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Bifurcation from the first eigenvalue of some nonlinear elliptic operators in Banach spaces

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1. Introduction

Let us consider an abstract operator equation

$$Au = \lambda Bu, \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter and A, B are operators acting from a certain Banach space into its dual. In the papers by Idogawa and Ôtani [9] and Chan et al. [4], the existence of the first variational and simple eigenvalue λ_1 of Eq. (1.1) is proved if A and B are single-valued subdifferentials of certain positive, convex functionals f^1 and f^2, respectively. Under some additional assumptions, they proved that the problem has a positive solution if and only if $\lambda = \lambda_1$. They provide examples of quasilinear elliptic boundary value problems on the bounded domain $\Omega \subset \mathbb{R}^N$, with smooth boundary $\partial\Omega$. In our paper we show that a slight modification of the assumptions in [4] allows us to extend their results also for problems in unbounded domains (including the case of $\Omega = \mathbb{R}^N$). Putting some additional assumptions on A and B, which seems to be natural for a wide class of quasilinear equations, we prove that there is a neighbourhood of λ_1, which does not contain any other eigenvalue than λ_1. Finally, under the assumption

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that A and B are the Fréchet derivatives of the functionals f^1 and f^2, respectively, we prove a global bifurcation result for the operator equation

$$Au = \lambda Bu + F(\lambda, u), \tag{1.2}$$

where the nonlinear operator F represents "higher-order" terms.

A special fact that the operators A and B need not be odd should be emphasized. In this spirit our result generalises in an essential way related results for the p-Laplacian (see [7] and the references therein).

As an example we can provide weak solvability of the nonlinear eigenvalue problem in $\mathbb{R}^N$

$$-\operatorname{div} a(x, u, \nabla u) + c(x, u, \nabla u) = \lambda b(x, u) + g(\lambda, x, u),$$

where the assumptions on a, b, c and g are specified in Section 4.

Our paper is organized as follows. In Section 2, we reformulate the assumptions of Chan et al. [4] with the modification which allows us to deal with the unbounded domain Ω. We also prove that λ_1 is isolated in the above-mentioned sense, if some extra conditions on the operators A and B are required. Section 3 deals with an abstract bifurcation result based on the change of the value of the degree when λ crosses λ_1. In Section 4, we give a typical application.

Notation. We denote by $B_R(0)$ the open ball of $\mathbb{R}^N$ with center 0 and radius R. $\langle .,. \rangle_V$ denotes the *duality pairing* between the spaces V^*, V. The symbols L^p, $\|\,.\,\|_p$, $1 \le p \le \infty$ and $\mathscr{D}^{1,p}$, are used in the place of $L^p(\mathbb{R}^N)$, $\mathscr{D}^{1,p}(\mathbb{R}^N)$ and $\|\,.\,\|_{L^p(\mathbb{R}^N)}$, respectively. We denote by $\to$ and $\rightharpoonup$ the strong and the weak convergence, respectively.

2. The first eigenvalue of abstract elliptic operators

Let $\Omega \subset \mathbb{R}^N$ be a domain (bounded, unbounded or possibly equal to $\mathbb{R}^N$). Let V be a real reflexive Banach space with norm $\|\,.\,\|_V$, with the dual space V^* and the duality pairing $\langle .,. \rangle_V$. Denote by $\Phi(V)$ the family of all lower semicontinuous convex functionals f from V into $(-\infty, \infty]$, such that $D(f) := \{u \in V; f(u) < \infty\} \neq \emptyset$. The subdifferential ∂f of f at u is defined by

$$\partial f(u) := \{h \in V^*; f(v) - f(u) \ge \langle h, v - u \rangle_V, \text{ for any } v \in D(f)\},$$

with the domain $D(\partial f) := \{u \in V; \partial f(u) \neq \emptyset\}$. Assume that $\partial f : V \to V^*$ is a single-valued operator. Note that if $f \in \Phi(V)$ is Fréchet differentiable, then $\partial f(u)$ is the Fréchet derivative of f at u. Let $\mathscr{D} := C_0^\infty(\Omega)$, $\mathscr{D}^+ := \{u \in \mathscr{D};\ u(x) \ge 0, \text{ for all } x \in \Omega\}$. Let V_i, $i = 1, 2$ be real reflexive Banach spaces of functions defined in Ω and denote

$$V_i^+ := \{u \in V_i;\ u(x) \ge 0, \text{ a.e.} \in \ \Omega\}, \quad i = 1, 2.$$

