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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SOME NONLINEARLY DAMPED WAVE EQUATIONS ON $\mathbb{R}^N$

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Abstract. We discuss the asymptotic behavior of solutions of the nonlinearly damped wave equation

$$u_{tt} + \delta|u_t|^{m-1}u_t - \phi(x)\Delta u = \lambda u|u|^{\beta-1}, \quad x \in \mathbb{R}^N,\ t \geq 0,$$

with the initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$, $\delta > 0$ and $(\phi(x))^{-1} = g(x)$ is a positive function lying in $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, for some p. We prove blow-up of solutions when the source term dominates over the damping, and the initial energy is assumed to be positive. We also discuss global existence energy decay of solutions.

1. Introduction

The aim of this work is to study the asymptotic behavior of solutions of the following semilinear hyperbolic Cauchy problem

$$u_{tt} + \delta|u_t|^{m-1}u_t - \phi(x)\Delta u = \lambda u|u|^{\beta-1}, \quad x \in \mathbb{R}^N,\ t \geq 0, \tag{1.1}$$

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$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 3$, $m > 1$ and $\delta > 0$. Throughout the paper we assume that ϕ (and g) satisfies the following hypothesis

($\mathcal{G}$) $\phi(x) > 0$, for all $x \in \mathbb{R}^N$, $(\phi(x))^{-1} =: g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $1 < m \leq (N+2)/(N-2)$.

In addition to the principal condition $\mathcal{G}$ we shall use the following hypotheses for g and the exponent β,

($\mathcal{G}_1$) $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $1 < \beta \leq N/(N-2)$,
($\mathcal{G}_2$) $(N+2)/N \leq \beta \leq N/(N-2)$.

Equations with non-constant coefficient arise in many phenomena of mathematical physics involving wave propagation in nonhomogeneous medium (see [1], [17], [26], [29], [30]). We refer also to [2], [7], [11], [12] for equations of parabolic type. The particular case treated here includes functions of the form

$$\phi(x) \sim c_0 + \varepsilon|x|^\gamma, \quad \varepsilon > 0,\ \gamma > 0,\ c_0 > 0,$$

resembling phenomena of *rapidly varying wave speed* around the constant speed c_0 (see [30] and [7], [11], [12] for the physical description in parabolic equations). There is a growing interest on the study of the interaction of nonlinear damping and source terms on the long time behavior of solutions of wave equations. In the case of bounded domains and $\phi(x) =$ constant we refer to the works [9] and [10]. In [9] it is proved that global existence occurs, when the damping term dominates over the source term, while blow-up appears in the opposite situation and under the assumption of sufficiently negative initial energy. In [10] global existence results are given for sufficiently small initial data, but there are not any relations between the exponents of the source and damping. The corresponding blow-up result is proved under the assumption of *positive initial energy* and sufficiently small values of damping, i.e. $E(0) < E_\delta < E_1$. The value E_1 denotes the depth of the potential well introduced in [24] (see also [23]). Global non-existence and blow-up results are extended to the case of *negative initial energy* in the work [21] concerning abstract quasilinear equations. In this work, among other applications (e.g. mean curvature and polyharmonic operators), it is treated the case of *variable bounded diffusion coefficient* in bounded domains. The case $0 \leq E(0) < E_1$ is discussed first in [25], for *linearly damped* quasilinear equations and bounded domains. In the work [29] the results of [25] are extended to most of the applications presented in [21] in bounded domains (including the models of bounded variable coefficients).

The problem becomes more complicated in the case of unbounded domains as in general, the equation *does not give rise to coercive or compact operators.* Important contributions to this direction are contained in the works [20], [22],

[25], [27], [28]. The works [20], [27] extend the result of [21] for the problem (1.1)–(1.2) and negative initial energy, with $\phi(x)$ = constant, involving "mass" and "mass-free" nonlinearities, respectively. Generally speaking, the "mass" case implies coercivity (see [25]), a necessary assumption for the extension of the results in unbounded domains. In the "mass-free" ("non-coercive") case it is necessary to assume initial data having compact support, which implies the *finite speed of propagation property.* In [22], a method using this assumption is developed, to extend the result of [25] in the "mass free", nonlinearly damped semilinear equation. In the recent work [28], it is treated the linear damped equation. Global existence is proved for sufficiently small initial data when the source exponent is in $((N+2)/N, N/(N-2))$ while blow-up holds when the source is in $(1,(N+2)/N)$ and initial data having positive average. Moreover, it is made the important observation that the support of the solution is concetrated in a ball much smaller than $|x| < t+K$ where K is the radius of the support of the initial data.

We would like to mention that evolution equations, involving diffusion coefficients $\phi(x) \to c_{\pm} > 0$, as $x \to \pm\infty$, are functionally formulated in the classical Sobolev space setting (e.g. see [8], [26]). In the special case of problem (1.1)–(1.2) treated here, *it is unclear a priori in which function spaces solutions might lie.*

