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# Multiplicity and regularity results for some quasilinear elliptic systems on $\mathbb{R}^N$

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## 1. Introduction

In this paper, we study the multiplicity and regularity of the solutions of the quasilinear elliptic system

$$\begin{aligned} -\Delta_p u &= \lambda a(x)|u|^{\gamma-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u, \quad x \in \mathbb{R}^N, \\ -\Delta_q v &= \lambda d(x)|v|^{\delta-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v, \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

$$u(x) > 0, v(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u(x) = 0, \tag{1.2}$$

where the $p$-Laplacian operator $\Delta_p u$ is $\Delta_p u =: div(|\nabla u|^{p-2}\nabla u)$. In addition, we assume that $1 < p < N$, $1 < q < N$, $\alpha \geqslant 0$ and $\beta \geqslant 0$. For the positive constants $\alpha$, $\beta$, $p$, $q$ and $N$ we consider that the following inequality is valid:

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1, \tag{1.3}$$

where $p^*$ and $q^*$ are the critical Sobolev exponents: $p^* = Np/(N-p)$ and $q^* = Nq/(N-q)$. Throughout this paper we assume that the following general hypothesis is satisfied.

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($\mathscr{H}$) $2 \leqslant \gamma \leqslant p^*$, $2 \leqslant \delta \leqslant q^*$, $a \in L^{p^*/(p^*-\gamma)}(\mathbb{R}^N)$, $d \in L^{q^*/(q^*-\delta)}(\mathbb{R}^N)$ and $b \in L^{\omega}(\mathbb{R}^N)$, where $\omega = p^*q^*/[p^*q^* - (\alpha+1)p^* - (\beta+1)q^*]$. Moreover, $a$, $d$ and $b$ are smooth functions at least of $C^{0,\eta}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $\eta \in (0,1)$.

In certain cases the coefficients $a$, $d$ and $b$ will satisfy some extra hypothesis, which is described as follows. Let the function $h : \mathbb{R}^N \to \mathbb{R}$. We say that $h$ satisfies the hypothesis

($\mathscr{H}_\infty$) if $h \in L^\infty(\mathbb{R}^N)$ and tends uniformly to zero, as $|x| \to \infty$, in the sense

$$\lim_{R\to\infty} \|h\|_{L^\infty(\mathbb{R}^N - B_R)} = 0,$$

($\mathscr{H}_+$) if there exists $\Omega \subseteq \mathbb{R}^N$, with $|\Omega| > 0$ such that $h(x) > 0$ for every $x \in \Omega$.

We also deal with the equation

$$-\Delta_p u = \lambda g(x)|u|^{\gamma-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

$$u(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \tag{1.5}$$

where $1 < p < N$, $2 \leqslant \gamma \leqslant p^*$ and $\gamma \neq p$. Throughout the paper we assume that $g$ satisfies the following condition:

($\mathscr{G}$) $g$ is at least a $C^{0,\eta}_{\mathrm{loc}}(\mathbb{R}^N)$-function, for some $\eta \in (0,1)$ and $g \in L^{p^*/(p^*-\gamma)}(\mathbb{R}^N)$.

Such kinds of systems have been studied by many authors both in bounded and unbounded domains. We mention the papers [1,3,6] and the references therein.

This paper is organized as follows: in Section 2, we recall the homogeneous space $\mathscr{D}^{1,p}(\mathbb{R}^N)$, we introduce the necessary operators and establish their basic characteristics. In Section 3, we prove the existence of at least one solution for Eqs. (1.4) and (1.5). The regularity of the solutions is also studied. In the bounded domain case, the problem is studied by Ôtani [4], where the existence was proved by means of a subdifferential. The critical case $\gamma = p$ gives rise to an eigenvalue problem, for which there is an extensive literature. We refer to the works [2,3], and the references therein. In Section 4, we investigate the existence of nonsemitrivial solutions for system (1.1) and (1.2). To this end, we use the results of Section 3 on Eqs. (1.4) and (1.5). In this section, we extend earlier results on the bounded domain (see works [1,9]) and complete the study, done in [3], for this problem on $\mathbb{R}^N$ concerning the range of the exponents. In Section 5, we answer the regularity question raised in [3], extending it to a wider class of systems. This is done by adapting Moser's iterative scheme to systems on $\mathbb{R}^N$. Furthermore, we may consider the regularity results as an extension of those obtained by [9] to all $\mathbb{R}^N$.

*Notation*: For simplicity, we use the symbol $\|.\|_p$ for the norm $\|.\|_{L^p(\mathbb{R}^N)}$ and $\mathscr{D}^{1,p}$ for the space $\mathscr{D}^{1,p}(\mathbb{R}^N)$. $B_R$ and $B_R(c)$ will denote the balls in $\mathbb{R}^N$ of centres and $c$, respectively, and radius $R$. In addition, the Lebesgue measure of a set $\Omega \subset \mathbb{R}^N$ will be denoted by $|\Omega|$. An equality introducing definition is denoted by =:. The integral symbol $\int$ without any indication will be used for integration on all of $\mathbb{R}^N$.

## 2. Space and operator settings

It is going to be proved that the natural space setting for our problem is the space $Z = D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$, with the norm $\|\tilde{z}\|_Z = \|u\|_{1,p} + \|v\|_{1,q}$, where $\tilde{z} = (u, v)$. The space $\mathscr{D}^{1,p}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the norm

$$\|u\|_{\mathscr{D}^{1,p}(\mathbb{R}^N)} =: \left( \int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x \right)^{1/p}.$$

It is known that $\mathscr{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{Np/(N-p)}(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\}$ and that there exists $K_0 > 0$, such that for all $u \in \mathscr{D}^{1,p}(\mathbb{R}^N)$

$$\|u\|_{L^{Np/(N-p)}} \leqslant K_0 \|u\|_{\mathscr{D}^{1,p}}. \tag{2.1}$$

The space $\mathscr{D}^{1,p}$ is a *reflexive Banach space*. For more details, we refer to [3]. Our approach is based on the following generalized Poincare's inequality.

