frac{2}{N+2}}. \end{aligned} \tag{2.15}$$

The function

$$\mathrm{Q}(x,t) = \int_{\mathbb{R}^N} \mathrm{H}(x,y,t) g^{\frac{N+2}{4}}(y) dy,$$

satisfies in the weak sense the properties

$$\left(\phi(x)\Delta_x - \frac{\partial}{\partial t}\right) \mathrm{Q}(x,t) \;=\; 0, \tag{2.16}$$

$$\lim_{|x| \to \infty} \mathrm{Q}(x,t) \;=\; 0, \; t \in (0,\infty), \tag{2.17}$$

$$\mathrm{Q}(x,0) \;=\; g^{\frac{N-2}{4}}(x). \tag{2.18}$$

From (2.16), (2.17), we obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^N} g(x)\mathrm{Q}^2(x,t)dx \leq 0,$$

which with (2.18) implies that

$$\int_{\mathbb{R}^N} g(x)\mathrm{Q}^2(x,t)dx \leq \int_{\mathbb{R}^N} g(x)\mathrm{Q}^2(x,0)dx = \int_{\mathbb{R}^N} g^{\frac{N}{2}}(x)dx. \tag{2.19}$$

By using (2.1), (2.15), (2.19) and (2.14), we derive the inequality

$$\begin{aligned} ||g||_{N/2}^{-1}\mathrm{h}^{\frac{N+2}{N}}(t) &\leq C_E \int_{\mathbb{R}^N} g(x)\int_{\mathbb{R}^N} |\bigtriangledown_y \mathrm{H}(x,y,t)|^2 dydx = -\frac{C_E}{2}\frac{\partial \mathrm{h}}{\partial t}, \quad \text{or} \\ \mathrm{h}^{-\frac{N+2}{N}}\,d\mathrm{h} &\leq -\frac{2}{C_E||g||_{N/2}}\,dt. \end{aligned} \tag{2.20}$$

Integration of (2.20) shows that

$$\sum_{i=1}^{\infty} \mathrm{e}^{-2\mu_i t} = \mathrm{h}(t) \leq \left(\frac{4}{C_E N||g||_{N/2}}\right)^{-\frac{N}{2}} t^{-\frac{N}{2}}.$$

For $t = N/4\mu_j$, the last inequality implies that

$$\begin{aligned} j\mathrm{e}^{\frac{-N}{2}} \leq \sum_{i=1}^{\infty}\exp\left(-\frac{N\mu_i}{2\mu_j}\right) &\leq \mu_j^{\frac{N}{2}}\left(\frac{N}{4}\right)^{-\frac{N}{2}}\left(\frac{4}{C_E N||g||_{N/2}}\right)^{-\frac{N}{2}} \\ &\leq \mu_j^{\frac{N}{2}}\left(\frac{1}{C_E||g||_{N/2}}\right)^{-\frac{N}{2}}. \diamond \end{aligned}$$

We use the *evolution triple* (2.5), to give the following definition of *weak solution* for the problem (1.1)-(1.2).

Definition 2.3. *Let η satisfy $(\mathcal{H})$ and $\{u_0, u_1\} \in \mathcal{X}_0$. A* weak solution *of (1.1)-(1.2) is a function $u(x,t)$ such that*
(i) $u \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)]$, $u_t \in L^2[0,T;L^2_g(\mathbb{R}^N)]$, $u_{tt} \in L^2[0,T;\mathcal{D}^{-1,2}(\mathbb{R}^N)]$,
(ii) for all $v \in C_0^\infty([0,T]\times\mathbb{R}^N)$, satisfies the generalized formula

$$\begin{aligned} \int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g}\,d\tau &+ \delta\int_0^T (u_t(\tau), v(\tau))_{L^2_g}\,d\tau + \int_0^T\int_{\mathbb{R}^N} \bigtriangledown u(\tau)\bigtriangledown v(\tau)\,dxd\tau \\ &+ \lambda\int_0^T (f(u(\tau)),\, v(\tau))_{L^2_g}\,d\tau = \int_0^T (\eta, v)_{L^2_g(\mathbb{R}^N)}\,d\tau. \end{aligned}$$

Remark 2.4. *We may see by using a density argument, that the generalized formula (2.21) is satisfied for every $v \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)]$. Moreover it can be shown that the above Definition 2.3 of the weak solution implies that*

$$u \in C[0,T;\ \mathcal{D}^{1,2}(\mathbb{R}^N)] \ \ \textit{and}\ \ u_t \in C[0,T;\ L^2_g(\mathbb{R}^N)].$$

Finally, we recall the basic results proved in [17].

