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Global attractor for the weakly damped driven Schrödinger equation in $H^2(\mathbb{R})$

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Abstract. We discuss the asymptotic behaviour of the Schrödinger equation

$$iu_t + u_{xx} + i\alpha u - k\sigma(|u|^2)u = f, \ \ x \in \mathbb{R}, \ \ t \geq 0, \quad \alpha, \ k > 0$$

with the initial condition $u(x, 0) = u_0(x)$. We prove existence of a global attractor in $H^2(\mathbb{R})$, by using a decomposition of the semigroup in weighted Sobolev spaces to overcome the noncompactness of the classical Sobolev embeddings.

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1 Introduction

The aim in this note is to show existence of global attractor for the initial value problem

$$iu_t + u_{xx} + i\alpha u - k\sigma(|u|^2)u = f, \ \ \alpha, k > 0, \ x \in \mathbb{R}, \ t \geq 0, \tag{1.1}$$

$$u(x, 0) = u_0(x). \tag{1.2}$$

The zero order dissipation term ($\alpha > 0$) is considered as the weak damping. We assume that the function σ satisfy the growth condition

$$\sigma(s) \leq cs^{\gamma-1}, \quad \text{for some } \ c > 0, \ \ 0 \leq \gamma < \infty. \tag{$\mathcal{C}$}$$

The Cauchy poblem for the nonlinear Schrödinger equation and the asymptotic behaviour of solutions have been treated by many authors. We refer on the monograph [6] and on [7], [12], [15], [21], [24], [27] for results on existence, non-existence and blow-up of solutions. Questions on the existence of global attractors for the problem (1.1)–(1.2) have been treated first by J.M. Ghidaglia [10], where the existence of a maximal attractor in the weak topologies of $H^1(0,L)$ and $H^2(0,L)$ is proved. In [2] it is proved that there exist a maximal attractor in the strong topology of $H^1(\Omega)$, when Ω is a bounded domain of $\mathbb{R}^2$. Existence of global attractors in $H^1(\Omega)$, when Ω is a bounded interval, is also given in [13], based on a specific decomposition of the semigroup. In [13] important results for the regularity of the attractor are also included. In fact it is shown that, if $f \in C^\infty$ the global attractor is in $H^k(\Omega)$, for any $k \geq 2$. In general, when the equations are considered in unbounded domains the difficulties arise by the lack of compacteness of the embeddings of the classical Sobolev spaces. However, various methods and techniques have been introduced by several authors, to overcome this difficulty. In [4] weighted Sobolev spaces have been introduced for the study of attractors for the parabolic equations defined in all of $\mathbb{R}^N$. Nevertheless, one has to restrict initial data on weighted spaces, when the work is done directly on this functional setting. For results on the existence of global attractors in weighted spaces we also mention the papers [1] and [8]. In [5] a method using the energy equation is introduced that yields existence of attractors in classical Sobolev spaces. This energy method is also used in [11] for the study of the KdV equation and in [2], to obtain compactness of the trajectories for the problem (1.1)–(1.2) in $H^1(\Omega)$, where Ω is a bounded subset of $\mathbb{R}^N$. In [28] an energy equation is derived and the energy method is applied for the proof of existence of globall atractor in $H^2_{per}(\mathbb{R})$. Using Strichartz estimates E. Feiresl [9] has proved existence of the global attractor in the classical energy space for semilinear wave equations in $\mathbb{R}^N$. A later approach is based on the use of homogeneous Sobolev spaces and their compact embeddings in the appropriate weighted-L^2 spaces. This general space setting (more general than the classical one) is introduced in our joint works [17, 18, 19, 20] for the study of a semilinear wave equation in $\mathbb{R}^N$ and presented in details in [16]. In [23] the existence of the global attractor in the strong topology of $H^1(\mathbb{R}^N)$, $N \leq 3$ for the problem (1.1)–(1.2) is proved. This is achieved by using certain weighted Sobolev spaces as an intermediate tool. In the present work using a decomposition $\mathcal{S}(t) = \mathcal{S}_1(t) + \mathcal{S}_2(t)$ of the semigroup, weighted Sobolev spaces are used to achieve uniform compactness for $\mathcal{S}_2(t)$ and hence existence of a global attractor *in the more regular space* $H^2(\mathbb{R})$. Note also that no further restrictions were imposed on the initial conditions (1.2) and the exterior force f. The choice of a standard type nonlinearity and the dimension $n = 1$ are made just for technical reasons. The results achieved can be extended to higher dimensions to equations involving more general nonlinearities (see [10], [23]). We would like to mention that the same result may also be proved by using the method developed in [28], since for the application of the energy method it is not necessary to consider compact embeddings of the function spaces involved. The paper is organized as follows. In

Section 2, we present some preliminary facts to be used in the sequel. In Section 3, the proof of the existence of an absorbing set in $H^2(\mathbb{R})$ is developed. In the final Section 4, the construction of the global attractor is presented.

