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GLOBAL EXISTENCE AND BLOW-UP RESULTS FOR SOME NONLINEAR WAVE EQUATIONS ON $\mathbb{R}^N$

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Abstract. We discuss the asymptotic behaviour of solutions of the semilinear hyperbolic problem

$$u_{tt} + \delta u_t - \phi(x)\Delta u = \lambda\, u|u|^{\beta-1}, \quad x \in \mathbb{R}^N, \;\; t \geq 0,$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$, $\delta \geq 0$ and $(\phi(x))^{-1} = g(x)$, a positive function belonging to $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Under certain conditions we prove the global existence of solutions. Also we examine blow-up in finite time when the initial data are sufficiently large. The space setting of the problem is the energy space $\mathcal{X}_0 = \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$, where L^2_g is an appropriate weighted Hilbert space; see Section 2.

1. Introduction. In this work we study the following semilinear hyperbolic Cauchy problem:

$$u_{tt} + \delta u_t - \phi(x)\Delta u \;=\; \lambda\, u|u|^{\beta-1}, \quad x \in \mathbb{R}^N, \;\; t \geq 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with initial conditions $u_0(x),\ u_1(x)$ in appropriate function spaces and $\delta \geq 0$. Models of this type are of interest in applications in various areas of mathematical physics (see [1, 30, 39]), as well as in geophysics and ocean acoustics, where, for example, the coefficient $\phi(x)$ represents the speed of sound at the point $x \in \mathbb{R}^N$ (see [13]). Throughout the paper we assume that the functions ϕ and $g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfy the following condition:

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($\mathcal{G}$) $\phi(x) > 0$, for all $x \in \mathbb{R}^N$, $(\phi(x))^{-1} =: g(x)$ is $\mathcal{C}^{0,\gamma}(\mathbb{R}^N)$-smooth, for some $\gamma \in (0,1)$ and $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (for functions ϕ of this type, e.g. polynomial-like, we refer to [39, p. 632]).

The questions of global existence, nonexistence and blow-up of solutions of the Cauchy problem for nonlinear wave equations have been treated by many authors; see [3], [5], [8], [14], [25], [26], [27], [32], etc. We refer also to the review papers [15], [35] and to the monographs [29], [34] for a survey of results and a long list of references. In [7] and [9] wave equations involving nonlinear damping and source terms are discussed. In general, global existence happens, when the damping terms dominate over the source terms, while blow-up appears in the opposite situation and under the assumption that the initial data is sufficiently large (i.e., when initial energy is assumed to be sufficiently negative). In [9] it is shown that for sufficiently small initial data global existence can be obtained, even when the influence of the source term is stronger than that of the damping term. In both works [7] and [9] the spatial domain is assumed to be bounded. On the other hand, in [22] the problem is considered in the whole of $\mathbb{R}^N$ and the method of *modified potential well* is used to construct the global solutions. In the papers [7], [9], and [22] the coefficient $\phi(x) = 1$, which makes possible the treatment of the equations in the classical Sobolev space setting. In the works [23], [24] and [41] decay properties of solutions of wave equations, involving weighted dissipative terms, are discussed. Recently H.A. Levine, S.R. Park, P. Pucci, J. Serrin and G. Todorova in [16, 17, 18, 28, 36, 37, 38] studied global existence and nonexistence of solutions for both the bounded and unbounded domain cases and nonlinear damping. In [17] and [37] nonexistence occurs for all negative initial energies (and not only sufficiently negative). In [28] nonexistence results for abstract evolution equations have been obtained, when the initial data possesses positive initial energy.

In this paper, problem (1.1)–(1.2) is considered in the homogeneous Sobolev space setting. This choice seems to be effective for the treatment of the difficulties of noncompactness arising in unbounded domains and the degeneracy induced by the nonconstant coefficient ϕ. In the same space environment, existence of finite-dimensional invariant sets for the problem (1.1)–(1.2) is discussed in the papers [11], [12]. All these results are presented in detail in the dissertation [10].

The presentation of this paper goes as follows: In Section 2 we discuss some useful properties of the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$

and imbedding relations with some weighted L^p spaces. Section 3 is devoted to the discussion of global solutions for (1.1)–(1.2). In Section 4 we obtain blow-up results for the solutions of the problem (1.1)–(1.2), for all negative initial energies. Finally, in Section 5 some global existence and blow-up results are presented for the solutions of the undamped equation.

Notation: We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for *simplicity* we use the symbols L^p, $1 \leq p \leq \infty$ and $\mathcal{D}^{1,2}$, for the spaces $L^p(\mathbb{R}^N)$ and $\mathcal{D}^{1,2}(\mathbb{R}^N)$, respectively; here $|| \cdot ||_p$ denotes the norm $|| \cdot ||_{L^p(\mathbb{R}^N)}$. Also sometimes differentiation with respect to time is denoted by a dot over the function. The constants C and c are considered in a generic sense.

2. Preliminary results. For later use, we briefly mention here some facts, notation and results from our earlier joint paper [11]. The space setting for the initial conditions and the solutions of the problem (1.1), (1.2) is the product space $\mathcal{X}_0 = \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$. The space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the "energy norm" $||u||_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} | \bigtriangledown u|^2 dx$. It is well known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \bigtriangledown u \in (L^2(\mathbb{R}^N))^N \right\}$$

and that $\mathcal{D}^{1,2}$ is embedded continuously in $L^{\frac{2N}{N-2}}$; i.e., there exists $k > 0$ such that

$$||u||_{\frac{2N}{N-2}} \leq k||u||_{\mathcal{D}^{1,2}}. \tag{2.1}$$

We shall frequently use the following generalized version of Poincaré's inequality:

$$\int_{\mathbb{R}^N} | \bigtriangledown u|^2 \, dx \geq \alpha \int_{\mathbb{R}^N} g u^2 \, dx, \tag{2.2}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha =: k^{-2}||g||^{-1}_{N/2}$ (see [6, Lemma 2.1]). It has been shown that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. The space $L^2_g(\mathbb{R}^N)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product

$$(u, v)_{L^2_g(\mathbb{R}^N)} =: \int_{\mathbb{R}^N} guv \, dx. \tag{2.3}$$

Clearly, $L^2_g(\mathbb{R}^N)$ is a separable Hilbert space. Moreover, we have the following:

Lemma 2.1. *Let $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the embedding $\mathcal{D}^{1,2} \subset L^2_g$ is compact.*

Proof. For the proof we refer to [11] and [4]. $\square$

Hence we are able to construct the *evolution triple*, which is necessary for our problem, namely

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.4}$$

where all the embeddings are compact and dense. We also need the following four lemmas, which describe embedding relations among weighted Lebesgue and homogeneous Sobolev spaces.