**Lemma 2.1.** *Suppose $g \in L^{N/p}(\mathbb{R}^N)$. Then there exists $\alpha > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla u|^p \, \mathrm{d}x \geqslant \alpha \int_{\mathbb{R}^N} |g||u|^p \, \mathrm{d}x, \tag{2.2}$$

*for all $u \in \mathscr{D}^{1,p}$.*

We introduce the operators $J_1, J_2, D_1, D_2, B_1, B_2 : Z \to Z^*$ in the following way:

$$(J_1(u,v),(w,z))_Z =: \frac{\alpha+1}{p} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla w,$$

$$(J_2(u,v),(w,z))_Z =: \frac{\beta+1}{q} \int_{\mathbb{R}^N} |\nabla v|^{q-2} \nabla v \nabla z,$$

$$(D_1(u,v),(w,z))_Z =: \frac{\gamma+1}{p} \int_{\mathbb{R}^N} a(x)|u|^{\gamma-2} u w,$$

$$(D_2(u,v),(w,z))_Z =: \frac{\delta+1}{p} \int_{\mathbb{R}^N} d(x)|v|^{\delta-2} v z,$$

$$(B_1(u,v),(w,z))_Z =: \int_{\mathbb{R}^N} b(x)|u|^{\alpha-1}|v|^{\beta+1} u w,$$

$$(B_2(u,v),(w,z))_Z =: \int_{\mathbb{R}^N} b(x)|u|^{\alpha+1}|v|^{\beta-1} v z.$$

The next lemma establishes some basic properties for the above operators. Its proof is given in [6, Lemma 2.2].

**Lemma 2.2.** *The operators $J_i, D_i, B_i$, $i = 1, 2$, are well defined. In addition, $J_i$, $i = 1, 2$, are continuous and the operators $D_i, B_i$, $i = 1, 2$, are compact.*

We say that $(u,v)$ is a *weak solution* of system (1.1) if $(u,v)$ is a critical point of the functional $A : Z \to \mathbb{R}$, defined by

$$A(u,v) =: \lambda\frac{\alpha+1}{p}\int|\nabla u|^p + \lambda\frac{\beta+1}{q}\int|\nabla v|^q - \lambda\frac{\alpha+1}{\gamma}\int a(x)|u|^\gamma$$
$$- \lambda\frac{\beta+1}{\delta}\int d(x)|v|^\delta - \lambda\int b(x)|u|^{\alpha+1}|v|^{\beta+1}.$$

Since $A(|u|,|v|) = A(u,v)$, if $(u,v)$ is a critical point of $A$, then the same is true for $(|u|,|v|)$. So we may consider that $u(x) \geqslant 0$ and $v(x) \geqslant 0$. Finally, the next definition will be proved to be of great importance for our study, so we describe it.

**Definition 2.3.** We say that a functional $A : Z \to \mathbb{R}$ *satisfies the* $(PS)$ *condition*, if every sequence $\{(u_n, v_n)\} \subset Z$ such that $A(u_n, v_n)$ is bounded and $A'(u_n, v_n) \to 0$ in $Z$, as $n \to \infty$, is relatively compact in $Z$.

## 3. The equation $-\Delta_p u = \lambda g(x)|u|^{\gamma-2}u$

In this section, we prove the existence of nontrivial solutions for Eqs. (1.4) and (1.5) and state under certain conditions the regularity of these solutions. The natural space setting for (1.4) is the space $\mathscr{D}^{1,p}(\mathbb{R}^N)$. We define the fuctional $\hat{A} : \mathscr{D}^{1,p} \to \mathbb{R}$, by

$$\hat{A}(u) =: \frac{1}{p}\int|\nabla u|^p - \frac{\lambda}{\gamma}\int g(x)|u|^\gamma.$$

The fact that $\hat{A}$ is well defined and is continuously Fréchet differentiable may be obtained by a standard procedure.

**Lemma 3.1.** *For any $\lambda \in \mathbb{R}$, the functional $\hat{A}$ satisfies the $(PS)$ condition.*

**Proof.** Let the sequence $\{u_n\} \subset \mathscr{D}^{1,p}$ be such that $\hat{A}(u_n)$ is bounded and $\hat{A}'(u_n) \to 0$ in $\mathscr{D}^{1,p}$, as $n \to \infty$. Then we consider the relation

$$\hat{A}(u_n) - \left(\hat{A}'(u_n), \frac{u_n}{p}\right) = \lambda\left(\frac{1}{p} - \frac{1}{\gamma}\right)\int g(x)|u_n|^\gamma$$

and follow the steps of [6, Lemma 2.3] to obtain the conclusion of the lemma. □

**Theorem 3.2.** *Let $2 \leqslant \gamma < p^*$, $\gamma \neq p$ and $\lambda g(x)$ satisfies $(\mathscr{H}_+)$. Then Eq.* (1.4) *admits at least one nontrivial solution $u_0$, such that $u_0(x) \geqslant 0$ for all $x \in \mathbb{R}^N$. Moreover,*
(a) *if $\gamma < p$, then $\hat{A}(u_0) < 0$, and*
(b) *if $\gamma > p$, then $\hat{A}(u_0) > 0$.*

**Proof.** (a) Let $\gamma < p$. Then $\hat{A}(u)$ is bounded from below, for all $u \in D^{1,p}$ and $\hat{A}(t\phi) < 0$, for $t$ which is small enough and $\phi \in C_0^\infty(\Omega)$. Hence, by a global minimization argument we obtain the existence of a nontrivial solution $u_0$, such that $\hat{A}(u_0) < 0$.

(b) Let $\gamma > p$. Then for every $u_n \in D^{1,p}$ such that $\|u_n\|_{1,p} = r$, where $r$ is fixed and small enough, we have that $\hat{A}(u) > k > 0$, for some $k = k(p, \gamma, g(x), K_0)$. Hence, by the mountain pass theorem we obtain the existence of a nontrivial solution $u_0$, such that $\hat{A}(u_0) > 0$. □

**Remark 3.3.** We have to note that if $\lambda g(x)$ does not satisfy condition $(\mathscr{H}_+)$ then Eq. (1.4) admits no nontrivial solution.

In the remaining part of this section, we shall prove the $C^{1,a}_{\mathrm{loc}}$ regularity as well as the asymptotic behaviour of the solutions of Eq. (1.4).