Assume that

$$\begin{array}{lll} (V_i) & \overline{\mathscr{D}^{V_1}} = V_1 & \hookrightarrow V_2 \subset L^1_{\mathrm{loc}}(\Omega), \\ (V_i^+) & \overline{\mathscr{D}^{+V_1}} = V_1^+ & \hookrightarrow V_2^+, \end{array}$$

where the symbol $\hookrightarrow$ is used to denote continuous embeddings. Let Ω_n, $n \in \mathbb{N}$ be a sequence of bounded subdomains of Ω satisfying the property

(Ω_n) $\overline{\Omega_n} \subset \Omega_{n+1} \subset \overline{\Omega_{n+1}} \subset \cdots \subset \Omega$, and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. Moreover, for any compact set $K \subset \Omega$, there exists $n \in \mathbb{N}$, such that $K \subset \Omega_n$.

For any $n \in \mathbb{N}$, introduce the operator $P_n : L^1_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\Omega)$, defined by

$$P_n z(x) = \begin{cases} z(x) & \text{if } x \in \Omega_n, \\ 0 & \text{if } x \in \Omega \setminus \Omega_n. \end{cases}$$

We impose the following hypothesis on $f^i \in \Phi(V_i)$, $i = 1, 2$:

(A1) (i) $A = \partial f^1$, $B = \partial f^2$, $B_n = \partial f^2(P_n(.))$;

(ii) $D(f^i) = V_i$, $i = 1, 2$;

(iii) P_n maps V_1 into V_2, for all $n \in \mathbb{N}$.

(A2) (i) $R(|u|) \le R(u) := f^1(u)/f^2(u)$, for all $u \in V_1$;

$(\text{i})_n$ $R_n(|u|) \le R_n(u) := f^1(u)/f^2(P_n u)$, for all $u \in V_1$;

(ii) $f^i(u) \ge 0$, for all $u \in V_i$, $i = 1, 2$, and $f^2(u) = 0$, if and only if $u = 0$;

(iii) there exists $w \in V_1$, with $w \ne 0$, such that $\lambda_1 = R(w) = \inf\{R(u);\ u \in V_1, u \ne 0\} < +\infty$;

$(\text{iii})_n$ there exists $w_n \in V_1$, with $w_n \ne 0$, such that $\lambda_1^n = R_n(w_n) = \inf\{R_n(u);\ u \in V_1, u \ne 0\} < +\infty$.

(A3) There exists $\alpha > 1$ such that $f^i(tu) = t^\alpha f^i(u)$, for all $u \in V_i$, $i = 1, 2$, and for all $t > 0$.

(A4) (i) $f^1(u \vee v) + f^1(u \wedge v) \le f^1(u) + f^1(v)$, for all $u, v \in V_1^+$;

(ii) $f^2(u \vee v) + f^2(u \wedge v) \ge f^2(u) + f^2(v)$, for all $u, v \in V_2^+$; where $(u \vee v)(x) = \max\{u(x), v(x)\}$, $(u \wedge v)(x) = \min\{u(x), v(x)\}$.

(A5) f^1 is strictly convex.

(A6) If $0 \le z \le u$, with $z, u \in D(B)$, then $B(z) \le B(u)$ holds in the sense of distributions.

(A7) (i) Every nonnegative nontrivial solution u of the problem

$$Au = \lambda B u, \tag{AE$_\lambda$}$$

belongs to $C(\Omega)$ and satisfies $u(x) > 0$, for all $x \in \Omega$;

$(\text{i})_n$ Every nonnegative nontrivial solution u of the approximation problems

$$Au = \lambda^n B_n u, \tag{AE$_{\lambda^n}$}$$

belongs to $L^\infty_{\text{loc}}(\Omega)$, for any $n \in \mathbb{N}$.

Under hypotheses (A1)–(A7) we have the following general result from Chan et al. [4].

Theorem 2.1. *The number λ_1 is the first eigenvalue of* $(\text{AE})_\lambda$, *it is simple and* $(\text{AE})_\lambda$ *has a positive solution if and only if $\lambda = \lambda_1$. The number λ_1^n is the first eigenvalue of problem* $(\text{AE})_{\lambda^n}$, *for any $n \in \mathbb{N}$ and* $(\text{AE})_{\lambda_1^n}$ *has nontrivial nonnegetive solutions with $\lambda_1^n \searrow \lambda_1$, as $n \nearrow \infty$.*

This assertion is proved for a bounded domain Ω in paper [4]. However, inspecting the proof in [4] we also get the same result for unbounded domain Ω satisfying hypothesis (Ω_n).