Theorem 2.5. *Suppose that the constants $T > 0$, $\delta > 0$ and the initial conditions*

$$u_0(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N) \ \ \textit{and}\ \ u_1(x) \in L^2_g(\mathbb{R}^N), \tag{2.21}$$

are given. Then for the problem (1.1)-(1.2) there exists a (weak) solution such that

$$u \in C[0,T;\ \mathcal{D}^{1,2}(\mathbb{R}^N)] \ \ \textit{and}\ \ u_t \in C[0,T;\ L^2_g(\mathbb{R}^N)].$$

Furthermore, the (weak) solution is unique if (i) $N = 3, 4$ or (ii) $N \geq 5$ and f' satisfies $(\mathcal{F}_\infty)$.

Theorem 2.6. *Let g satisfy $(\mathcal{G})$, η satisfy $(\mathcal{H})$, and f satisfy $(\mathcal{F})$ and $(\mathcal{F}_\infty)$. If*

$$\lambda < \min\left(\frac{\alpha^{1/2}\delta}{4c^*}, \left(\frac{\alpha\mu_1}{8}\right)^{1/2}\frac{1}{c^*}\right), \tag{2.22}$$

the dynamical system associated to the problem (1.1), (1.2), possesses a global attractor $\mathcal{A} = \omega(\mathcal{B}_0)$, which is compact, connected and maximal among the functional invariant sets in $\mathcal{X}_0$.

Note that in the absence of an external force $\eta(x)$, the existence of an absorbing set in $\mathcal{X}_0$ may be shown for all $\lambda > 0$, if the functions g, f satisfy the following *pseudocoercivity hypothesis*

$$\liminf_{||\phi||_{\mathcal{D}^{1,2}}\to\infty} \frac{\int_{\mathbb{R}^N} g(x)F(\phi)dx}{||\phi||^2_{\mathcal{D}^{1,2}}} \geq 0,$$

$$\liminf_{||\phi||_{\mathcal{D}^{1,2}}\to\infty} \frac{\int_{\mathbb{R}^N} g(x)f(\phi)\phi dx - C\int_{\mathbb{R}^N} g(x)F(\phi)dx}{||\phi||^2_{\mathcal{D}^{1,2}}} \geq 0,$$

for some $C > 0$, where $F(s) = \int_0^s f(s)ds$.

3. **The Hausdorff Dimension of the Global Attractor in $\mathcal{X}_0$.** In this section we prove that the global attractor given in Theorem 2.6, is finite dimensional. We follow the standard methods appearing in [15], [31]. By Theorem 2.5 we may associate to the problem (1.1)-(1.2), the semigroup of operators $\mathcal{S}(t) : \mathcal{X}_0 \longmapsto \mathcal{X}_0$ by

$$\mathcal{S}(t) : \{u_0, u_1\} \longmapsto \{u(t), u_t(t)\}.$$

Denote by $\varepsilon_0 = \min(\delta/4, \mu_1/2\delta)$. Then for any $\varepsilon \in (0, \varepsilon_0]$, we consider the semigroup of operators $\mathcal{S}_\varepsilon(t) := \mathcal{R}_\varepsilon\mathcal{S}(t)\mathcal{R}_{-\varepsilon}$, defined by

$$\mathcal{S}_\varepsilon(t) : \{u_0, v_1 = u_1 + \varepsilon u_0\} \longmapsto \{u(t), v(t) = u_t(t) + \varepsilon u(t)\}.$$

The operator $\mathcal{R}_\alpha, \alpha \in \mathbb{R}$, is an isomorphism of $\mathcal{X}_0$, given by the formula

$$\mathcal{R}_\alpha : \{x, y\} \to \{x, y + \alpha x\}, \quad \text{for any } x, y \in \mathcal{X}_0.$$

It is easy to see that the new operators $\mathcal{S}_\varepsilon(t)$ form a group, and any result concerning the semigroup $\mathcal{S}(t)$ can be directly assigned to $\mathcal{S}_\varepsilon(t)$, i.e. if $\mathcal{A}$ is the maximal attractor defined by Theorem 2.6 for $\mathcal{S}(t)$, then $\mathcal{R}_\varepsilon\mathcal{A}$ is the maximal attractor for $\mathcal{S}_\varepsilon(t)$. To prove the first result of this section, which is the differentiability of the semigroup $\mathcal{S}(t)$, we need two lemmas. The first lemma is a restatement of an interpolation result, phrased as follows.