Lemma 2.2. *Let $g \in L^{\frac{2N}{2N-pN+2p}}(\mathbb{R}^N)$. Then we have the following continuous embedding: $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^p_g(\mathbb{R}^N)$, for all $1 \leq p \leq 2N/N-2$.*

Proof. The lemma is a consequence of Hölder's inequality. In fact,

$$\begin{aligned} \int_{\mathbb{R}^N} gu^p\,dx &\leq \Big(\int_{\mathbb{R}^N} g^a\,dx\Big)^{\frac{1}{a}}\Big(\int_{\mathbb{R}^N} |u|^{pb}\,dx\Big)^{\frac{1}{b}} \\ &\leq \Big(\int_{\mathbb{R}^N} g^a\,dx\Big)^{\frac{1}{a}}\Big(\int_{\mathbb{R}^N} |\bigtriangledown u|^2\,dx\Big)^{\frac{p}{2}}, \end{aligned}$$

where $a = 2N/(2N-pN+2p)$ and $b = 2N/(N-2)p$. $\square$

Remark 2.3. *The assumption of Lemma 2.2 is satisfied under hypothesis $(\mathcal{G})$, if $p \geq 2$.*

Lemma 2.4. *Let g satisfy condition $(\mathcal{G})$. If $1 \leq q < p < p^* = 2N/N-2$, then there exists $C_0 > 0$ such that the weighted inequality*

$$||u||_{L^p_g} \leq C_0||u||^{1-\theta}_{L^q_g}||u||^{\theta}_{\mathcal{D}^{1,2}} \tag{2.5}$$

is valid for all $\theta \in (0,1)$ which satisfy the relation $1/p = (1-\theta)/q + \theta/p^*$.

Proof. We get relation (2.5) by using the weighted interpolation inequality

$$||u||_{L^p_g} \leq ||u||^{1-\theta}_{L^q_g}||u||^{\theta}_{L^{p^*}_g}$$

(see [21] or [31]) and inequality (2.1). Here $C_0 = k^\theta$. $\square$

Lemma 2.5. *Assume that $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the continuous embedding $L^p_g(\mathbb{R}^N) \subset L^q_g(\mathbb{R}^N)$ is true for any $1 \le q \le p < \infty$.*

Proof. Using Hölder's inequality again we get

$$\int_{\mathbb{R}^N} gu^q \, dx \le \Big(\int_{\mathbb{R}^N} (g^\sigma)^a \, dx \Big)^{\frac{1}{a}} \Big(\int_{\mathbb{R}^N} (g^\tau |u|^q)^b \, dx \Big)^{\frac{1}{b}},$$

where $a = p/(p-q)$ and $b = p/q$. Hence for $\sigma = (p-q)/p$ and $\tau = q/p$ we obtain the embedding inequality $||u||_{L^q_g} \le C_* ||u||_{L^p_g}$, where $C_* = ||g||_1^{(p-q)/pq}$. □

Lemma 2.6. *Assume that $1 < a,\ b,\ c < \infty,\ s \in [0,\ c^{-1})$ and $a^{-1} + b^{-1} + c^{-1} = 1$. Then for every $u \in L^a_g,\ v \in L^b_g,\ w \in L^c_g$ and every $K > 0$ we have the inequality*

$$\Big| \int_{\mathbb{R}^N} guvw \, dx \Big| \le K^{s-c^{-1}} ||w||_{L^c_g} \Big(||u||^a_{L^a_g} + ||v||^b_{L^b_g} + K \Big)^{1-s}.$$

Proof. The proof is a direct application of [17, Lemma 4.1]. □

In order to deal with (1.1)–(1.2), we need information concerning the properties of the operator $-\phi\Delta$. We consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \quad x \in \mathbb{R}^N, \tag{2.6}$$

without boundary conditions. Since for every u, v in $C^\infty_0(\mathbb{R}^N)$

$$(-\phi\Delta u,\ v)_{L^2_g} = \int_{\mathbb{R}^N} \triangledown u \triangledown v \, dx, \tag{2.7}$$

and $L^2_g(\mathbb{R}^N)$ is defined as the closure of $C^\infty_0(\mathbb{R}^N)$ with respect to the inner product (2.3), we may consider equation (2.6) as an operator equation:

$$A_0 u = \eta, \ \ A_0 : D(A_0) \subseteq L^2_g(\mathbb{R}^N) \to L^2_g(\mathbb{R}^N), \quad \text{for any } \ \eta \in L^2_g(\mathbb{R}^N). \tag{2.8}$$

Relation (2.7) implies that the operator $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C^\infty_0(\mathbb{R}^N)$ being symmetric. Let us note that the operator A_0 is not symmetric in the standard Lebesgue space $L^2(\mathbb{R}^N)$, because of the appearance of $\phi(x)$. For comments of the same nature on a similar model in the case of a bounded weight we refer to [30, pages 185–187]. From (2.2) and equation (2.7) we have

$$(A_0 u,\ u)_{L^2_g} \ge \alpha ||u||^2_{L^2_g}, \ \text{ for all } \ u \in D(A_0). \tag{2.9}$$

From (2.7) and (2.9) we conclude that A_0 is a symmetric, strongly monotone operator on $L^2_g(\mathbb{R}^N)$. Hence, the Friedrichs extension theorem (see [40]) is applicable. The energy scalar product given by (2.7) is

$$(u,\ v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx,$$

and the energy space is the completion of $D(A_0)$ with respect to $(u,\ v)_E$. It is obvious that the energy space X_E is the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The *energy extension* $A_E = -\phi\Delta$ of A_0, namely

$$-\phi\Delta : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.10}$$

is defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. For every $\eta \in \mathcal{D}^{-1,2}(\mathbb{R}^N)$ the equation (2.6) has a unique solution. Define $D(A)$ to be the set of all solutions of the equation (2.6), for arbitrary $\eta \in L^2_g(\mathbb{R}^N)$. The *Friedrichs extension* A of A_0 is the restriction of the energy extension A_E to the set $D(A)$. The operator A is self-adjoint and therefore graph-closed. Its domain, $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u,\ v)_{D(A)} = (u,\ v)_{L^2_g} + (Au,\ Av)_{L^2_g}, \ \text{ for all } \ u,\ v \in D(A).$$

The norm induced by the scalar product $(u,\ v)_{D(A)}$ is

$$||u||_{D(A)} = \Big\{ \int_{\mathbb{R}^N} g|u|^2 \ dx + \int_{\mathbb{R}^N} \phi|\Delta u|^2 \ dx \Big\}^{\frac{1}{2}},$$

which is equivalent to the norm $||Au||_{L^2_g} = \big\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2 \ dx \big\}^{\frac{1}{2}}$. A consequence of the compactness of the embeddings in (2.4) is that for the eigenvalue problem

$$-\phi(x)\Delta u = \mu u, \quad x \in \mathbb{R}^N, \tag{2.11}$$

there exists a complete system of eigensolutions $\{w_n,\ \mu_n\}$ with the following properties:

$$\begin{cases} -\phi\Delta w_j = \mu_j w_j, & j = 1, 2, \ldots, \quad w_j \in D^{1,2}(\mathbb{R}^N), \\ 0 < \mu_1 \le \mu_2 \le \cdots, & \mu_j \to \infty, \qquad \text{as } j \to \infty. \end{cases} \tag{2.12}$$

It can be shown, as in [6], that every solution of (2.11) is such that

$$u(x) \to 0, \quad \text{as} \quad |x| \to \infty \tag{2.13}$$

uniformly with respect to x. Finally, we give the definition of *weak solutions* for the problem (1.1)–(1.2).