In works [14], [15], [16] it is proved that the problem (1.1)–(1.2) (with weak dissipation) is naturally formulated in a *homogeneous Sobolev and weighted L^p-space setting.* It is the appropriate functional setting to overcome the difficulties that arise from the appearance of the "unbounded" diffusion coefficient and the consideration of the problem on $\mathbb{R}^N$. These works constitute the first complete application of the homogeneous Sobolev spaces, for the study of time dependent problems in unbounded domains, which have been used in the past for the study of nonlinear elliptic equations (see [5], [6] and the references therein). In [15] the linear damping case ($m = 1$) is considered. The results concern blow-up of solutions for all negative initial energies and global existence for sufficiently small initial data. Here we extend the blow-up result in the case of positive initial energy. In the case of nonlinear damping, we also prove global existence and energy decay. A difference of the results presented here with those of [20], [22], [25], [27], [28], is that *it is not necessary to assume compactness of the support of the initial data. This is a consequence of the appearance of the function* $\phi(x) \to \infty$ *(see the discussion in* [21]*). This observation combined with the fact that the classical energy space is included in the energy space defined by* (1.1)–(1.2)*, shows that the problem is solvable for a wide class of initial data.*

The work is organized as follows. In Section 2, we recall the basic properties of the energy space and give the functional formulation of the problem. In Section 3, we first prove that solution blows-up in finite time in the case where

$(\mathcal{G}_1)$ is satisfied, the initial energy $\mathcal{E}^*(0) < P_0$ for some $P_0 > 0$ and the initial condition u_0 is large in an appropriate norm The existence of global solutions is proved, under the hypothesis $(\mathcal{G}_2)$, the initial energy been bounded as above and the ininitial condition u_0 is sufficiently small (in the sense that it is in a potential well). In addition, if $(\mathcal{G}_1)$ is satisfied the energy decay of solutions is also derived. As in [10], we don't assume any ordering relation between the exponents m and β. In Section 4 we prove the corresponding blow-up result for the linearly damped equation. In this case we may relax the condition $(\mathcal{G}_1)$ assumed also in [15], for the blow-up result for negative initial energies, by assuming condition $(\mathcal{G}_2)$.

Notation. We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for simplicity we use the symbols $\mathcal{D}^{1,2}$, L^p, $1 \le p \le \infty$, for the spaces $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, respectively, $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$.

2. Preliminaries

This section is a brief description of the space setting on which problem (1.1)–(1.2) is formulated. For the details and proofs of the results, we refer to [5], [15]. The space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the "energy norm" $\|u\|_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} |\bigtriangledown u|^2\,dx$. It is well known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2N/(N-2)}(\mathbb{R}^N) : \bigtriangledown u \in (L^2(\mathbb{R}^N))^N\}$$

and that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is embedded continuously in $L^{2N/(N-2)}(\mathbb{R}^N)$, i.e. there exists $k > 0$ such that

$$\|u\|_{2N/(N-2)} \le k\|u\|_{\mathcal{D}^{1,2}}.$$

In [5, Lemma 2.1] it is proved that the *generalized Poincaré's inequality*

$$\int_{\mathbb{R}^N} |\bigtriangledown u|^2\,dx \ge \alpha \int_{\mathbb{R}^N} gu^2\,dx, \tag{2.1}$$

holds for all $u \in C_0^\infty(\mathbb{R}^N)$ and $g \in L^{N/2}(\mathbb{R}^N)$, where $\alpha =: k^{-2}\|g\|_{N/2}^{-1}$. It is shown that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. The space $L_g^2(\mathbb{R}^N)$, defined to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product $(u,v)_{L_g^2} =: \int_{\mathbb{R}^N} guv\,dx$, is a separable Hilbert space. Moreover, the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^2(\mathbb{R}^N)$ is compact. The following lemmas will be used in the sequel.

Lemma 2.1. *Let $g \in L^{2N/(2N-pN+2p)}(\mathbb{R}^N)$. Then we have the following continuous embedding*

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^p(\mathbb{R}^N),$$

for all $1 \le p \le 2N/(N-2)$.

Remark 2.2. Let us note that the assumption of Lemma 2.1 is satisfied under the hypothesis $(\mathcal{G})$, if $2 \le p \le 2N/(N-2)$.

LEMMA 2.3. *Assume that* $1 < a, b, c < \infty$, $s \in [0, c^{-1})$ *and* $a^{-1}+b^{-1}+c^{-1} = 1$. *Then for every* $u \in L^a_g$, $v \in L^b_g$, $w \in L^c_g$ *and every* $K > 0$ *we have the inequality*

$$\left| \int_{\mathbb{R}^N} guvw\, dx \right| \le K^{s-c^{-1}} \|w\|_{L^c_g} (\|u\|^a_{L^a_g} + \|v\|^b_{L^b_g} + K)^{1-s}.$$

LEMMA 2.4. *Assume that* $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then the following continuous embedding* $L^p_g(\mathbb{R}^N) \subset L^q_g(\mathbb{R}^N)$ *is true, for any* $1 \le q \le p < \infty$.

In [15] it is proved that the operator $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^N)$ is a symmetric, strongly monotone operator on $L^2_g(\mathbb{R}^N)$. By applying the Friedrichs extension theorem (see [31]), we construct the *energetic extension* $A_E = -\phi\Delta$ of A_0, defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The *Friedrichs extension* A of A_0 is a self-adjoint operator and its domain $D(A)$, is a Hilbert space endowed with the norm

$$\|u\|_{D(A)} = \left\{ \int_{\mathbb{R}^N} g|u|^2\, dx + \int_{\mathbb{R}^N} \phi|\Delta u|^2\, dx \right\}^{1/2},$$

which is equivalent to the norm

$$\|Au\|_{L^2_g} = \left\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2\, dx \right\}^{1/2}.$$

The compact and dense embeddings

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N) \subset D^{-1,2}(\mathbb{R}^N),$$

will serve as the evolution triple for (1.1)–(1.2). Finally, we give the definition of the *weak solution* for the problem (1.1)–(1.2).