We shall denote by $u_1 = u_1(x) > 0$ the eigenfunction associated with λ_1 and we normalize it as $\|u_1\|_{V_1} = 1$. For $w \in V_1$, we set $\Omega_w^+ := \{x \in \Omega;\ w(x) > 0\}$, $\Omega_w^- := \{x \in \Omega;\ w(x) < 0\}$ and $w^+(x) := (w \vee 0)(x)$, $w^-(x) := (w \wedge 0)(x)$. We further impose the following assumptions.

(V1) For any $w \in V_i$, we have $w^+ \in V_i^+$ and $-w^- \in V_i^+$ for $i = 1, 2$. Moreover, for any fixed number $\kappa > 0$ and for any sequence $\{u_n\} \subset V_1$, such that $u_n \rightharpoonup \mp u_1$ in V_1, we have $\operatorname{meas}(\Omega_{u_n}^{\pm} \cap B_k(0)) \to 0$, where $B_k(0) := \{x \in \mathbb{R}^N; |x| < k\}$.

(A8) There exist $c_0 > 0$ and $c_1 > 0$ such that

(i) $\langle Au, u^{\pm}\rangle_{V_1} \geq c_0 \langle Au^{\pm}, u^{\pm}\rangle_{V_1}$, for all $u \in V_1$;

(ii) $\langle Au, u\rangle_{V_1} \geq c_1 \|u\|_{V_1}^{\alpha}$, for all $u \in V_1$.

(B1) f^2 is weakly sequentially continuous as a functional from V_1 into $\mathbb{R}$, i.e., if $u_n \rightharpoonup u$ in V_1, then $f^2(u_n) \to f^2(u)$.

(B2) For any $v \in V_2$, $v^+ \neq 0$, there exists $c_2 = c_2(v^+) > 0$, such that

$$\langle Bv, v^+\rangle_{V_2} \leq c_2(v^+)\|v^+\|_{V_2}^{\alpha}.$$

Moreover, the following implication holds: for any $\varepsilon > 0$ there exists $\delta > 0$ and $k > 0$, such that meas $(\Omega_v^+ \cap B_k(0)) < \delta$, implies that $c_2(v^+) < \varepsilon$. Similarly, for $v \in V_2$, with $v^- \neq 0$.

Proposition 2.2. *The number λ_1 is an isolated eigenvalue of problem* $(\mathrm{AE})_\lambda$ *in the following sense*: *there exists $\eta > 0$, such that the interval $(-\infty, \lambda_1 + \eta)$ does not contain any other eigenvalue than λ_1.*

Proof. Assume the contrary, i.e., there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_1$ and $u_n \in V_1$ with $\|u_n\|_{V_1} = 1$, satisfying

$$Au_n = \lambda_n Bu_n. \tag{2.1}$$

Then $\lambda_n > \lambda_1$, due to Theorem 2.1 and (by passing to a subsequence, if necessary) we may assume that $u_n \rightharpoonup u_0$ in V_1. Note that the functional $f^1(u) := 1/p\langle \partial f^1(u), u\rangle_{V_1}$ is weakly lower semicontinuous (it is lower semicontinuous and convex). Hence

$$\langle Au_0, u_0\rangle_{V_1} \leq \liminf_{n\to\infty} \langle Au_n, u_n\rangle_{V_1}. \tag{2.2}$$

It follows from (B1) that

$$\lambda_n \langle Bu_n, u_n\rangle_{V_1} \to \lambda_1 \langle Bu_0, u_0\rangle_{V_1}, \tag{2.3}$$

and from (A8)(ii) we get that

$$\langle Au_n, u_n\rangle_{V_1} \geq c_1 \|u_n\|_{V_1}^{\alpha} = c_1 > 0. \tag{2.4}$$