Definition 2.7. A weak solution of (1.1)–(1.2) is a function $u(x,t)$ such that
(i) $u \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)],\ u_t \in L^2[0,T;L^2_g(\mathbb{R}^N)],\ u_{tt} \in L^2[0,T;\mathcal{D}^{-1,2}(\mathbb{R}^N)]$,
(ii) For all $v \in C_0^\infty([0,T]\times\mathbb{R}^N)$, u satisfies the generalized formula

$$\begin{aligned}&\int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g}\, d\tau + \delta \int_0^T (u_t(\tau), v(\tau))_{L^2_g}\ d\tau\\ &+ \int_0^T \int_{\mathbb{R}^N} \bigtriangledown u(\tau) \bigtriangledown v(\tau)\ dx\, d\tau - \lambda \int_0^T (f(u(\tau)),\, v(\tau))_{L^2_g}\, d\tau = 0,\end{aligned} \tag{2.14}$$

where $f(s) = |s|^{\beta-1}s$, and
(iii) u satisfies the initial conditions

$$u(x,0) = u_0(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u_t(x,0) = u_1(x) \in L^2_g(\mathbb{R}^N).$$

Remark 2.8. Using a density argument, we may see that the generalized formula (2.14) is satisfied for every $v \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)]$. By the compactness and density of the embeddings in the evolution triple (2.4) we see that, as in [11, Proposition 3.2], the above Definition 2.7 of weak solutions implies that $u \in C[0,T;\ \mathcal{D}^{1,2}(\mathbb{R}^N)]$ and $u_t \in C[0,T;\ L^2_g(\mathbb{R}^N)]$.

3. Global existence results.

In this section we prove that under certain assumptions on the initial data, solutions exist globally in the energy space $\mathcal{X}_0$. To this end, in addition to the principal condition $(\mathcal{G})$ in the introduction, we shall use the following additional hypotheses for the function g and the nonlinearity exponent β: $(\mathcal{G}_1)$ $g \in L^1(\mathbb{R}^N)$ and $1 < \beta \le \frac{N}{N-2}$, for all $N \ge 3$. $(\mathcal{G}_2)$ $N \ge 3$ and $\frac{N+2}{N} \le \beta \le \frac{N}{N-2}$. $(\mathcal{G}_3)$ $N = 3,4$ and $\frac{N+4}{N} \le \beta \le \frac{N}{N-2}$.

Let us note that since $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ by hypothesis $(\mathcal{G})$, then any g satisfying hypothesis $(\mathcal{G}_1)$ belongs to all spaces $L^p(\mathbb{R}^N)$, for $p \in [1,+\infty)$. First we give the following local existence result.

Proposition 3.1. *Let g, β, N satisfy conditions $(\mathcal{G}_1)$ or $(\mathcal{G}_2)$. Suppose that the constants $\delta > 0$, $\lambda < \infty$ and the initial conditions*

$$u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \textit{and} \quad u_1 \in L^2_g(\mathbb{R}^N), \tag{3.1}$$

are given. Then for sufficiently small $T > 0$ the problem (1.1)–(1.2) *admits a unique (weak) solution such that $u \in C[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)]$ and $u_t \in C[0,T;L^2_g(\mathbb{R}^N)]$.*

Proof. (a) *Local Existence of the Restricted Problem on B_R.* First we prove an existence result for the problem

$$\begin{aligned} u_{tt} + \delta u_t - \phi(x)\Delta u &= \lambda\, u|u|^{\beta-1}, \quad (x,t) \in B_R \times (0,T),\\ u(x,0) = u_0(x), \quad u_t(x,0) &= u_1(x), \qquad\qquad x \in B_R,\\ u(x,t) &= 0, \qquad\qquad (x,t) \in \partial B_R \times (0,T), \end{aligned} \tag{3.2}$$

where $u_0 \in \mathcal{D}^{1,2}(B_R)$ and $u_1 \in L^2_g(B_R)$. Let $Z =: \{z,\ z_t\} \in C[0,T;\ \mathcal{X}_0(B_R)]$ be given. In order to obtain solutions for (3.2) we first consider the following nonhomogeneous problem:

$$\begin{aligned} u_{tt} + \delta u_t - \phi(x)\Delta u &= \lambda\, z|z|^{\beta-1}, \quad (x,t) \in B_R \times (0,T),\\ u(x,0) = u_0(x), \quad u_t(x,0) &= u_1(x), \qquad\qquad x \in B_R,\\ u(x,t) &= 0, \qquad\qquad (x,t) \in \partial B_R \times (0,T), \end{aligned} \tag{3.3}$$

where $u_0 \in \mathcal{D}^{1,2}(B_R)$ and $u_1 \in L^2_g(B_R)$. Existence of a unique (weak) solution for problem (3.3) can be obtained by using Faedo-Galerkin approximations (see [11, Lemma 3.1]).

For $Z \in C[0,T;\ \mathcal{X}_0(B_R)]$ we define the mapping $\mathcal{T} : C[0,T;\ \mathcal{X}_0(B_R)] \to C[0,T;\ \mathcal{X}_0(B_R)]$ by $U = \mathcal{T}(Z)$, where $U = \{u,\ u_t\}$ is the unique solution of equation (3.3). It is clear that the map $\mathcal{T}$ is well defined. Next, we show that $\mathcal{T}$ maps the ball $\mathcal{B}_M$ to itself, where $\mathcal{B}_M =: \{\Psi \in \mathcal{X}_{0,T} : \sup_{0\le t\le T} ||\Psi(\cdot,\ t)|| \le M\}$ and the space $\mathcal{X}_{0,T}$ is defined by $\mathcal{X}_{0,T} =: \{\Psi \in C[0,T;\mathcal{X}_0(B_R)] : \Psi(0,\ \cdot) = \{\psi_0, \psi_1\} \in \mathcal{X}_0(B_R)\}$.

