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Existence results for quasilinear elliptic systems in $\mathbb{R}^N$ *

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Abstract

We prove existence results for the quasilinear elliptic system

$$-\Delta_p u = \lambda a(x)|u|^{\gamma-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u,$$
$$-\Delta_q v = \lambda d(x)|v|^{\delta-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v\,,$$

where γ and δ may reach the critical Sobolev exponents, and the coefficient functions a, b, and d may change sign.

For the unperturbed system ($a = 0$, $b = 0$), we establish the existence and simplicity of a positive principal eigenvalue, under the assumption that $u(x) > 0$, $v(x) > 0$, and $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u(x) = 0$.

1 Introduction

In this paper we study the existence of nontrivial solutions of the quasilinear elliptic system containing the p-Laplacian operator, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$,

$$\begin{aligned}-\Delta_p u &= \lambda a(x)|u|^{\gamma-2}u + \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u,\\ -\Delta_q v &= \lambda d(x)|v|^{\delta-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v,\end{aligned} \tag{1.1}$$

for all x in $\mathbb{R}^N$, where $1 < p < N$, $1 < q < N$, $\alpha \geq 0$ and $\beta \geq 0$. For α, β, p, q, and N we distinguish the following cases:

i) $\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1$,

ii) $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$,

iii) $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$,

iv) $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, where p^* and q^* are the critical Sobolev exponents $p^* = \frac{Np}{N-p}$ and $q^* = \frac{Nq}{N-q}$.

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Throughout this paper we assume the following hypothesis

(H) $2 \leq \gamma \leq p^*$, $2 \leq \delta \leq q^*$, and $a \in L^{p^*/(p^*-\gamma)}(\mathbb{R}^N)$, $d \in L^{q^*/(q^*-\delta)}(\mathbb{R}^N)$, and $b \in L^{\omega}(\mathbb{R}^N)$, where $\omega = p^*q^*/[p^*q^* - (\alpha+1)p^* - (\beta+1)q*]$. Also we assume that a, d and b are smooth functions of at least $C^{0,\eta}_{\rm loc}(\mathbb{R}^N)$ with $\eta \in (0,1)$.

In certain cases the coefficients a, d, and b will be assumed to lie in certain function spaces:
$\mathcal{H}_\infty$ which consists of functions $h : \mathbb{R}^N \to \mathbb{R}$ in $L^\infty(\mathbb{R}^N)$, and $h(x)$ tends uniformly to zero as $|x| \to \infty$, in the sense

$$\lim_{R\to\infty} \|h\|_{L^\infty(\mathbb{R}^N - B_R)} = 0\,.$$

$\mathcal{H}_+$ which consists of functions $h : \mathbb{R}^N \to \mathbb{R}$ such that there exists $\Omega \subseteq \mathbb{R}^N$, with $|\Omega| > 0$, and $h(x) > 0$, for all $x \in \Omega$.

The operator $-\Delta_p$ turns up in many mathematical settings; e.g., Non-Newtonian fluids, reaction-diffusion problems, porous media, astronomy, etc. (see for example [3]). Problems including this operator for bounded domains have been studied in [1, 5, 8, 14], and for unbounded domains in [2, 6, 9, 11].

This paper is organized as follows. In Section 2, we define the homogeneous space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and establish the characteristics of the basic operators to be used later. The cases i), ii), iii), and iv) stated above are studied in the Sections 3, 4, 5, and 6, respectively, where we prove the existence of at least one solution for the system (1.1). In Section 5, we study the eigenvalue problem

$$\begin{gathered}
-\Delta_p u = \lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u, \\
-\Delta_q v = \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v, \\
u(x) > 0, \quad v(x) > 0, \quad \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} u(x) = 0,
\end{gathered} \tag{1.2}$$

for all x in $\mathbb{R}^N$, where p, q, and b satisfy the Hypothesis **(H)**. Moreover, when b satisfies the extra hypothesis

(B) $b(x) \geq 0$, for every $x \in \mathbb{R}^N$ and $b(x) \not\equiv 0$,

we prove the existence of a positive principal eigenvalue for (1.2) and establish the simplicity of this eigenvalue.

This work extends to unbounded domains the results in [5, 14]. Since the coefficients a, d, and b may change sign and γ, δ may approach the critical Sobolev exponents, we therefore are studying problems that extend the results obtained in [9]. For a discussion on the critical Sobolev exponent, we refer the reader to [10].

Notation. For simplicity we use the symbol $\|.\|_p$ for the norm $\|.\|_{L^p(\mathbb{R}^N)}$, and $\mathcal{D}^{1,p}$ for the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$. B_R will denote the ball in $\mathbb{R}^N$ of center zero and radius R. Also the Lebesgue measure of a set $\Omega \subset \mathbb{R}^N$ will be denoted by $|\Omega|$. Equalities introducing definitions are denoted by =:. The integral symbol $\int$ without any indication will be used for integration over the whole space $\mathbb{R}^N$.

2 Space and Operator Settings

It is going to be proved that the natural space setting for our problem is the space $Z = D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$, with the norm $\|\tilde{z}\|_Z = \|u\|_{1,p} + \|v\|_{1,q}$, where $\tilde{z} = (u,v)$. The space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} =: \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p}.$$

It is known that

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \left\{u \in L^{\frac{Np}{N-p}}(\mathbb{R}^N) : \nabla u \in \left(L^p(\mathbb{R}^N)\right)^N\right\}$$

and that there exists $K_0 > 0$ such that, for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$

$$\|u\|_{L^{\frac{Np}{N-p}}} \le K_0 \|u\|_{\mathcal{D}^{1,p}}. \tag{2.1}$$

The space $\mathcal{D}^{1,p}$ is a reflexive Banach space. For more details, see [9]. Our approach is based on the following generalized Poincare's inequality.