**Theorem 3.4.** *Let* $g \in L^\infty(\mathbb{R}^N)$ *and suppose that* $u \in \mathscr{D}^{1,p}$ *is a solution of* (1.4), *for some* $\lambda \in \mathbb{R}$. *Then* $u \in L^\infty(\mathbb{R}^N)$ *and* $u(x)$ *decays uniformly to zero, as* $|x| \to +\infty$.

**Proof.** The proof is based on the classical Moser's iteration scheme as it was adapted by Ôtani for the bounded domain case in [4, Theorem II]. Let $k \in \mathbb{N}$ and $L = (|\lambda| \|g\|_\infty)^{1/p} K_0$. Then we introduce the sequences

$$\gamma_{k+1} =: \gamma_k^* p^*/p, \quad \gamma_k^* =: \gamma_k - \gamma + p, \quad \gamma_1 =: p^*, \tag{3.1}$$

$$L_{k+1} =: L^{p/\gamma_k^*} (\gamma_k - \gamma + 1)^{-1/\gamma_k^*} (\gamma_k^*/p)^{p/\gamma_k^*} L_k^{\gamma_k/\gamma_k^*}, \quad L_1 = \|u\|_{p^*}. \tag{3.2}$$

We claim that, for every $k \in \mathbb{N}$, the following estimate is true:

$$\|u\|_{\gamma_k} \leqslant L_k. \tag{3.3}$$

For $k = 1$, inequality (3.3) is obvious. We suppose that estimate (3.3) holds for some $k$. We define, for $n \in \mathbb{N}$, the $C^1$ real functions $\psi_n$, as

$$\psi_n(t) =: \begin{cases} t, & |t| \leqslant n, \\ n+1, & |t| \geqslant n+2 \end{cases}, \quad 0 \leqslant \psi_n'(t) \leqslant 1. \tag{3.4}$$

Setting $u_n = \psi_n(u)$ we obtain that $|u_n|^{l-2} u_n$ belongs to $D^{1,p} \cap L^\infty$, for all $l \in [2, +\infty)$. Multiplying Eq. (1.4) by $|u_n|^{\gamma_k - \gamma} u_n$ and integrating over $\mathbb{R}^N$, we derive

$$(\gamma_k - \gamma + 1) \int |\nabla u|^p \psi_n'(u) |u_n|^{\gamma_k - \gamma} = \lambda \int g(x) |u_n|^{\gamma_k - \gamma + 1} |u|^{\gamma - 1}. \tag{3.5}$$

The definition of $u_n$ implies that

$$\lambda \int g(x) |u_n|^{\gamma_k - \gamma + 1} |u|^{\gamma - 1} \leqslant |\lambda| \int |g(x)| |u|^{\gamma_k} \leqslant |\lambda| \|g\|_\infty \|u\|_{\gamma_k}^{\gamma_k}. \tag{3.6}$$

On the other hand, from (3.4) and (2.1) it follows that:

$$\begin{aligned} (\gamma_k - \gamma + 1) \int |\nabla u|^p \psi_n'(u) |u_n|^{\gamma_k - \gamma} &\geqslant (\gamma_k - \gamma + 1) \int |\nabla u_n|^p |u_n|^{\gamma_k - \gamma} \\ &\geqslant (\gamma_k - \gamma + 1)(p/\gamma_k^*)^p \int |\nabla(|u_n|^{\gamma_k^*/p})|^p \\ &\geqslant K_0^{-p} (\gamma_k - \gamma + 1)(p/\gamma_k^*)^p \||u_n|^{\gamma_k^*/p}\|_{p^*}^p. \end{aligned} \tag{3.7}$$

Hence from relations (3.5)–(3.7) we deduce

$$\|u_n\|_{\gamma_{k+1}}^{\gamma_k^*} = \||u_n|^{\gamma_k^*/p}\|_{p^*}^p \leqslant |\lambda|\|g\|_\infty K_0^{-p}(\gamma_k - \gamma + 1)^{-1}(p/\gamma_k^*)^p L_k^{\gamma_k},$$

which implies that

$$\|u_n\|_{\gamma_{k+1}} \leqslant L_{k+1}.$$

Letting $n \to \infty$, we prove (3.3), for any $k \in \mathbb{N}$. Setting

$$\zeta = p^* \log L(p^* - \min\{p^*(\gamma - p)/(p^* - p), 0\}),$$

we get the following estimate:

$$L_k \leqslant (p^*/p)^{k-1} L_1 + \{\zeta((p^*/p) - 1) \\ + p^* \log(p^*/p)\}((p^*/p)^{k-1} - 1)/((p^*/p) - 1)^2.$$

Then the solutions of (1.4) satisfy the following $L^\infty$ estimate:

$$\|u\|_\infty \leqslant \lim_{k \to +\infty} \|u\|_{\gamma_k} \leqslant \mathrm{e}^d, \tag{3.8}$$

where $d = [L_1 + \{\zeta((p^*/p) - 1) + p^* \log(p^*/p)\}/((p^*/p) - 1)]/(p^* - (p^*/p))$. By Serrin [5, Theorem 1], there exists a constant $C = C(N, \gamma_2)$, such that for any solution $u \in D^{1,p}$ of the equation

$$-\Delta_p u = f,$$

the following estimate is true:

$$\sup_{y \in B_1(x)} |u(y)| \leqslant C\{\|u\|_{L^p(B_2(x))} + \|f\|_{L^{\gamma_2}(B_2(x))}\}.$$

Since the sequence $\gamma_k$ is increasing and $\gamma_k \to \infty$, as $k \to \infty$, there exists some $k \in \mathbb{N}$ such that $q = \gamma_k/(\gamma - 1) \geqslant \gamma_2$. Then for any solution of Eq. (1.4) we have

$$\sup_{y \in B_1(x)} |u(y)| \leqslant C\{\|u\|_{L^{\gamma_1}(B_2(x))} + |\lambda|\|g\|_\infty \||u|^{\gamma-1}\|_{L^q(B_2(x))}^{1/(\gamma-1)}\}.$$

Since $|u|^{\gamma-1}$ belongs to $L^q(\mathbb{R}^N)$, we may conclude that $u$ decays uniformly to zero, as $|x| \to +\infty$. □

**Corollary 3.5.** *If* $u(x)$ *is a solution of* (1.4), *then* $u \in C^{1,a}(B_r)$, *for any* $r > 0$ *and* $a = a(r) \in (0, 1)$.