For $Z \in \mathcal{B}_M$, we multiply equation (3.3) by gu_t and integrate with respect to time and space on the set $(0,t) \times B_R$, for some $t \in (0,T]$, to obtain

$$\begin{aligned} \frac{1}{2}||U(\cdot,\ t)||^2_{\mathcal{X}_0(B_R)} \quad &- \quad \frac{1}{2}||U(\cdot,\ 0)||^2_{\mathcal{X}_0(B_R)} + \delta \int_0^t ||u_t(\cdot,\ s)||^2_{L^2_g(B_R)}\, ds\\ &\le \quad \lambda \int_0^t \int_{B_R} |g(x)|z|z|^{\beta-1} u_t\, dx\, ds. \end{aligned} \tag{3.4}$$

The positivity of the quantity $\delta \int_0^t ||u_t(\cdot,\ s)||^2_{L^2_g(B_R)}\ ds$ implies that

$$\frac{1}{2}||U(\cdot,t)||^2_{\mathcal{X}_0(B_R)} - \frac{1}{2}||U(\cdot,0)||^2_{\mathcal{X}_0(B_R)} \le \lambda \int_0^t ||Z||^{\beta}_{\mathcal{X}_0(B_R)}||U||_{\mathcal{X}_0(B_R)}\, ds. \quad (3.5)$$

We use the assumption on Z and relation (3.5) to obtain

$$||U(\cdot,\ t)||_{C[0,T;\ \mathcal{X}_0(B_R)]} \le C||U(\cdot,\ 0)||_{C[0,T;\ \mathcal{X}_0(B_R)]} + C_4(\lambda)\ M^{\beta}T.$$

Choosing T sufficiently small and M sufficiently large, depending on the norm of the initial data, we have $\sup_{0\le t\le T} ||U(\cdot,\ t)||_{\mathcal{X}_0(B_R)} \le M$; i.e., $U \in \mathcal{B}_M$. The next step is to show that $\mathcal{T}$ is a contraction. Let $Z,\ Z^* \in \mathcal{X}_{0,T}$ such that $U = \mathcal{T}(Z),\ U^* = \mathcal{T}(Z^*)$, and consider the difference $W =: U - U^* = \{w,\ w_t\} = \{u - u^*, u_t - u_t^*\}$, which satisfies the equation

$$w_{tt} + \delta w_t - \phi(x)\Delta w = \lambda(z|z|^{\beta-1} - z^*|z^*|^{\beta-1}), \quad (x,t) \in B_R \times (0,T). \quad (3.6)$$

Following the procedure above, for the right-hand side of equation (3.6), we get the estimates

$$\begin{aligned}
&\Big| \int_{B_R} g(x)(z|z|^{\beta-1} - z^*|z^*|^{\beta-1})(u_t - u_t^*)\ dx\Big| \\
\le & C\int_{B_R} |g(x)|\ |z - z^*|(|z|^{\beta-1} + |z^*|^{\beta-1})|u_t - u_t^*|\ dx \\
\le & C||g||_{\infty}^{\frac{N-2}{2N}}||z - z^*||_{\frac{2N}{N-2}}||u_t - u_t^*||_{L^2_g(B_R)}\big\{||z||^{\beta-1}_{L_g^{N(\beta-1)}(B_R)} + ||z^*||^{\beta-1}_{L_g^{N(\beta-1)}(B_R)}\big\} \\
\le & C||z - z^*||_{\mathcal{D}^{1,2}}||u_t - u_t^*||_{L^2_g(B_R)}\left\{||z||^{\beta-1}_{\mathcal{D}^{1,2}(B_R)} + ||z^*||^{\beta-1}_{\mathcal{D}^{1,2}(B_R)}\right\}. \qquad (3.7)
\end{aligned}$$

From relations (3.6) and (3.7) we have

$$\begin{aligned}
||W(\cdot,\ t)||^2_{\mathcal{X}_0(B_R)} \quad \le \quad & C(\lambda) \int_0^t (||Z||^{\beta-1}_{\mathcal{X}_0(B_R)} + ||Z^*||^{\beta-1}_{\mathcal{X}_0(B_R)})||W(\cdot,\ s)||_{\mathcal{X}_0} \\
& \times||Z(\cdot,\ s) - Z^*(\cdot,\ s)||_{\mathcal{X}_0(B_R)}\ ds,
\end{aligned}$$

which is equivalent to the inequality

$$||\mathcal{T}(Z) - \mathcal{T}(Z^*)||_{C[0,T;\ \mathcal{X}_0(B_R)]} \le 2C(\lambda)\ M^{\beta-1}T||Z - Z^*||_{C[0,T;\ \mathcal{X}_0(B_R)]}.$$

For $T < C^{-1}(\lambda)\ M^{1-\beta}$ the map $\mathcal{T}$ is a contraction. Then the result of existence for (3.2) is a direct consequence of the contraction mapping theorem.

(b) *Extension of Solutions to* $\mathbb{R}^N$. For $R \geq R_0,\ R \in \mathbb{N}$, with $\{u_0, u_1\} \in C_0^\infty(B_R) \times C_0^\infty(B_R)$ such that $supp(u_0) \subset B_{R_0}$ and $supp(u_1) \subset B_{R_0}$, we consider the approximating problem

$$\begin{aligned} u^R_{tt} + \delta u^R_t - \phi(x)\Delta u^R &= \lambda f(u^R), \quad (x,t) \in B_R \times (0,T), \\ u^R(x,0) = u_0(x), \quad u^R_t(x,0) &= u_1(x), \qquad x \in B_R, \\ u^R(x,t) &= 0, \qquad (x,t) \in \partial B_R \times (0,T), \end{aligned} \tag{3.8}$$

with $f(s) = s|s|^{\beta-1}$. The existence result in (a) holds for (3.8). We get that u^R is bounded in $C[0,T;\ \mathcal{D}^{1,2}(B_R)]$ and u^R_t is bounded in $C[0,T;\ L^2_g(B_R)]$, independent of R. Since, for any Banach space X, the following continuous embedding $C[0,T;\ X] \subset L^p[0,T;\ X]$ is valid, for all $1 \leq p < \infty$, we have that $u^R,\ u^R_t$ remain bounded in $L^2[0,T;\ \mathcal{D}^{1,2}(B_R)]$ and in $L^2[0,T;\ L^2_g(B_R)]$, respectively. We extend u^R, as

$$\tilde{u}^R(x,t) =: \begin{cases} u^R(x,t), \text{if} & |x| \leq R, \\ 0, & \text{otherwise}. \end{cases}$$

