Bifurcation Results for the Mean Curvature Equations Defined on all $\mathbb{R}^N$

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Abstract. We prove certain local bifurcation results for the mean curvature problem

$$- \nabla \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = \lambda\, f(x,u), \quad x \in \mathbb{R}^N.$$

This is achieved by applying standard local bifurcation theory. The use of certain equivalent weighted and homogeneous Sobolev spaces was proved to be crucial.

Keywords: Mean Curvature Equations, Nonlinear Spectral Theory, Local Bifurcation, Indefinite Weight, Homogeneous Sobolev Spaces, Weighted L^p-Spaces, Unbounded Domain.

1. Introduction

In this paper we study the bifurcation of a continuum of solutions for the mean curvature equation

$$- \nabla \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = \lambda\, g(x) f(u), \quad x \in \mathbb{R}^N, \tag{1}$$

$$u(x) \geq 0, \text{ almost everywhere in } \mathbb{R}^N. \tag{2}$$

It is proved that the continuum of solutions bifurcates from the positive principal eigenvalue of the linearized problem

$$-\Delta u = \lambda\, g(x)\, u, \quad x \in \mathbb{R}^N, \tag{3}$$

$$u(x) > 0, \text{ for all } x \in \mathbb{R}^N, \tag{4}$$

where $\lambda \in \mathbb{R}$ and $2 < N \leq 5$. We are not going to discuss the case of dimensions 1 and 2. It seems that other approaches would be necessary for treating these cases. The general hypotheses, which will be assumed throughout the paper, are the following

$(\mathcal{G})$ g is a smooth function, at least of type $C^{1,\alpha}(\mathbb{R}^N)$, for some $\alpha \in (0,1)$, such that $g \in L^{N/2}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ and there exists $\Omega^+ \subset \mathbb{R}^N$ of positive measure, i.e., $|\Omega^+| > 0$, such that $g(x) > 0$, for all $x \in \Omega^+$.

$(\mathcal{F})$ $f : \mathbb{R} \longmapsto [0, \infty)$ is a smooth function, such that $f(0) = 0$, $f'(0) > 0$ and $f(s) > 0$, for all $s \neq 0$. Also, $f', f'' \in L^\infty(\mathbb{R})$ and there exists $k^* > 0$, such that $|f(s)| \leq k^* s$, for all $s \in \mathbb{R}^+$.

Furthermore, for the weight function g, we assume the following hypotheses

$(\mathcal{G}^+)$ $g(x) \geq 0$, for all $x \in \mathbb{R}^N$,

$(\mathcal{G}^-)$ there exists $\Omega^- \subset \mathbb{R}^N$ with $|\Omega^-| > 0$ such that $g(x) < 0$, for all $x \in \Omega^-$.

The mean curvature equation, which is connected with the least surface problem, has a variety of applications both in pure mathematics and in natural sciences, see for example (G. Gilbarg and N. S. Trudinger, 1983), (E. Zeidler, 1986) and (E. Zeidler, 1985).

From the mathematical point of view, problem (1) is of great interest because of it's non-uniformity considered as an elliptic equation. So the corresponding operator is not (strongly or weakly) coercive in the standard reflexive Sobolev spaces.

For the bounded domain case, we refer to the works (C. O. Horgan, L. E. Payne and G. A. Philippin, 1995) (L. E. Payne and G. A. Philippin, 1994), for a priori estimates and maximum principle. Existence results have been obtained in (E. S. Noussair, C. A. Swanson and Y. Yianfu, 1993) using a barrier method. In the paper (A. Greco, 1998) the existence of radial solutions is proved. In (M. Nakao, 1990) the existence of global bifurcation was proved. Here the problem is studied in the classical space setting. So, the "blowing up" technique, was possible to be applied (see (E. Zeidler, 1986)).

In the unbounded domain case we mention the existence results of (E. S. Noussair, C. A. Swanson, 1993), where a barrier method was used. For a priori estimates and maximum principle we refer to the work (L. Caffarelli, N. Garofalo and F. Segala, 1994) in all $\mathbb{R}^N$. Also, estimates were obtained for a corresponding parabolic equation in the paper (M. Nakao and Y. Ohara, 1996). Nonexistence results for positive solutions appear in a recent paper by (W. Allegretto, 2000), where a generalized type of Picone's identity is used.