**Proof.** The proof is a consequence of Theorem 3.4 and the results of Tolksdorf [7].

**Remark 3.6.** If in addition to the hypothesis of Theorem 3.2 we have that $g \in L^\infty(\mathbb{R}^N)$, the corresponding solutions of (1.4) are strictly positive. This is a direct consequence of Vázquez' Maximum Principle [8].

## 4. Multiplicity

Before we give the multiplicity results, we recall some existence results obtained in [6]. First we assume that $\alpha, \beta, p$ and $q$ satisfy one of the following hypothesis:

(I) $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$,

(II) $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$,

(III) $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$,

(IV) $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$.

For each hypothesis we have to distinguish several cases.

(I) Suppose that there exist $p_1,\ q_1 \in \mathbb{R}^+$, such that $p_1 < p,\ q_1 < q$ and $\frac{\alpha+1}{p_1} + \frac{\beta+1}{q_1} = 1$. Then the following cases are assumed.

(i) $\gamma < p,\ \delta < q$ and $a(x), d(x), b(x)$ have the same sign at every $x \in \mathbb{R}^N$,
(ii) $\gamma < p,\ \delta < q$ so that $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} < 1$ and $\lambda b(x) > 0$,
(iii) $\gamma < p,\ \delta < q$ so that $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} > 1$ and $\lambda b(x) < 0$,
(iv) $\gamma > p,\ \delta > q$ and $\lambda b(x) < 0$,
(v) $\gamma < p,\ \delta < q$ so that $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} = 1$,
(vi) $\gamma < p_1,\ \lambda a(x) > 0,\ \delta < q_1$ and $\lambda d(x) > 0$,
(vii) $\gamma < p_1,\ \lambda a(x) > 0,\ \delta > q_1$ and $\lambda d(x) < 0$,
(viii) $\gamma > p_1,\ \lambda a(x) < 0,\ \delta < q_1$ and $\lambda d(x) > 0$,
(ix) $\gamma > p_1,\ \lambda a(x) < 0,\ \delta > q_1$ and $\lambda d(x) < 0$.

The next theorem states the existence results obtained in [6, Theorem 3.3].

**Theorem 4.1.** (a) *Let one of the hypotheses* (i), (iii) *and* (v)–(viii) *be satisfied, and in addition* $\lambda a(x)$ *or* $\lambda d(x)$ *satisfy* $(\mathscr{H}_+)$, *or assume that hypothesis* (ix) *is satisfied and in addition,* $\lambda b(x)$ *satisfies* $(\mathscr{H}_+)$, *or hypothesis* (ii) *is satisfied, then problem* (1.1) *has at least one nonnegative* (*componentwise*) *solution.*

(b) *If hypothesis* (iv) *is satisfied, and in addition* $\lambda a(x)$ *or* $\lambda d(x)$ *satisfy* $(\mathscr{H}_+)$, *then problem* (1.1) *has at least one nonnegative* (*componentwise*) *solution.*

(II) Suppose that there exist $p_1,\ q_1 \in \mathbb{R}^+$, such that $p_1 > p,\ q_1 > q$ and $(\alpha+1)/p_1 + (\beta+1)/q_1 = 1$. Then the following cases are assumed:

(i) $p \leqslant \gamma,\ q \leqslant \delta$ and $a(x), d(x), b(x)$ have the same sign at every $x \in \mathbb{R}^N$,
(ii) $\gamma > p,\ \delta > q$ so that $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} < 1$ and $\lambda b(x) < 0$,
(iii) $\gamma > p,\ \delta > q$ so that $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} > 1$ and $\lambda b(x) > 0$,
(iv) $\gamma < p,\ \delta < q$ and $\lambda b(x) > 0$,
(v) $\gamma > p,\ \delta > q$ so that $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} = 1$,
(vi) $\gamma < p_1,\ \lambda a(x) < 0,\ \delta < q_1$ and $\lambda d(x) < 0$,
(vii) $\gamma < p_1,\ \lambda a(x) < 0,\ \delta > q_1$ and $\lambda d(x) > 0$,
(viii) $\gamma > p_1,\ \lambda a(x) > 0,\ \delta < q_1$ and $\lambda d(x) < 0$,
(ix) $\gamma > p_1,\ \lambda a(x) > 0,\ \delta > q_1$ and $\lambda d(x) > 0$.

The next theorem states the existence results obtained in [6, Theorem 4.3].

**Theorem 4.2.** (a) *Let one of the hypotheses* (i), (ii), (v) *and* (ix) *be satisfied, and in addition* $\lambda a(x)$ *or* $\lambda d(x)$ *satisfy* $(\mathscr{H}_+)$, *or assume that one of the hypotheses* (iii) *and* (vii)–(ix) *is satisfied, then problem* (1.1) *has at least one nonnegative* (*componentwise*) *solution.*

(b) *If the hypothesis* (vi) *is satisfied and in addition* $\lambda b(x)$ *satisfies* $(\mathscr{H}_+)$, *then problem* (1.1) *has at least one nonnegative* (*componentwise*) *solution.*

(III) The following cases are assumed:
(i) If $\lambda < 0$,
(ii) If $\lambda < \lambda_1$ and $a(x)d(x) \geqslant 0$, almost everywhere on $\mathbb{R}^N$.
The next theorem states the existence results obtained in [6, Theorem 5.6].

**Theorem 4.3.** *If one of the cases* (i) *or* (ii) *is satisfied and* $\lambda a(x)$ *or* $\lambda d(x)$ *satisfy* $(\mathscr{H}_+)$, *then problem* (1.1) *has at least one nonnegative* (*componentwise*) *solution.*

(IV) The following cases are assumed:
(i) $\gamma < p$, $\delta < q$ and $\lambda b(x) < 0$,
(ii) $\gamma > p$, $\delta > q$ and $\lambda b(x) > 0$,
(iii) $\lambda a(x) < 0$ and $\lambda d(x) < 0$.
The next theorem states the existence results obtained in [6, Theorem 6.3].

**Theorem 4.4.** (a) *Let one of the cases* (i) *and* (ii) *be satisfied and in addition,* $\lambda a(x)$ *or* $\lambda d(x)$ *satisfy* $(\mathscr{H}_+)$, *or*

(b) *let hypothesis* (iii) *be satisfied and in addition* $\lambda b(x)$ *satisfies* $(\mathscr{H}_+)$, *and* $\gamma < p^*$, $\delta < q^*$.