Lemma 3.1. *Let $0 < \delta_0 < \frac{2}{N-2}$. Then we have that*

$$||u||_{L_g^{2(\delta_0+1)}} \leq C||u||_{\mathcal{D}^{1,2}}.$$

Proof From [25], [29] we have the interpolation inequality

$$||u||_{L_g^a} \leq ||u||^{1-\rho}_{L_g^b}||u||^\rho_{L_g^{p^*}},$$

where $b \leq a \leq p^* = 2N/(N-2)$ and $\rho \in (0,1)$, satisfy the relation $1/a = (1-\rho)/b + \rho/p^*$. We set $a = 2(\delta_0 + 1)$, $b = 2$ and we find $\rho = N\delta_0/2(\delta_0 + 1)$. This value of ρ is in $(0, 1)$ if $0 < \delta_0 < 2/(N-2)$. From (2.1), (2.2) we have that

$$\begin{aligned} ||u||_{L_g^{2(\delta_0+1)}} \leq ||u||^{1-\rho}_{L_g^2}||u||^\rho_{L_g^{p^*}} &\leq (\alpha^{-1})^{1-\rho}||u||^{1-\rho}_{\mathcal{D}^{1,2}}||g||^\rho_\infty||u||^\rho_{\frac{2N}{N-2}} \\ &= (k^2||g||_{N/2})^{1-\rho}||g||^\rho_\infty k^\rho||u||_{\mathcal{D}^{1,2}}. \end{aligned}$$

So, the lemma is proved with $C = k^{2-\rho}||g||_{N/2}^{1-\rho}||g||_{\infty}^{\rho}$. $\diamond$

Lemma 3.2. *Assume that either (i) $N = 3, 4$ or (ii) $N \geq 5$ and f satisfies $(\mathcal{F}_\infty)$. Then the linearized problem of (1.1)-(1.2) around the solution u*

$$\begin{aligned} V_{tt} + \delta V_t - \phi(x)\Delta V + \lambda f'(u)V &= 0, \quad x \in \mathbb{R}^N, \ t \in [0,T], \\ V(x,0) = z \in \mathcal{D}^{1,2}(\mathbb{R}^N), \qquad & V_t(x,0) = \omega \in L_g^2(\mathbb{R}^N). \end{aligned}$$

has a unique (weak) solution.

Proof Following ideas from Lemmas 4.2, 4.3 in [17], we are able to prove that the linearized problem (3.1), posses a unique solution V such that $V \in C[\mathbb{R}, \mathcal{D}^{1,2}(\mathbb{R}^N)]$ and $V_t \in C[\mathbb{R}, L_g^2(\mathbb{R}^N)]$. $\diamond$

The differentiability result for the semigroup $\mathcal{S}(t)$ has as follows.

Lemma 3.3. *Assume that f satisfy conditions $(\mathcal{F}_\infty)$ and $(\mathcal{F}_\delta)$. Then for any $t > 0$, the mapping $\mathcal{S}(t)$ is Fréchet differentiable on $\mathcal{X}_0$. The differential at $\phi_0 = \{u_0, u_1\}$ is the linear operator $\mathcal{L}(t, \phi_0)$ on $\mathcal{X}_0$ given by*

$$\mathcal{L}(t, \phi_0) : \{z, \omega\} \longmapsto \{V(t), V_t(t)\},$$

where V denotes the solution of the first variation problem (3.1).

Proof There exists $R > 0$ and $\{z, \omega\} \in \mathcal{X}_0$, such that for $\phi_0 = \{u_0, u_1\}$, $\tilde{\phi_0} = \{u_0 + z, u_1 + \omega\}$, we have $||\phi_0||_{\mathcal{X}_0} \leq R$, $||\tilde{\phi_0}||_{\mathcal{X}_0} \leq R$. Next, we consider the associated solutions $\phi = \mathcal{S}(t)\phi_0 = \{u(t), u_t(t)\}$, $\tilde{\phi} = \mathcal{S}(t)\tilde{\phi_0} = \{\tilde{u}(t), \tilde{u}_t(t)\}$ and their difference $w = u - \tilde{u}$, which satisfies the initial value problem

$$\begin{aligned} w_{tt} + \delta w_t - \phi(x)\Delta w \quad &+ \quad \lambda[f(u) - f(\tilde{u}] = 0, \\ w(x,0) = z, \qquad & w_t(x,0) = \omega. \end{aligned} \tag{3.1}$$