Recently some works have appeared dealing with bifurcation phenomena of semilinear and quasilinear equations and systems on $\mathbb{R}^N$. We may mention among others the papers (K. J. Brown and N. M. Stavrakakis, 1996; P. Drábek and Y. X. Huang, 1997; N. M. Stavrakakis and N. B. Zographopoulos, 1999; N. M. Stavrakakis and N. B. Zographopoulos, 2000). For a rather complete discussion on such kind of methods we refer to (N. B. Zographopoulos, 2000).

To be able to carry out our study and especially to apply the bifurcation methods, we introduce certain equivalent weighted and homogeneous Sobolev spaces. This is done in Section 2. Also we briefly state results, to be used later, concerning the existence, of the first positive principal eigenvalue λ_1 of the linearized problem (3) - (4). In Section 3, we deal with some basic properties of the mean curvature operator. Finally, in Section 4, we prove the existence of a local continuum of nonnegative solutions of problem (1) - (2) branching out from the first eigenvalue of the linearized problem (3) - (4).

REMARK 1.1. *We want to mention that the restriction in the dimension:* $2 < N \leq 5$ *is imposed in order to obtain the differentiability of the operators, which correspond to the right hand side of the equation (1) (see Lemma 4.2) and not because of the mean curvature operator. Also this restriction is necessary for the proof of some useful estimates. For similar thinks in the semilinear elliptic problem we refer to the work of (*K. J. Brown and N. M. Stavrakakis, 1996*).*

Notation. For simplicity we use the symbol $||.||_2$ for the norm $||.||_{L^2(\mathbb{R}^N)}$ and $\mathcal{D}^{1,2}$ for the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. B_R and $B_R(c)$ will denote the balls in $\mathbb{R}^N$ of radius R and centers zero and c, respectively. Also the Lebesgue measure of a set $\Omega \subset \mathbb{R}^N$ will be denoted by $|\Omega|$. An equality introducing definition is denoted by $=:$. Integration in all of $\mathbb{R}^N$ will be denoted with the integral symbol $\int$ without any indication.

2. Space Setting - The Linearized Problem

The natural space setting for the eigenvalues of the linear elliptic problem (3)-(4), as we show next, will be the energy space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, i.e., the closure of the $C_0^\infty(\mathbb{R}^N)$ functions with respect to the energy norm

$$||u||_{\mathcal{D}^{1,2}} =: \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{1/2}.$$

It is known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\right\}$$

and that there exists $K_0 > 0$, such that, for every $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$||u||_{\frac{2N}{N-2}} \leq K_0 \, ||u||_{\mathcal{D}^{1,2}}, \tag{1}$$

and $D^{1,2}(\mathbb{R}^N)$ is a reflexive Banach space.

Our approach here is based on the following generalized Poincaré inequality.

LEMMA 2.1. *Suppose that* $g \in L^{N/2}(\mathbb{R}^N)$. *Then there exists* $a = 1/K||g||_{N/2}^{1/2} > 0$ *such that*

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq a \int_{\mathbb{R}^N} |g||u|^2\, dx,$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Thus if $g \in L^{N/2}(\mathbb{R}^N)$ and $\alpha > 0$ is as in Lemma 2.1, we may define an inner product on $C_0^\infty(\mathbb{R}^N)$ by

$$< u, v >=: \int_{\mathbb{R}^N} \nabla u \nabla v\, dx - \frac{a}{2} \int_{\mathbb{R}^N} guv\, dx.$$

Next we define the weighted Sobolev space $\mathcal{V}$ to be the completion of C_0^∞ functions with respect to the above inner product. The space $\mathcal{V}$ depends on the function g; it is natural to expect that $\mathcal{V}$ grows as $|g|$ becomes smaller. However, under condition $(\mathcal{G})$ it is proved (see (K. J. Brown and N. M. Stavrakakis, 1996)) that $\mathcal{V}$ is independent of the function g. In fact, the space $\mathcal{V}$ is characterized by the following lemma.

LEMMA 2.2. *Suppose* $g \in L^{N/2}(\mathbb{R}^N)$. *Then* $\mathcal{V} = \mathcal{D}^{1,2}$.

Thus we may henceforth suppose that the norm in $\mathcal{V}$, coincides with the norm $||.||_{\mathcal{D}^{1,2}}$ and that the inner product in $\mathcal{V}$ is given by

$$< u, v >= \int_{\mathbb{R}^N} \nabla u \nabla v\, dx.$$

Following a standard procedure, we define the bilinear form

$$\beta(u, v) =: \int_{\mathbb{R}^N} guv\, dx,$$

for every $u, v \in \mathcal{V}$. Since $\mathcal{V} \subseteq L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ we obtain that β is *bounded* in $\mathcal{V}$. Hence by Riesz Representation Theorem we may define a bounded linear operator M, such that

$$< Mu, v >=: \beta(u, v), \text{ for every } u, v \in \mathcal{V}.$$

It is standard to check the following result.