DEFINITION 2.5. A *weak solution* of (1.1)–(1.2) is a function $u(x,t)$ such that

(i) $u \in L^2[0,T; \mathcal{D}^{1,2}(\mathbb{R}^N)]$, $u_t \in L^2[0,T; L^2_g(\mathbb{R}^N)]$, $u_{tt} \in L^2[0,T; D^{-1,2}(\mathbb{R}^N)]$ with $u_t \in L^{m+1}_g([0,\ T) \times \mathbb{R}^N)$,

(ii) for all $v \in C_0^\infty([0,T] \times \mathbb{R}^N)$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g}\, d\tau + \delta \int_0^T (h(u_t(\tau)), v(\tau))_{L^2_g}\, d\tau + \int_0^T \int_{\mathbb{R}^N} \nabla u(\tau) \nabla v(\tau)\, dx\, d\tau - \lambda \int_0^T (f(u(\tau)), v(\tau))_{L^2_g}\, d\tau = 0, \tag{2.2}$$

where $f(s) = |s|^{\beta-1}s$, $h(s) = |s|^{m-1}s$, and

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u_t(x,0) = u_1(x) \in L^2_g(\mathbb{R}^N).$$

Remark 2.6. We may see by using a density argument, that the generalized formula (2.2) is satisfied for every $v \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)]$. Note that in case of bounded domain Ω, $H_0^1(\Omega) \equiv \mathcal{D}^{1,2}(\Omega)$, while $H^1(\mathbb{R}^N) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, by the definition of g we have that $L^2(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N)$. Therefore although weighted space is involved, the solvability of (1.1)–(1.2) is obtained for a more general class of initial data in contrast to the weighted spaces environment used in [3].

3. The nonlinearly damped equation $(m > 1)$

In this section we prove that in the nonlinear damping case solutions blow-up in finite time, under the assumption of positive initial energy and sufficiently large initial data u_0. On the other hand, if the initial condition u_0 is in a modified potential well it is shown that solutions exist globally and energy decays. First we give the following local existence result.

Proposition. *Let g satisfy conditions $(\mathcal{G}_1)$ or $(\mathcal{G}_2)$. Suppose that the constants $\delta > 0$, $\lambda < \infty$ and the initial conditions*

$$u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{and} \quad u_1 \in L^2_g(\mathbb{R}^N),$$

are given. Then there exists $T > 0$ such that the problem (1.1)–(1.2) *admits a unique* (*weak*) *solution with*

$$u \in C[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)] \quad \text{and} \quad u_t \in C[0,T;L^2_g(\mathbb{R}^N)] \cap L^{m+1}_g((0,T)\times\mathbb{R}^N).$$

Proof. The proof follows the lines of [15]: For the solvability of the problem (1.1)–(1.2) restricted on the ball B_R of $\mathbb{R}^N$, we combine density arguments similar to those of [9, Proposition 2.1] and estimates of [15, Proposition 3.1(a)]. The extension of solution to all of $\mathbb{R}^N$, as $R \to \infty$, is obtained exactly as in [15, Proposition 3.1(b)]. $\square$

Blow up of solutions. Multiplying equation (1.1) by gu_t and integrating over $\mathbb{R}^N$ we obtain the energy relation

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2_g} + \delta\|u_t\|^{m+1}_{L^{m+1}_g} + \frac{1}{2}\frac{d}{dt}\|u\|^2_{\mathcal{D}^{1,2}} = \frac{\lambda}{\beta+1}\frac{d}{dt}\int_{\mathbb{R}^N} g(x)|u|^{\beta+1}\,dx. \tag{3.1}$$

The energy of the problem (1.1)–(1.2) is defined as

$$\mathcal{E}^*(t) =: \frac{1}{2}\|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^2_{\mathcal{D}^{1,2}} - \frac{\lambda}{\beta+1}\int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\,dx. \tag{3.2}$$

From (3.1), (3.2) it is easily seen that, for every $t \in [0,T)$, $\mathcal{E}^*(t)$ is a nonincreasing function of t and

$$\mathcal{E}^*(t) + \delta\int_0^t \|u_t(s)\|^{m+1}_{L^{m+1}_g}\,ds = \mathcal{E}^*(0). \tag{3.3}$$

From Lemma 2.1 and assumption $(\mathcal{G})$ we have that $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L_g^{\beta+1}(\mathbb{R}^N)$, for $1 < \beta \leq N/(N-2)$, i.e.,

$$\|u\|_{L_g^{\beta+1}} \leq C_g \|u\|_{\mathcal{D}^{1,2}}, \tag{3.4}$$

where $C_g = \|g\|_a^{1/(\beta+1)}$, with $a = 2N/(N(\beta-1)+2(\beta+1))$. Using the energy (3.2) and the embedding inequality (3.4) we get that

$$\mathcal{E}^*(t) \geq \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^2 - \frac{\lambda}{\beta+1}\|u|_{L_g^{\beta+1}}^{\beta+1} \geq \frac{1}{2C_g}\|u\|_{L_g^{\beta+1}}^2 - \frac{\lambda}{\beta+1}\|u\|_{L_g^{\beta+1}}^{\beta+1}. \tag{3.5}$$

Denoting by $p \equiv p(t) = \|u(t)\|_{L_g^{\beta+1}}$, we consider the functional

$$P(p) = \frac{1}{2C_g}p^2 - \frac{\lambda}{\beta+1}p^{\beta+1}. \tag{3.6}$$

There exists a unique maximum value P_0 for the function P at a point p_0, where

$$P_0 = \lambda\left(\frac{1}{\lambda C_g^2}\right)^{(\beta+1)/(\beta-1)} \frac{\beta-1}{2(\beta+1)} \quad \text{and} \quad p_0 = \left(\frac{1}{\lambda C_g^2}\right)^{1/(\beta-1)}.$$