Now, it follows from (2.1)–(2.4) that $u_0 \neq 0$ and

$$Au_0 = \lambda_1 Bu_0,$$

i.e., u_0 is the eigenfunction associated with λ_1. We may assume without loss of generality, that $u_0 = u_1$, i.e., $u_n \rightharpoonup u_1$ in V_1. Then it follows from (V1) that, for any fixed $\kappa > 0$, we have that

$$\text{meas}\,(\Omega^-_{u_n} \cap B_\kappa(0)) \to 0, \tag{2.5}$$

as $n \to \infty$. On the other hand, from (2.1), (A8), (B2) and $V_1 \hookrightarrow V_2$ we have that

$$\begin{aligned} c_0 c_1 \|u_n^-\|^\alpha_{V_1} &\le \langle Au_n, u_n^-\rangle_{V_1} = \lambda_n \langle Bu_n, u_n^-\rangle_{V_1} \\ &\le \tilde{c}c_2(u_n^-)\|u_n^-\|^\alpha_{V_2} \le \hat{c}c_2(u_n^-)\,\|u_n^-\|^\alpha_{V_1}. \end{aligned} \tag{2.6}$$

Since $\Omega^-_{u_n} \neq \emptyset$ by Theorem 2.1, relation (2.6) implies that $c_2(u_n^-) \ge \text{const} > 0$, for any $n \in \mathbb{N}$. But this contradicts relation (2.5) and condition (B2). □

3. Bifurcation from the first eigenvalue for abstract elliptic operators

We shall consider an abstract bifurcation problem of the form

$$Au = \lambda Bu + F(\lambda, u), \tag{3.1}$$

where A, B are the operators studied above and $F(\lambda, .)$ represents "higher-order" terms with respect to A and B. Our main tool will be the degree theory for generalized monotone mappings satisfying condition (S_+) (see, e.g., [3,11,14]) and the global bifurcation result of Rabinowitz [10]. In order to apply the degree theory, we have to strengthen the assumptions on A and B in the following sense:

(A, B) f^1 and f^2 are Fréchet differentiable in V_1 and V_2, respectively; $\partial f^1 : V_1 \to V_1^*$ is bounded and demicontinuous, $\partial f^2 : V_2 \to V_2^*$ is compact.

(A9) f^1 is uniformly convex in the sense that the following implication holds: for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $u, v \in V_1$ with $f^1(u) \le 1$, $f^1(v) \le 1$, and $\|u - v\|_{V_1} \ge \varepsilon$, it implies that

$$f^1(\tfrac{1}{2}(u+v)) \le \tfrac{1}{2}(f^1(u) + f^1(v)) - \delta.$$

(F) For any fixed $\lambda \in \mathbb{R}$, the operator $F(\lambda, .) : V_1 \to V_1^*$ is compact and

$$\lim_{\|u\|_{V_1} \to \infty} \frac{F(\lambda, u)}{\|u\|^{\alpha-1}_{V_1}} = 0,$$

holds uniformly for λ in bounded intervals of $\mathbb{R}$.

Remark 3.1. Remind that the Fréchet differentiability of f^i, $i = 1, 2$, implies that $A = \partial f^1$ and $B = \partial f^2$ are the corresponding Fréchet derivatives. In particular, the compactness of ∂f^2 implies (B1) (see [13, Corollary 41.9]).

The basic assertion is the following:

Lemma 3.2. *The operator* $V_1 \to V_1^*$, defined by

$$u \mapsto Au - \lambda Bu - F(\lambda, u), \tag{3.2}$$

satisfies condition (S_+).

Proof. It is well known (see, e.g., [14]) that, every compact perturbation of an operator satisfying condition (S_+) satisfies also condition (S_+). So due to assumptions (A,B) and (F), it is sufficient to prove that A satisfies condition (S_+). Let as assume that $u_n \rightharpoonup u$ in V_1, and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_{V_1} \le 0. \tag{3.3}$$

Since $f^1(u_n) - f^1(u) \le \langle Au_n, u_n - u\rangle_{V_1}$ we get from inequality (3.3) that

$$\limsup_{n\to\infty} f^1(u_n) \le f^1(u). \tag{3.4}$$

On the other hand, the weak lower semicontinuity of f^1 yields

$$\liminf_{n\to\infty} f^1(u_n) \ge f^1(u). \tag{3.5}$$

Hence we get from relations (3.4) and (3.5) that

$$\lim_{n\to\infty} f^1(u_n) = f^1(u). \tag{3.6}$$

Let us denote by $\mu_n := \max\{f^1(u_n), f^1(u)\}$. Then due to relation (3.4) we get that $\mu_n \to f^1(u)$, as $n \to \infty$. Set

$$v_n := \frac{u_n}{(\mu_n)^{1/\alpha}}, \quad v := \frac{u}{(f^1(u))^{1/\alpha}},$$

i.e., $v_n \rightharpoonup v$ in V_1, and

$$f^1(v) \le \liminf_{n\to\infty} f^1(\tfrac{1}{2}(v + v_n)). \tag{3.7}$$

The homogeneity of f^1 implies that $f^1(v) = 1$, $f^1(v_n) \le 1$, and the convexity of f^1 yields that

$$f^1(\tfrac{1}{2}(v + v_n)) \le \tfrac{1}{2}(f^1(v) + f^1(v_n)). \tag{3.8}$$

Then relations (3.7) and (3.8) imply that

$$f^1(\tfrac{1}{2}(v + v_n)) \to 1,$$

which together with condition (A9) yields $v_n \to v$ in V_1. But from here we directly get that $u_n \to u$ in V_1. $\square$