LEMMA 2.3. *Suppose that* $g \in L^{N/2}(\mathbb{R}^N)$. *Then operator* M *is selfadjoint and compact.*

To simplify notation, without loss of generality, we assume that $f'(0) = 1$. So (3) is exactly the linearization of (1). By means of classical spectral methods the following results are proved concerning the existence, positivity and principality of the first eigenvalue.

THEOREM 2.4. *a) Assume that the function g satisfies hypothesis $(\mathcal{G}^+)$. Then problem (3)–(4) admits a positive principal eigenvalue given by*

$$\lambda_1 =: \inf_{<Mu,u>=1} ||u||^2_{\mathcal{D}^{1,2}}.$$

b) Assume that function g satisfies hypothesis $(\mathcal{G}^-)$. Then problem (3)–(4) admits two principal eigenvalues given by

$$\lambda_1^+ =: \inf_{<Mu,u>=1} ||u||^2_{\mathcal{D}^{1,2}},$$

$$\lambda_1^- =: -\inf_{<Mu,u>=-1} ||u||^2_{\mathcal{D}^{1,2}}.$$

The associated eigenfunction ϕ $(\phi^+,\ \phi^-$ respectively), belongs (belong) to the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and is a classical solution of the problem (3)–(4).

Having in mind the application of the bifurcation theory for the study of the mean curvature problem (1)–(2), information concerning the dimension of the eigenspace associated to the principal eigenvalues of the linearized problem (3)–(4) are of basic importance. The main results in this direction, needed in the rest of the paper, can be stated as follows

THEOREM 2.5. *Assume that the function g satisfies hypothesis $(\mathcal{G}^+)$ ($(\mathcal{G}^-)$, respectively). Then we have:*
(i) the eigenspace corresponding to the principal eigenvalue λ_1, $(\lambda^+$, λ^- respectively) is of dimension 1,
(ii) λ_1 $(\lambda^+,\ \lambda^-$ respectively) is the only eigenvalue of problem (3)–(4), to which corresponds a positive eigenfunction.

Proof The proof of this theorem is long and technical.

REMARK 2.6. *The algebraic and the geometric multiplicities of the eigenvalues of the problem under discussion are equal since by Lemma 2.3 the operator M is compact and selfadjoint (see (*E. Zeidler, 1986*)).*

The proofs of all results presented in this section are given in detail in (K. J. Brown and N. M. Stavrakakis, 1996).

3. Properties of the Mean Curvature Operator

In this section we state some basic properties of the operators corresponding to problem (1)–(2). We introduce the operator $A : D^{1,2} \to D^{1,2}$ and the functional $F : D^{1,2} \to \mathbb{R}$, as follows:

$$\langle A(u), v\rangle =: \int \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}}\, dx,$$

$$F(u) =: \int (\sqrt{1+|\nabla u|^2} - 1)\, dx.$$

THEOREM 3.1. *The operator A and the functional F are well defined. Moreover, holds that*

$$F(u) \leq \langle A(u), u\rangle, \tag{1}$$

for every $u \in D^{1,2}$.

Proof The result for the operator A is implied by the inequality

$$\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} \leq |\nabla u|.$$

On the other hand, from the inequality

$$\sqrt{1+|\nabla u|^2} - 1 = \frac{|\nabla u|^2}{1+\sqrt{1+|\nabla u|^2}} \leq \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}},$$

we deduce the relation (1). ◇

The next two lemmas state some abstract estimates needed in the sequel.

LEMMA 3.2. *For any positive real numbers a, b the following inequalities hold*

$$\left|\frac{a}{\sqrt{1+a^2}} - \frac{b}{\sqrt{1+b^2}}\right| \leq |a-b|, \tag{2}$$

$$\left(\frac{a}{\sqrt{1+a^2}} - \frac{b}{\sqrt{1+b^2}}\right)(a-b) \geq \left(\frac{a}{\sqrt{1+a^2}} - \frac{b}{\sqrt{1+b^2}}\right)^2, \tag{3}$$

Proof Following some simple calculations we obtain that

$$\left|\frac{a}{\sqrt{1+a^2}} - \frac{b}{\sqrt{1+b^2}}\right| = \frac{a+b}{\sqrt{1+a^2}\,\sqrt{1+b^2}\,(a\sqrt{1+b^2}+b\sqrt{1+a^2})}\,|a-b|.$$