Using ideas from [17, Proposition 3.2 (c)], we may obtain the following estimate

$$\begin{aligned} ||\phi(t) - \tilde{\phi}(t)||_{\mathcal{X}_0}^2 &= ||u(t) - \tilde{u}(t)||_{\mathcal{D}^{1,2}}^2 + ||u_t(t) - \tilde{u}_t(t)||_{L_g^2}^2 \\ &\leq \exp{(Ct)} \left\{ ||z||_{\mathcal{D}^{1,2}}^2 + ||\omega||_{L_g^2}^2 \right\}. \end{aligned} \tag{3.2}$$

Consider the difference $W = \tilde{u} - u - V$. We easily get that W is a solution of the initial value problem

$$\begin{aligned} W_{tt} + \delta W_t &- \phi(x)\Delta W + \lambda f'(u)W = \lambda F, \\ W(x,0) &= 0, \\ W_t(x,0) &= 0, \end{aligned}$$

where $F = f(u) - f(\tilde{u}) - f'(u)(u - \tilde{u})$. By application of the mean value theorem, we have that for $\tau \in [0,1]$,

$$F = \int_0^1 [f'(\tau\tilde{u} + (1-\tau)u) - f'(u)](u - \tilde{u})d\tau. \tag{3.3}$$

We multiply equation (3.3) by gW_t and we integrate over $\mathbb{R}^N$, to obtain the equality

$$\frac{1}{2}\frac{d}{dt}\left\{ ||W||_{\mathcal{D}^{1,2}}^2 + ||W_t||_{L_g^2}^2 \right\} + \delta||W_t||_{L_g^2} = \lambda \int_{\mathbb{R}^N} g[F - f'(u)W]W_t \, dx. \tag{3.4}$$

Using hypotheses $(\mathcal{F}_\delta)$, $(\mathcal{F}_\infty)$ and (3.3), we have the following estimate, for the right-hand side of (3.4),

$$\begin{aligned}
&\left|\lambda \int_{\mathbb{R}^N} g[F - f'(u)W]W_t\, dx\right| \leq \\
\leq\ & \lambda \int_{\mathbb{R}^N} g\,|f'(u)|\,|W|\,|W_t|\,dx + \lambda \int_{\mathbb{R}^N} g\,|F|\,|W_t|\,dx \\
\leq\ & C_1||W||_{L^2_g}||W_t||_{L^2_g} + C_2||W_t||_{L^2_g}\left\{\int_{\mathbb{R}^N} g\,|u-\tilde{u}|^{2(\delta_0+1)}\,dx\right\}^{1/2} \\
\leq\ & C_3||W||_{\mathcal{D}^{1,2}}||W_t||_{L^2_g} + C_2||W_t||_{L^2_g}||u-\tilde{u}||^{\delta_0+1}_{L_g^{2(\delta_0+1)}} \\
\leq\ & C_4\left\{||W||^2_{\mathcal{D}^{1,2}} + ||W_t||^2_{L^2_g}\right\} + C_5||u-\tilde{u}||^{2(\delta_0+1)}_{L_g^{2(\delta_0+1)}} \qquad (3.5)
\end{aligned}$$

Using (3.2), (3.4), (3.5), Lemma 3.1 and Gronwall's Lemma, we deduce that

$$\begin{aligned}
\left\{||W||^2_{\mathcal{D}^{1,2}} + ||W_t||^2_{L^2_g}\right\} &\leq C_6\exp(C_4t) \times \int_0^t ||u(s)-\tilde{u}(s)||^{2+2\delta_0}_{\mathcal{D}^{1,2}}\,dt \\
&\leq C_6\exp(C_7t) \times \left\{||z||^2_{\mathcal{D}^{1,2}} + ||\omega||^2_{L^2_g}\right\}^{1+\delta_0}. \qquad (3.6)
\end{aligned}$$