2 Preliminaries

In this section, we recall some basic results related to the functional setting of the problem (1.1)–(1.2). We consider the differential operator

$$Au = -iu_{xx}, \tag{2.3}$$

on the Hilbert space $L^2(\mathbb{R})$ with domain of definition $D(A) = H^2(\mathbb{R})$. We have that the operator iA is self adjoint in $L^2(\mathbb{R})$ (see also [25, Lemma 5.2, Chapt 7]). Then we have the following local existence result ([6]).

Theorem 2.1 *Let $T > 0$, $u_0 \in H^1(\mathbb{R})$ and $f \in L^2(\mathbb{R})$ and assume that condition $(\mathcal{C})$ holds. Then there exists a unique solution for the problem* (1.1)–(1.2) *such that*

$$u \in C([0,T], H^1(\mathbb{R})) \cap C^1([0,T], H^{-1}(\mathbb{R})).$$

In addition, if $u_0 \in H^2(\mathbb{R})$ then

$$u \in C([0,T], H^2(\mathbb{R})) \cap C^1([0,T], L^2(\mathbb{R})).$$

We can define by Theorem 2.1 the semigroup of operators

$$\begin{aligned} \mathcal{S}(t) : H^2(\mathbb{R}) &\to H^2(\mathbb{R}), \text{ such that} \\ \mathcal{S}(t) : u_0 &\to u(t), \ \ t \geq 0. \end{aligned}$$

To study the properties of the semigroup $\mathcal{S}(t)$ we use weighted Sobolev spaces introduced in [4]. For $w(x) = (1 + |x|^2)^\gamma$, $\gamma > 0$, the weighted-L^2 space is defined by

$$L^2_w(\mathbb{R}) := \left\{ u : \mathbb{R} \to \mathbb{C} \ : \ \|u\|^2_{L^2_w} := \int_{\mathbb{R}} g|u|^2 \, dx < \infty \right\}. \tag{2.4}$$

The weighted Sobolev spaces are introduced as follows

$$H^m_w(\mathbb{R}) := \left\{ u \in L^2_w(\mathbb{R}) \ : \ \|u\|^2_{H^m_w} := \sum_{|a| \leq m} \|\partial^a u\|^2_{L^2_w} < \infty \right\}, \tag{2.5}$$

where ∂^a is the usual multiindex notation. For $g(x)$ as above we have the following compactness lemma (see also [4, Lemma 2.16]) that will be used in the sequel.

Lemma 2.3 *The embedding $H^3(\mathbb{R}^N) \cap H^2_w(\mathbb{R}^N) \hookrightarrow H^2(\mathbb{R}^N)$ is compact.*

Proof. For completeness, we just sketch the proof. Set $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Let $\mathcal{F}$ be a bounded set in $H^3(\mathbb{R}^N) \cap H^2_w(\mathbb{R}^N)$. We may easily see that for every $u \in \mathcal{F}$

$$\|u\|_{H^2(\mathbb{R}^N \setminus B_R)} \leq C_{\mathcal{F}}(1+R)^{-\gamma}. \tag{2.6}$$

Consider the set

$$\mathcal{F}^R := \{u \in \mathcal{F} : u|_{B_R}\}.$$

Since $H^3(B_R) \hookrightarrow H^2(B_R)$, compactly, there exists a finite covering of $\epsilon/2$-balls of $H^3(B_R)$ for the set $\mathcal{F}^R \subset H^2(B_R)$. From (2.6) we see that an ϵ-covering of the set $\mathcal{F}$ in $H^2(\mathbb{R}^N)$ can be obtained by choosing R large enough such that $C_{\mathcal{F}}(1+R)^{-\gamma} \leq \epsilon/2$. □

To estimate several quantities, we shall use the weight function $g(x) = (1+\epsilon_0|x|^2)^{\gamma}$, for some $\epsilon_0 \in (0,\ 1]$. We have the following equivalence of norms

$$C_{\epsilon_0}^{-1}\|g^{\frac{1}{2}}u\|_2 \leq \|u\|_{L^2_w} \leq C_{\epsilon_0}\|g^{\frac{1}{2}}u\|_2. \tag{2.7}$$

3 Existence of absorbing set in $H^2(\mathbb{R})$

In this section we show the existence of an absorbing ball in $H^2(\mathbb{R})$ by obtaining uniform in time estimates. We start with the following lemma.