So that $\tilde{u}^R,\ \tilde{u}^R_t$ remain bounded in the above spaces with B_R replaced by $\mathbb{R}^N$. Using the assumptions on β, we may easily check that $f(u^R)$ is bounded in $L^2[0,T;\ L^2_g(\mathbb{R}^N)]$. From the relations (2.10) and (3.8) we obtain (as in [19, Remark 8.2, page 265]), that u^R_{tt} is bounded in $L^2[0,T;\ \mathcal{D}^{-1,2}(B_R)]$. Lemma 2.1 applied to [33, Lemma 4 (ii)] implies that $\tilde{u}^R$ is relatively compact in $C[0,T;\ L^2_g(\mathbb{R}^N)]$. Therefore we get $\tilde{u}^R \to \tilde{u}$, in $L^2[0,T; L^2_g(\mathbb{R}^N)]$. Hence we may extract a subsequence of $\tilde{u}^R$, denoted by $\tilde{u}^{R_m}$, such that

$$\begin{array}{lll} \tilde{u}^{R_m} & \rightharpoonup \tilde{u}, & \text{in} \quad L^2[0,T;\ \mathcal{D}^{1,2}(\mathbb{R}^N)], \\ \tilde{u}^{R_m}_t & \rightharpoonup \tilde{u}_t, & \text{in} \quad L^2[0,T;\ L^2_g(\mathbb{R}^N)], \\ \tilde{u}^{R_m}_{tt} & \rightharpoonup \tilde{u}_{tt}, & \text{in} \quad L^2[0,T;\ \mathcal{D}^{-1,2}(\mathbb{R}^N)], \\ f(\tilde{u}^{R_m}) & \rightharpoonup f(\tilde{u}), & \text{in} \quad L^2[0,T;\ L^2_g(\mathbb{R}^N)]. \end{array} \tag{3.9}$$

Following the arguments in [11, Proposition 3.2 and Theorem 3.3] we may see that $\tilde{u}$ defines a unique weak solution of (1.1)–(1.2) with initial data satisfying (3.1). □

To obtain global existence, we adapt the method of *modified potential well*, as developed by Payne and Sattinger ([27]) and generalized to all of $\mathbb{R}^N$ by Nakao and Ono in [22]. To this end we consider the potential well

$$\mathcal{W} =: Int\big\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\ :\ \mathcal{K}(u) =: ||u||^2_{\mathcal{D}^{1,2}} - \lambda ||u||^{\beta+1}_{L^{\beta+1}_g} \geq 0\big\},$$

where *Int* B denotes the interior of set B. It is easily seen that 0 is in $\mathcal{W}$. Indeed, from Lemma 2.4, the Poincaré inequality (2.2) and hypothesis $(\mathcal{G}_1)$ we have

$$\begin{aligned} ||u||^{\beta+1}_{L^{\beta+1}_g} &\leq C||u||^{(1-\theta)(\beta+1)}_{L^2_g}||u||^{\theta(\beta+1)}_{\mathcal{D}^{1,2}} \\ &\leq C||u||^{(1-\theta)(\beta+1)}_{L^2_g}||u||^{\theta(\beta+1)-2}_{\mathcal{D}^{1,2}}||u||^2_{\mathcal{D}^{1,2}} \leq \frac{C_0}{\alpha}||u||^{\beta-1}_{\mathcal{D}^{1,2}}||u||^2_{\mathcal{D}^{1,2}}. \end{aligned}$$

Therefore, for any $\lambda \in \mathbb{R}^+$, we obtain

$$\mathcal{K}(u) \geq \big(1 - \frac{\lambda C_0}{\alpha}||u||^{\beta-1}_{\mathcal{D}^{1,2}}\big)||u||^2_{\mathcal{D}^{1,2}}.$$

It is obvious that, if $||u||_{\mathcal{D}^{1,2}}$ is sufficiently small, then $\mathcal{K}(u) \geq 0$ and 0 is in $\mathcal{W}$. Also consider the functional

$$\mathcal{J}(u) =: \frac{1}{2}||u||^2_{\mathcal{D}^{1,2}} - \frac{\lambda}{\beta+1}\int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\,dx. \tag{3.10}$$

By the definition of $\mathcal{W}$ we have that

$$\mathcal{J}(u) \geq \frac{\beta-1}{2(\beta+1)}||u||^2_{\mathcal{D}^{1,2}}, \quad \text{for every} \quad u \in \mathcal{W}. \tag{3.11}$$

Multiply equation (1.1) by gu_t and integrate over $\mathbb{R}^N$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u_t(t)||^2_{L^2_g} + \delta||u_t(t)||^2_{L^2_g} &+ \frac{1}{2}\frac{d}{dt}||u(t)||^2_{\mathcal{D}^{1,2}} \\ &= \frac{\lambda}{\beta+1}\frac{d}{dt}\int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\,dx. \end{aligned} \tag{3.12}$$

The energy of the problem is defined as

$$\begin{aligned} \mathcal{E}^*(u(t),u_t(t)) = \mathcal{E}^*(t) &=: \frac{1}{2}||u_t(t)||^2_{L^2_g} + \frac{1}{2}||u(t)||^2_{\mathcal{D}^{1,2}} \\ &\quad -\frac{\lambda}{\beta+1}\int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\,dx. \end{aligned} \tag{3.13}$$

Let us note that $\mathcal{E}^*(u,u_t) \geq 0$ if $u \in \bar{\mathcal{W}}$ and $u \notin \bar{\mathcal{W}}$ if $\mathcal{E}^*(u,u_t) < 0$. Lemma 2.2 and Proposition 3.1 imply that the functional $\mathcal{E}^*(t)$ is well defined. From equation (3.12) and definition (3.13), it is easy to obtain that

$\dot{\mathcal{E}}^*(t) = -\delta||u_t(t)||^2_{L^2_g} \leq 0$. Therefore, $\mathcal{E}^*(t)$ is a nonincreasing function of t; i.e.,

$$\mathcal{E}^*(t) \leq \mathcal{E}^*(0), \quad \text{for every} \quad t \in [0,\ T). \tag{3.14}$$

The global existence result is given in the following theorem.

Theorem 3.2. *Let condition* $(\mathcal{G}_3)$ *be satisfied and* $u_0 \in W$. *Assume that the initial data satisfy* (3.1) *and they are sufficiently small in the sense that*

$$\mathcal{E}^*(0) \leq \big(\frac{1}{C_0 \lambda \mu_0^{p_1}}\big)^{\frac{1}{p_2}}, \tag{3.15}$$

where $p_1 = \frac{2(\beta+1)-N(\beta-1)}{2}$ *and* $p_2 = \frac{N\beta-N-4}{4}$. *Then the* (*weak*) *solution of* (1.1)–(1.2) *is such that* $u \in C([0,\infty); \mathcal{D}^{1,2}(\mathbb{R}^N))$ *and* $u_t \in C([0,\infty); L^2_g(\mathbb{R}^N))$.