Relation (3.6) is equivalent to

$$||\tilde{\phi}(t) - \phi(t) - \{V(t), V_t(t)\}||^2_{\mathcal{X}_0} \leq C_6\exp\left(C_7t\right) \times ||\{z,\omega\}||^{2+2\delta_0}_{\mathcal{X}_0},\ \ i.e$$

$$\frac{||\mathcal{S}(t)(\phi_0+h) - \mathcal{S}(t)(\phi_0) - \mathcal{L}(t,\phi_0)h||^2_{\mathcal{X}_0}}{||h||^2_{\mathcal{X}_0}} \to 0\ \ in\ \mathcal{X}_0$$

as $h = \{z,\omega\} \to 0$ and the lemma is proved. $\diamond$

For the proof of the main result, we shall use the following lemma, which gives additional information for the global attractor $\mathcal{A}$. We define $\mathcal{X}_1 := D(A) \times \mathcal{D}^{1,2}$.

Lemma 3.4. *Let the functions f and η satisfy conditions $(\mathcal{F}_\infty)$ and $(\mathcal{H})$ respectively. Then the global attractor $\mathcal{A}$ is included and is bounded in the space $\mathcal{X}_1$.*

Sketch of Proof We differentiate equation (1.1) with respect to time. Using $(\mathcal{F}_\infty)$, we may see that $f'(u)u_t \in C_b(\mathbb{R},\ L^2_g(\mathbb{R}^N)$. Therefore, we may apply arguments similar to those used for the proof of [17, Lemma 4.7] to obtain that $\{u_t,\ u_{tt}\} \in C_b(\mathbb{R},\ \mathcal{X}_0)$. Then equation (1.1) implies that $-\phi\Delta u \in C_b(\mathbb{R},\ L^2_g(\mathbb{R}^N))$, i.e, $u \in C_b(\mathbb{R},\ D(A))$. By following the lines of [31, Theorem 3.2, pg 210], we obtain the result. $\diamond$.

Setting $\theta = \mathcal{R}_\varepsilon\phi = \{u, v = u_t + \varepsilon u\}$, we may rewrite the problem (1.1)- (1.2), as a first order evolution equation of the form

$$\theta_t = B(\theta) = -A_\varepsilon\theta - \lambda b(\theta) + \tilde{\eta},$$

where $\theta = \{u,v\},\ b(\theta) = \{0, f(u)\},\ \tilde{\eta} = \{0,\eta\}$ and

$$A_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ -\phi\Delta - \varepsilon(\delta-\varepsilon)I & (\delta-\varepsilon)I \end{pmatrix}.$$

Here I denotes the identity mapping. Also for simplicity of the presentation, when we write the vector $\{u,v\}$ we mean the transposed form of it, i.e., $\{u,v\}^T$. By the above notation, the first variation equation (3.1) has the form

$$\begin{aligned}
\Theta_t = B'(\theta)\Theta &= -A_\varepsilon\Theta - \lambda b'(\theta)\Theta, \\
\Theta(0) &= \xi, \qquad (3.7)
\end{aligned}$$

where $\Theta = \{V, V_t + \varepsilon V\}$, $b'(\theta)\Theta = \{0, f'(u)V\}$, $\xi = \{z, \omega\} \in \mathcal{X}_0$. Let ξ_k, $k = 1, 2, ..., m$ initial values in $\mathcal{X}_0$ and $\Theta_k(t)$ the (unique) solutions of (3.7) with $\Theta_k(0) = \xi_k$. Recall that in the generalized Liouville formula

$$\begin{aligned} |\Theta_1(t) \wedge ... \wedge \Theta_m(t)|_{\wedge^m \mathcal{X}_0} &= |\xi_1 \wedge ... \wedge \xi_m|_{\wedge^m \mathcal{X}_0} \times \\ &\times \exp \int_0^t Tr(B'(\mathcal{S}_\varepsilon(s)\theta_0) \circ Q_m(s))\, ds, \end{aligned} \tag{3.8}$$

the $m-$trace $Tr(B'(\mathcal{S}_\varepsilon(s)\theta_0) \circ Q_m(s))$, provides information for the evolution of the $m-$dimensional volumes, transported along $\mathcal{S}_\varepsilon(t)\theta_0$, by the first variation equation. We denote by $Q_m(t)$ the orthogonal projector in $\mathcal{X}_0$ onto the subspace spanned by $\Theta_1(t), ..., \Theta_m(t)$. We also denote by

$$\Phi_j(t) = \{z_j, \omega_j\}, \quad j = 1, ..., m,$$

an orthonormal basis of $\text{span}\{\Theta_1(t), ..., \Theta_m(t)\} = Q_m(t)$. We have that