Lemma 3.1 *Let $f \in L^2(\mathbb{R})$. Then there exists a constant M_1 independent of t such that as $t \to \infty$*

$$\|u(t)\|_2^2 \leq M_1$$

Proof. We multiply (1.1) by $\overline{u}$, we integrate over $\mathbb{R}$ and we keep the imaginary parts. Then we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \alpha\|u\|_2^2 = Im(\overline{u},\ f) \leq \frac{\alpha}{2}\|u\|_2^2 + \frac{\|f\|_2^2}{2\alpha}$$

and the result comes by application of the Gronwall's lemma. □

Lemma 3.2 *Let $f \in L^2(\mathbb{R})$. Then there exists a constant M_2 independent of t such that as $t \to \infty$*

$$\|u_x(t)\|_2^2 \leq M_2$$

Proof. Now we multiply equation (1.1) by $-\overline{u_t} - \alpha\overline{u}$. This time we keep the real parts and we get the equation

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|u_x\|_2^2 + k\Sigma(|u|^2) + 2Re(f,\ \overline{u})\} + k\alpha\Sigma(|u|^2)\\ &\quad+\alpha\|u_x\|_2^2 + \alpha Re(f,\ \overline{u}) = 0.\end{aligned}$$

where $\Sigma(s) = \int_0^s \sigma(r)dr$. We have the inequality

$$\frac{1}{2}\frac{d}{dt}J_1(u) + \alpha J_1(u) \leq \frac{\alpha}{2}\|f\|_2^2 + \frac{1}{2\alpha}\|u\|_2^2, \tag{3.8}$$

where

$$J_1(u) = \|u_x\|_2^2 + \Sigma(|u|^2) + 2Re(f,\ \overline{u}).$$

We use estimate of Lemma 3.1 and Gronwall's inequality to obtain the result. □

Lemma 3.3 *Let $f \in L^2(\mathbb{R})$. Suppose also that*

$$k < \frac{\alpha}{3c_*cM}, M = \max\{M_1^{2\gamma+1}, M_2^{2\gamma+1}\} \tag{3.9}$$

Then there exists a constant M_3 independent of t such that as $t \to \infty$,

$$\|u_{xx}(t)\|_2^2 \leq M_3.$$

Proof. We multiply (1.1) by $\overline{u}_{xxt} + \alpha\overline{u}_{xx}$, integrate over $\mathbb{R}$ and keep the real parts to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|u_{xx}\|_2^2 - 2kRe\int_{\mathbb{R}}\sigma(|u|^2)u\overline{u}_{xx}dx - 2Re\int_{\mathbb{R}}f\overline{u}_{xx}dx\right\} \\
&\quad +\alpha\|u_{xx}\|_2^2 - \alpha kRe\int_{\mathbb{R}}\sigma(|u|^2)u\overline{u}_{xx}dx - \alpha Re\int_{\mathbb{R}}f\overline{u}_{xx}dx \\
&= -kRe\int_{\mathbb{R}}\sigma'(|u|^2)|u|^2u_t\,\overline{u}_{xx}dx - kRe\int_{\mathbb{R}}\sigma'(|u|^2)u^2\overline{u}_t\,\overline{u}_{xx}dx \\
&\quad -kRe\int_{\mathbb{R}}\sigma(|u|^2)u_t\overline{u}_{xx}dx.
\end{aligned} \tag{3.10}$$

To estimate the integral terms of the right-hand side of (3.10), we insert equation (1.1). By using $(\mathcal{C})$, we observe that

$$\begin{aligned}
k\int_{\mathbb{R}}|\sigma(|u|^2)|\,|u_t|\,|u_{xx}|dx \ \leq\ & ck\int_{\mathbb{R}}|u|^{\gamma_1}|u_{xx}|^2dx \\
& + \alpha ck\int_{\mathbb{R}}|u|^{\gamma_2}|u_{xx}|dx + ck^2\int_{\mathbb{R}}|u|^{\gamma_3}|u_{xx}|dx \\
& + ck\int_{\mathbb{R}}|u|^{\gamma_1}\,|f|\,|u_{xx}|dx,
\end{aligned}$$

where $\gamma_j(\gamma) > 0$, $j = 1, 2, 3$. Using Gagliardo-Nirenberg inequality

$$\|u\|_\infty \leq c_*\|u\|_2^{\frac{1}{2}}\|u_x\|_2^{\frac{1}{2}}, \ \ u \in L^2(\mathbb{R}) \cap H^2(\mathbb{R}),$$