Lemma 2.1 *Suppose $g \in L^{N/p}(\mathbb{R}^N)$. Then there exists $\alpha > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \ge \alpha \int_{\mathbb{R}^N} |g||u|^p dx, \tag{2.2}$$

for all $u \in \mathcal{D}^{1,p}$.

We introduce the operators $J_1, J_2, D_1, D_2, B_1, B_2 : Z \to Z^*$ in the following way

$$\begin{aligned}
(J_1(u,v),(w,z))_Z &=: \frac{\alpha+1}{p}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla w,\\
(J_2(u,v),(w,z))_Z &=: \frac{\beta+1}{q}\int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \nabla z,\\
(D_1(u,v),(w,z))_Z &=: \frac{\gamma+1}{p}\int_{\mathbb{R}^N} a(x)|u|^{\gamma-2}uw,\\
(D_2(u,v),(w,z))_Z &=: \frac{\delta+1}{p}\int_{\mathbb{R}^N} d(x)|v|^{\delta-2}vz,\\
(B_1(u,v),(w,z))_Z &=: \int_{\mathbb{R}^N} b(x)|u|^{\alpha-1}|v|^{\beta+1}uw,\\
(B_2(u,v),(w,z))_Z &=: \int_{\mathbb{R}^N} b(x)|u|^{\alpha+1}|v|^{\beta-1}vz.
\end{aligned}$$

Lemma 2.2 *The operators $J_i, D_i, B_i, i = 1,2$, are well defined. Also $J_i, i = 1,2$, are continuous and the operators D_i, B_i, $i = 1,2$, are compact.*

Proof. The fact that $J_i, D_i, B_i,\ i=1,2$, are well defined is proved in [11]. The continuity of $J_i,\ i=1,2$, and the proof of compactness for $B_i,\ i=1,2$, follows the same lines as in [6] and [9]. We prove the compactness of D_1. The analogous holds for the operator D_2.

Let (u_n, v_n) be a bounded sequence in Z. Hence (u_n, v_n) converges weakly (up to a subsequence) to (u_0, v_0) in Z, i.e., $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,p}$ and $v_n \rightharpoonup v_0$ in $\mathcal{D}^{1,q}$, as $n \to \infty$. For $(w,z) \in Z$ it follows that

$$|(D_1(u_n, v_n),(w,z)) - (D_1(u_0,v_0),(w,z))| \le I + J,$$

where

$$I =: \int |a(x)| \left| |u_n|^{\gamma-2} - |u_0|^{\gamma-2} \right| |u_n||w|$$

and

$$J =: \int |a(x)||u_n - u_0||u_0|^{\gamma-2}|w|.$$

For $R>0$ we write $I = I_1 + I_2$ where

$$I_1 =: \int_{B_R} |a(x)| \left| |u_n|^{\gamma-2} - |u_0|^{\gamma-2} \right| |u_n||w|$$

and

$$I_2 =: \int_{\mathbb{R}^N - B_R} |a(x)| \left| |u_n|^{\gamma-2} - |u_0|^{\gamma-2} \right| |u_n||w|.$$

Applying Hölder inequality to I_1, we obtain

$$I_1 \le \|a(x)\|_{L^{\frac{p^*}{p^*-\gamma}}(B_R)} \| |u_n|^{\gamma-2} - |u_0|^{\gamma-2} \|_{L^{\frac{p^*}{\gamma-2}}(B_R)} \|u_n\|_{L^{p^*}(B_R)} \|w\|_{L^{p^*}(B_R)}.$$

Since $\{u_n\}$ is a bounded sequence in $D^{1,p}(\mathbb{R}^N)$ it is also bounded in $L^{p^*}(B_R)$. So, passing to a subsequence if necessary, we have $u_n \rightharpoonup u_0$ in $L^{p'}(B_R)$, as $n \to \infty$, for any $1 \le p' \le p^*$. Then, we have that $|u_n|^{\gamma-2} \to |u_0|^{\gamma-2}$ in $L^{\frac{p^*}{\gamma-2}}(B_R)$. So that, for n large enough, we obtain $I_1 < \epsilon$. Applying Hölder inequality to I_2 we obtain

$$\begin{aligned} I_2 \quad &\le \quad \|a(x)\|_{L^{\frac{p^*}{p^*-\gamma}}(\mathbb{R}^N - B_R)} \| |u_n| - |u_0| \|_{L^{p^*}(\mathbb{R}^N - B_R)} \times \\ &\quad \times \|u_n\|^{\gamma-2}_{L^{p^*}(\mathbb{R}^N - B_R)} \|w\|_{L^{p^*}(\mathbb{R}^N - B_R)} < \epsilon, \end{aligned}$$

for R sufficiently large. Therefore, $I < 2\epsilon$. Similarly we prove that $J < 2\epsilon$.

All the above arguments may be extended to the critical case of $\gamma = p^*$ and $\delta = q^*$, as long as a and d belong to $\mathcal{H}_\infty$ and the lemma is proved. $\diamondsuit$

We say that (u,v) is a *weak solution* of the system (1.1) if (u,v) is a critical point of the functional $A : Z \to \mathbb{R}$, defined by

$$\begin{aligned} A(u,v) \quad &=: \quad \lambda \frac{\alpha+1}{p} \int |\nabla u|^p + \lambda \frac{\beta+1}{q} \int |\nabla v|^q - \lambda \frac{\alpha+1}{\gamma} \int a(x)|u|^\gamma \\ &\quad -\lambda \frac{\beta+1}{\delta} \int d(x)|v|^\delta - \lambda \int b(x)|u|^{\alpha+1}|v|^{\beta+1}. \end{aligned}$$

Since $A(|u|,|v|)=A(u,v)$, if (u,v) is a critical point of A, then the same happens for $(|u|,|v|)$. Hence we may consider that $u(x)\geq 0$ and $v(x)\geq 0$.

