JOURNAL OF DIFFERENTIAL EQUATIONS **157**, 183–205 (1999)
Article ID jdeq.1999.3618, available online at http://www.idealibrary.com on IDEAL®

Existence of a Global Attractor for Semilinear Dissipative Wave Equations on $\mathbb{R}^N$

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Received July 30, 1998

We consider the semilinear hyperbolic problem $u_{tt}+\delta u_t-\phi(x)\,\Delta u+\lambda f(u)=\eta(x)$, $x\in\mathbb{R}^N$, $t>0$, with the initial conditions $u(x,0)=u_0(x)$ and $u_t(x,0)=u_1(x)$ in the case where $N\geqslant 3$ and $(\phi(x))^{-1}:=g(x)$ lies in $L^{N/2}(\mathbb{R}^N)$. The energy space $\mathcal{X}_0=\mathcal{D}^{1,2}(\mathbb{R}^N)\times L^2_g(\mathbb{R}^N)$ is introduced, to overcome the difficulties related with the non-compactness of operators which arise in unbounded domains. We derive various estimates to show local existence of solutions and existence of a global attractor in $\mathcal{X}_0$. The compactness of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N)\subset L^2_g(\mathbb{R}^N)$ is widely applied. © 1999 Academic Press

Key Words: dynamical systems; attractors; hyperbolic equations; nonlinear problems; semigroups; unbounded domains; generalised Sobolev spaces.

1. INTRODUCTION

Our aim is to study the following semilinear hyperbolic initial value problem

$$u_{tt}+\delta u_t-\phi(x)\,\Delta u+\lambda f(u)=\eta(x), \qquad x\in\mathbb{R}^N, \quad t>0, \tag{1.1}$$

$$u(x,0)=u_0(x) \qquad \text{and} \qquad u_t(x,0)=u_1(x), \qquad x\in\mathbb{R}^N, \tag{1.2}$$

with the initial conditions $u_0(x)$, $u_1(x)$ in appropriate function spaces. Models of this type arise mainly in wave phenomena of various areas in mathematical physics (see [2, 29, 36]) as well as in geophysics and ocean acoustics, where, for example, the coeficient $\phi(x)$ represents the speed of sound at the point $x\in\mathbb{R}^N$ (see [19]). Throughout the paper we assume that the functions ϕ, g, η: $\mathbb{R}^N\to\mathbb{R}$ and f: $\mathbb{R}\to\mathbb{R}^+$ satisfy the following conditions:

0022-0396/99 $30.00

($\mathscr{G}$) $\phi(x) > 0$, for all $x \in \mathbb{R}^N$, $(\phi(x))^{-1} := g(x)$ is $\mathscr{C}^{0,\gamma}(\mathbb{R}^N)$-smooth, for some $\gamma \in (0, 1)$ and $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (for functions ϕ of this type, e.g., polynomial like, we refer to [36, p. 632]),

($\mathscr{H}$) $\eta \in L^2_g(\mathbb{R}^N)$,

($\mathscr{F}$) $f: \mathbb{R} \to \mathbb{R}^+$ is a smooth function such that $f(0) = 0$. Furthermore, $|f(s)| \leqslant c^* |s|$ and $|f'(s)| \leqslant c_2 |s|$, where c^*, c_2 are positive constants.

In some cases we shall use an extra condition on f, which is

($\mathscr{F}_\infty$) f' is in $L^\infty(\mathbb{R})$.

In the bounded domain case the problem is studied by many researchers, for an extensive literature we refer to the monographs of A. V. Babin and M. I. Vishik [3], J. K. Hale [17], O. A. Ladyzenskaha [23] and R. Temam [35]. For the unbounded domain case there is a recent rapidly growing interest. Among others we refer to the works of Ph. Brenner [7] on strong global solutions of some nonlinear hyperbolic equations, E. Feiresl [12, 13, 14] on asymptotic behaviour and compact attractors for semilinear damped wave equations on $\mathbb{R}^N$, A. I. Komech, and B. R. Vainberg [21] on asymptotic stability of stationary solutions to nonlinear wave and Klein–Gordon Equations, and T. Motai [27] on energy decay problems for wave equations with nonlinear dissipative term in $\mathbb{R}^N$. J. Shatal and M. Struwe in [31, 32, 34] discussed questions of existence regularity and well-posedness for semilinear wave equations with no damping. Recently H. A. Levine, S. R. Park, P. Pucci, and J. Serrin in [24, 25, 28] studied global existence and nonexistence of solutions for both the bounded and unbounded domain case. For existence results concerning the steady state problem our work is based on the papers [9, 10] and the references therein.

The paper is organised as follows. In Section 2 we discuss the space setting of the problem and the necessary embeddings for constructing the evolution triple. In Section 3 by means of the standard Faedo–Galerkin approximation we prove existence and uniqueness of solutions for the initial value problem. In Section 4 we prove the existence of a global attractorfor the dynamical system defined from the semigroup generated by the problem.