*Then problem* (1.1) *has at least one nonnegative* (*componentwise*) *solution.*

Throughout this section, we assume that at least one of the quantities $\lambda a(x)$ or $\lambda d(x)$ satisfy condition $(\mathscr{H}_+)$. Under this assumption, Theorem 3.2 implies that there exist at least one semitrivial solution for system (1.1). If both $a(x)$ and $d(x)$ satisfy $(\mathscr{H}_+)$, we have at least two semitrivial solutions. Let us note that if on the contrary $\lambda a(x) < 0$ and $\lambda d(x) < 0$ (simultaneously) the solutions which are obtained are not semitrivial. This occurs under case (ix) in the first hypothesis, case (vi) in the second hypothesis and case (iii) in the fourth hypothesis.

**Theorem 4.5.** *Let the hypothesis of Theorems* 4.1, 4.3 *and* 4.4 *hold such that* $\gamma < p$ *and* $\delta < q$. *Furthermore, we assume that at least one of the following conditions is satisfied.*

(a) *If* $a(x)$, $d(x)$, $b(x)$ *satisfy* $(\mathscr{H}_+)$ *at the same* $\Omega \subseteq \mathbb{R}^N$,

(b) *If* $\lambda b(x)$ *satisfies* $(\mathscr{H}_+)$ *and* $\beta + 1 < \delta$ *or* $\alpha + 1 < \gamma$ *depending on which one of the functions* $a(x)$ *and* $d(x)$ *satisfies* $(\mathscr{H}_+)$, *respectively,*

(c) *If* $\lambda b(x)$ *does not satisfy* $(\mathscr{H}_+)$ *and* $\beta + 1 > \delta$ *or* $\alpha + 1 > \gamma$ *depending on which one of the functions* $a(x)$ *and* $d(x)$ *satisfies* $(\mathscr{H}_+)$, *respectively.*

*Then system* (1.1) *has at least two or three nonnegative* (*componentwise*) *solutions and at least one of them is not semitrivial.*

**Proof.** (a) Let $a(x)$ satisfy $(\mathscr{H}_+)$ and $(u_0,0)$ a semitrivial solution of (1.1). Then for each $v \in D^{1,q}$ and $t>0$

$$A(u_0,tv)=\frac{\alpha+1}{p}\int|\nabla u_0|^p+t^q\frac{\beta+1}{q}\int|\nabla v|^q-\lambda\frac{\alpha+1}{\gamma}\int a(x)|u_0|^\gamma$$
$$-t^\delta\lambda\frac{\beta+1}{\delta}\int d(x)|v|^\delta-t^{\beta+1}\lambda\int b(x)|u_0|^{\alpha+1}|v|^{\beta+1}.$$

If case (i) of hypothesis (I) holds, we have that $A(u_0,t\phi)<A(u_0,0)$, for $t$ which is small enough, where $\phi\in C_0^\infty(\Omega)$. This implies that $inf\{A(u,v);(u,v)\in Z\}<A(u_0,0)$. In the same way, if $(0,v_0)$ is a semitrivial solution of system (1.1), we obtain $inf\{A(u,v);(u,v)\in Z\}<A(0,v_0)$. Then from the proof of Theorem 4.1 we deduce the existence of a solution for system (1.1), which is not semitrivial, and the multiplicity result follows.

Similarly, the result may be obtained for the other hypothesis and cases. □

The following lemma gives an inequality result, to be used later.

**Lemma 4.6.** *Let $\alpha,\beta,p$ and $q$ be positive real numbers, such that*

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}\geqslant 1. \tag{4.1}$$

*Then $\alpha+\beta+2>\min\{p,q\}$. Moreover, if (4.1) holds as an equality, then we have* $\max\{p,q\}>\alpha+\beta+2>\min\{p,q\}$.

**Proof.** Let $p>q$, so $p=xq$ for some $x>1$. Inequality (4.1) implies that $\alpha+1\geqslant x[q-(\beta+1)]$, and the result follows. If (4.1) holds as an equality, assuming that $\alpha+\beta+2\geqslant p$ we conclude that $x\leqslant 1$, which is a contradiction. Hence, the proof is complete. □

**Theorem 4.7.** *Let the hypothesis of Theorem 4.3 hold such that $\gamma>p$ and $\delta>q$. Furthermore, we assume that at least one of the following conditions is satisfied.*

(a) $p>q$, *$\lambda d(x)$ satisfies $(\mathscr{H}_+)$ and $\alpha+\beta+2<\delta$,*

(b) $p<q$, *$\lambda a(x)$ satisfies $(\mathscr{H}_+)$ and $\alpha+\beta+2<\gamma$.*

*Then system (1.1) has at least two or three nonnegative (componentwise) solutions and at least one of them is not semitrivial.*

**Proof.** (a) Let $0<\lambda<\lambda_1$ and $p>q$. Since $\lambda d(x)$ satisfies condition $(\mathscr{H}_+)$, then for some $\lambda=\lambda_0$ we have a semitrivial solution $(0,v_0)$, for system (1.1), i.e.,

$$\int|\nabla v_0|^q=\lambda_0\int d(x)|v_0|^\delta. \tag{4.2}$$

From (4.2) and the definition of $A(u,v)$, for $u\in D^{1,p}$ we obtain

$$A(tu,tv_0)=t^p\frac{\alpha+1}{p}\int|\nabla u|^p-t^\gamma\lambda_0\frac{\alpha+1}{\gamma}\int a(x)|u|^\gamma$$
$$+t^\delta\lambda_0(\beta+1)\left(\frac{1}{q}-\frac{1}{\delta}\right)\int d(x)|v_0|^\delta-t^{\alpha+\beta+2}\lambda_0\int b(x)|u|^{\alpha+1}|v_0|^{\beta+1}.$$