Proof. We shall show that the local solution given by Proposition 3.1 is in the modified potential well $\mathcal{W}$, as long as it exists. We argue by contradiction. Assume that there exists some time $T^* > 0$, such that $u(t) \in \mathcal{W}$, where $0 \leq t < T^*$ and $u(T^*) \in \partial\mathcal{W}$. Then $\mathcal{K}(u(T^*)) = 0$ and $u(T^*) \neq 0$. We multiply equation (1.1) by gu and integrate over $\mathbb{R}^N$, to get the equation

$$\begin{aligned} \frac{d}{dt}(u(t),\ u_t(t))_{L^2_g} \quad &- \quad ||u_t(t)||^2_{L^2_g} + \frac{\delta}{2}\frac{d}{dt}||u(t)||^2_{L^2_g} \\ &+ \quad ||u||^2_{\mathcal{D}^{1,2}} - \lambda \int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\, dx = 0. \end{aligned} \tag{3.16}$$

We integrate over $[0,\ t]$, for some $t \in [0,T)$, to get the inequality

$$\begin{aligned} \delta||u(t)||^2_{L^2_g} &\leq \delta||u(0)||^2_{L^2_g} + 2|(u(t),u_t(t))_{L^2_g}| + 2(u_0,u_1)_{L^2_g} + 2\int_0^t ||u_t(s)||^2_{L^2_g} ds \\ &\leq \delta||u(0)||^2_{L^2_g} + 2\left(\frac{\delta}{4}||u(t)||^2_{L^2_g} + \frac{1}{\delta}||u_t(t)||^2_{L^2_g}\right) \\ &+ 2(u_0,\ u_1)_{L^2_g} + 2\int_0^t ||u_t(s)||^2_{L^2_g}\, ds, \end{aligned} \tag{3.17}$$

by applying Young's inequality for $\epsilon = \delta/2$. Since $u(t)$ is in $\mathcal{W}$, we have from (3.11) and (3.12)

$$\frac{1}{2}||u_t(s)||^2_{L^2_g} + \delta\int_0^t ||u_t(s)||^2_{L^2_g}\, ds \leq \mathcal{E}^*(0). \tag{3.18}$$

Then from (3.17) and (3.18) we get the estimate

$$||u(t)||^2_{L^2_g} \leq \frac{2}{\delta}\big\{\delta||u(0)||^2_{L^2_g} + 2(u_0, u_1)_{L^2_g} + \frac{4}{\delta}\mathcal{E}^*(0)\big\} =: \mu_0^2. \tag{3.19}$$

Using Lemma 2.4 and relation (3.19) we obtain the inequality

$$\begin{aligned} ||u(t)||^{\beta+1}_{L^{\beta+1}_g} &\leq C_0\mu_0^{(\beta+1)(1-\theta)}||u(t)||^{(\beta+1)\theta}_{\mathcal{D}^{1,2}} \\ &\leq C_0\mu_0^{(\beta+1)(1-\theta)}\mathcal{J}(u(t))^{\frac{(\beta+1)\theta}{2}-1}||u(t)||^2_{\mathcal{D}^{1,2}} \\ &\leq C_0\mu_0^{(\beta+1)(1-\theta)}\mathcal{E}^*(0)^{\frac{(\beta+1)\theta}{2}-1}||u(t)||^2_{\mathcal{D}^{1,2}}, \end{aligned} \tag{3.20}$$

where $\theta = \frac{N(\beta-1)}{2(\beta+1)}$ according to Lemma 2.4, $p_1 = (\beta+1)(1-\theta)$, $p_2 = \frac{(\beta+1)\theta}{2} - 1$ and p_1, p_2 are positive by hypothesis $(\mathcal{G}_3)$. Setting $\delta_1 = C_0\mu_0^{p_2}\mathcal{E}^*(0)^{p_1}$, inequality (3.20) implies, for $t = T^*$, that

$$\mathcal{K}(u(T^*)) \geq (1 - \lambda\delta_1)||u(T^*)||^2_{\mathcal{D}^{1,2}} > 0, \tag{3.21}$$

under the assumption that $\lambda < \frac{1}{\delta_1}$ (which is equivalent to the relation (3.15)), and the contradiction is achieved. $\square$

4. Blow-up of solutions. In this section we prove that solutions of the problem (1.1)–(1.2) blow up in finite time if we consider negative initial energy. As in [27] we have the following lemma.

Lemma 4.1. *Let the mapping $t \to \rho(t) =: ||u(\cdot,\ t)||^2_{L^2_g}$, where u is the weak solution of the problem* (1.1)–(1.2). *Then $\dot\rho(t)$ is Lipschitz and $\ddot\rho(t)$ exists, for almost all $t \in [0, T]$.*

Proof. ¿From the weak formulation of the problem (1.1)–(1.2), we have

$$(u_t, v)_{L^2_g}\big|^{t_2}_{t_1} = \int_{t_1}^{t_2}\Big\{(u_t,\ v_t)_{L^2_g} - \int_{\mathbb{R}^N} \nabla u \nabla v\, dx \\ - \delta(u_t,\ v)_{L^2_g} + \lambda\int_{\mathbb{R}^N} g(x)u|u|^{\beta-1}v\, dx\Big\}\, ds,$$

for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Next we introduce the function

$$R(t,\ s) =: \int_{\mathbb{R}^N} g(x)u(x,\ t)u(x,\ s)\, dx.$$

We have

$$\dot{\rho}(t) = \big(\frac{\partial}{\partial t}R(t,\ s) + \frac{\partial}{\partial s}R(t,\ s)\big)_{t=s} = 2\int_{\mathbb{R}^N} g u_t u\, dx.$$

Therefore,

$$\begin{aligned}\dot{\rho}(t_1) - \dot{\rho}(t_2) &= 2\int_{t_1}^{t_2}\Big\{||u_t||^2_{L^2_g} - \delta(u_t,\ u)_{L^2_g} - ||u||^2_{\mathcal{D}^{1,2}} \\ &\quad + \lambda\int_{\mathbb{R}^N} g(x)|u|^{\beta+1}\, dx\Big\}\, ds.\end{aligned} \tag{4.1}$$

Since u is a weak solution of the problem (1.1)–(1.2) the integrant in (4.1) is bounded. Hence $\dot{\rho}(t)$ is Lipschitz continuous. Furthermore, we get

$$\ddot{\rho}(t) = 2\left\{||u_t||^2_{L^2_g} - \delta(u_t,\ u)_{L^2_g} - ||u||^2_{\mathcal{D}^{1,2}} + \lambda\int_{\mathbb{R}^N} g(x)|u|^{\beta+1}\, dx\right\} \tag{4.2}$$

almost everywhere in $[0,\ T]$, and the proof is completed. □

The main result is contained in the following theorem.