It follows from the previous Lemma 3.2 that the degree of the mapping (3.2) with respect to a bounded set $D \subset V_1$ and $0 \in V_1^*$, i.e.,

$$\mathrm{Deg}\,[A - \lambda B - F(\lambda, .);\ D,\ 0]$$

is well defined if $Au - \lambda Bu - F(\lambda, u) \ne 0$, for any $u \in \partial D$ (see e.g., [3,11]). The following assertion allows us to apply the global bifurcation results of Rabinowitz's type.

Proposition 3.3. *Let λ_1 be as in Proposition 2.2. Then there exists $\eta > 0$ and $\rho > 0$ such that, for the ball $B_\rho(0) := \{u \in V_1;\ \|u\|_{V_1} < \rho\}$ we have*

$$\mathrm{Deg}\,[A - \lambda B - F(\lambda, .);\ B_\rho(0),\ 0] = 1 \quad \textit{if } \lambda < \lambda_1, \tag{3.9}$$

and

$$\mathrm{Deg}\,[A-\lambda B-F(\lambda,.);\ B_\rho(0),\ 0]=-1 \quad \textit{if}\ \lambda\in(\lambda_1,\lambda_1+\eta). \tag{3.10}$$

Proof. Due to the assumption (F) we have

$$\mathrm{Deg}\,[A-\lambda B-F(\lambda,.);\ B_\rho(0),\ 0]=\mathrm{Deg}\,[A-\lambda B;\ B_\rho(0),\ 0],$$

when $\rho>0$ is small enough and λ belongs to a bounded interval. So it suffices to prove that

$$\mathrm{Deg}\,[A-\lambda B;\ B_\rho(0),\ 0]=1 \quad \text{if}\ \lambda<\lambda_1 \tag{3.11}$$

and

$$\mathrm{Deg}\,[A-\lambda B;\ B_\rho(0),\ 0]=-1 \quad \text{if}\ \lambda\in(\lambda_1,\lambda_1+\eta). \tag{3.12}$$

To prove (3.11) and (3.12) we adapt the method developed in [5–7]. Consider the functional $\Phi_\lambda : V_1\to\mathbb{R}$ defined by

$$\Phi_\lambda(u):=f^1(u)-\lambda f^2(u)=\frac{1}{p}\langle Au,u\rangle_{V_1}-\frac{\lambda}{p}\langle Bu,u\rangle_{V_1}.$$

The variational characterization of λ_1 (see hypothesis (A2) (iii)) implies that, for $\lambda<\lambda_1$, we have

$$\langle\Phi'_\lambda(u),u\rangle_{V_1}>0 \quad \text{for all}\ u\in V_1,\ u\neq 0,$$

from where we get (see e.g., [11])

$$\mathrm{Deg}\,[\Phi'_\lambda;\ B_\rho(0),\ 0]=1 \quad \text{for any}\ \rho>0.$$

Hence assertion (3.11) is proved. Let us consider now a real nonnegative C^1–function $\psi:\mathbb{R}\to\mathbb{R}$, defined by

$$\psi(t)=\begin{cases}0 & \text{for}\quad t\le K,\\ \text{strictly convex} & \text{for}\quad t\in(K,3K),\\ \dfrac{2\eta}{\lambda_1}(t-2K) & \text{for}\quad t\ge 3K,\end{cases}$$

for $K>0$ large enough, to be defined later (see (3.19)). Fix $\lambda\in(\lambda_1,\lambda_1+\eta)$ and set

$$\Psi_\lambda(u):=\Phi_\lambda(u)+\psi(f^1(u)).$$

Then $\langle\Psi'_\lambda(u),v\rangle_{V_1}=0$, for any $v\in V_1$, if and only if

$$\langle Au,v\rangle_{V_1}-\frac{\lambda}{1+\psi'(f^1(u))}\langle Bu,v\rangle_{V_1}=0 \quad \text{for any}\ v\in V_1. \tag{3.13}$$