The proof of the inequality (2) is completed by the following relation

$$a+b \leq a\sqrt{1+b^2}+b\sqrt{1+a^2}.$$

Relation (3) may be written equivalently as

$$\left(\frac{a}{\sqrt{1+a^2}}-\frac{b}{\sqrt{1+b^2}}\right)\left[(a-b)-\left(\frac{a}{\sqrt{1+a^2}}-\frac{b}{\sqrt{1+b^2}}\right)\right] \geq 0. \quad (4)$$

The result for the inequality (4) is an immediate consequence of the fact that

$$\left(\frac{a}{\sqrt{1+a^2}}-\frac{b}{\sqrt{1+b^2}}\right)(a-b) \geq 0,$$

for every $a, b \in \mathbb{R}^+$. $\diamondsuit$

Notation For the simplification of the representation we denote by $N(u)$ the quantity

$$N(u) =: 1+|\bigtriangledown u|^2.$$

A generalization of the previous lemma is the following result.

LEMMA 3.3. *For every* $u, v \in D^{1,2}$, *the following inequalities hold*

$$\left|\frac{\nabla u}{\sqrt{N(u)}}-\frac{\nabla v}{\sqrt{N(v)}}\right| \leq |\bigtriangledown u-\nabla v|, \quad (5)$$

$$\left(\frac{\nabla u}{\sqrt{N(u)}}-\frac{\nabla v}{\sqrt{N(v)}}\right)\cdot(\nabla u-\nabla v) \geq \left(\frac{|\bigtriangledown u|}{\sqrt{N(u)}}-\frac{|\bigtriangledown v|}{\sqrt{N(v)}}\right)^2, \quad (6)$$

$$\left|\frac{1}{(N(u))^{3/2}}-\frac{1}{(N(v))^{3/2}}\right| \leq k|\bigtriangledown u-\nabla v|, \quad (7)$$

where $k>0$ *is independent of* u *and* v.

Proof Raising to the square both sides of inequality (5) and making some calculations, we obtain that

$$2\left[N(u)\,N(v)-\sqrt{N(u)\,N(v)}\right]\bigtriangledown u\cdot\nabla v \leq |\bigtriangledown u|^4 N(v)+|\bigtriangledown v|^4 N(u).$$

Also, from relation (2) we have that

$$\left|\frac{|\bigtriangledown u|}{\sqrt{N(u)}}-\frac{|\bigtriangledown v|}{\sqrt{N(v)}}\right| \leq \Big||\bigtriangledown u|-|\bigtriangledown v|\Big| \leq |\bigtriangledown u-\nabla v|.$$

Since $\nabla u\cdot\nabla v \leq |\bigtriangledown u||\bigtriangledown v|$ we get inequality (5). With a similar procedure we may prove that relation (6) may be written as follows

$$\frac{|\bigtriangledown u|^2}{\sqrt{N(u)}}-\frac{|\bigtriangledown u|^2}{N(u)}+\frac{|\bigtriangledown v|^2}{\sqrt{N(v)}}-\frac{|\bigtriangledown v|^2}{N(v)} \geq \left[\frac{\sqrt{N(v)}+\sqrt{N(u)}-2}{\sqrt{N(v)}\,\sqrt{N(u)}}\right]\nabla u\cdot\nabla v.$$

Also, from (5) we deduce that

$$\Big(\frac{|\bigtriangledown u|}{\sqrt{N(u)}} - \frac{|\bigtriangledown v|}{\sqrt{N(v)}}\Big)(|\bigtriangledown u| - |\bigtriangledown v|) \geq \Big(\frac{|\bigtriangledown u|}{\sqrt{N(u)}} - \frac{|\bigtriangledown v|}{\sqrt{N(v)}}\Big)^2.$$

Again the fact that $\bigtriangledown u \cdot \bigtriangledown v \leq |\bigtriangledown u||\bigtriangledown v|$ implies inequality (6). Finally, concerning inequality (7) we have that

$$\begin{aligned} & \Big| \frac{1}{(N(u))^{3/2}} - \frac{1}{(N(v))^{3/2}} \Big| \\ = & \Big| \frac{(N(u))^3 - (N(v))^3}{[(N(u))^{3/2} + (N(v))^{3/2}]\,[(N(u))\,(N(v))]^{3/2}} \Big| \\ \leq & \frac{|\bigtriangledown u - \bigtriangledown v|\,(|\bigtriangledown u| + |\bigtriangledown v|)\,[(N(u))^2 + (N(u))(N(v)) + (N(v))^2]}{[(N(u))^{3/2} + (N(v))^{3/2}]\,[(N(u))\,(N(v))]^{3/2}}. \end{aligned}$$

The last result and inequality $|\bigtriangledown u| + |\bigtriangledown v| \leq (N(u))^{1/2}\,(N(v))^{1/2}$, which holds for every u, v in $D^{1,2}$, imply relation (7). $\diamondsuit$

Next we prove that the operator A is Fréchet differentiable.