$$Tr(B'(\mathcal{S}_\varepsilon(s)\theta_0) \circ Q_m(s)) = \sum_{j=1}^m (B'(\theta(s))\Phi_j(s),\ \Phi_j(t))_{\mathcal{X}_0}, \tag{3.9}$$

where $(\{p, q\},\ \{\tilde{p}, \tilde{q}\})_{\mathcal{X}_0} = (p,\ \tilde{p})_{\mathcal{D}^{1,2}} + (q,\ \tilde{q})_{L^2_g}$, is the inner product in $\mathcal{X}_0$ for $p,\ \tilde{p} \in \mathcal{D}^{1,2}$ and $q,\ \tilde{q} \in L^2_g$. Under the above notations, we have the following main result of this section.

Theorem 3.5. *Let conditions* $(\mathcal{F}_\infty)$ *and* $(\mathcal{F}_\delta)$ *and (2.22) be fulfilled. Then there exist* m *such that* $\frac{1}{m}\sum_{j=1}^m \mu_j^{-1} \leq \frac{\rho_1^2}{2\gamma^2}$, *for which, the Hausdorff dimension of the global attractor* $\mathcal{X}$ *in* $\mathcal{X}_0$, *of the dynamical system associated with problem (1.1)-(1.2), is less than or equal to* m, *and the fractal dimension is less than or equal to* $\frac{4m}{3}$.

Proof From Lemma 3.4, we have that $\mathcal{R}_\varepsilon \mathcal{A} \subset \mathcal{X}_1$. Using this fact and condition $(\mathcal{F}_\infty)$, we may write

$$\begin{aligned} & (B'(\theta(s))\Phi_j,\ \Phi_j)_{\mathcal{X}_0} = (-A_\varepsilon \Phi_j,\ \Phi_j)_{\mathcal{X}_0} + (-b'(\theta)\Phi_j,\ \Phi_j)_{\mathcal{X}_0} \\ = & - (\{\varepsilon z_j - \omega_j,\ -\phi\Delta z_j - \varepsilon(\delta-\varepsilon)z_j + (\delta-\varepsilon)\omega_j\},\ \{z_j,\ \omega_j\})_{\mathcal{X}_0} \\ & - (\{0,\ \lambda f'(u)z_j\},\ \{z_j,\ \omega_j\})_{\mathcal{X}_0} \\ = & - \varepsilon||z_j||^2_{\mathcal{D}^{1,2}} + (z_j,\ \omega_j)_{\mathcal{D}^{1,2}} - (-\phi\Delta z_j,\ \omega_j)_{L^2_g} + \varepsilon(\delta-\varepsilon)(z_j,\ \omega_j)_{L^2_g} \\ & - (\delta-\varepsilon)(\omega_j,\ \omega_j)_{L^2_g} - \lambda(f'(u)z_j,\ \omega_j)_{L^2_g} \\ \leq & - \rho_1\left\{||z_j||^2_{\mathcal{D}^{1,2}} + ||\omega_j||^2_{L^2_g}\right\} + \lambda||f'||_\infty \int_{\mathbb{R}^N} g\,|z_j|\,|\omega_j| dx, \end{aligned} \tag{3.10}$$

whith $\rho_1 = \varepsilon/2$, $0 \leq \varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = \min(\delta/4, \mu_1/2\delta)$ ([17, Lemma 4.1]). Setting $\gamma = \lambda||f'||_\infty$, and applying Young's inequality to the relation (3.10), we have that

$$\begin{aligned} (B'(\theta(s))\Phi_j,\ \Phi_j)_{\mathcal{X}_0} &\leq -\rho_1\left\{||z_j||^2_{\mathcal{D}^{1,2}} + ||\omega_j||^2_{L^2_g}\right\} \\ &+ \frac{\rho_1}{2}||\omega_j||^2_{L^2_g} + \frac{\gamma^2}{2\rho_1}||z_j||^2_{L^2_g} \\ &\leq -\frac{\rho_1}{2}\left\{||z_j||^2_{\mathcal{D}^{1,2}} + ||\omega_j||^2_{L^2_g}\right\} + \frac{\gamma^2}{2\rho_1}||z_j||^2_{L^2_g}. \end{aligned} \tag{3.11}$$