Assume that $\Psi'_\lambda(u)=0$ in V_1^*. Due to the definition of ψ we have

$$\frac{\lambda}{1+\psi'(f^1(u))}<\lambda_1+\eta. \tag{3.14}$$

Then assumption $\Psi'_\lambda(u)=0$ in V_1^* and Proposition 2.2 imply that either $u=0$ or it follows from relations (3.13) and (3.14) that

$$\frac{\lambda}{1+\psi'(f^1(u))}=\lambda_1 \tag{3.15}$$

and u is an eigenfunction associated to λ_1. Due to the fact that $0 < \psi'(f^1(u)) < \eta/\lambda_1$, we get $f^1(u) \in (K, 3K)$. Since f^1 is homogeneous and λ_1 is simple, there exists $t_1 > 0$ and $t_2 < 0$ such that either $u = t_1 u_1$ or $u = t_2 u_1$. (Note that in the case of an even function f^1, then we have $t_1 = -t_2$.) Hence the only possible critical points of Ψ_λ are 0, $t_1 u_1$, $t_2 u_1$. On the other hand, Ψ_λ is a weakly lower semicontinuous functional – this follows from the convexity and continuity of $f^1(.) + \psi(f^1(.))$ and weak continuity of f^2. Let us prove that Ψ_λ is coercive, i.e.,

$$\lim_{\|u\|_{V_1} \to \infty} \Psi_\lambda(u) = \infty$$

and bounded from below. Indeed, using the variational characterization of λ_1 (see (A2)(iii)), we have

$$\begin{aligned}
\Psi_\lambda(u) &= f^1(u) - \lambda f^2(u) + \psi(f^1(u)) \\
&= f^1(u) - \lambda_1 f^2(u) + (\lambda_1 - \lambda) f^2(u) + \psi(f^1(u)) \\
&\geq \frac{\lambda_1 - \lambda}{\lambda_1} f^1(u) + \frac{2\eta}{\lambda_1}(f^1(u) - 2K) \to \infty \quad \text{as } \|u\|_{V_1} \to \infty,
\end{aligned}$$

due to (A8)(ii). Hence Ψ_λ achieves its global minimum on V_1. Clearly, this minimum must be negative (since $\lambda_1 < \lambda$). But $f^1(t_1 u_1) = \lambda_1 f^2(t_1 u_1)$, $f^1(t_2 u_1) = \lambda_1 f^2(t_2 u_1)$ and $f^1(t_1 u_1) = f^1(t_2 u_1)$ (due to (3.15) and the strong monotonicity of ψ). Hence also $f^2(t_1 u_1) = f^2(t_2 u_1)$, so $t_1 u_1$, $t_2 u_1$ are the points where the global minimum of Ψ_λ is achieved. Note that both $t_1 u_1$ and $t_2 u_1$ are isolated critical points of Ψ_λ. Therefore (see e.g., [11]), for $\kappa > 0$ small enough we have

$$\operatorname{Deg}[\Psi'_\lambda;\ B_\kappa(t_1 u_1),\ 0] = \operatorname{Deg}[\Psi'_\lambda;\ B_\kappa(t_2 u_1),\ 0] = 1, \tag{3.16}$$

where $B_\kappa(t_i u_1) := \{u \in V_1\colon \|u - t_i u_1\|_{V_1} < \kappa\}$, $i = 1, 2$. On the other hand, we have

$$\begin{aligned}
\langle \Psi'_\lambda(u), u\rangle_{V_1} &= p[f^1(u) - \lambda f^2(u) + \psi'(f^1(u)) f^1(u)] \\
&= p\left[f^1(u) - \lambda_1 f^2(u) + \psi'(f^1(u))\left(f^1(u) - \frac{\lambda - \lambda_1}{\psi'(f^1(u))} f^2(u)\right)\right] \\
&\geq \frac{2\eta p}{\lambda_1}\left(f^1(u) - \frac{\lambda_1}{2} f^2(u)\right) \to \infty \quad \text{as } \|u\|_{V_1} \to \infty,
\end{aligned}$$

due to (A2)(iii) and (A8)(ii). Hence, taking $R > 0$ large enough we have

$$\operatorname{Deg}[\Psi'_\lambda;\ B_R(0),\ 0] = 1. \tag{3.17}$$

Now the additivity property of the degree and relations (3.16), (3.17) yield that, by taking $\rho > 0$ small enough, we get