Lemma 2.3 *Let (u_n,v_n) be a bounded sequence in Z such that $A(u_n,v_n)$ is bounded and $A'(u_n,v_n)\to 0$, as $n\to\infty$. Then (u_n,v_n) has a convergent subsequence.*

Proof. Since the sequence (u_n,v_n) is bounded in Z we may consider that there is a subsequence (denoted again by (u_n,v_n)), which is weakly convergent in Z. Moreover, we have that

$$\begin{aligned}
&(A'(u_n,v_n)-A'(u_m,v_m),(u_n-u_m,v_n-v_m))\\
&= (\alpha+1)\int(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m)(\nabla u_n-\nabla u_m)\,dx\\
&\quad+(\beta+1)\int(|\nabla v_n|^{q-2}\nabla v_n-|\nabla v_m|^{q-2}\nabla v_m)(\nabla v_n-\nabla v_m)\,dx\\
&\quad-(\alpha+1)\int a(x)(|u_n|^{\gamma-2}u_n-|u_m|^{\gamma-2}u_m)(u_n-u_m)\,dx\\
&\quad-(\beta+1)\int d(x)(|v_n|^{\delta-2}v_n-|v_m|^{\delta-2}v_m)(v_n-v_m)\,dx\\
&\quad-(\alpha+1)\int b(x)(|u_n|^{\alpha-1}|v_n|^{\beta+1}u_n-|u_m|^{\alpha-1}|v_m|^{\beta+1}u_m)(u_n-u_m)\,dx\\
&\quad-(\beta+1)\int b(x)(|u_n|^{\alpha+1}|v_n|^{\beta-1}v_n-|u_m|^{\alpha+1}|v_m|^{\beta-1}v_m)(v_n-v_m)\,dx\,.
\end{aligned}$$

From the compactness of the operators B_i, D_i, $i=1,2$, we obtain (passing to a subsequence, if necessary) that

$$\begin{aligned}
&(\alpha+1)\int(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m)(\nabla u_n-\nabla u_m)+\\
&(\beta+1)\int(|\nabla v_n|^{q-2}\nabla v_n-|\nabla v_m|^{q-2}\nabla v_m)(\nabla v_n-\nabla v_m) \quad\to\quad 0\,,
\end{aligned}$$

which implies (see [6]) that (u_n,v_n) converges strongly in Z. $\diamondsuit$

In what follows the Palais-Smale (PS) condition will be proved to be a crucial tool for our study, so we describe it.

Definition 2.4 *We say that a functional $A:Z\to\mathbb{R}$ satisfies the (PS) condition, if every sequence $\{(u_n,v_n)\}\subset Z$ such that $A(u_n,v_n)$ is bounded and $A'(u_n,v_n)\to 0$ in Z, as $n\to\infty$, is relatively compact in Z.*

3 The Case $\frac{\alpha+1}{p}+\frac{\beta+1}{q}<1$

Throughout this section we consider that there exist real numbers p_1 and q_1 such that $p_1\in(1,p)$, $q_1\in(1,q)$ and $\frac{\alpha+1}{p_1}+\frac{\beta+1}{q_1}=1$.

Lemma 3.1 *Assume that any one of the following cases is satisfied*

i) $\gamma < p$, $\delta < q$ *and* $a(x), d(x), b(x)$ *have the same sign at every* $x \in \mathbb{R}^N$

ii) $\gamma < p$, $\delta < q$ *so that* $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} < 1$, *and* $\lambda b(x) > 0$

iii) $\gamma < p$, $\delta < q$ *so that* $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} > 1$, *and* $\lambda b(x) < 0$

iv) $\gamma > p$, $\delta > q$, *and* $\lambda b(x) < 0$,

v) $\gamma < p$, $\delta < q$ *so that* $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} = 1$

vi) $\gamma < p_1$, $\lambda a(x) > 0$, $\delta < q_1 and \lambda d(x) > 0$

vii) $\gamma < p_1$, $\lambda a(x) > 0$, $\delta > q_1$ *and* $\lambda d(x) < 0$

viii) $\gamma > p_1$, $\lambda a(x) < 0$, $\delta < q_1$ *and* $\lambda d(x) > 0$

ix) $\gamma > p_1$,$\lambda a(x) < 0$, $\delta > q_1$ *and* $\lambda d(x) < 0$.

Then the functional $A(u, v)$ *satisfies the (PS) condition.*

Proof. According to Lemma 2.3 it suffices to prove that the sequence (u_n, v_n) is bounded in Z. For case i) we have

$$\begin{aligned} & A(u_n, v_n) - (A'(u_n, v_n), (\frac{u_n}{p}, \frac{v_n}{q})) \\ = \;& \lambda\Big[-(\alpha+1)(\frac{1}{\gamma} - \frac{1}{p}) \int a(x)|u_n|^\gamma - (\beta+1)(\frac{1}{\delta} - \frac{1}{q}) \int d(x)|u_n|^\delta \\ & -(1 - \frac{\alpha+1}{p} - \frac{\beta+1}{q}) \int b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\Big]. \end{aligned}$$

Since $A(u_n, v_n)$ is bounded and $A'(u_n, v_n) \to 0$, as $n \to \infty$, we obtain that all the quantities $(D_i(u_n, v_n), (u_n, v_n))$ and $(B_i(u_n, v_n), (u_n, v_n))$, $i = 1, 2$, are bounded. So $(J_i(u_n, v_n), (u_n, v_n))$, $i = 1, 2$, are bounded too. Then the conclusion follows from the definition of the space Z. For the cases ii)-v) we have that

$$\begin{aligned} & A(u_n, v_n) - (A'(u_n, v_n), (\frac{u_n}{\gamma}, \frac{v_n}{\delta})) \\ = \;& (\alpha+1)(\frac{1}{p} - \frac{1}{\gamma}) \int |\nabla u_n|^p + (\beta+1)(\frac{1}{q} - \frac{1}{\delta}) \int |\nabla v_n|^q \\ & -\lambda\,(1 - \frac{\alpha+1}{\gamma} - \frac{\beta+1}{\delta}) \int b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}. \end{aligned} \tag{3.1}$$