Theorem 4.2. *Let condition ($\mathcal{G}_1$) hold. Moreover, we assume that* $0 < \delta$, $\lambda < \infty$ *and*

$$\mathcal{E}^*(0) < 0. \tag{4.3}$$

Then (weak) solutions of (1.1)–(1.2) *blow up in finite time.*

Proof. We shall use here the energy $\mathcal{E}(t) = -\mathcal{E}^*(t)$. Condition (4.3) implies that the initial data are chosen such that $0 < \mathcal{E}(0)$. From equation (3.12) and definition (3.13) we see that $\mathcal{E}(t)$ is a nondecreasing function of t. Using (3.12) and (4.3) we obtain the inequality

$$0 < \mathcal{E}(0) \leq \mathcal{E}(t) \leq \frac{\lambda}{\beta+1}||u(t)||^{\beta+1}_{L^{\beta+1}_g}. \tag{4.4}$$

As in [17] (see also [7]) we shall use the functional

$$\mathcal{F}(t) =: \mu\mathcal{E}(t)^{1-\alpha} + \dot{\rho}(t) \tag{4.5}$$

where $\mu,\ a$ are positive constants to be fixed later. Using relations (3.13), (4.2) and (4.5) we observe that

$$\begin{aligned}\dot{\mathcal{F}}(t) &= \mu(1-\alpha)\mathcal{E}(t)^{-\alpha}\dot{\mathcal{E}}(t) + 4||u_t(t)||^2_{L^2_g} + 4\mathcal{E}(t) \\ &+ \frac{2\lambda(\beta-1)}{\beta+1}\int_{\mathbb{R}^N} g(x)|u|^{\beta+1}\, dx - 2\delta\int_{\mathbb{R}^N} g(x)u(t)u_t(t)\, dx.\end{aligned} \tag{4.6}$$

Applying Lemma 2.6 for $u,\ v=u_t,\ w=\varepsilon$ for some $\varepsilon>0$ and $a=\beta+1,\ b=2,\ c=2(\beta+1)/(\beta-1),\ s=0,\ K=\mathcal{E}(t)$ and using inequality (4.4), we obtain for the last term of relation (4.6) the following estimate:

$$\begin{aligned}
&\delta\Big|\int_{\mathbb{R}^N} g(x)u(t)u_t(t)\,dx\Big| \le \varepsilon\delta\mathcal{E}(t)^{\frac{1-\beta}{2(\beta+1)}}||g||_1^{\frac{\beta-1}{2(\beta+1)}}\times\\
&\times\big\{||u(t)||_{L_g^{\beta+1}}^{\beta+1}+||u_t(t)||_{L_g^2}^2+\mathcal{E}(t)\big\}\\
&\le \varepsilon\delta\mathcal{E}(0)^{\frac{1-\beta}{2(\beta+1)}}||g||_1^{\frac{\beta-1}{2(\beta+1)}}\times\big\{||u(t)||_{L_g^{\beta+1}}^{\beta+1}+||u_t(t)||_{L_g^2}^2+\mathcal{E}(t)\big\}. \qquad (4.7)
\end{aligned}$$

By relation (4.4) the quantity $\mu(1-\alpha)\mathcal{E}^{-\alpha}(t)\dot{\mathcal{E}}(t)$ is nonnegative. Then relations (4.6) and (4.7) imply that

$$\begin{aligned}
\dot{\mathcal{F}}(t) &\ge 2\Big\{\frac{\lambda(\beta-1)}{\beta+1}-\varepsilon\delta\mathcal{E}(0)^{\frac{1-\beta}{2(\beta+1)}}||g||_1^{\frac{\beta-1}{2(\beta+1)}}\Big\}||u(t)||_{L_g^{\beta+1}}^{\beta+1} \qquad (4.8)\\
&+2\Big\{2-\varepsilon\delta\mathcal{E}(0)^{\frac{1-\beta}{2(\beta+1)}}||g||_1^{\frac{\beta-1}{2(\beta+1)}}\Big\}||u_t(t)||_{L_g^2}^2+2\Big\{2-\varepsilon\delta\mathcal{E}(0)^{\frac{1-\beta}{2(\beta+1)}}||g||_1^{\frac{\beta-1}{2(\beta+1)}}\Big\}\mathcal{E}(t).
\end{aligned}$$

We require

$$\begin{aligned}
K_1 &= \frac{\lambda(\beta-1)}{\beta+1}-\varepsilon\delta\mathcal{E}(0)^{\frac{1-\beta}{2(\beta+1)}}||g||_1^{\frac{\beta-1}{2(\beta+1)}}>0 \ \text{ and}\\
K_2 &= 2-\varepsilon\delta\mathcal{E}(0)^{\frac{1-\beta}{2(\beta+1)}}||g||_1^{\frac{\beta-1}{2(\beta+1)}}>0.
\end{aligned}$$

For the above requirements to be satisfied, we choose

$$\varepsilon<\min\Big\{\frac{\mathcal{E}(0)^{\frac{\beta-1}{2(\beta+1)}}\lambda(\beta-1)}{\delta(\beta+1)||g||_1^{\frac{\beta-1}{2(\beta+1)}}},\ \frac{2\mathcal{E}(0)^{\frac{\beta-1}{2(\beta+1)}}}{\delta||g||_1^{\frac{\beta-1}{2(\beta+1)}}}\Big\}.$$

Then we get the inequality

$$\dot{\mathcal{F}}(t)\ge K_3\Big\{||u_t(t)||_{L_g^2}^2+\mathcal{E}(t)+||u||_{L_g^{\beta+1}}^{\beta+1}\Big\}>0,\quad K_3=\min\{K_1,\ K_2\}. \qquad (4.9)$$

Moreover, by choosing μ sufficiently large (i.e., $\mu>-\dot{\rho}(0)\mathcal{E}(0)^{\alpha-1}$ if $\dot{\rho}(0)<0$) we obtain from (4.4) and (4.5) that $\mathcal{F}(t)>\mathcal{F}(0)>0$; i.e., $\mathcal{F}$ is a strictly increasing function of t. Finally, we shall show, for some constants $C>0$ and $\gamma>1$, that

$$\dot{\mathcal{F}}(t)\ge C\mathcal{F}^{\gamma}(t). \qquad (4.10)$$

Set $\gamma = 1/(1-\alpha)$. We again apply Lemma 2.6, but for $s = \alpha$ (instead of $s = 0$ as above) to get the inequality

$$|\dot{\rho}(t)| \leq 2\mathcal{E}(t)^{\alpha - \frac{\beta-1}{2(\beta+1)}} ||g||_1^{\frac{\beta-1}{2(\beta+1)}} \left\{ ||u(t)||^{\beta+1}_{L_g^{\beta+1}} + ||u_t(t)||^2_{L_g^2} + \mathcal{E}(t) \right\}^{\frac{1}{\gamma}}. \quad (4.11)$$