Next, we state the following two conditions, to be used in the sequel

$$\mathcal{E}^*(0) < P_0, \tag{3.7}$$

$$p(0) > p_0. \tag{3.8}$$

Following the lines of [25], it can be shown that the assumptions (3.7) and (3.8) imply the inequalities

$$p(t) \geq p_0 \quad \text{and} \quad \|u(t)\|_{\mathcal{D}^{1,2}}^2 \geq \frac{1}{C_g^2}p_0^2, \quad \text{for every } t \in [0,T). \tag{3.9}$$

THEOREM 3.2. *Let us suppose that hypothesis* $(\mathcal{G}_1)$ *and conditions* (3.7)–(3.8) *are satisfied. We also assume that* $m < \beta$. *Then the solution of problem* (1.1)–(1.2) *blows-up in finite time.*

PROOF. As in [29], we define the functional

$$F(t) =: \mathcal{E}_1^* - \mathcal{E}^*(0) + \delta \int_0^t \|u_t(s)\|_{L_g^{m+1}}^{m+1}\, ds \quad \text{for some } \mathcal{E}_1^* \in (\mathcal{E}^*(0), P_0). \tag{3.10}$$

From (3.2), (3.3), (3.7)–(3.8) and (3.10) we get that the functional $F(t)$ is increasing, positive function of t and

$$\frac{\lambda}{\beta+1}\|u\|_{L_g^{\beta+1}}^{\beta+1} \geq \mathcal{E}_1^* - \mathcal{E}^*(t) = F(t) \geq F(0) > 0. \tag{3.11}$$

Let $0<\varepsilon<\beta-1$. We multiply (1.1) by gu and integrate over $\mathbb{R}^N$ to obtain

$$\begin{aligned}\frac{d}{dt}(u(t),u_t(t))_{L^2_g} &= \|u_t(t)\|^2_{L^2_g} - \delta(|u_t(t)|^{m-1}u_t(t),u(t))_{L^2_g}\|u(t)\|^2_{\mathcal{D}^{1,2}} \\ &\quad + \lambda\int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\,dx \\ &\quad + (\beta+1-\varepsilon)\mathcal{E}^*(t)-(\beta+1-\varepsilon)\mathcal{E}^*(t).\end{aligned} \tag{3.12}$$

We insert (3.2) and (3.11) in (3.12) to get the inequality

$$\begin{aligned}\frac{d}{dt}(u,\ u_t)_{L^2_g} \geq \frac{C_1(\varepsilon)}{2}\|u_t\|^2_{L^2_g} + \frac{C_2(\varepsilon)}{2}\|u\|^2_{\mathcal{D}^{1,2}} + C_3(\varepsilon)\|u\|^{\beta+1}_{L^{\beta+1}_g} \\ - \delta(|u_t|^{m-1}u_t,\ u)_{L^2_g} + C_4(\varepsilon)F(t) - C_4(\varepsilon)\mathcal{E}^*_1,\end{aligned} \tag{3.13}$$

where $C_1(\varepsilon)=\beta+3-\varepsilon$, $C_2(\varepsilon)=\beta-1-\varepsilon$, $C_3(\varepsilon)=\varepsilon\lambda/(\beta+1)$, $C_4(\varepsilon)=\beta+1-\varepsilon$. From (3.9) and (3.13) it follows that

$$\begin{aligned}\frac{d}{dt}(u(t),u_t(t))_{L^2_g} &\geq \frac{C_1(\varepsilon)}{2}\|u_t(t)\|^2_{L^2_g} + C_3(\varepsilon)\|u(t)\|^{\beta+1}_{L^{\beta+1}_g} \\ &\quad + C_4(\varepsilon)F(t) - \delta(|u_t(t)|^{m-1}u_t(t),\ u(t))_{L^2_g} \\ &\quad + C_4(\varepsilon)\left\{\frac{C_2(\varepsilon)}{C_4(\varepsilon)}\frac{1}{2C_g^2}p_0^2 - \mathcal{E}^*_1\right\}.\end{aligned} \tag{3.14}$$

We estimate the fourth term on the right-hand side of (3.14) as follows: We apply Hölder's inequality with exponents $p=\beta(m+1)/(\beta-m)$ and $q=\beta(m+1)/m(\beta+1)$ to get that

$$\|h(u_t)\|_{L^{\beta_*}_g} = \left(\int_{\mathbb{R}^N} g^{p^{-1}}g^{q^{-1}}|u_t|^{m\beta_*}\,dx\right)^{\beta_*^{-1}} \leq M_2\|u_t\|^m_{L^{m+1}_g}, \tag{3.15}$$

where $M_2=\|g\|_1^{p^{-1}}$ and $\beta_*=(\beta+1)/\beta$ is the Hölder conjugate of $\beta+1$. Hence using (3.10), we obtain that

$$\begin{aligned}|\delta(h(u_t(t)),&u(t))_{L^2_g}| \\ &\leq \delta\|h(u_t(t))\|_{L^{\beta_*}_g}\|u(t)\|_{L^{\beta+1}_g} \leq \delta M_2\|u_t(t)\|^m_{L^{m+1}_g}\|u(t)\|_{L^{\beta+1}_g} \\ &\leq \delta M_2\|u_t(t)\|^m_{L^{m+1}_g}\{\|u(t)\|^{\beta+1}_{L^{\beta+1}_g}\}^{1/(m+1)}\{\|u(t)\|^{\beta+1}_{L^{\beta+1}_g}\}^{-k},\end{aligned} \tag{3.16}$$