So as in the case i) we have that the sequence (u_n, v_n) is bounded in Z. Finally, for the cases vi)-ix) we have that

$$\begin{aligned} & A(u_n, v_n) - (A'(u_n, v_n), (\frac{u_n}{p_1}, \frac{v_n}{q_1})) \\ = \;& (\alpha+1)(\frac{1}{p} - \frac{1}{p_1}) \int |\nabla u_n|^p + (\beta+1)(\frac{1}{q} - \frac{1}{q_1}) \int |\nabla v_n|^q \\ & -\lambda(\alpha+1)(\frac{1}{\gamma} - \frac{1}{p_1}) \int a(x)|u_n|^\gamma - \lambda(\beta+1)(\frac{1}{\delta} - \frac{1}{q_1}) \int d(x)|u_n|^\delta. \end{aligned}$$

Hence the sequence (u_n, v_n) is again bounded in the space Z and the lemma is proved. $\diamondsuit$

Remark 3.2 *We have to notice that the results of Lemma 3.1 are independent of λ in the case i), of the signs of $a(x)$ and $d(x)$ in the cases ii)-iv), of the sign of $b(x)$ in the cases vi)-ix) and of λ and the signs of $a(x)$, $d(x)$ and $b(x)$ in the case v).*

We prove the existence of a solution for the system (1.1) by a global minimization method or by the mountain pass theorem, depending on the hypothesis.

Theorem 3.3 *a) Let one of the hypotheses i), iii), v)-viii) be satisfied, and in addition $\lambda a(x)$ or $\lambda d(x)$ be in $\mathcal{H}_+$, or let the hypothesis ix) be satisfied and in addition $\lambda b(x)$ be in $\mathcal{H}_+$, or hypothesis ii) be satisfied, then (1.1) has at least one nonnegative (componentwise) solution.*

b) If the hypothesis iv) is satisfied, and in addition $\lambda a(x)$ or $\lambda d(x)$ are in $\mathcal{H}_+$, then (1.1) has at least one nonnegative (componentwise) solution.

Proof. a) Let hypothesis i) be satisfied. Applying Hölder, Young inequalities and the relation (2.1) to $A(u,v)$ we derive

$$\begin{aligned} A(u,v) \quad \geq \quad & \frac{\alpha+1}{p}\|u\|_{1,p}^p + \frac{\beta+1}{q}\|v\|_{1,q}^q - \lambda K_0^\gamma \frac{\alpha+1}{\gamma}\|a(x)\|_{\frac{p^*}{p^*-\gamma}}\|u\|_{1,p}^\gamma \\ & -\lambda K_0^\delta \frac{\beta+1}{\delta}\|d(x)\|_{\frac{q^*}{q^*-\delta}}\|v\|_{1,q}^\delta - \lambda K_0^{p_1}\frac{p_1}{\alpha+1}\|b(x)\|_\omega \|u\|_{1,p}^{p_1} \\ & -\lambda K_0^{q_1}\frac{q_1}{\beta+1}\|b(x)\|_\omega\|v\|_{1,q}^{q_1}. \end{aligned}$$

From this inequality we may observe that $A(u,v)$ is bounded from below. Suppose that $(u_1, v_1) \in Z$ and $\phi \in C_0^\infty(\Omega)$.Then

$$\begin{aligned} A(t^{1/p}u_1, t^{1/q}v_1) \quad = \quad & \lambda t\frac{\alpha+1}{p}\int |\nabla u_1|^p + \lambda t \frac{\beta+1}{q}\int |\nabla v_1|^q \\ & -\lambda t^{\frac{\gamma}{p}}\frac{\alpha+1}{\gamma}\int a(x)|u_1|^\gamma - \lambda t^{\frac{\delta}{q}}\frac{\beta+1}{\delta}\int d(x)|v_1|^\delta \\ & -\lambda t^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}\int b(x)|u_1|^{\alpha+1}|v_1|^{\beta+1}. \end{aligned} \tag{3.2}$$

Setting $(u_1, v_1) = (\phi, 0)$, if $\lambda a(x) > 0$ or $(u_1, v_1) = (0, \phi)$, if $\lambda d(x) > 0$, and getting t to be sufficiently small, we have that $A(u_0, v_0) < 0$, where $(u_0, v_0) = (t^{\frac{1}{p}}\phi, 0)$ or $(u_0, v_0) = (0, t^{\frac{1}{q}}\phi)$. This means that $M = \inf\{A(u,v) : (u,v) \in Z\} < 0$. Therefore, Ekeland's variational principle [7] and Lemma 3.1 imply the existence of a solution of (1.1), such that $A(u,v) < 0$. Similarly, we obtain the same result for all other hypotheses of the case a).

b) Let hypothesis vi) be satisfied. As in part a) we derive

$$\begin{aligned} A(u,v) \quad \geq \quad & \frac{\alpha+1}{p}\|u\|_{1,p}^p + \frac{\beta+1}{q}\|v\|_{1,q}^q - \lambda K_0^\gamma \frac{\alpha+1}{\gamma}\|a(x)\|_{\frac{p^*}{p^*-\gamma}}\|u\|_{1,p}^\gamma \\ & -\lambda K_0^\delta \frac{\beta+1}{\delta}\|d(x)\|_{\frac{q^*}{q^*-\delta}}\|v\|_{1,q}^\delta. \end{aligned}$$

This inequality implies that for a given $r > 0$ sufficiently small, there exists a constant $k = k(\alpha, \beta, \gamma, \delta, p, q, a(x), d(x), K_0)$, such that $A(u,v) > k > 0$, for all $(u,v) \in Z$ with $\|(u,v)\|_Z = r$.