and Lemmas 3.1 and 3.2, we deduce that

$$
\begin{aligned}
ck\int_{\mathbb{R}}|u|^{\gamma_1}|u_{xx}|^2dx &\leq ck\|u\|_\infty^{\gamma_1}\|u_{xx}\|_2^2 \\
&\leq c_*ck\left\{\frac{1}{2}\|u\|_2^{\gamma_1}+\frac{1}{2}\|u_x\|_2^{\gamma_1}\right\}\|u_{xx}\|_2^2 \\
&\leq C_*\left\{\frac{M_1^{\gamma_1}}{2}+\frac{M_2^{\gamma_1}}{2}\right\}\|u_{xx}\|_2^2 \\
&\leq \frac{C_*M}{2}\|u_{xx}\|_2^2, \quad M=\max\{M_1^{\gamma_1},\, M_2^{\gamma_1}\} \\
ck\int_{\mathbb{R}}|u|^{\gamma_1}\,|f|\,|u_{xx}|dx &\leq \alpha_0\|u_{xx}\|_2^2+c_1(\alpha,k,M)\|f\|_2^2,
\end{aligned}
$$

where $C_* = c_*ck$ and α_0 depends on α. The rest of the integrals that appear in (3.10) can be estimated similarly. From this procedure we may derive the inequality

$$\frac{1}{2}\frac{d}{dt}J_2(u)+\alpha_*J_2(u)\leq M_*,$$

$$J_2(u)=\|u_{xx}\|_2^2-2kRe\int_{\mathbb{R}}\sigma(|u|^2)u\overline{u}_{xx}dx-2Re\int_{\mathbb{R}}f\overline{u}_{xx}dx,$$

taking into account that $\alpha_*=\alpha/2-(3c_*ckM)/2>0$ which justifies assumption (3.9). The constant M_* depends on $\alpha, c_*, k, M, \|f\|_2$. The lemma is proved by applying Gronwall's ineqality in (3). □

Lemma 3.4 *Let $f\in L^2(\mathbb{R})$. Then there exists an absorbing set in $H^2(\mathbb{R})$, for the semigroup $\mathcal{S}(t)$ defined by the problem* (1.1)–(1.2).

Proof. By using the uniform in time estimates of Lemmas 3.1, 3.2 and 3.3 we may show that there exists a time t_0 and some $\rho_0>0$, such that for every fixed $\rho_1\geq\rho_0$,

$$\|u(t)\|_{H^2}^2\leq\rho_1^2, \text{ for every } t\geq t_0.$$

The ball $B_{H^2}(0,\,\rho_1)$, defines an absorbing set on $H^2(\mathbb{R})$. □

4 Construction of the global attractor

In this section we prove that the semigroup associated with the problem (1.1)–(1.2) posses a global attractor in $H^2(\mathbb{R})$. The aim is to verify the classical theorem on existence of attractors (see [26, Theorem 1.1]. Motivated by [23], we decompose the solution of the original problem as $u = \varphi + \psi$, where φ, ψ satisfy the system of equations

$$\begin{aligned} &i\varphi_t + \varphi_{xx} + i\alpha\varphi = k\sigma(|u|^2)u - i\delta\psi_{xx} + (f - f^R), \\ &\varphi(x,0) = u_0,\ f^R \in C_0^\infty(\mathbb{R}) \end{aligned} \tag{4.1}$$

$$\begin{aligned} &i\psi_t + \psi_{xx} + i\alpha\psi - i\delta\psi_{xx} = f^R, \\ &\psi(x,0) = 0. \end{aligned} \tag{4.2}$$

Concerning the solution of problem (4.2), we have the following lemma.

Lemma 4.1 *The solution* ψ *of the initial value problem* (4.2) *is in* L^∞ $[\mathbb{R}^+; H^m(\mathbb{R})], m \geq 0$.

Proof. Denote by $\psi_{(m)}$ the m-order (weak) derivative of the function ψ. We multiply (4.2) by $(-1)^m\overline{\psi}_{(2m)}$ and keep imaginary parts to obtain the equation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\psi_{(m)}\|_2^2 + \alpha\|\psi_{(m)}\|_2^2 + \delta\|\psi_{(m+1)}\|_2^2 \\ &\quad = (-1)^m Im \int_{\mathbb{R}} f^R\overline{\psi}_{(2m)}\, dx. \end{aligned} \tag{4.3}$$