THEOREM 3.4. *The operator A is Fréchet differentiable in $D^{1,2}$ with*

$$\langle A'(u)\,h, v\rangle = \int \frac{\bigtriangledown h}{(1 + |\bigtriangledown u|^2)^{3/2}} \bigtriangledown v\, dx,$$

for every u, h and v in $D^{1,2}$.

Proof In order to prove that A is Fréchet differentiable, we prove that there exists the Gâteaux derivative of A in $D^{1,2}$ as a continuous operator (see (M. S. Berger, 1977, Theorem 2.1.13)). Setting

$$A(u, th) = A(u + th) - A(u) - tA'(u)h,$$

we have that

$$\begin{aligned} ||A(u, th)||^2 &= \langle A(u + th) - A(u) - tA'(u)h, A(u, th)\rangle \\ &\leq \int \Big[\frac{\bigtriangledown u + t \bigtriangledown h}{\sqrt{N(u + th)}} - \frac{\bigtriangledown u}{\sqrt{N(u)}} - \frac{t \bigtriangledown h}{(N(u))^{3/2}}\Big]^2 dx. \end{aligned}$$

Since the following relations hold

$$\Big|\frac{\bigtriangledown u + t \bigtriangledown h}{\sqrt{N(u + th)}} - \frac{\bigtriangledown u}{\sqrt{N(u)}} - \frac{t \bigtriangledown h}{(N(u))^{3/2}}\Big| \leq 2(|\bigtriangledown u| + |\bigtriangledown h|)$$

and

$$\lim_{t \to 0} \Big|\frac{\bigtriangledown u + t \bigtriangledown h}{\sqrt{N(u + th)}} - \frac{\bigtriangledown u}{\sqrt{N(u)}} - \frac{t \bigtriangledown h}{(N(u))^{3/2}}\Big| = \Big|\frac{\bigtriangledown u}{\sqrt{N(u)}} - \frac{\bigtriangledown u}{\sqrt{N(u)}}\Big| = 0,$$

by Lebesgue's Dominated Convergence Theorem we have that

$$\lim_{t \to 0} ||A(u, th)||^2 = 0.$$

Hence, the operator A is Gâteaux differentiable in $D^{1,2}$.

In order to prove that the operator $A'(u)h$ is continuous with respect to u, it suffices to prove that it is continuous for every $h \in C_0^\infty$. Assume that $u,\ v,\ \phi\ \in D^{1,2}$, then we have

$$\left|\langle A'(u)h - A'(v)h, \phi\rangle\right| \le \int \left|\frac{1}{(N(u))^{3/2}} - \frac{1}{(N(v))^{3/2}}\right| |\bigtriangledown h| |\bigtriangledown \phi|\, dx. \quad (8)$$

¿From (7), (8) and Hölder's inequality, we obtain that

$$\left|\langle A'(u)h - A'(v)h, \phi\rangle\right| \le c||u - v||_{1,2}|| \bigtriangledown h||_\infty ||\phi||_{1,2},$$

for some positive constant c and the proof is completed. $\diamondsuit$

COROLLARY 3.5. *The operator A is continuous in $D^{1,2}$.*

The connection between the operator A and the functional F, is given by the next result.

THEOREM 3.6. *The operator A is a potential operator. A potential for A is the functional F.*

Proof We notice that for the Fréchet derivative A' of A holds that

$$\langle A'(u)v, w\rangle = \langle v, A'(u)w\rangle, \quad \text{for every } u, v, w \in D^{1,2}. \quad (9)$$