where $k=1/(m+1)-1/(\beta+1)$ and $0<k<1$, when $\beta>m$. Using relations (3.11) and (3.16) we obtain that

$$\begin{aligned}|\delta(h(u_t(t)),&\ u(t))_{L^2_g}| \\ &\leq \delta M_2\|u_t(t)\|^m_{L^{m+1}_g}\times\{\|u(t)\|^{\beta+1}_{L^{\beta+1}_g}\}^{1/(m+1)}\left\{\frac{\beta+1}{\lambda}\right\}^{-k}F(t)^{-k}.\end{aligned} \tag{3.17}$$

Now Young's inequality for some $\varepsilon_0 > 0$, implies that

$$|\delta(h(u_t(t)), u(t))_{L^2_g}| \le \{C_5 \varepsilon_0^{m+1} \|u(t)\|^{\beta+1}_{L^{\beta+1}_g} + \delta^{(m+1)/m} \varepsilon_0^{-(m+1)/m} \|u_t(t)\|^{m+1}_{L^{m+1}_g}\} F(t)^{-k}. \tag{3.18}$$

Let some $0 < k^* < k$. From (3.11) and (3.18) it holds that

$$\begin{aligned}
|\delta(h(u_t(t)),u(t))_{L^2_g}| &\le C_5 \varepsilon_0^{m+1} \|u(t)\|^{\beta+1}_{L^{\beta+1}_g} F(t)^{-k} \\
&\quad + \delta^{(m+1)/m} \varepsilon_0^{-(m+1)/m} \|u_t(t)\|^{m+1}_{L^{m+1}_g} F(t)^{k^*-k} F(t)^{-k^*} \\
&\le C_5 \varepsilon_0^{m+1} \|u(t)\|^{\beta+1}_{L^{\beta+1}_g} F(0)^{-k} \\
&\quad + \delta^{(m+1)/m} \varepsilon_0^{-(m+1)/m} \|u_t(t)\|^{m+1}_{L^{m+1}_g} F(0)^{k^*-k} F(t)^{-k^*} \\
&\le C_5 \varepsilon_0^{m+1} \|u(t)\|^{\beta+1}_{L^{\beta+1}_g} F(0)^{-k} \\
&\quad + \delta^{(m+1)/m} \varepsilon_0^{-(m+1)/m} F'(t) F(0)^{k^*-k} F(t)^{-k^*}.
\end{aligned} \tag{3.19}$$

A combination of estimates (3.14) and (3.19) with the fact that the last term on the right-hand side of (3.14) is positive (by assumption (3.9)), shows the inequality

$$\begin{aligned}
\frac{d}{dt}(u(t),u_t(t))_{L^2_g} &\ge \frac{C_1(\varepsilon)}{2} \|u_t(t)\|^2_{L^2_g} + \{C_3(\varepsilon) - C_5 \varepsilon_0^{m+1} F(0)^{-k}\} \|u(t)\|^{\beta+1}_{L^{\beta+1}_g} \\
&\quad + C_4(\varepsilon) F(t) - \varepsilon_0^{-(m+1)/m} \delta^{(m+1)/m} F(0)^{k^*-k} F(t)^{-k^*} F'(t).
\end{aligned} \tag{3.20}$$

As in [9] for some $\mu > 0$ to be chosen later, we use the functional

$$G(t) =: \mu F(t)^{1-k^*} + (u_t(t),\ u(t))_{L^2_g}.$$

Differentiating $G(t)$ and using (3.20) we see that

$$\begin{aligned}
G'(t) &\ge \{\mu(1-k^*) - \varepsilon_0^{-(m+1)/m} \delta^{(m+1)/m} F(0)^{k^*-k}\} F(t)^{-k^*} F'(t) \\
&\quad + \{C_3(\varepsilon) - C_5 \varepsilon_0^{m+1} F(0)^{-k}\} \|u(t)\|^{\beta+1}_{L^{\beta+1}_g} \\
&\quad + C_4(\varepsilon) F(t) + \frac{C_1(\varepsilon)}{2} \|u_t(t)\|^2_{L^2_g}.
\end{aligned} \tag{3.21}$$

Choosing μ sufficiently large, we may find values for $\varepsilon_0 > 0$, such that the following inequalities hold simultaneously.

$$\begin{aligned}
K_1 &= \mu(1-k^*) - \varepsilon_0^{-(m+1)/m} \delta^{(m+1)/m} F(0)^{k^*-k} > 0, \\
K_2 &= C_3(\varepsilon) - C_5 \varepsilon_0^{m+1} F(0)^{-k} > 0.
\end{aligned}$$

Hence, from (3.21), we obtain that

$$G'(t) \geq K_4\{\|u_t(t)\|^2_{L^2_g} + F(t) + \|u(t)\|^{\beta+1}_{L^{\beta+1}_g}\} > 0, \tag{3.22}$$

for some constant $K_4 > 0$. Therefore, $G(t) > G(0) > 0$, i.e. G is a strictly increasing function of t. We set $\gamma = 1/(1-k^*)$. Moreover, from Lemma 2.3 (where $u = u$, $v = u_t$, $w = 1$, $a = \beta+1$, $b = 2$, $c = 2(\beta+1)/(\beta-1)$, $s = k^*$, $K = F(t)$),

$$\begin{aligned} &|(u,u_t)_{L^2_g}| \\ &\quad \leq F(t)^{k^*-(\beta-1)/2(\beta+1)}\|g\|_1^{(\beta-1)/2(\beta+1)}\{\|u\|^{\beta+1}_{L^{\beta+1}_g} + \|u_t\|^2_{L^2_g} + F(t)\}^{1/\gamma}. \end{aligned} \tag{3.23}$$