Notation. We *denote* by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for *simplicity reasons* we use the symbols L^p, $1 \leqslant p \leqslant \infty$, $\mathscr{D}^{1,2}$, respectively, for the spaces $L^p(\mathbb{R}^N)$, $\mathscr{D}^{1,2}(\mathbb{R}^N)$, respectively; $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$. By $\mathscr{L}(V, W)$ we denote the space of linear operators from V to W. Also sometimes differentiation with respect to time is denoted by a dot over the function. The constants C or c are considered in a generic sense. The end of the proofs is marked by ∎.

2. SPACE SETTING: FORMULATION OF THE PROBLEM

As we will see the space setting for the initial conditions and the solutions of our problem is the product space $\mathcal{X}_0 = \mathcal{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$. By $\mathcal{D}^{1,2}(\mathbb{R}^N)$ we define the closure of the $C_0^\infty(\mathbb{R}^N)$ functions with respect to the "energy norm" $\|u\|_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$. It is well known (see [22, Proposition 2.4]) that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2N/(N-2)}(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\}$$

and that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ can be embedded continuously in $L^{2N/(N-2)}(\mathbb{R}^N)$, i.e., there exists $k > 0$ such that

$$\|u\|_{2N/(N-2)} \leqslant k \, \|u\|_{\mathcal{D}^{1,2}}. \tag{2.1}$$

The following generalised version of Poincaré's inequality is essential.

LEMMA 2.1. *Suppose that* $g \in L^{N/2}(\mathbb{R}^N)$.*Then there exists* $\alpha > 0$ *such that*

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geqslant \alpha \int_{\mathbb{R}^N} g u^2 \, dx, \tag{2.2}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Proof. The proof is based on the fact that $g \in L^{N/2}(\mathbb{R}^N)$ (see [9, Lemma 2.1]). It is found that $\alpha = k^{-2} \, \|g\|^{-1}_{N/2}$. ∎

It can be shown (see [9, Lemma 2.2]) that $\mathcal{D}^{1,2}$ is a separable Hilbert space. Next we introduce the weighted Lebesque space $L^2_g(\mathbb{R}^N)$ to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product

$$(u, v)_{L^2_g} =: \int_{\mathbb{R}^N} g u v \, dx.$$

Clearly, $L^2_g(\mathbb{R}^N)$ is a separable Hilbert space. The following lemma is crucial for the analysis of the problem. The complete proof can be found in the work [6].

LEMMA 2.2. *Suppose that* $g \in L^{N/2} \cap L^\infty$. *Then* $\mathcal{D}^{1,2}$ *is compactly embedded in* L^2_g.

Sketch of the Proof. Let $\{u_n\}$ be a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then there exists a constant $k^* > 0$ such that for all positive integers m, n and any $R > 0$ we have

$$\int_{\mathbb{R}^N} g(u_n^2 - u_m^2) \, dx$$
$$\leqslant k^* \{ \|g(u_n - u_m)\|_{L^{2N/(N+2)}(\mathbb{R}^N \setminus B_R)} + \|g(u_n - u_m)\|_{L^{2N/(N+2)}(B_R)} \}.$$

Let $\varepsilon > 0$ be chosen arbitrarily. Since $\{u_n\}$ is a bounded sequence in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $g \in L^{N/2}(\mathbb{R}^N)$, we may choose R_0 sufficiently large, so that by a diagonalization procedure we have

$$\begin{aligned}\int_{\mathbb{R}^N} g(u_n^2 - u_m^2)\,dx \\ &\leqslant k^*\{\|g(u_n - u_m)\|_{L^{2N/(N+2)}(\mathbb{R}^N \setminus B_{R_0})} + \|g(u_n - u_m)\|_{L^{2N/(N+2)}(B_{R_0})}\} \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for m and n sufficiently large. Therefore $\{u_n\}$ is a Cauchy sequence in $L_g^2(\mathbb{R}^N)$. ∎

So we are able to construct the necessary *evolution triple* for the space setting of our problem, which is

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L_g^2(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.3}$$

where all the embeddings are compact and dense. Next we consider the equation

$$-\phi(x)\,\Delta u(x) = \eta(x), \qquad x \in \mathbb{R}^N, \tag{2.4}$$

without boundary condition. It is easy to see that for every u, v in $C_0^\infty(\mathbb{R}^N)$

$$(-\phi\Delta u, v)_{L_g^2} = \int_{\mathbb{R}^N} \nabla u\,\nabla v\,dx. \tag{2.5}$$

By the definition of the space $L_g^2(\mathbb{R}^N)$ and (2.5) it is natural to consider Eq. (2.4) as an operator equation

$$A_0 u = \eta, \qquad A_0 \colon D(A_0) \subseteq L_g^2(\mathbb{R}^N) \to L_g^2(\mathbb{R}^N), \tag{2.6}$$

where $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^N)$ and $\eta \in L_g^2(\mathbb{R}^N)$. Relation (2.5) implies that the operator A_0 is symmetric. Let us note that the operator A_0 is not symmetric in the standard Lebesque space $L^2(\mathbb{R}^N)$. For comments of the same nature on a similar model in the case of a bounded weight we refer to [29, pp. 185–187]. From Lemma 2.2 and Eq. (2.5) we have that