$$\operatorname{Deg}[\Psi'_\lambda;\ B_\rho(0),\ 0] = -1. \tag{3.18}$$

Due to the definition of ψ for $\rho > 0$ small enough, so that $\rho < K$, we have

$$\Psi'_\lambda(u) = \Phi'_\lambda(u), \tag{3.19}$$

for any $u \in B_\rho(0)$. Then (3.12) follows from (3.18) and (3.19). $\square$

Let us define the space $E := \mathbb{R} \times V_1$ equipped with the norm

$$\| (\lambda, u) \|_E = (|\lambda|^2 + \|u\|_{V_1})^{1/2} \quad \text{for } (\lambda, u) \in E. \tag{3.20}$$

Let C be a connected set in E with respect to the topology induced by norm (3.20) and $C \subset \{(\lambda, u) \in E; (\lambda, u) \text{ solves } (3.1)\}$. Then C is called a *continuum of nontrivial solutions* of (3.1). We say that $\lambda_1 \in \mathbb{R}$ is a *global bifurcation point* of (3.1) in the sense of Rabinowitz, if there is a continuum C of nontrivial solutions of (3.1) such that $(\lambda_1, 0) \in \overline{C}$ (closure of C in E) and C is either unbounded in E or there is an eigenvalue λ_0 of $Au = \lambda Bu$ such that $\lambda_0 > \lambda_1$ and $(\lambda_0, 0) \in \overline{C}$.

Theorem 3.4. *Let λ_1 be as in Proposition* 2.2. *Then λ_1 is a global bifurcation point of* (3.1) *in the sense of Rabinowitz.*

Proof. The proof relies on the jump of the Leray–Schauder dergee when λ crosses λ_1 as proved in Proposition 3.3. Then we can implement the proof of the original Rabinowitz's result from [10]. □

Remark 3.5. Let us emphasize that the essential ingredients for the proof of Proposition 3.3 (and Theorem 3.4) are the following properties of λ_1:

- λ_1 is the first variational eigenvalue of $Au = \lambda Bu$,
- λ_1 is simple, and
- λ_1 is isolated (in the sense of Proposition 2.2).

Due to these facts the assertion of Theorem 3.4 holds true also for some operators $B = \partial f^2$, for which f^2 might change sign. However, in this case the above properties of λ_1 can be derived using other tools than Theorem 2.1 (see, e.g. [7, Chapters 3 and 4]).

4. An application

Consider the following boundary value problem:

$$\begin{aligned} &-\operatorname{div}\{a_1(x)|\nabla u^+|^{p-2}\nabla u^+ + a_2(x)|\nabla u^-|^{p-2}\nabla u^-\} \\ &\quad = \lambda\{b_1(x)|u^+|^{p-2}u^+ + b_2(x)|u^-|^{p-2}u^-\}, \quad x \in \mathbb{R}^N \end{aligned} \tag{4.1}$$

$$\lim_{|x|\to\infty} u(x) = 0, \quad u(x) > 0, \ x \in \mathbb{R}^N. \tag{4.2}$$

Boundary value problems, where quasilinear elliptic operators, like the p-Laplacian $-\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, are present, arise both from pure mathematics, e.g., in the theory of quasiregular and quasiconformal mappings (see [12] and the references therein), as well as from a variety of applications, e.g. steady flows of non-Newtonian fluids, reaction–diffusion problems, flow through porous media, fracture at bimaterial interface, nonlinear elasticity, glaceology, petroleum extraction, astronomy, etc. (see [1,2]).

In problem (4.1), (4.2) we have that $\Omega = \mathbb{R}^N$ and take the approximating sets as $\Omega_n := B_n(0) = \{x \in \mathbb{R}; |x| < n\}$. Let $N > p > 1$. We assume that

$$V_1 = \mathscr{D}^{1,p}(\mathbb{R}^N) = \overline{C_0^{\infty}}^{\|\nabla u\|_p}, \qquad V_2 = L^{p^*}(\mathbb{R}^N) \quad \text{with } p^* = \frac{Np}{N-p}.$$