Note that $k^* \in (0,(\beta-1)/2(\beta+1))$, since $k^* \in (0,k)$. From the definition of $G(t)$ and (3.23), we have that

$$|(u(t),u_t(t))_{L^2_g}|^\gamma \leq K_5\{\|u(t)\|^{\beta+1}_{L^{\beta+1}_g} + \|u_t(t)\|^2_{L^2_g} + F(t)\}, \tag{3.24}$$

where $K_5 = \{F(0)^{k^*-(\beta-1)/(2(\beta+1))}\|g\|_1^{(\beta-1)(2(\beta+1))}\}^\gamma$. From relation (3.24) we get that

$$\begin{aligned} G(t)^\gamma &\leq 2^{\gamma-1}(\mu^{\gamma-1}F(t) + |(u(t),u_t(t))_{L^2_g}|^\gamma) \\ &\leq K_6\{\|u_t(t)\|^2_{L^2_g} + F(t) + \|u(t)\|^{\beta+1}_{L^{\beta+1}_g}\} \leq \frac{K_6}{K_4}\dot{F}(t), \end{aligned} \tag{3.25}$$

where $K_6 = \max\{(2\mu)^{\gamma-1},(2\mu)^{\gamma-1}K_5\}$. The last inequality (3.25) implies that $G'(t) \geq CG(t)^\gamma$. Well known arguments (see [4, Theorem 4.2]), imply that $G'(t)$ cannot be global in time and the proof is completed. □

Global existence and energy decay. Consider the *potential well*

$$\mathcal{W} =: \operatorname{Int}\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \mathcal{K}(u) =: \|u\|^2_{\mathcal{D}^{1,2}} - \lambda\|u\|^{\beta+1}_{L^{\beta+1}_g} \geq 0\},$$

where $\operatorname{Int} B$ denotes the interior of the set B. It is easily seen that $\mathcal{K}(u) \geq 0$ for small $u \in \mathcal{D}^{1,2}$ and $0 \in \mathcal{W}$ For the details we refer to [15, p. 164–165].

Theorem 3.3. *Let condition* (3.7) *and hypothesis* ($\mathcal{G}_2$) *be fulfilled. Assume that* $u_0 \in \mathcal{W}$ *and*

$$\lambda < \left\{C_g^{\beta+1}\left(\frac{2(\beta+1)P_0}{\beta-1}\right)^{(\beta-1)/2}\right\}^{-1} =: \lambda_0^{-1}. \tag{3.26}$$

Then

(i) *the unique* (*weak*) *solution of* (1.1)–(1.2) *is such that*

$$u \in C([0,\infty);\mathcal{D}^{1,2}(\mathbb{R}^N)) \quad \textit{and} \quad u_t \in C([0,\infty);L^2_g(\mathbb{R}^N)).$$

(ii) *In addition, if condition* $(\mathcal{G}_1)$ *is satisfied, the solution decays in time, i.e.*

$$\lim_{t\to\infty} \|u(t)\|^2_{\mathcal{D}^{1,2}} = \lim_{t\to\infty} \|u_t(t)\|^2_{L^2_g} = 0.$$

PROOF. (i) Assume that there exists some time $T^* > 0$, such that $u(t) \in \mathcal{W}$, where $0 \le t < T^*$ and $u(T^*) \in \partial\mathcal{W}$. Then $\mathcal{K}(u(T^*)) = 0$ and $u(T^*) \neq 0$. Since $u(t) \in \mathcal{W}$ we may see that

$$\mathcal{J}(u(t)) =: \frac{1}{2}\|u(t)\|^2_{\mathcal{D}^{1,2}} - \frac{\lambda}{\beta+1}\|u(t)\|^{\beta+1}_{L^{\beta+1}_g} \ge \frac{\beta-1}{2(\beta+1)}\|u(t)\|^2_{\mathcal{D}^{1,2}}, \tag{3.27}$$

for $t \in [0, T)$. Furthermore, from (3.3) and (3.7) we have

$$\mathcal{J}(u(t)) \le \mathcal{E}^*(t) \le \mathcal{E}^*(0) < P_0, \quad \text{for all } t \in [0,\ T). \tag{3.28}$$

Using (3.4), (3.27) and (3.28) we obtain the inequality

$$\|u(t)\|^{\beta+1}_{L^{\beta+1}_g} \le C_g^{\beta+1}(\|u(t)\|^2_{\mathcal{D}^{1,2}})^{(\beta-1)/2}\|u(t)\|^2_{\mathcal{D}^{1,2}} \le \lambda_0\|u(t)\|^2_{\mathcal{D}^{1,2}}, \tag{3.29}$$

for $t \in [0,\ T)$. For $t = T^*$ we have

$$\mathcal{K}(u(T^*)) \ge (1-\lambda\lambda_0)\|u(T^*)\|^2_{\mathcal{D}^{1,2}} > 0, \tag{3.30}$$

if $\lambda < 1/\lambda_0$ (which justifies assumption (3.26)). This contradiction implies that $T = \infty$. Also from (3.4), (3.27) and (3.28) we get the estimates every $t \in [0, \infty)$,

$$\|u(t)\|^2_{\mathcal{D}^{1,2}} + \|u_t(t)\|^2_{L^2_g} \le 2\frac{\beta+1}{\beta-1}P_0, \tag{3.31}$$

$$\int_0^t \|u_t(s)\|^{m+1}_{L^{m+1}_g}\, ds \le \frac{P_0}{\delta}. \tag{3.32}$$