The hypothesis of the theorem and Lemma 4.6 imply that $A(tu,\ tv_0) < 0$, for $t$ which is small enough. Then we deduce, that there exists an $r$, such that

$$\inf_{\|(u,v)\|_Z \leqslant r} A(u,v) < 0.$$

From the last inequality we derive that for $\varepsilon > 0$ there exists a $\delta > 0$, such that $-\varepsilon \leqslant A(u,v)$, for every $(u,v) \in Z$, with $r - \delta \leqslant \|(u,v)\|_Z \leqslant r$. So, for some $r' < r$, and for all $(u,v)$ with $r' \leqslant \|(u,v)\|_Z \leqslant r$ it holds that

$$A(u,v) \geqslant \frac{1}{2} \inf_{\|(u,v)\|_Z \leqslant r} A(u,v). \tag{4.3}$$

Let $(u_n, v_n)$ be a minimizing sequence of $inf\{A(u,v) : (u,v) \in \bar{B}(0,r)\}$. From (4.3) we may assume that $(u_n, v_n) \in \bar{B}(0,r')$. The ball $\bar{B}(0,r)$ equipped with the metric

$$dist((u,v),(w,z)) = \|\nabla u - \nabla w\|_p + \|\nabla v - \nabla z\|_q,$$

becomes a complete metric space. Let $\delta_n > 0$ be some sequence of positive real numbers, such that $\delta_n \to 0$, as $n \to \infty$. Then by Ekeland's variational principle we may assume that, every $(u_n, v_n)$ is a minimizer for the set

$$\{A(u,v) + \delta_n(\|\nabla u - \nabla w\|_p + \|\nabla v - \nabla z\|_q) : (u,v) \in \bar{B}(0,r)\}.$$

This implies that $A'(u_n, v_n) \to 0$, as $n \to \infty$. Hence, we deduce that $(u_n, v_n)$ is a (PS) sequence for the functional $A$. Since $A(u,v)$ satisfies (PS) condition, we have the existence of a solution $(u^*, v^*)$ for system (1.1), such that

$$A(u^*, v^*) = \inf_{\|(u,v)\|_Z \leqslant r} A(u,v) < 0.$$

Moreover, we may note that the solution, derived in this way, is a nonsemitrivial one, since every semitrivial solution must satisfy $A(u_0, 0) > 0$ or $A(0, v_0) > 0$. In a similar way the conclusion follows for (b). □

Finally, we give a multiplicity result of hypothesis (II).

**Theorem 4.8.** *Let the hypothesis of Theorem* 4.2 *hold such that* $\gamma > p$ *and* $\delta > q$. *Also assume that* $b(x)$ *satisfies* $(\mathscr{H}_+)$ *and that* $\alpha + \beta + 2 < \min\{\gamma, \delta\}$. *Furthermore, we assume that at least one of the following conditions is satisfied.*

(a) $\alpha+\beta+2 < \max\{p,q\}$, *and if* $p < q$ *then* $\lambda a(x)$ *satisfies* $(\mathscr{H}_+)$ *or if* $p > q$ *then* $\lambda d(x)$ *satisfies* $(\mathscr{H}_+)$,

(b) $\lambda a(x)$ *and* $\lambda d(x)$ *satisfies* $(\mathscr{H}_+)$.

*Then system* (1.1) *has at least two or three nonnegative* (*componentwise*) *solutions and at least one of them is not semitrivial.*

**Proof.** The proof follows the lines of Theorem 4.7.

## 5. Regularity results

In this last section, we prove some regularity results for the solutions of system (1.1). More precisely, we are going to prove the $L^\infty$ character of the solutions, then the regularity results easily follow. Throughout this section, we assume in addition that the functions $a, b, d$ satisfy the following hypothesis:

$$b \in L^{\omega_1} \cap L^{\infty}, \quad a \in L^{q^*/(\beta+1)} \cap L^{\omega_2} \quad \text{and} \quad d \in L^{p^*/(\alpha+1)} \cap L^{\omega_3},$$

where

$$\omega_2 = \frac{p^* q^*}{p^*(\beta+1) + q^*(\alpha+1-\gamma)} \quad \text{and} \quad \omega_3 = \frac{p^* q^*}{q^*(\alpha+1) + p^*(\beta+1-\delta)}.$$

Let $\alpha$, $\beta$ satisfy the strict inequality

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1. \tag{5.1}$$

Let us note that (5.1) corresponds to hypotheses (I)–(III). In addition, we assume that $\alpha$, $\beta$ satisfy the inequalities

$$\frac{p^*}{p}\left(1 - \frac{\beta+1}{q^*}\right) > 1 \quad \text{and} \quad \frac{q^*}{q}\left(1 - \frac{\alpha+1}{p^*}\right) > 1. \tag{5.2}$$

Furthermore, for $\gamma$, $\delta$ we assume that

$$\gamma < p^* \frac{\beta+1}{q^*} + (\alpha+1) \quad \text{and} \quad \delta < q^* \frac{\alpha+1}{p^*} + (\beta+1). \tag{5.3}$$

Here, we are going to extend to the system the procedure followed in Theorem 3.4 for the equation. For $k \in \mathbb{N}$, we introduce the sequences $\eta_k$, $L_k$, $\theta_k$, $M_k$, with $\eta_1 = p^*$, $L_1 = \|u\|_{p^*}$, $\theta_1 = q^*$, $M_1 = \|v\|_{q^*}$, such that

$$\eta_{k+1} =: \eta_k^* p^*/p, \quad \eta_k^* =: \eta_k - \tilde{\eta}_k + p,$$

$$\tilde{\eta}_k =: \frac{\eta_k(\beta+1)}{q^*} + (\alpha+1), \quad \chi_k = \frac{\eta_k}{\tilde{\eta}_k - \gamma},$$

$$L_{k+1} =: L^{p/\eta_k^*}(\eta_k - \tilde{\eta}_k + 1)^{-1/\eta_k^*}(\eta_k^*/p)^{p/\eta_k^*} \max\{\|a\|_{\chi_k}^{\chi_k/\eta_k^*}, L_k^{\eta_k/\eta_k^*}, M_1^{\theta_1/\eta_k^*}\},$$

$$\theta_{k+1} =: \theta_k^* q^*/q, \quad \theta_k^* =: \theta_k - \tilde{\theta}_k + q,$$

$$\tilde{\theta}_k =: \frac{\theta_k(\alpha+1)}{p^*} + (\beta+1), \quad \psi_k = \frac{\theta_k}{\tilde{\theta}_k - \delta},$$

$$M_{k+1} =: L^{q/\theta_k^*}(\theta_k - \tilde{\theta}_k + 1)^{-1/\theta_k^*}(\theta_k^*/q)^{q/\theta_k^*} \max\{\|d\|_{\psi_k}^{\psi_k/\theta_k^*}, M_k^{\theta_k/\theta_k^*}, L_1^{\eta_1/\theta_k^*}\},$$

where $L = (|\lambda| \cdot \max\{1, \|b\|_{\infty}\})^{1/p} K_0$ and $M = (|\lambda| \cdot \max\{1, \|b\|_{\infty}\})^{1/q} K_0$.