We have that

$$\left|\int_{\mathbb{R}} f^R\overline{\psi}_{(2m)}\, dx\right| \leq \frac{1}{2\delta}\|f^R_{(m-1)}\|_2^2 + \frac{\delta}{2}\|\psi_{(m+1)}\|_2^2 \tag{4.4}$$

Relations (4.3), (4.4) and Gronwall's inequality imply the estimate

$$\|\psi_{(m)}\|_2^2 \leq c(\delta)(1 - exp(-\alpha t))\|f^R_{(m-1)}\|_2^2.$$

In the case $m = 0$, $f^R \in H^{-1}(\mathbb{R})$ and $\|f^R_{(-1)}\|_2 \equiv \|f^R\|_{H^{-1}}$. □

Next lemma shows that the solution ψ of the initial value problem (4.2) is bounded, uniformly in time in $H^2_w(\mathbb{R})$.

Lemma 4.2 *The solution* ψ *of the initial value problem* (4.2) *is in* $L^\infty[\mathbb{R}^+; H^2_w(\mathbb{R})]$.

Proof. We multiply (4.2) by $g\overline{\psi}$ and keep imaginary parts. The obtained equation is

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|g^{1/2}\psi\|_2^2 + \alpha\|g^{1/2}\psi\|_2^2 + \delta\|g^{1/2}\psi_x\|_2^2 = \delta Re Z_1(\psi) \\ &\quad + Im Z_1(\psi) - Im Z_2(\psi), \end{aligned} \tag{4.5}$$

where $Z_1(\psi) = \int_{\mathbb{R}} g_x \psi_x \overline{\psi} dx$ and $Z_2(\psi) = \int_{\mathbb{R}} g f^R \overline{\psi} dx$. We observe that the following estimates hold

$$|Im Z_1(\psi)| \leq c\epsilon_0 \int_{\mathbb{R}} g|\psi_x||\psi|\, dx \leq \frac{\alpha}{6}\|g^{1/2}\psi\|_2^2 + c_1{\epsilon_0}^2\|g^{1/2}\psi_x\|_2^2, \tag{4.6}$$

$$|Im Z_2(\psi)| \leq \frac{\alpha}{6}\|g^{1/2}\psi\|_2^2 + c_2{\epsilon_0}^2\|g^{1/2} f^R\|_2^2. \tag{4.7}$$

From (4.5), (4.6) and (4.7) we get for ϵ_0 sufficiently small

$$\frac{1}{2}\frac{d}{dt}\|g^{1/2}\psi\|_2^2 + \frac{\alpha}{2}\|g^{1/2}\psi\|_2^2 + \delta_0\|g^{1/2}\psi_x\|_2^2 \leq c\|g^{1/2} f^R\|_2^2, \delta_0 > 0.$$

Since f^R has compact support the term $\|g^{1/2} f^R\|_2^2$ is bounded. We apply Gronwall's lemma to obtain that

$$\|g^{1/2}\psi\|_2^2 \leq c_3(1 - exp(-\alpha t))\|g^{1/2} f^R\|_2^2. \tag{4.8}$$

Multiply equation (4.2) by $-g_{\epsilon_0}\overline{\psi}_{xx}$, keep again imaginary parts to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|g^{1/2}\psi_x\|_2^2 + \alpha\|g^{1/2}\psi_x\|_2^2 + \delta\|g^{1/2}\psi_{xx}\|_2^2 + iIm Z_3(\psi,\psi_t)\\ &\quad +\alpha Re Z_4(\psi) = -Im Z_5(\psi),\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}Z_3(\psi,\psi_t) &= \int_{\mathbb{R}} \psi_t \overline{\psi}_x g_x dx, Z_4(\psi) = \int_{\mathbb{R}} \psi \overline{\psi}_x g_x dx,\\ Z_5(\psi) &= \int_{\mathbb{R}} g f^R \overline{\psi}_{xx} dx.\end{aligned}$$

Inserting equation (4.2) in $Z_3(\psi,\psi_t)$ we get

$$\begin{aligned}iIm Z_3(\psi,\psi_t) &= -\alpha Re\int_{\mathbb{R}} g_x \psi \overline{\psi}_x dx - Im\int_{\mathbb{R}} g_x \psi_{xx}\overline{\psi}_x dx\\ &\delta Re\int_{\mathbb{R}} g_x \psi_{xx}\overline{\psi}_x dx + \int_{\mathbb{R}} g_x f^R \overline{\psi}_x dx.\end{aligned} \tag{4.10}$$

All the integral terms that appear in (4.10) can be estimated by using Young's inequality as in (4.6), (4.7) and eventually