Therefore, in order to prove that A is a potential operator, it suffices to prove that the mapping $(t, s) \mapsto \langle A'(w + tu + sv)x, y\rangle$ is continuous in $[0, 1] \times [0, 1]$, for every $u, v, w, x, y \in D^{1,2}$ (see (E. Zeidler, 1986, Proposition 41.5)). Assume that $(t_i, s_i) \in [0, 1] \times [0, 1]$, $i = 1, 2$. Then we have that

$$\left|\langle A'(\bar{u})\, x, y\rangle - \langle A'(\bar{v})\, x, y\rangle\right| \le \int \left|\frac{1}{(N(\bar{u}))^{3/2}} - \frac{1}{((N(\bar{v}))^{3/2}}\right| |\bigtriangledown x|\, |\bigtriangledown y|\, dx,$$

where

$$\bar{u} = \bigtriangledown w + t_1 \bigtriangledown u + s_1 \bigtriangledown v \quad \text{and} \quad \bar{v} = \bigtriangledown w + t_2 \bigtriangledown u + s_2 \bigtriangledown v.$$

Following the same procedure as in the proof of Theorem 3.4, we may derive that

$$\left|\frac{1}{(N(\bar{u}))^{3/2}} - \frac{1}{((N(\bar{v}))^{3/2}}\right| \le C\, |(t_1 - t_2) \bigtriangledown u + (s_1 - s_2) \bigtriangledown v|\, |\bar{u} + \bar{v}|,$$

where $C = \frac{(N(\bar{u}))^2+(N(\bar{u}))(N(\bar{v}))+(N(\bar{v}))^2}{((N(\bar{u}))^{3/2}+(N(\bar{v}))^{3/2})\,(N(\bar{u}))^{3/2}\,(N(\bar{v}))^{3/2}}$. From the following inequality

$$\left|\frac{1}{(N(\bar{u}))^{3/2}} - \frac{1}{((N(\bar{v}))^{3/2}}\right| |\bigtriangledown x| \, |\bigtriangledown y| \leq 2 |\bigtriangledown x| \, |\bigtriangledown y|,$$

Lebesgue's Dominated Convergence Theorem implies that the mapping $(t,s) \mapsto \langle A'(w+tu+sv)x, y\rangle$ is continuous in $[0,1]\times[0,1]$, so that A is a potential operator. It is clear that a potential for A is the functional F. $\diamondsuit$

Another, important property of the operator A and so for the functional F, is stated in the next lemma.

THEOREM 3.7. *The operator* A *is strictly monotone, i.e.,*

$$\langle A(u) - A(v), u - v\rangle > 0,$$

for every u, v *in* $D^{1,2}$, *with* $u \neq v$.

Proof Assume that, for some u, v in $D^{1,2}$, holds that

$$\langle A(u) - A(v), u - v\rangle = 0.$$

Then, relation (6) implies that

$$\int \left(\frac{\bigtriangledown u}{\sqrt{N(u)}} - \frac{\bigtriangledown v}{\sqrt{N(v)}} \right)^2 dx = 0.$$

Hence, we obtain that $A(u) = A(v)$, almost everywhere in $\mathbb{R}^N$. Making some simple calculations, we conclude that $u = v$, almost everywhere in $\mathbb{R}^N$ and the proof is completed. $\diamondsuit$

COROLLARY 3.8. *The functional* F *is strictly convex in* $D^{1,2}$.

Proof Result is an immediate consequence of (E. Zeidler, 1986, Proposition 25.10).

4. Local Bifurcation Results

In this section we state some existence results for the nonlinear problem (1)–(2) near the point $(\lambda_1, 0)$, proving the existence of a local branch of solutions bifurcating from the trivial solution. To apply local bifurcation theory we introduce the nonlinear operator $P : \mathbb{R}\times\mathcal{V} \to \mathcal{V}$ through the relation

$$< P(\lambda, u), \phi >= \int_{\mathbb{R}^N} \frac{1}{\sqrt{N(u)}} \bigtriangledown u \bigtriangledown \phi \, dx - \lambda \int_{\mathbb{R}^N} g f(u)\phi \, dx, \qquad (1)$$

for every $\phi \in \mathcal{V}$, where $<,>$ denotes the inner product in $\mathcal{D}^{1,2}$.

LEMMA 4.1. *The operator P is well defined by (1).*

Proof For fixed $u \in \mathcal{D}^{1,2}$ we define the functional

$$\Phi(\phi) =: \int_{\mathbb{R}^N} \frac{1}{\sqrt{N(u)}} \bigtriangledown u \bigtriangledown \phi \, dx - \lambda \int_{\mathbb{R}^N} g f(u) \phi \, dx,$$

where $\phi \in \mathcal{D}^{1,2}$. Since the function f satisfies hypothesis $(\mathcal{F})$, then $f(u) \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and for some positive constant K_1 holds that

$$\begin{aligned} |\Phi(\phi)| &\leq || \bigtriangledown u||_2 \, || \bigtriangledown \phi||_2 + |\lambda| \, ||g||_{N/2} \, ||f(u)||_{\frac{2N}{N-2}} \, ||\phi||_{\frac{2N}{N-2}} \\ &\leq K_1(|| \bigtriangledown u||_2 + |\lambda| \, ||g||_{N/2} \, ||f(u)||_{\frac{2N}{N-2}}) \, ||\phi||_{\mathcal{V}}. \end{aligned}$$