The next lemma states some basic properties for the sequences $\eta_k$ and $\theta_k$.

**Lemma 5.1.** *Let conditions* (1.3) *and* (5.2) *hold. Then* $\eta_k$ *and* $\theta_k$ *have the following properties*:
(i) $\eta_k$ *and* $\theta_k$ *are increasing*,
(ii) $\eta_k \to \infty$ *and* $\theta_k \to \infty$, *as* $k \to \infty$,
(iii) $\eta_k - \tilde{\eta}_k > 0$ *and* $\theta_k - \tilde{\theta}_k > 0$.

**Proof.** We give the proof for $\eta_k$. In a similar way, the conclusions follow for $\theta_k$.

(i) Since $\eta_1 = p^*$, from (5.2) and the definition of $\eta_k$, it follows that

$$\eta_1^* = p^*\left(1 - \frac{\beta+1}{q^*}\right) - (\alpha+1) + p > p.$$

Hence, $\eta_2 > p^* = \eta_1$. Suppose that $\eta_k > \eta_{k-1}$, for some $k \in \mathbb{N}$. From the fact that

$$\eta_{k+1} = \left\{\eta_k\left(1 - \frac{\beta+1}{q^*}\right) - (\alpha+1) + p\right\}\frac{p^*}{p}, \tag{5.4}$$

we obtain $\eta_{k+1} > \eta_k$. Hence, by induction $\eta_k$ is an increasing sequence.

(ii) From (5.4) we may express $\eta_k$ in terms of $k$, as

$$\eta_k = \varepsilon^{k-1}\eta_1 + \sum_{i=0}^{k-2}\varepsilon^i\xi = \varepsilon^{k-1}\eta_1 + \xi\frac{\varepsilon^{k-2}-1}{\varepsilon - 1},$$

where

$$\varepsilon = \frac{p^*}{p}\left(1 - \frac{\beta+1}{q^*}\right) \quad \text{and} \quad \xi = \frac{p^*}{p}(p - (\alpha+1)).$$

Letting $k \to \infty$, and using (5.2), we derive that $\eta_k \to \infty$.

(iii) Since $\eta_k$ is an increasing sequence, the conclusion follows directly from (5.2). □

In the following lemma, we give the connection between the solutions of system (1.1) and the sequences we introduce above.

**Lemma 5.2.** *Let* $(u, v)$ *be a solution of system* (1.1). *For every* $k \in \mathbb{N}$, $u$ *belongs to* $L^{\eta_k}(\mathbb{R}^N)$, $v$ *belongs to* $L^{\theta_k}(\mathbb{R}^N)$ *and satisfy*

$$\|u\|_{\eta_k} \leqslant L_k \quad \textit{and} \quad \|v\|_{\theta_k} \leqslant M_k. \tag{5.5}$$

**Proof.** We study only the case of $u$. The case of $v$ may be treated similarly. Proceeding by induction, we see that for $k = 1$, inequalities (5.5) are obvious. We suppose that (5.5) hold for some $k$. We recall the construction of $\psi_n$ from (3.4) and set $u_n = \psi_n(u)$. Multiplying the first equation of system (1.1) by $|u_n|^{\eta_k - \tilde{\eta}_k}u_n$ and integrating we obtain

$$(\eta_k - \tilde{\eta}_k + 1)\int |\nabla u|^p \psi_n'(u)|u_n|^{\eta_k - \tilde{\eta}_k} = \lambda\int a(x)|u_n|^{\eta_k - \tilde{\eta}_k + 1}|u|^{\gamma-1}$$

$$+ \lambda\int b(x)|u_n|^{\eta_k - \tilde{\eta}_k + 1}|u|^{\alpha}|v|^{\beta+1}. \tag{5.6}$$

From the definition of $u_n$ and $v_n$ and from the fact that

$$\frac{\eta_k - \tilde{\eta}_k + \alpha + 1}{\eta_k} + \frac{\beta + 1}{q^*} = 1,$$

we obtain

$$\begin{aligned} \lambda \int b(x)|u_n|^{\eta_k - \tilde{\eta}_k + 1}|u|^{\alpha}|v|^{\beta+1} &\leqslant |\lambda|\|b\|_\infty \int |u|^{\eta_k - \tilde{\eta}_k + \alpha + 1}|v|^{\beta+1} \\ &\leqslant |\lambda|\|b\|_\infty \left(\int |u|^{\eta_k}\right)^{(\eta_k - \tilde{\eta}_k + \alpha + 1)/\eta_k} \left(\int |v|^{q^*}\right)^{(\beta+1)/q^*} \\ &\leqslant |\lambda|\|b\|_\infty \max\{\|u\|_{\eta_k}^{\eta_k}, \|v\|_{q^*}^{q^*}\}. \end{aligned} \tag{5.7}$$

Next, we note that

$$\frac{1}{\chi_k} + \frac{\eta_k - \tilde{\eta}_k + \gamma}{\eta_k} = 1.$$

Hence

$$\begin{aligned} \lambda \int a(x)|u_n|^{\eta_k - \tilde{\eta}_k + 1}|u|^{\gamma - 1} &\leqslant |\lambda| \left(\int |a(x)|^{\chi_k}\right)^{1/\chi_k} \left(\int |u|^{\eta_k}\right)^{(\eta_k - \tilde{\eta}_k + \gamma)/\eta_k} \\ &\leqslant |\lambda| \max\{\|a(x)\|_{\chi_k}^{\chi_k}, \|u\|_{\eta_k}^{\eta_k}\}. \end{aligned} \tag{5.8}$$

As in (3.7) we derive

$$(\eta_k - \tilde{\eta}_k + 1) \int |\nabla u|^p \psi_n'(u)|u_n|^{\eta_k - \tilde{\eta}_k} \geqslant K_0^{-p}(\eta_k - \tilde{\eta}_k + 1)(p/\eta_k^*)^p \| |u_n|^{\eta_k^*/p} \|_{p^*}^p. \tag{5.9}$$