Moreover, we suppose that $a_i, b_i \in L^{\infty}(\mathbb{R}^N)$, $b_i \in L^{N/p}(\mathbb{R}^N)$, and $a_i(x) \geq \rho > 0, b_i(x) \geq \rho > 0$, for some $\rho > 0$, $i = 1,2$. Then we may consider as

$$f^1(u) := \frac{1}{p}\int_{\mathbb{R}^N} [a_1(x)|\nabla u^+|^p + a_2(x)|\nabla u^-|^p]\,\mathrm{d}x,$$

$$f^2(u) := \frac{1}{p}\int_{\mathbb{R}^N} [b_1(x)|u^+|^p + b_2(x)|u^-|^p]\,\mathrm{d}x.$$

So the weak formulation of problem (4.1), (4.2) is of the following type:

$$Au = \lambda Bu \quad \text{in } V_1^*,$$

where the operators A and B are defined by

$$\langle Au, v\rangle_{V_1} := \int_{\mathbb{R}^N} [a_1(x)|\nabla u^+|^{p-2}\nabla u^+\nabla v + a_2(x)|\nabla u^-|^{p-2}\nabla u^-\nabla v]\,\mathrm{d}x,$$

$$\langle Bu, v\rangle_{V_1} := \int_{\mathbb{R}^N} [b_1(x)|u^+|^{p-2}u^+ v + b_2(x)|u^-|^{p-2}u^- v]\,\mathrm{d}x,$$

for all v in V_1. It follows from Theorem 2.1, Proposition 2.2 and standard compactness argument that problem (4.1), (4.2) has the first eigenvalue $\lambda_1 > 0$, which is simple, isolated with

$$\lambda_1 = \min_{u\in\mathscr{D}^{1,p},\, u\neq 0} \frac{f^1(u)}{f^2(u)} \tag{4.3}$$

and the minimum in (4.3) is achieved at some strictly positive function $u \in \mathscr{D}^{1,p}(\mathbb{R}^N)$. We may notice that the verification of all assumptions follows the same reasoning as that in Section 4.1 of paper [9]. The decay of u follows from Serrin's estimate (see [7, Theorem 1.10] or [8, Theorem 2.4]).

Let us consider a function $f : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying Carathéodory's conditions, i.e., $f(.,x,.)$ is continuous, for a.e. $x \in \Omega$ and $f(\lambda,.,s)$ is measurable, for all $(\lambda,s) \in \mathbb{R}^2$. Assume that there is a constant γ with $p < \gamma < p^*$ and a function $\rho(x) \geq 0, \rho \in L^{\gamma_1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}^N)$, with

$$\gamma_1 = \frac{p^*}{p^* - (\gamma+1)} = \frac{Np}{Np - (\gamma+1)(N-p)}$$

such that

$$|f(\lambda,x,s)| \leq \Lambda\rho(x)\,|s|^{\gamma-1},$$

for any $s \in \mathbb{R}$, a.e. $x \in \Omega$ and λ from a certain interval I (here $\Lambda = \Lambda(I)$). Then the Nemytskij operator $F(\lambda, .)$ generated by f, i.e.,

$$\langle F(\lambda, u), v\rangle_{V_1} = \int_{\mathbb{R}^N} f(\lambda, x, u)v \, dx,$$

defines a compact map from $\mathscr{D}^{1,p}(\mathbb{R}^N)$ into $\mathscr{D}^{-1,p^*}(\mathbb{R}^N)$ which satisfies

$$\lim_{\|u\|_{\mathscr{D}^{1,p}} \to \infty} \frac{F(\lambda, u)}{\| u \|^p_{\mathscr{D}^{1,p}}} = 0.$$

So, from Theorem 3.4 we get a global bifurcation result for the nonlinear problem in $\mathbb{R}^N$:

$$\begin{aligned} &-\mathrm{div}\{a_1(x)|\nabla u^+|^{p-2}\nabla u^+ + a_2(x)|\nabla u^-|^{p-2}\nabla u^-\} \\ &\quad = \lambda\{b_1(x)|u^+|^{p-2}u^+ + b_2(x)|u^-|^{p-2}u^-\} + f(\lambda, x, u), \quad x \in \mathbb{R}^N, \\ &\lim_{|x|\to\infty} u(x) = 0, \quad u(x) > 0, \ x \in \mathbb{R}^N. \end{aligned} \tag{4.4}$$

Using the bootstraping argument (see e.g. [7, Proposition 4.1]) we may even show that $u \in L^r(\mathbb{R}^N)$, $p^* \le r \le +\infty$, where u is any nontrivial solution to the problem (4.4). Then the regularity result of Tolksdorf [12] implies that $u \in C^1_{\mathrm{loc}}(\mathbb{R}^N)$.

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