So Φ is a bounded linear functional. Hence by the Riesz Representation Theorem we may define P as in (1). $\diamondsuit$

LEMMA 4.2. *The operator P defined by (1) is continuous and for $N = 3, 4, 5$, is Fréchet differentiable with continuous Fréchet derivatives, given by*

$$\begin{aligned} < P_u(\lambda, u)\phi, \psi > &= \textstyle\int_{\mathbb{R}^N} \frac{1}{(N(u))^{3/2}} \bigtriangledown \phi \bigtriangledown \psi \, dx - \lambda \int_{\mathbb{R}^N} g f'(u) \phi \psi \, dx, \\ < P_\lambda(\lambda, u), \phi > &= - \textstyle\int_{\mathbb{R}^N} g f(u) \phi \, dx, \\ < P_{\lambda u}(\lambda, u)\phi, \psi > &= - \textstyle\int_{\mathbb{R}^N} g f'(u) \phi \psi \, dx, \end{aligned}$$

for all $\phi, \psi \in \mathcal{D}^{1,2}$.

Proof We refer to the proof of Theorem 3.4. See also (K. J. Brown and N. M. Stavrakakis, 1996). $\diamondsuit$

Next we have the following local result.

THEOREM 4.3. (Local Bifurcation) *There exists $\epsilon_0 > 0$ and continuous functions $\eta : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ and $\psi : (-\epsilon_0, \epsilon_0) \to [\phi]^\perp$ such that $\eta(0) = \lambda_1$, $\psi(0) = 0$ and every nontrivial solution of $P(\lambda, u) = 0$ in a small neighborhood of $(\lambda_1, 0)$ is of the form $(\lambda_\epsilon, u_\epsilon) = (\eta(\epsilon), \epsilon\phi + \epsilon\psi(\epsilon))$.*

Proof We shall prove that the operator P satisfies all the hypotheses of the local bifurcation theorem, see (M. Crandall and P. H. Rabinowitz, 1971).

(i) The operator $P_u(\lambda_1, 0)$ is linear, compact, selfadjoint and holds that $P_u(\lambda_1, 0)\phi = 0$, if and only if $\phi \in \mathcal{V}$ is a solution of the (3).

Therefore $N(P_u(\lambda_1,0)) = [\phi]$, where ϕ is the principal eigenvalue of (3). So $\psi \in R(P_u(\lambda_1,0))$, if and only if there exists $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $<\psi,\phi> = <P_u(\lambda_1,0)w,\phi>$. But selfadjointness of $P_u(\lambda_1,0)$ implies that

$$<P_u(\lambda_1,0)w,\phi> = <w, P_u(\lambda_1,0)\phi> = <w,0> = 0.$$

Hence $R(P_u(\lambda_1,0)) = [\phi]^{\perp}$.

(ii) Let $w \in N(P_u(\lambda_1,0)) \bigcap R(P_u(\lambda_1,0))$. Then $P_u(\lambda_1,0)w = 0$ and there exists $\psi \in \mathcal{D}^{1,2}$, such that $<P_u(\lambda_1,0)\psi, w> = <w,w>$. Again the selfadjointness of the operator $P_u(\lambda_1,0)$ implies that $0 = <\psi, P_u(\lambda_1,0)w> = <w,w>$. So

$$<w,w> = ||w||_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^N} |\nabla w|^2\, dx\right)^{1/2} = 0.$$

We conclude that $w \equiv 0$, almost everywhere in $\mathbb{R}^N$, i.e.,

$$N(P_u(\lambda_1,0)) \bigcap R(P_u(\lambda_1,0)) = \{0\}.$$

Also, it is easy to see that

$$N(P_u(\lambda_1,0)) \oplus R(P_u(\lambda_1,0)) = \mathcal{D}^{1,2}(\mathbb{R}^N).$$

(iii) Finally, we have that $P_{\lambda u}(\lambda_1,0)\phi \notin R(P_u(\lambda_1,0))$ (transversality condition), since

$$<P_{\lambda u}(\lambda_1,0)\phi,\phi> = -\int_{\mathbb{R}^N} g\,\phi^2 dx < 0$$

and the proof is completed. $\diamondsuit$

In the remaining part of this section we give some results concerning the sign of the solutions of (1) near the bifurcation point $(\lambda_1, 0)$.