Suppose that $(u_1, v_1) \in Z$ and $\phi \in C_0^\infty(\Omega)$. Then from relation (3.2) setting $(u_1, v_1) = (\phi, 0)$ or $(u_1, v_1) = (0, \phi)$, whether $\lambda a(x) > 0$ or $\lambda d(x) > 0$, respectively, and getting t to be sufficiently large we have that $A(u_0, v_0) < 0$, where $(u_0, v_0) = (t^{\frac{1}{p}}\phi, 0)$ or $(u_0, v_0) = (0, t^{\frac{1}{q}}\phi)$, respectively and $(u_0, v_0) \notin B_r(0)$. Since the functional A satisfies the (PS) condition, we deduce from the Mountain Pass Theorem [15, Theorem 44.D] the existence of a critical point $(u,v) \in Z$ for A such that $A(u,v) \geq k$. $\diamondsuit$

4 The Case $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1$

Throughout this section we assume that there exist real numbers p_1 and q_1 such that $p_1 > p$, $q_1 > q$ and $\frac{\alpha+1}{p_1} + \frac{\beta+1}{q_1} = 1$.

Lemma 4.1 *Assume that one of the following conditions is satisfied*

i) $p \leq \gamma$, $q \leq \delta$ *and* $a(x), d(x), b(x)$ *have the same sign at every* $x \in \mathbb{R}^N$

ii) $\gamma > p$, $\delta > q$ *so that* $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} < 1$ *and* $\lambda b(x) < 0$

iii) $\gamma > p$, $\delta > q$ *so that* $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} > 1$ *and* $\lambda b(x) > 0$

iv) $\gamma < p$, $\delta < q$ *and* $\lambda b(x) > 0$

v) $\gamma > p$, $\delta > q$ *so that* $\frac{\alpha+1}{\gamma} + \frac{\beta+1}{\delta} = 1$

vi) $\gamma < p_1$, $\lambda a(x) < 0$, $\delta < q_1$ *and* $\lambda d(x) < 0$

vii) $\gamma < p_1$, $\lambda a(x) < 0$, $\delta > q_1$ *and* $\lambda d(x) > 0$

viii) $\gamma > p_1$, $\lambda a(x) > 0$, $\delta < q_1$ *and* $\lambda d(x) < 0$

ix) $\gamma > p_1$, $\lambda a(x) > 0$, $\delta > q_1$ *and* $\lambda d(x) > 0$.

Then the functional $A(u,v)$ *satisfies the (PS) condition.*

The proof follows the same lines as in Lemma 3.1.

Remark 4.2 *We have to notice that the result is independent of* λ *in case i), of the signs of* $a(x)$ *and* $d(x)$ *in cases ii)-iv), of the sign of* $b(x)$ *in cases vi)-ix) and of* λ *and the signs of* $a(x)$, $d(x)$ *and* $b(x)$ *in case v).*

Now we prove the existence of a solution to (1.1), using the mountain pass theorem.

Theorem 4.3 *a) Let one of the hypothesis i), ii), v), ix) be satisfied, and in addition $\lambda a(x)$ or $\lambda d(x)$ be in $\mathcal{H}_+$, or let one of the hypotheses iii), vii)-ix) be satisfied, then the problem (1.1) has at least one nonnegative (componentwise) solution.*

b) If the hypothesis vi) is satisfied and in addition $\lambda b(x)$ is in $\mathcal{H}_+$, then the problem (1.1) has at least one nonnegative (componentwise) solution.

Proof. a) Let hypothesis i) be satisfied. Applying Hölder, Young inequalities and relation (2.1) to $A(u,v)$ we derive

$$\begin{aligned} A(u,v) \quad \geq \quad & \frac{\alpha+1}{p}\|u\|_{1,p}^p + \frac{\beta+1}{q}\|v\|_{1,q}^q - \lambda K_0^\gamma \frac{\alpha+1}{\gamma}\|a(x)\|_{\frac{p^*}{p^*-\gamma}}\|u\|_{1,p}^\gamma \\ & -\lambda K_0^\delta \frac{\beta+1}{\delta}\|d(x)\|_{\frac{q^*}{q^*-\delta}}\|v\|_{1,q}^\delta - \lambda K_0^{p_1}\frac{p_1}{\alpha+1}\|b(x)\|_\omega \|u\|_{1,p}^{p_1} \\ & -\lambda K_0^{q_1}\frac{q_1}{\beta+1}\|b(x)\|_\omega\|v\|_{1,q}^{q_1}. \end{aligned}$$

Then for a given $r > 0$ sufficiently small, there exists a constant k depending on α, β, γ, δ, p, q, $a(x)$, $b(x)$, $d(x)$, and K_0 such that $A(u,v) > k > 0$, for $\|(u,v)\|_Z = r$.

Suppose that $(u_1,v_1) \in Z$ and $\phi \in C_0^\infty(\Omega)$. Then from relation (3.2) setting $(u_1,v_1) = (\phi,0)$ or $(u_1,v_1) = (0,\phi)$, whether $\lambda a(x) > 0$ or $\lambda d(x) > 0$, respectively, and getting t to be sufficiently large we have that $A(u_0,v_0) < 0$, where $(u_0,v_0) = (t^{\frac{1}{p}}\phi, 0)$ or $(u_0,v_0) = (0, t^{\frac{1}{q}}\phi)$, respectively and $(u_0,v_0) \notin B_r(0)$. Since the functional A satisfies the (PS) condition, we deduce from the Mountain Pass Theorem the existence of a critical point $(u,v) \in Z$ for A such that $A(u,v) \geq k$.

b) As in part a) we derive the inequality

$$\begin{aligned} A(u,v) \quad \geq \quad & \frac{\alpha+1}{p}\|u\|_{1,p}^p + \frac{\beta+1}{q}\|v\|_{1,q}^q - \lambda K_0^{p_1}\frac{p_1}{\alpha+1}\|b(x)\|_\omega\|u\|_{1,p}^{p_1} \\ & -\lambda K_0^{q_1}\frac{q_1}{\beta+1}\|b(x)\|_\omega\|v\|_{1,q}^{q_1}. \end{aligned}$$

Then for a given $r > 0$ sufficiently small, there exists a constant k depending on α, β, p, q, $b(x)$, and K_0 such that $A(u,v) > k > 0$ for all $(u,v) \in Z$ with $\|(u,v)\|_Z = r$.