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|g^{1/2}\psi_x\|_2^2 + \frac{\alpha}{2}\|g^{1/2}\psi_x\|_2^2 + \delta_1\|g^{1/2}\psi_{xx}\|_2^2\\ &\quad \leq c\|g^{1/2} f^R\|_2^2, \quad \text{or}\end{aligned}$$

$$\|g^{1/2}\psi_x\|_2^2 \leq c_4(1 - exp(-\alpha t))\|g^{1/2} f^R\|_2^2. \tag{4.11}$$

We proceed by multiplying equation (4.2) by $g\overline{\psi}_{xxxx}$ keep imaginary parts, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g^{1/2}\psi_{xx}\|_2^2 + \alpha\|g^{1/2}\psi_{xx}\|_2^2 + \delta\|g^{1/2}\psi_{xxx}\|_2^2 + ImI_1(\psi,\psi_t)\\ -ImI_2(\psi,\psi_t) + \alpha ReI_3(\psi) - \alpha ReI_4(\psi)\\ -ImI_5(\psi) + \delta ReI_5(\psi) - ImI_6(\psi)\\ -ImI_7(\psi),\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}I_1(\psi,\psi_t) &= \int_{\mathbb{R}} i\psi_{tx}g_x\overline{\psi}_{xx}dx,\ \ I_2(\psi,\psi_t) = \int_{\mathbb{R}} i\psi_t g_x\overline{\psi}_{xxx}dx,\\ I_3(\psi) &= \int_{\mathbb{R}} g_x\psi_x\overline{\psi}_{xx}dx,\ \ I_4(\psi) = \int_{\mathbb{R}} g_x\psi\overline{\psi}_{xxx}dx,\\ I_5(\psi) &= \int_{\mathbb{R}} g_x\psi_{xx}\overline{\psi}_{xxx}dx,\ \ I_6(\psi) = \int_{\mathbb{R}} g_x f^R\overline{\psi}_{xxx}dx,\\ I_7(\psi) &= \int_{\mathbb{R}} g f_x^R\overline{\psi}_{xxx}dx.\end{aligned}$$

In order to estimate the integral $I_1(\psi,\psi_t)$, we differentiate equation (4.2) in x (see [26]). Then ψ_x satisfies the equation

$$i\psi_{tx} + i\alpha\psi_x + \psi_{xxx} - i\delta\psi_{xxx} = f_x^R. \tag{4.13}$$

We insert 4.13 in $I_1(\psi,\psi_t)$ to get

$$\begin{aligned}I_1(\psi) = -\alpha Re\int_{\mathbb{R}} g_x\psi_x\overline{\psi}_{xx}dx - Im\int_{\mathbb{R}} g_x\psi_{xxx}\overline{\psi}_{xx}dx\\ +\delta Re\int_{\mathbb{R}} g_x\psi_{xxx}\overline{\psi}_{xx}dx + Im\int_{\mathbb{R}} g_x f_x^R\overline{\psi}_{xx}dx.\end{aligned}$$

Similarly, we insert equation (4.2) in the integral term $I_2(\psi,\psi_t)$, and we have that

$$\begin{aligned}I_2(\psi) = \alpha Re\int_{\mathbb{R}} g_x\psi\overline{\psi}_{xxx}dx + Im\int_{\mathbb{R}} g_x\psi_{xx}\overline{\psi}_{xxx}dx\\ -\delta Re\int_{\mathbb{R}} g_x\psi_{xx}\overline{\psi}_{xxx}dx - Im\int_{\mathbb{R}} g_x f^R\overline{\psi}_{xxx}dx.\end{aligned}$$

All the integrals in (4.12) can be estimated by appropriate use of Young's inequality. As an example, we give the estimate

$$\begin{aligned}\left|Im\int_{\mathbb{R}} g_x\psi_{xx}\overline{\psi}_{xxx}dx\right| &\le c\epsilon_0\int_{\mathbb{R}} |g|\,|\psi_{xx}|\,|\psi_{xxx}|dx\\ &\le c(\alpha)\|g^{1/2}\psi_{xx}\|_2^2 + \epsilon_0^2 c(\delta)\|g^{1/2}\psi_{xxx}\|_2^2.\end{aligned}$$

for sufficiently small ϵ_0. The result is the following inequality

$$\frac{1}{2}\frac{d}{dt}\|g^{1/2}\psi_{xx}\|_2^2 + \frac{\alpha}{2}\|g^{1/2}\psi_{xx}\|_2^2 + \delta_2\|g^{1/2}\psi_{xxx}\|_2^2 \leq M, \tag{4.14}$$