THEOREM 4.4. *Let $(\lambda_\epsilon, u_\epsilon)$ be the solutions of equation (1)–(2), as they are defined in Theorem 4.3. Then there exists $\epsilon_0 > 0$, such that $u_\epsilon(x) \geq 0$, almost everywhere in $\mathbb{R}^N$ and $\epsilon \in (0,\epsilon_0]$.*

Proof We proceed by contradiction, i.e., suppose that the assertion of the theorem is not true. Let (λ_n, u_n) be a sequence of solutions of (1), as they are defined in Theorem 4.3, such that $(\lambda_n, u_n) \to (\lambda_1, 0)$. We denote by $\tilde{u}_n, = u_n/||u_n||$ the normalization of u_n. Then there exists a subsequence of $\tilde{u}_n$, which again we denote by $\tilde{u}_n$, such that it converges weakly in $D^{1,2}$ to some function $\tilde{u}_0$. From equation (1) we have that

$$\int \frac{|\nabla u_n|^2}{\sqrt{N(u_n)}}\, dx = \lambda_n \int g(x)\, f(u_n)\, u_n\, dx. \qquad (2)$$

The equation (2), from Lemma 4.2, can be written as

$$\int |\nabla \tilde{u}_n|^2\, dx + O(||u_n||^2) = \lambda_n \int g(x)\, \tilde{u}_n^2\, dx + O(||u_n||^2). \tag{3}$$

The compactness of the operator M (see Lemma 2.3) implies that (3) in the limit gives

$$\int |\nabla \tilde{u}_0|^2\, dx = \lambda_1 \int g(x)\, \tilde{u}_0^2\, dx.$$

So, the function $\tilde{u}_0$ must be the eigenfunction ϕ.

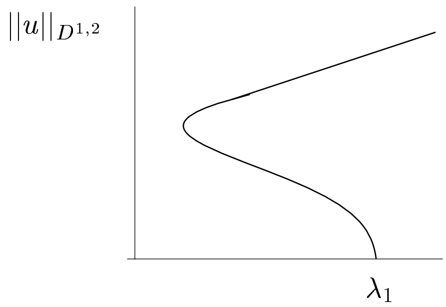

Figure 1. Bifurcation Diagram

Next we introduce the following notation $\mathcal{U}_n^- =: \{x \in \mathbb{R}^N : \tilde{u}_n(x) < 0\}$. It is clear that

$$\int_{\mathcal{U}_n^-} |\nabla \tilde{u}_n^-|^2\, dx + O(||u_n||^2_{\mathcal{U}_n^-}) = \lambda_n \int_{\mathcal{U}_n^-} g(x)|\tilde{u}_n^-|^2\, dx + O(||u_n||^2_{\mathcal{U}_n^-}).$$

Hölder's inequality and relation (1), imply that

$$||\tilde{u}_n^-||^2_{1,2} - O(||u_n||^2_{\mathcal{U}_n^-}) \leq c_0 ||g(x)||_{L^{N/2}(\mathcal{U}_n^-)} ||\tilde{u}_n^-||^2_{1,2} + O(||u_n||^2_{\mathcal{U}_n^-}),$$

or, equivalently

$$c_1 \leq ||g(x)||_{L^{N/2}(\mathcal{U}_n^-)}, \tag{4}$$

where the constant c_1 is independent from u_n. Then we deduce the existence of a constant $M_0 > 0$, large enough, such that

$$|(\mathcal{U}_n^- \cap B_M(0)| \geq c_2,$$

for all $M > M_0$, and the constant $c_5 > 0$ is independent from λ_n and u_n. Following a similar procedure as in (P. Drábek and Y. X. Huang, 1997, Lemma 2.3), which is based on the Egorov's Theorem, we conclude by contradiction that the functions $\tilde{u}_n$ are nonnegative, for n large enough. Hence, holds that $u_n \geq 0$, for all $(\lambda, u_n) \in C \cap B_\eta(\lambda_1, 0)$, with $\eta > 0$, small enough. So we have a bifurcation diagramm from the principal eigenvalue λ_1, which may look like the one in Figure 1. $\diamondsuit$

REMARK 4.1. *We notice that all the arguments, obtained in this work, are applicable to the bounded domain case and similar results may be obtained.*

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