Suppose that $\phi \in C_0^\infty$ and $(u_1,v_1) = (\phi,\phi)$. Then from relation (3.2), where p, q are substituted by p_1, q_1, respectively, we get that $A(t^{\frac{1}{p_1}}\phi, t^{\frac{1}{q_1}}\phi) < 0$ and $(t^{\frac{1}{p_1}}\phi, t^{\frac{1}{q_1}}\phi) \notin B_r(0)$ for t sufficiently large. Since the functional A satisfies the (PS) condition, we deduce from the Mountain Pass Theorem the existence of a critical point $(u,v) \in Z$ for A such that $A(u,v) \geq k$. ♢

5 The Case $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$

Throughout this section we assume that $b(x)$ satisfies hypothesis $(\mathcal{B})$ and $\gamma < p$, $\delta < q$ or $\gamma > p$, $\delta > q$, simultaneously. We introduce the functionals $\tilde{A}$ and $\tilde{B}$

by

$$\tilde{A}(u,v) = \tfrac{\alpha+1}{p}\int_{\mathbb{R}^N}|\nabla u|^p + \tfrac{\beta+1}{q}\int_{\mathbb{R}^N}|\nabla v|^q,$$
$$\tilde{B}(u,v) = \int_{\mathbb{R}^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}.$$

These two functionals are well defined and continuously F-differentiable. Also $\tilde{A}$ is weakly lower semicontinuous. Furthermore, if $(u_n,v_n)\rightharpoonup(u_0,v_0)$ in Z then $\tilde{B}(u_n,v_n)\to\tilde{B}(u_0,v_0)$ and if $\tilde{B}'(u,v)=0$ then $\tilde{B}(u,v)=0$ (see [9, Lemma 5.1]). So all requirements of [4, Theorem 6.3.2] are satisfied.

Theorem 5.1 *The system (1.2) admits a principal eigenvalue given by*

$$\lambda_1 = \inf_{\tilde{B}(u,v)=1}\tilde{A}(u,v). \tag{5.1}$$

Moreover, there exists an eigenfunction which is nonnegative (componentwise) everywhere in $\mathbb{R}^N$.

Next we state some properties for the solutions of system (1.2), and its principal eigenvalue.

Remark 5.2 *If we assume, in addition, that $b\in L^\infty(\mathbb{R}^N)$. Then, as in [9, Theorem 5.2], we obtain that each component of the solutions of (1.2) is of class $C^{1,\alpha}(B_R)$, for any $R>0$, where $\alpha=\alpha(R)\in(0,1)$. Then Vasquez' Maximum Principle [13] implies the existence of a strictly positive (componentwise) eigenfunction corresponding to λ_1.*

Following the lines of [2, Theorem 2.7] we have the following result.

Theorem 5.3 *The eigenspace corresponding to the principal eigenvalue λ_1 is of dimension 1 and λ_1 is the only eigenvalue of the system (1.2), to which corresponds a positive (componentwise) eigenfunction.*

Remark 5.4 *Notice that the system is homogeneous in the sense that if (u,v) is substituted by $(t^{1/p}u, t^{1/q}v)$ then in both sides of the equations we get t, which then cancels. On the other hand, for this system it is proved in [2, Theorem 2.7] and [9, Theorem 6.2] that if (u,v) is a solution, then any other solution may be written as $(c_1u, {c_1}^{p/q}v)$. So there is a unique arbitrary constant c_1, which produces the one dimensional eigenspace. This eigenspace is not a linear set and it is even not homogeneous in the Cartesian product with components u and v, as it is in the case of the linear Laplacian equation and even in the case of the homogeneous p-Laplacian equation. We imagine this set as a parabola with power p/q in the space of two dimensions.*

Remark 5.5 *An open question, which in our opinion is far from being trivial, is under which hypothesis the positive principal eigenvalue λ_1 of Theorem 5.3 is isolated. This question applies also to the system in [9].*

By the next lemma we establish the (PS) condition for the functional A.

Lemma 5.6 *Assume that any one of the following conditions is satisfied: a) $\lambda<0$, b) $\lambda<\lambda_1$ and $a(x)d(x)\geq 0$ almost everywhere on $\mathbb{R}^N$. Then the functional $A(u,v)$ satisfies the (PS) condition.*

Proof. According to Lemma 2.3 it suffices to prove that the sequence (u_n, v_n) is bounded in the space Z.

Under assumption a), we have

$$\begin{aligned} & A(u_n, v_n) - (A'(u_n, v_n), (\frac{u_n}{\gamma}, \frac{v_n}{\delta})) \\ = \; & (\alpha+1)(\frac{1}{p} - \frac{1}{\gamma}) \int |\nabla u_n|^p + (\beta+1)(\frac{1}{q} - \frac{1}{\delta}) \int |\nabla v_n|^q \\ & -\lambda(1 - \frac{\alpha+1}{\gamma} - \frac{\beta+1}{\delta}) \int b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}. \end{aligned}$$

Then the boundedness of (u_n, v_n) in Z is easily obtained.