where M depends on $\|g^{1/2}f^R\|_2, \|g^{1/2}f_x\|_2, \alpha, \delta$. Clearly, using (2.7) and inequalities (4.8), (4.11) and (4.14) we obtain the final estimate, which is

$$\|\psi(t)\|^2_{H^2_w} \leq M_1.$$

□

From Lemma 4.1 we have that $\psi \in L^\infty[\mathbb{R}^+; H^3(\mathbb{R})]$. Since the embedding $L^\infty[\mathbb{R}^+; H^3(\mathbb{R})] \hookrightarrow L^\infty[\mathbb{R}^+; H^2(\mathbb{R})]$ is continuous and $u \in L^\infty[\mathbb{R}^+; H^2(\mathbb{R})]$ by Lemma 3.4, we have that for the solution $\phi = u - \psi$ of the initial value problem (4.1) it holds that

$$\|\varphi\|_{L^\infty[\mathbb{R}^+;H^2(\mathbb{R})]} \leq \|u\|_{L^\infty[\mathbb{R}^+;H^2(\mathbb{R})]} + \|\psi\|_{L^\infty[\mathbb{R}^+;H^2(\mathbb{R})]}.$$

More precisely it is possible to obtain the estimate given by the following lemma.

Lemma 4.3 *Let condition* (3.9) *be satisfied. Then for the solution of the problem* (4.1) *there exist* α_0, $K^* > 0$ *such that*

$$\frac{d}{dt}J_3(\varphi) + \alpha_0 J_3(\varphi) \leq K^*, \tag{4.15}$$

where $J_3(u)$ *is defined as*

$$\begin{aligned} J_3(\varphi) = \|\varphi_{xx}\|_2^2 - 2kRe\int_{\mathbb{R}} \sigma(|u|^2)u\overline{\varphi}_{xx}dx \\ -2\delta Im\int_{\mathbb{R}} \psi_{xx}\overline{\varphi}_{xx}dx - 2Re\int_{\mathbb{R}} F\overline{\varphi}_{xx}dx. \end{aligned}$$

Proof. Set $F = f - f^R$. Following the same procedure as in Lemma 3.3 we obtain the equation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}J_3(\varphi) + \alpha\|\varphi_{xx}\|_2^2 - \alpha kRe\int_{\mathbb{R}} \sigma(|u|^2)u\overline{\varphi}_{xx}dx \\ &-\alpha\delta Im\int_{\mathbb{R}} \psi_{xx}\overline{\varphi}_{xx}dx - \alpha Re\int_{\mathbb{R}} F\overline{\varphi}_{xx}dx = -\delta Im\int_{\mathbb{R}} \psi_{xxt}\overline{\varphi}_{xx}dx \\ &-kRe\int_{\mathbb{R}} \sigma'(|u|^2)u^2\overline{u}_t\,\overline{\varphi}_{xx}dx - kRe\int_{\mathbb{R}} \sigma(|u|^2)u_t\overline{\varphi}_{xx}dx \\ &-kRe\int_{\mathbb{R}} \sigma'(|u|^2)|u|^2u_t\,\overline{\varphi}_{xx}dx. \end{aligned} \tag{4.16}$$

Differentiate (4.13) in x. Then ψ_{xx} satisfy equation

$$i\psi_{txx} + i\alpha\psi_{xx} + \psi_{xxxx} - i\delta\psi_{xxxxx} = f^R_{xx}. \tag{4.17}$$

Hence we may obtain that

$$\begin{aligned}\delta \int_{\mathbb{R}} |\psi_{xxt}|\,|\varphi_{xx}|dx \;&\leq\; \alpha\delta \int_{\mathbb{R}} |\psi_{xx}|\,|\varphi_{xx}|dx \\ &\quad +(\delta+\delta^2)\int_{\mathbb{R}} |\psi_{xxxx}|\,|\varphi_{xx}|dx \;+\delta\int_{\mathbb{R}} |f^R_{xx}|\,|\varphi_{xx}|dx \\ &\leq\; \alpha_1\|\varphi_{xx}\|_2^2 + C_1\|\psi_{xxxx}\|_2^2 + C_2\|f^R_{xx}\|_2^2\end{aligned}$$

and the constants α_1, C_1, C_2 depend on α, δ. Note that $\|\psi_{xxxx}\|_2$ is bounded by Lemma 4.1. The rest of the indefinite-sign integrals of (4.16) can be estimated exactly as in Lemma 3.3. Therefore

$$k\int_{\mathbb{R}} |\sigma(|u|^2)|\,|u_t|\,|\varphi_{xx}|dx \;\leq\; \alpha_1\|\varphi_{xx}\|_2^2 + C_3(\rho_1).$$