Clearly for every $\alpha \in (0,\ (\beta-1)/2(\beta+1))$, we obtain from (4.4) and (4.11) that

$$|\dot{\rho}(t)|^{\gamma} \leq K_4 \left\{ ||u_t(t)||^2_{L_g^2} + \mathcal{E}(t) + ||u||^{\beta+1}_{L_g^{\beta+1}} \right\}, \quad (4.12)$$

where $K_4 = \{2\mathcal{E}(0)^{\alpha - \frac{\beta-1}{2(\beta+1)}} ||g||_1^{\frac{\beta-1}{2(\beta+1)}}\}^{\gamma}$. Then from (4.12), we get

$$\begin{aligned} \mathcal{F}^{\gamma}(t) &\leq 2^{\gamma-1}\left(\mu^{\gamma-1}\mathcal{E}(t) + |\dot{\rho}(t)|^{\gamma}\right) \leq K_5\{||u_t(t)||^2_{L_g^2} + \mathcal{E}(t) + ||u||^{\beta+1}_{L_g^{\beta+1}}\} \\ &\leq \frac{K_5}{K_3}\dot{\mathcal{F}}(t), \quad K_5 = \max\{(2\mu)^{\gamma-1},\ (2\mu)^{\gamma-1}K_4\}, \quad (4.13) \end{aligned}$$

which implies (4.10). Then the result is obtained by applying the classical blow-up argument of [3, Theorem 4.2]. □

5. The equation without damping. In this section we treat the following problem without dissipation ($\delta = 0$):

$$\begin{aligned} u_{tt}(x,t) - \phi(x)\Delta u(x,t) &= \lambda u(x,t)|u(x,t)|^{\beta-1},\ x \in \mathbb{R}^N,\ t \geq 0 \quad (5.1) \\ u(x,0) = u_0(x), &\qquad u_t(x,0) = u_1(x),\ x \in \mathbb{R}^N. \quad (5.2) \end{aligned}$$

With the properties of the space setting described above, we show that an analogue of [3, Theorem 4.1] holds. The energy for the problem is given by (3.13), and we consider again the mapping $\rho(t) = ||u(t)||^2_{L_g^2}$.

Theorem 5.1. *Let g, β, and N satisfy conditions ($\mathcal{G}_1$) or ($\mathcal{G}_2$). Suppose that $0 < \lambda < \infty$ and that initial conditions*

$$u_0(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N) \ \ and \ \ u_1(x) \in L_g^2(\mathbb{R}^N) \quad (5.3)$$

are given. Then for sufficiently small $T > 0$ the problem (5.1)–(5.2) admits a unique (weak) solution such that $u \in C[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)]$ and $u_t \in C[0,T;L_g^2(\mathbb{R}^N)]$. Furthermore, assume that the initial data satisfy $\mathcal{E}^*(0) < 0$, $\dot{\rho}(0) > 0$ and condition ($\mathcal{G}_1$) holds. Then the solution of (5.1)–(5.2) blows up in finite time.

Proof. The existence result can be obtained as in Proposition 3.1. For the rest, it is not hard to see that if u is a solution of (5.1)–(5.2), then

$$\mathcal{E}^*(t) = \mathcal{E}^*(0). \tag{5.4}$$

Using the approximation argument of the previous section we have

$$\ddot{\rho}(t) = 2\{||u_t||^2_{L^2_g} - ||u||^2_{\mathcal{D}^{1,2}} + \lambda \int_{\mathbb{R}^N} g(x)|u|^{\beta+1}\, dx\}. \tag{5.5}$$

From Lemma 2.5 we obtain

$$\int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1} dx \geq ||g||_1^{\frac{1-\beta}{2}} \rho(t)^{\frac{\beta+1}{2}}. \tag{5.6}$$

Therefore, from (3.13), (5.4) and (5.6) we get the inequality

$$\ddot{\rho}(t) \geq \frac{2\lambda(\beta-1)}{\beta+1} \int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\, dx - 4\mathcal{E}^*(0) \geq c_1 \rho(t)^{\frac{\beta+1}{2}} - 4\mathcal{E}^*(0), \tag{5.7}$$

where $c_1 = \frac{\beta+1}{2\lambda(\beta-1)}||g||_1^{\frac{1-\beta}{2}}$. Since $\mathcal{E}^*(0) < 0$, relation (5.7) implies that $\dot{\rho}(t),\ \rho(t)$ are nondecreasing (positive) functions of t. We multiply (5.7) by $\dot{\rho}$ and integrate to get the inequality $\frac{1}{2}\dot{\rho}^2(t) \geq \frac{2C_1}{\beta+3}\rho^{\frac{\beta+3}{2}}(t) + C$, which is equivalent to $\int_{\rho(0)}^{\rho(t)} \frac{1}{\sqrt{\sigma(t)}} \geq t$, for all $t \in [0,\ T]$, where $\sigma(t) = C + \frac{2C_1}{\beta+3}\rho^{\frac{\beta+3}{2}}(t)$. Then following the argument in [3, Theorem 4.2] we get the result. $\square$

Theorem 5.2. *Assume that condition $(\mathcal{G}_3)$ is satisfied. Moreover, assume that $u_0 \in W$ and that the initial data satisfy conditions* (5.3) *and are sufficiently small in the sense that*

$$\mathcal{E}^*(0) < \{\frac{1}{\lambda}\}^{\frac{4}{4\beta+N(\beta-1)}} \{\frac{\beta-1}{2(\beta+1)||g||_{N/2}\kappa^2}\}^{2\frac{N+2-\beta(N-2)}{4\beta+N(\beta-1)}}. \tag{5.8}$$

Then the unique (*weak*) *solution of* (5.1)–(5.2) *is such that*

$$u \in C([0,\infty);\ \mathcal{D}^{1,2}(\mathbb{R}^N)) \ \ \textit{and} \ \ u_t \in C([0,\infty);\ L^2_g(\mathbb{R}^N)).$$

Proof. We argue as in Theorem 3.2. For $0 \leq t < T^*$, we write (5.4) as

$$\frac{1}{2}||u_t(t)||^2_{L^2_g} + \mathcal{J}(u(t)) = \mathcal{E}^*(0). \tag{5.9}$$

From (2.2) and (3.11) we obtain the estimate

$$||u(t)||^2_{L^2_g} \leq \frac{2(\beta+1)}{\alpha(\beta-1)}\mathcal{E}^*(0) =: \hat{\mu_0}^2. \tag{5.10}$$

We replace μ_0 with $\hat{\mu_0}$ in (3.20) and get the inequality

$$||u(t)||^{\beta+1}_{L^{\beta+1}_g} \leq \hat{\delta_1}||u(t)||^2_{\mathcal{D}^{1,2}},$$

where $\hat{\delta_1} = \left\{\frac{2(\beta+1)||g||_{N/2}\kappa^2}{\beta-1}\right\}^{\frac{N+2-\beta(N-2)}{2}}\mathcal{E}^*(0)^{\frac{4\beta+N(\beta-1)}{4}}$. Therefore under the assumption $\lambda < \left\{\frac{\beta-1}{2(\beta+1)||g||_{N/2}\kappa^2}\right\}^{\frac{N+2-\beta(N-2)}{2}}\left\{\frac{1}{\mathcal{E}^*(0)}\right\}^{\frac{4\beta+N(\beta-1)}{4}}$ (equivalent to (5.8)), we again get a contradiction. □

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