(ii) Consider the case $\beta > m$. To prove the decay of the energy norm of solutions at infinity, we integrate with respect to the time t relation (3.12), use Poincaré's inequality (2.1) and estimates (3.31), (3.32), to obtain

$$\begin{aligned}\int_0^t \mathcal{K}(u(s))\, ds &\le \int_0^t \|u_t(s)\|^2_{L^2_g}\, ds + |(u(t), u_t(t))_{L^2_g}| \\ &\quad + |(u_0, u_1)_{L^2_g}| + \delta\int_0^t (h(u_t(s)),\ u(t))_{L^2_g}\, ds \\ &\le M + \int_0^t \|u_t(s)\|^2_{L^2_g}\, ds + \delta\int_0^t (h(u_t(s)),\ u(s))_{L^2_g}\, ds,\end{aligned} \tag{3.33}$$

with $M = 4\min\{(\beta+1)/(2(\beta-1)), 1/(2\alpha)\}P_0$.

Using (3.4) and inequalities (3.15), (3.31) we estimate the last term of the right-hand side of (3.33), as follows

$$
\begin{aligned}
(3.34)\qquad \int_0^t (h(u_t(s)),u(s))_{L^2_g}\,ds &\le \int_0^t \|h(u_t(s))\|_{L_g^{\beta *}}\|u(s)\|_{L_g^{\beta+1}}\,ds\\
&\le M_2C_g\int_0^t \|u_t(s)\|^m_{L_g^{m+1}}\|u(s)\|_{\mathcal{D}^{1,2}}\,ds\\
&\le M_3 t^{1/(m+1)}\left(\int_0^t \|u_t(s)\|^{m+1}_{L_g^{m+1}}\,ds\right)^{m/(m+1)},
\end{aligned}
$$

with $M_3=(2P_0(\beta+1)/(\beta-1))^{1/2}M_2C_g$. Moreover, using Lemma 2.4, we see that

$$
\begin{aligned}
(3.35)\qquad \int_0^t \|u_t(s)\|^2_{L^2_g}\,ds &\le \left(\int_0^t ds\right)^{(m-1)/(m+1)}\left(\int_0^t \|u_t(s)\|^{m+1}_{L^2_g}\,ds\right)^{2/(m+1)}\\
&\le t^{(m-1)/(m+1)}\left(\int_0^t \|u_t(s)\|^{m+1}_{L_g^{m+1}}\,ds\right)^{2/(m+1)}.
\end{aligned}
$$

As in [10], since $(d/dt)\mathcal{E}^*(t)=-\delta\|u_t(t)\|^{m+1}_{L_g^{m+1}}$ it holds that

$$\frac{d}{dt}(1+t)\mathcal{E}^*(t)\le \mathcal{E}^*(t).$$

We integrate the last inequality over $[0,t]$ and use the relation

$$(\beta+1)\mathcal{J}(u(t))=\frac{\beta-1}{2}\|u(t)\|^2_{\mathcal{D}^{1,2}}+\mathcal{K}(u(t)),$$

to obtain that

$$
\begin{aligned}
(3.36)\qquad (1+t)\mathcal{E}^*(t)\le \mathcal{E}^*(0)&+\frac12\int_0^t \|u_t(s)\|^2_{L^2_g}\,ds\\
&+\frac{\beta-1}{2(\beta+1)}\int_0^t \|u(s)\|^2_{\mathcal{D}^{1,2}}\,ds+\frac{1}{\beta+1}\int_0^t \mathcal{K}(u(s))\,ds.
\end{aligned}
$$

Since inequality (3.30) holds for every $t\in[0,\infty)$, it follows from (3.36) that

$$(3.37)\qquad (1+t)\mathcal{E}^*(t)\le \mathcal{E}^*(0)+\frac12\int_0^t \|u_t(s)\|^2_{L^2_g}\,ds+M_4\int_0^t \mathcal{K}(u(s))\,ds,$$

where

$$M_4=\frac{\beta-1}{2(1-\lambda\lambda_0)(\beta+1)}+\frac{1}{\beta+1}.$$

We insert the estimates (3.7), (3.33)-(3.35) to (3.37) to obtain, for every $t\in[0,\infty)$,

$$(3.38)\qquad \mathcal{E}^*(t)\le M_5\frac{1}{1+t}+M_6\frac{t^{(m-1)/(m+1)}}{1+t}+M_7\frac{t^{1/(m+1)}}{1+t},$$

where $M_5=P_0+M_4M$, $M_6=(3/2)M_4(P_0/\delta)^{2/(m+1)}$, $M_7=M_4\delta(P_0/\delta)^{m/(m+1)}$. Finally, we let $t\to\infty$, in (3.38) to obtain that $\lim_{t\to\infty}\mathcal{E}^*(t)=0$.