By inserting these estimates in (4.16), inequality (4.15) may be derived, for an appropriate choice of the constant α_1. □

We consider next the following stationary problem

$$\begin{aligned}\phi^s_{xx} + i\alpha\phi^s = k|u|^2u + (f-f^R) - i\delta\psi_{xx} := \mathcal{F},\\ \phi^s \in H^2(\mathbb{R}).\end{aligned} \tag{4.18}$$

Lemmas 3.4, 4.3 and the assumptions on f, f^R imply that $\mathcal{F}\in L^2(\mathbb{R})$. Classical arguments on existence and regularity of solutions of linear elliptic equations (see [22, Chapter II]) show existence of solution for the problem (4.18). In fact, if we multiply equation (4.18) by $\overline{\phi^s}$ and $-\overline{\phi^s}_x$, keep imaginary parts and by $\overline{\phi^s}_{xx}$, keep real parts and add the resulting equations we shall obtain for ϕ^s the estimate

$$\begin{aligned}\|\phi^s\|_{H^2} \leq K_1(\alpha,\delta,k)\|\psi_{xx}\|_2^2 \\ +K_2(\alpha,k,\rho_1) + K_3(\alpha,\delta)\|f-f^R\|_2^2,\end{aligned}$$

Lemma 4.4 *The solution φ of problem* (4.1) *and the solution ϕ^s of the stationary problem* (4.18) *satisfy the estitmate*

$$\|\varphi-\phi^s\|_{H^2} = \|u_0-\varphi^s\|_{H^2}\exp(-ct), \;\; c>0$$

Proof. The difference $z := \varphi - \phi^s$ is the solution of equation

$$iz_t + z_{xx} + i\alpha z = 0. \tag{4.19}$$

Multiply (4.19) consecutively by $\overline{z}$, $-\overline{z}_{xx}$, $\overline{z}_{xxxx}$ and keep imaginary parts and add the resulting equations. Then we have that

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{H^2} + \alpha\|z\|^2_{H^2} = 0.$$

The result follows by an application of Gronwall's Lemma. □

Finally, we state the main result of this work.

Theorem 4.4 *If $f \in L^2(\mathbb{R})$, then the semigroup $S(t)$ posseses a global attractor $\mathcal{A}$ in $H^2(\mathbb{R})$.*

Proof. Let u_0 be in a bounded set $\mathcal{B}$ of $H^2(\mathbb{R})$. We decompose $\mathcal{S}(t)$ as $\mathcal{S}(t) = \mathcal{S}_1(t) + \mathcal{S}_2(t)$ where $\mathcal{S}_1(t)u_0 = \varphi(t)$ and $\mathcal{S}_2(t)u_0 = \psi(t)$, the solutions of problems (4.1) and (4.2) respectively. Lemmas 4.1 and 4.2 imply that there exist $t_0 > 0$ such that the set $\mathcal{O}_2 := \cup_{t \geq t_0} \mathcal{S}_2(t)\mathcal{B}$ is in a bounded set of $H^3(\mathbb{R}) \cap H^2_w(\mathbb{R})$. From Lemma 2.3 the set $\mathcal{O}_2$ is relatively compact in $H^2(\mathbb{R})$. Consider the set $\mathcal{O} = \mathcal{O}_2 + \phi^s$. We may write the solution $u(t)$ of the problem (1.1) as $u(t) = (\varphi(t) - \phi^s) + (\psi(t) + \phi^s)$. Since $dist\,(\mathcal{S}_1(t)u_0,\ \phi^s) \to 0$ as $t \to \infty$ (Lemma 4.4), and $\mathcal{S}_2(t)u_0 + \phi^s \in \mathcal{O}$, we obtain that

$$dist\,(\mathcal{S}(t)\mathcal{B},\ \mathcal{O}) \to 0, \ \text{ as } \ t \to \infty.$$

Moreover it is clear that the set $\mathcal{O}$ is relatively compact in $H^2(\mathbb{R})$. The results of [14], [26] imply the existence of a global attractor for the semigroup $S(t)$. □

Remark 4.5 It is may me possible, by repeating inductively the calculations of Lemmas 3.3, 4.2, using Lemma 4.1 and the compactness of the embedding $H^{m+1}(\mathbb{R}) \cap H^m_w(\mathbb{R}) \hookrightarrow H^m(\mathbb{R})$, to show existence of global attractor for (1.1)–(1.2) in $H^m(\mathbb{R})$, $m > 2$ if $u_0 \in H^m(\mathbb{R})$ and $f \in H^{m-2}(\mathbb{R})$.

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