Since $\| |u_n|^{\eta_k^*/p} \|_{p^*}^p = \|u_n\|_{\eta_{k+1}}^{\eta_k^*}$, inequality (5.9) implies

$$(\eta_k - \tilde{\eta}_k + 1) \int |\nabla u|^p \psi_n'(u)|u_n|^{\eta_k - \tilde{\eta}_k} \geqslant K_0^{-p}(\eta_k - \tilde{\eta}_k + 1)(p/\eta_k^*)^p \|u_n\|_{\eta_{k+1}}^{\eta_k^*}. \tag{5.10}$$

Applying (5.7), (5.9) and (5.10) to Eq. (5.6) we derive the estimates

$$\|u_n\|_{\eta_{k+1}} \leqslant L_{k+1}.$$

Finally, letting $n \to \infty$, we obtain estimate (5.5) for the solution $u$. □

With the help of the above lemma, we are ready to prove the main result of this section.

**Theorem 5.3.** *Let* $(u, v)$ *be a solution of* (1.1), *with the restrictions as stated above. Then* $u$ *and* $v$ *are uniformly bounded on* $\mathbb{R}^N$.

**Proof.** We claim that

$$\|u\|_\infty \leqslant \max\{1, \|a(x)\|_{q^*/(\beta+1)}^{\tilde{\gamma}}, \|v\|_{q^*}^{q^*}, e^d\}, \tag{5.11}$$

where $\tilde{\gamma} = q^*/[p^*(\beta+1)+q^*(\alpha+1)-q^*\gamma]$ and $d$ a real number to be fixed later. We suppose that there exists a subsequence of $\{k\}$, which we denote again with $\{k\}$, such that

$$L_k^{\eta_k} \geqslant \max\{\|a(x)\|_{\chi_k}^{\chi_k}, M_1^{\theta_1}\}.$$

Let $E_k =: \eta_k \log(L_k)$ and set $a = p^*/p$. By simple calculations and using Lemma 5.1, we derive

$$E_{k+1} \leqslant r_k + aE_k,$$

where $r_k = p^* \log(L\tilde{\eta}_k)$. It is now easy to obtain the following inequality for $E_k$.

$$E_k \leqslant a^{k-1}E_1 + r_{k-1} + ar_{k-2} + \cdots + a^{k-2}r_1. \tag{5.12}$$

The definition of $\eta_k^*$ and Lemma 5.1 imply that

$$\eta_k^* \leqslant a\varepsilon^{k-1}\left(\eta_1 - \xi^- \frac{\varepsilon^{k-2}-1}{\varepsilon^{k-1}(\varepsilon-1)}\right),$$

where $\xi^- = \min\{0, \xi\}$. Thus, $r_k$ can be estimated as

$$r_k \leqslant p^*(k-1)\log(\varepsilon) + p^* \log[aL(\eta_1 - \xi^-)].$$

Choosing $k$ to be sufficiently large and setting $b = p^* \log[aL(\eta_1 - \xi^-)]$ we deduce

$$r_k \leqslant p^*(k-1)\log(a) + b. \tag{5.13}$$

From (5.12) and (5.13) we derive

$$E_k \leqslant a^{k-1}E_1 + \{b(a-1) + p^* \log(a)\}(a^{k-1}-1)/(a-1)^2.$$

Furthermore, from Lemma 5.2 we obtain that

$$\|u\|_\infty \leqslant \limsup_{k\to\infty} \|u\|_{\eta_k} \leqslant \limsup_{k\to\infty} \mathrm{e}^{E_k/\eta_k} \leqslant \mathrm{e}^d, \tag{5.14}$$

where $d = [E_1 + \{b(a-1) + p^* \log(a)\}/(a-1)^2]/(p^* - a)$. Assume now that for a subsequence $k_n$ we have

$$M_1^{\theta_1/\eta_{k_n}} \geqslant \max\{L_{k_n}, \|a(x)\|_{\chi_{k_n}}^{\chi_{k_n}}\}.$$

Since $\eta_k$ is increasing, we derive

$$\|u\|_{\eta_{k_n}} \leqslant L_{k_n} \leqslant \|u\|_{q^*}^{q^*/\eta_{k_n}} \leqslant \max\{1, \|v\|_{q^*}^{q^*}\}. \tag{5.15}$$

On the contrary, if for a subsequence $k_n$ holds

$$\|a(x)\|_{\chi_{k_n}}^{\chi_{k_n}} \geqslant \max\{L_{k_n}, M_1^{\theta_1/\eta_{k_n}}\},$$

then from the definition of $\chi_k$, the Lebesgue Dominated Convergence Theorem and condition (5.3) we obtain

$$\|u\|_{\eta_{k_n}} \leqslant L_{k_n} \leqslant \|a(x)\|_{\chi_{k_n}}^{\chi_{k_n}/\eta_{k_n}} \leqslant \max\{1, \|a(x)\|_{q^*/(\beta+1)}^{\tilde{\gamma}}\}. \tag{5.16}$$

Thus, from estimates (5.14)–(5.16) we conclude (5.11), i.e., the uniform boundness of $u$. The analogous holds for $v$. □

To complete this section, we state some immediate consequences of Theorem 5.3, analogous to those of Section 3.

**Corollary 5.4.** *Let the conditions of Theorem* 5.3 *hold and* $(u,v)$ *be a solution of* (1.1) *then u and v decay uniformly to zero as* $|x| \to \infty$. *Moreover, both of them are of class* $C^{1,\alpha}(B_r)$, *for any* $r > 0$ *and* $\alpha = \alpha(r) \in (0,1)$.

**Proof.** It is a consequence of Theorem (5.3), the results of Serrin [5] and of Tolksdorf [7]. □

**Corollary 5.5.** *Let the conditions of Theorem* 5.3 *hold. Any solution for system* (1.1) *obtained by Theorems* 4.1–4.8 *is strictly positive* (*componentwise*).

**Proof.** It is a consequence of Theorem 5.3 and the results of Vázquez [8].

**Remark 5.6.** We note that all the results, obtained in this work, are applicable to the bounded domain case and similar results may be obtained. □

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