Under assumption b), the following relation implies that $(D_i(u_n, v_n), (u_n, v_n))$, $i = 1, 2$, are bounded.

$$\begin{aligned} & A(u_n, v_n) - (A'(u_n, v_n), (\frac{u_n}{p}, \frac{v_n}{q})) \\ = \; & \lambda\Big[-(\alpha+1)(\frac{1}{\gamma} - \frac{1}{p}) \int a(x)|u_n|^\gamma - (\beta+1)(\frac{1}{\delta} - \frac{1}{q}) \int d(x)|u_n|^\delta\Big]. \end{aligned}$$

We also notice that

$$\begin{aligned} A(u_n, v_n) \; \geq \; & \tilde{A}(u_n, v_n) - \lambda_1 \tilde{B}(u_n, v_n) - \lambda\frac{\alpha+1}{\gamma} \int a(x)|u_n|^\gamma \\ & -\lambda\frac{\beta+1}{\delta} \int d(x)|v_n|^\delta + (\lambda_1 - \lambda)\tilde{B}(u_n, v_n). \end{aligned}$$

Using the variational characterization of λ_1 and the above argument, we derive that $(B_i(u_n, v_n), (u_n, v_n))$, $i = 1, 2$, are also bounded. Then the conclusion follows. ◊

The next theorem shows that a solution to (1.1) can be obtained by global minimization if $\gamma < p$ and $\delta < q$, and by the mountain pass theorem if $\gamma > p$ and $\delta > q$.

Theorem 5.7 *If one of the hypotheses a) or b) of Lemma 5.6 is satisfied and $\lambda a(x)$ or $\lambda d(x)$ is in $\mathcal{H}_+$, then (1.1) has at least one nonnegative (componentwise) solution.*

Proof. Under assumption a), let $\lambda < 0$. Then from the definition of $A(u, v)$ we have

$$\begin{aligned} A(u, v) \; \geq \; & \frac{\alpha+1}{p}\|u\|_{1,p}^p + \frac{\beta+1}{q}\|v\|_{1,q}^q - \lambda K_0^\gamma \frac{\alpha+1}{\gamma}\|a(x)\|_{\frac{p^*}{p^*-\gamma}}\|u\|_{1,p}^\gamma \\ & -\lambda K_0^\delta \frac{\beta+1}{\delta}\|d(x)\|_{\frac{q^*}{q^*-\delta}}\|v\|_{1,q}^\delta. \end{aligned} \tag{5.2}$$

If $\gamma < p$ and $\delta < q$, then from (5.2) we obtain that $A(u, v)$ is bounded from below. As in the proof of part a) of Theorem 3.3, we have that $M = \inf\{A(u, v) :$

$(u,v) \in Z\} < 0$. Then, Ekeland's variational principle [7] and Lemma 5.6 imply the existence of a solution of (1.1) such that $A(u,v) < 0$. If $\gamma > p$ and $\delta > q$, then from (5.2), Lemma 5.6 and the Mountain Pass Theorem, as in the proof of b) in Theorem 3.3, we obtain the existence of a critical point $(u,v) \in Z$ for A, such that $A(u,v) \geq k$.

Under assumption b), suppose that $0 < \lambda < \lambda_1$ and $a(x)\,d(x) \geq 0$, almost everywhere on $\mathbb{R}^N$. Then from the definition of $A(u,v)$ we have

$$\begin{aligned} A(u,v) \quad \geq \quad & \left(1-\frac{\lambda}{\lambda_1}\right)\left(\frac{\alpha+1}{p}\|u\|_{1,p}^p + \frac{\beta+1}{q}\|v\|_{1,q}^q\right) \\ & -\lambda K_0^\gamma \frac{\alpha+1}{\gamma}\|a(x)\|_{\frac{p^*}{p^*-\gamma}}\|u\|_{1,p}^\gamma - \lambda K_0^\delta \frac{\beta+1}{\delta}\|d(x)\|_{\frac{q^*}{q^*-\delta}}\|v\|_{1,q}^\delta . \end{aligned} \tag{5.3}$$

If $\gamma < p$ and $\delta < q$, then from (5.3) we obtain that $A(u,v)$ is bounded from below. As in the previous case we obtain the existence of a solution of (1.1) such that $A(u,v) < 0$.
If $\gamma > p$ and $\delta > q$, then from (5.3), as in the previous case, we obtain the existence of a critical point $(u,v) \in Z$ for A, such that $A(u,v) \geq k > 0$. $\diamondsuit$

6 The Case $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$

Throughout this Section we assume that b belongs to $\mathcal{H}_\infty$. Following the arguments of Lemma 2.2 we prove that the operators B_i, $i = 1,2$, are well defined and compact.

Lemma 6.1 *Assume that one of the following conditions is satisfied*

i) $\gamma < p$, $\delta < q$, *and* $\lambda b(x) < 0$

ii) $\gamma > p$, $\delta > q$ *and* $\lambda b(x) > 0$

iii) $\lambda a(x) < 0$ *and* $\lambda d(x) < 0$.

Then the functional $A(u,v)$ *satisfies the (PS) condition.*

Proof. Since the operators B_i, $i = 1,2$, are compact, the results of Lemma 2.3 hold. So it suffices to prove that the sequence (u_n, v_n) is bounded in Z. Cases i), ii) are done as in Lemma 3.1, by considering the relation corresponding to (3.1) for the quantity

$$A(u_n,v_n) - (A'(u_n,v_n), (\frac{u_n}{\gamma}, \frac{v_n}{\delta})).$$

The same lines are followed for case iii), using the quantity

$$A(u_n,v_n) - (A'(u_n,v_n), (\frac{u_n}{p^*}, \frac{v_n}{q^*})).$$

Remark 6.2 *We have to notice that the above results are independent of the signs of $a(x)$ and $d(x)$ in cases i), ii) and of the sign of $b(x)$ and the values of γ and δ in case iii).*

The next theorem states that a solution for the system (1.1) may be obtained by global minimization under the hypothesis i), and by the Mountain Pass Theorem under the hypotheses ii) and iii).

Theorem 6.3 *Let one of the following two conditions be satisfied.*

a) Hypotheses i), ii) of Lemma 6.1 are satisfied, and $\lambda a(x)$ or $\lambda d(x)$ belong to $\mathcal{H}_+$

b) Hypothesis iii) is satisfied, and $\lambda b(x)$ belongs to $\mathcal{H}_+$, and $\gamma < p^*$, $\delta < q^*$.