Since Φ_j is an orthonormal basis of $Q_m\mathcal{X}_0$, we have from (3.9), (3.11), the inequality

$$\sum_{j=1}^{m}(B'(\theta(s))\Phi_j(s),\ \Phi_j(t))_{\mathcal{X}_0} \leq -\frac{m\rho_1}{2} + \frac{\gamma^2}{2\rho_1}\sum_{j=1}^{m}||z_j||^2_{L^2_g}. \tag{3.12}$$

In the previous section, we show that the operator $-\phi\Delta$ is compact with domain $D(A)$, and that the injection $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N)$ is compact. Therefore applying the result from [31, Chapter VI, Lemma 6.3], for any orthogonal family of elements $\{z_j,\ \omega_j\},\ j = 1, ..., m$ of $\mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$, we have that

$$\sum_{j=1}^{m}||z_j||^2_{L^2_g} \leq \sum_{j=1}^{m}\mu_j^{-1}. \tag{3.13}$$

Thus, by (3.12), (3.13), we have the following estimate

$$Tr(B'(\theta(t))\circ Q_m(t)) \leq -\frac{m\rho_1}{2} + \frac{\gamma^2}{2\rho_1}\sum_{j=1}^{m}\mu_j^{-1}. \tag{3.14}$$

We integrate (3.14) with respect to time, to obtain the relation

$$q_m := \limsup_{t\to 0}\frac{1}{t}\int_0^t Tr(B'(\mathcal{S}_\varepsilon(s)\theta_0)\circ Q_m(s))\ ds \leq -\frac{m\rho_1}{2} + \frac{\gamma^2}{2\rho_1}\sum_{j=1}^{m}\mu_j^{-1}.$$

From (2.7), we obtain that $\frac{1}{m}\sum_{j=1}^{m}\mu_j^{-1} \to 0$, as $m \to \infty$. Hence there exist $m \geq 1$, such that

$$\frac{1}{m}\sum_{j=1}^{m}\mu_j^{-1} \leq \frac{\rho_1^2}{2\gamma^2}. \tag{3.15}$$

Consequently, we have that

$$q_m \leq -\frac{m\rho_1}{2}\left(1 - \frac{\gamma^2}{2\rho_1 m}\sum_{j=1}^{m}\mu_j^{-1}\right) \leq -\frac{3m\rho_1}{4}.$$

For $i = 1, ..., m$,

$$(q_i)_+ \leq \frac{\gamma^2}{2\rho_1}\sum_{j=1}^{i}\mu_j^{-1} \quad\leq\quad \frac{\gamma^2}{2\rho_1}\sum_{j=1}^{m}\mu_j^{-1} \leq \frac{m\rho_1}{4}, \tag{3.16}$$

$$\max_{1\leq i\leq m-1}\frac{(q_i)_+}{|q_m|} \quad\leq\quad \frac{1}{3}. \tag{3.17}$$

Now, we apply the Constantin-Foias-Temam Theorem on the dimension of the attractor [31, Chapter V, Theorem 3.3], to complete the proof. $\diamond$

Proposition 2.2, allows for a more explicit estimate on the dimension, in terms of given parameters (see [31, Chapter VI, pg. 453-454], for the sine-Gordon equation). By using (2.9), we get that there exist $C(K) > 0$,

$$\sum_{j=1}^{m}\mu_j^{-1} \leq C(K)m^{1-\frac{2}{N}}. \tag{3.18}$$

We may consider $\rho_1 = \min(\delta/8, 4\delta/\mu_1)$. We replace $1/\rho_1$ by

$$\frac{1}{\hat{\rho}_1} = \frac{8}{\delta}\left(1 + \frac{\delta^2}{\mu_1}\right),$$

in (3.15), which with (3.18) implies that m is the first integer, such that

$$m \geq \left(\frac{2\gamma^2 C(K)}{\hat{\rho_1}^2}\right)^{\frac{N}{2}}, \ \gamma = \lambda ||f'||_\infty.$$

Remark 3.6. *In the functional set-up developed in* [17] *and in the present article, it is possible to follow the method of construction of exponential attractors introduced in* [9], [10]. *In fact, under the additional assumption that $f' \in L^\infty(\mathbb{R}^N)$, it can be proved that the semigroup $\mathcal{S}(t)$, satisfies the discrete squeezing property on an absorbing set of the energy space $\mathcal{X}_1 = D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.*

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E-mail address: `karan@aegean.gr`