In the case $\beta \leq m$ we use the inequality

$$(h(u_t(t)), u(t))_{L^2_g} \leq \|h(u_t(t))\|_{L_g^{(m+1)/m}} \|u(t)\|_{L_g^{m+1}} \leq C\|u_t(t)\|^{m+1}_{L_g^{m+1}} \|u(t)\|_{\mathcal{D}^{1,2}}$$

in order to obtain the estimate (3.34). □

REMARK 3.4. Under the assumptions of Theorem 3.3 for the initial data, global existence holds with out any relations between β, m (see also [10]). It is possible to show the global existence of solutions in the case $\beta \leq m$, if g satisfies $(\mathcal{G}_1)$, without any assumption on the initial condition u_0 and on the initial energy $\mathcal{E}^*(0)$, as in the works [9] and [20]. In the case $\delta = 0$ global existence for sufficiently small initial data is shown, when $N = 3, 4$ and $(N+4)/N \leq \beta \leq N/(N-2)$ (see [15, Theorem 5.2])

4. The linearly damped equation $(m = 1)$

In this section we prove blow-up of solutions when the initial energy is positive, in the case of the linearly damped equation. We note that in the present result, we may assume $(\mathcal{G}_2)$ instead of condition $(\mathcal{G}_1)$ of [15, Theorem 4.2])

THEOREM 4.1. *Let conditions* (3.7)–(3.8) *and hypothesis* $(\mathcal{G}_2)$ *be fulfilled. Then the solution of the problem* (1.1)–(1.2) *blows-up in finite time.*

PROOF. The result will be obtained by application of the Concavity Lemma introduced by Levine in [18] (see also [25]). For fixed t_0, T_0, we consider the functional

$$H(t) =: \|u(t)\|^2_{L^2_g} + \delta \int_0^t \|u(\tau)\|^2_{L^2_g}\, d\tau + \delta(T_0 - t)\|u_0\|^2_{L^2_g} + \gamma(t+t_0)^2, \tag{4.1}$$

where γ is a positive constant to be fixed later. We differentiate (4.1) with respect to t and integrate by parts to obtain

$$H'(t) = 2(u(t),\ u_t(t))_{L^2_g} + 2\delta \int_0^t (u(\tau), u_t(\tau)_{L^2_g}\, d\tau + 2\gamma(t+t_0). \tag{4.2}$$

Multiply (1.1) by gu and integrate over $\mathbb{R}^N$ to get

$$\begin{aligned} \frac{d}{dt}(u(t), u_t(t))_{L^2_g} - \|u_t(t)\|^2_{L^2_g} + \frac{\delta}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_g} \\ + \|u(t)\|^2_{\mathcal{D}^{1,2}} - \lambda \int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\, dx = 0. \end{aligned} \tag{4.3}$$

From (4.2), the definition of energy (3.2) and relation (4.3), we have

$$\frac{1}{2}H''(t) = \frac{\beta+3}{2}\|u_t(t)\|^2_{L^2_g} + \frac{\beta-1}{2}\|u(t)\|^2_{\mathcal{D}^{1,2}} - (\beta+1)\mathcal{E}^*(t) + \gamma. \tag{4.4}$$

We insert relation (3.3) (for $m = 1$) and inequality (3.9) to equation (4.4), to get

$$\frac{1}{2}H''(t) \geq \frac{\beta+3}{2}\|u_t(t)\|^2_{L^2_g} + \delta(\beta+1)\int_0^t \|u_t(\tau)\|^2_{L^2_g}\,d\tau + (\beta+1)\left\{\frac{\beta-1}{2(\beta+1)C_g^2}p_0^2 - \mathcal{E}^*(0)\right\} + \gamma. \tag{4.5}$$

Next, we choose $\eta =: 2\{(\beta-1)/(2(\beta+1)C_g^2)p_0^2 - \mathcal{E}^*(0)\}$, which is positive by assumptions (3.7)–(3.8). Then (4.5) becomes

$$H''(t) \geq (\beta+3)\{\|u_t(t)\|^2_{L^2_g} + \eta\} + 2\delta(\beta+1)\int_0^t \|u_t(\tau)\|^2_{L^2_g}\,d\tau. \tag{4.6}$$

Finally, we check the assumptions of the Concavity Lemma. Since $H(0) = 2(u_0, u_1)_{L^2_g} + 2\eta t_0$, one may choose t_0 sufficiently large so that $H'(0) > 0$. From inequality (4.6) we have that $H''(t) > 0$. This implies that $H'(t)$ is an increasing function of t. Therefore $H'(t) > H'(0) > 0$, i.e. $H(t)$ is also an increasing function of t. From the definition (4.1) we have that $H(t) > 0$. Consider the quantities

$$\begin{aligned}
\mathcal{A} &= \|u(t)\|^2_{L^2_g} + \int_0^t \|u(\tau)\|^2_{L^2_g}\,d\tau + \eta(t+t_0)^2,\\
\mathcal{B} &= \frac{1}{2}H'(t),\\
\mathcal{C} &= \|u_t(t)\|^2_{L^2_g} + \delta\int_0^t \|u_t(\tau)\|^2_{L^2_g}\,d\tau + \eta.
\end{aligned}$$

From (4.1) and (4.6) it holds that $A \leq H(t)$ and $(\beta+3)C \leq H''(t)$ on $[0, T_0]$. The arguments of [25], imply

$$H(t)H''(t) - aH'(t)^2 \geq 0,$$

for $a = (\beta+3)/4$, which which shows that solution blows-up in finite time. □

Remark 4.2. In the case of the linear damping, the global existence and energy decay results of Theorem 3.3 are also valid under the assumption $(\mathcal{G}_2)$.

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