Then (1.1) has at least one nonnegative (componentwise) solution.

Proof. a) S ince i) is satisfied, from the definition of $A(u,v)$ we have that

$$\begin{aligned} A(u,v) \quad \geq \quad & \frac{\alpha+1}{p}\|u\|_{1,p}^{p} + \frac{\beta+1}{q}\|v\|_{1,q}^{q} - \lambda K_0^{\gamma}\frac{\alpha+1}{\gamma}\|a(x)\|_{\frac{p^*}{p^*-\gamma}}\|u\|_{1,p}^{\gamma} \\ & -\lambda K_0^{\delta}\frac{\beta+1}{\delta}\|d(x)\|_{\frac{q^*}{q^*-\delta}}\|v\|_{1,q}^{\delta}, \end{aligned}$$

which implies that $A(u,v)$ is bounded from below. Using the same argument as in the proof of Theorem 3.3 for the case a), we obtain the existence of a solution of (1.1) such that $A(u,v) < 0$.

If hypothesis ii) is satisfied we have

$$\begin{aligned} A(u,v) \quad \geq \quad & \frac{\alpha+1}{p}\|u\|_{1,p}^{p} + \frac{\beta+1}{q}\|v\|_{1,q}^{q} - \lambda K_0^{\gamma}\frac{\alpha+1}{\gamma}\|a(x)\|_{\frac{p^*}{p^*-\gamma}}\|u\|_{1,p}^{\gamma} \\ & -\lambda K_0^{\delta}\frac{\beta+1}{\delta}\|d(x)\|_{\frac{q^*}{q^*-\delta}}\|v\|_{1,q}^{\delta} - \lambda K_0^{p^*}\frac{p^*}{\alpha+1}\|b(x)\|_{\infty}\|u\|_{1,p}^{p^*} \\ & -\lambda K_0^{q^*}\frac{q^*}{\beta+1}\|b(x)\|_{\infty}\|v\|_{1,q}^{q^*}. \end{aligned}$$

Following the lines of the proof of Theorem 3.3 b), we obtain the existence of a solution of (1.1) such that $A(u,v) \geq k > 0$.

b) Since hypothesis iii) is satisfied,

$$\begin{aligned} A(u,v) \quad \geq \quad & \frac{\alpha+1}{p}\|u\|_{1,p}^{p} + \frac{\beta+1}{q}\|v\|_{1,q}^{q} - \lambda K_0^{p^*}\frac{p^*}{\alpha+1}\|b(x)\|_{\infty}\|u\|_{1,p}^{p^*} \\ & -\lambda K_0^{q^*}\frac{q^*}{\beta+1}\|b(x)\|_{\infty}\|v\|_{1,q}^{q^*}. \end{aligned}$$

Replacing $(u,v) = (t^{\frac{1}{p^*}}u_1, t^{\frac{1}{q^*}}v_1)$ in (3.2), we obtain the existence of a solution to (1.1) such that $A(u,v) \geq k > 0$. $\diamondsuit$

References

[1] D. R. Adams, H. J. Nussenzveig Lopez *Nonlinear weakly elliptic* 2×2 *Systems of Variational Inequalities with Unilateral Obstacle Constraints*, Electron. J. Diff. Eqns. 1997(1997) No. 18, 1-20.

[2] W. Allegretto and Y. X. Huang *A Picone's Identity for the p-Laplacian and Applications*, Nonlinear Analysis TMA, Vol 32 (7), (1998), 819-830.

[3] C. Atkinson and C. R. Champion, *On some Boundary Value Problems for the Equation* $\nabla \cdot (F(|\nabla w|)\nabla w) = 0$, Proc. Roy. Soc. London A, 448 (1995), 269-279.

[4] M. S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977.

[5] J. Chabrowski, *On Multiple Solutions for Nonhomogeneous System of Elliptic Equations*, Revista Matemática Univ Complutense Madrid, Vol 9 (1), (1996), 207-234.

[6] P. Drábek and Y. X. Huang *Bifurcation problems for the p-Laplacian in* $\mathbb{R}^N$, Trans. Amer. Math. Soc. 349 (1), (1997), 171-188.

[7] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. 47 (1974), 324-357.

[8] J. Fleckinger, R. Manasevich and F. de Thelin, *Global Bifurcation from the First Eigenvalue for a System of p-Laplacians*, Math. Nachr. 182 (1996), 217-242.

[9] J. Fleckinger, R. Manasevich, N M Stavrakakis and F. de Thelin, *Principal Eigenvalues for some Quasilinear Elliptic Equations on* $\mathbb{R}^N$, Adv. Diff. Eqns., Vol 2 (6), (1997), 981-1003.

[10] Y. X. Huang, *Positive Solutions of Certain Elliptic Equations Involving Critical Sobolev Exponents*, Nonlinear Analysis TMA, 33, (1998), 617-636.

[11] N. M. Stavrakakis and N. B. Zographopoulos, *Bifurcation Results for some Quasilinear Elliptic Systems on* $\mathbb{R}^N$, Submitted

[12] P. Tolksdorf, *Regularity for a More General Class of Quasilinear Elliptic Equations*, J. Diff. Eqns., 51 (1984), 126-150.

[13] J. L. Vázquez, *A Strong Maximum Principle for some Quasilinear Elliptic Equations*, Appl. Math. Optim., 12 (1984), 191-202.

[14] J. Vélin, F. de Thélin, *Existence and Nonexistence of Nontrivial Solutions for Some Nonlinear Elliptic Systems*, Revista Matemática Univ. Complutense Madrid, 6(1) (1993), 153-194.

[15] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol III: Variational Methods and Optimization*, Springer-Verlag, Berlin, 1990.

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