$$(A_0 u, u)_{L_g^2} \geqslant \alpha\,\|u\|_{L_g^2}^2, \qquad \text{for all} \quad u \in D(A_0), \tag{2.7}$$

where $\alpha > 0$ is fixed given in Lemma 2.1, i.e., the operator A_0 is strongly monotone. Therefore the assumptions for the Friedrichs' extension theorem (see [37, Theorem 19.C]) are satisfied. By the evolution triple constructed in (2.3) we may define the energetic scalar product given by (2.5)

$$(u, v)_E = \int_{\mathbb{R}^N} \nabla u \, \nabla v \, dx$$

and the energetic space X_E is the completion of $D(A_0)$ with respect to $(u, v)_E$, i.e., the energetic space coincides with the homogeneous Sobolev space $\mathscr{D}^{1,2}(\mathbb{R}^N)$. The *energetic extension* $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta : \mathscr{D}^{1,2}(\mathbb{R}^N) \to \mathscr{D}^{-1,2}(\mathbb{R}^N),$$

is defined to be the duality mapping of $\mathscr{D}^{1,2}(\mathbb{R}^N)$ and for every $\eta \in \mathscr{D}^{-1,2}(\mathbb{R}^N)$ Eq. (2.4) has a unique solution. All the solutions u of the equation

$$A_E u = \eta, \qquad \eta \in L^2_g(\mathbb{R}^N),$$

form the set $D(A)$. The *Friedrichs' extension* A of A_0 is defined as the restriction of the energetic extension A_E to the set $D(A)$. The operator A is self-adjoint and therefore graph-closed. This implies that the set $D(A)$ is a Hilbert space with respect to the graph scalar product

$$(u, v)_{D(A)} = (u, v)_{L^2_g} + (Au, Av)_{L^2_g}, \qquad \text{for all} \quad u, v \in D(A).$$

The norm induced by the scalar product $(u, v)_{D(A)}$ is

$$\|u\|_{D(A)} = \left\{ \int_{\mathbb{R}^N} g \, |u|^2 \, dx + \int_{\mathbb{R}^N} \phi \, |\Delta u|^2 \, dx \right\}^{1/2},$$

which is equivalent to the norm

$$\|Au\|_{L^2_g} = \left\{ \int_{\mathbb{R}^N} \phi |\Delta u|^2 \, dx \right\}^{1/2}.$$

The weak formulation for the Eq. (2.4) is

$$\int_{\mathbb{R}^N} \nabla u \, \nabla v \, dx = \int_{\mathbb{R}^N} g\eta v \, dx, \qquad \text{for fixed} \quad v \in D^{1,2} \quad \text{and all} \quad u \in C_0^\infty.$$

It follows from the compactness of the embeddings in (2.3) that for the eigenvalue problem

$$-\phi(x) \, \Delta u = \mu u, \qquad x \in \mathbb{R}^N, \tag{2.8}$$

there exists a complete system of eigensolutions $\{w_n, \mu_n\}$ satisfying the following relations

$$\begin{cases} -\phi\,\Delta w_j = \mu_j w_j, & j=1,2,..., \quad w_j \in D^{1,2}(\mathbb{R}^N), \\ 0<\mu_1 \leqslant \mu_2 \leqslant ..., & \mu_j \to \infty, \quad \text{as} \quad j \to \infty. \end{cases} \tag{2.9}$$

Additional information concerning the asymptotic behaviour of the eigenfunctions of problem (2.8) can be obtained. In fact (see [9, Theorem 3.2]) every solution u of (2.8) is such that

$$u(x) \to 0, \qquad \text{as} \quad |x| \to \infty. \tag{2.10}$$

For the positive selfadjoint operator $A = -\phi\Delta$ we can define the fractional powers as follows. For every $s>0$, A^s is an unbounded selfadjoint operator in L^2_g, with domain $D(A^s)$ to be a dense subset in L^2_g. The operator A^s is strictly positive and injective. Also $D(A^s)$ endowed with the scalar product $(u, v)_{D(A^s)} = (A^s u, A^s v)_{L^2_g}$ becomes a Hilbert space. We write as usual, $V_{2s} = D(A^s)$ and we have the following identifications $D(A^{-1/2}) = \mathscr{D}^{-1,2}$, $D(A^0) = L^2_g$, and $D(A^{1/2}) = \mathscr{D}^{1,2}$. Moreover, the mapping

$$A^{s/2}\colon V_x \mapsto V_{x-s} \tag{2.11}$$

is an isomorphism. Furthermore, as a consequence of the relation (2.3) the injection $D(A^{s_1}) \subset D(A^{s_2})$ is compact and dense, for every $s_1, s_2 \in \mathbb{R}$, $s_1 > s_2$. For more, see Henry [18, pp. 24–30].

In the space setting described above, we give the following definition of *weak solution* for the problem (1.1)–(1.2).

DEFINITION 2.3. A *weak solution* of (1.1)–(1.2) is a function $u(x,t)$ such that

(i) $u \in L^2[0, T; \mathscr{D}^{1,2}(\mathbb{R}^N)]$, $u_t \in L^2[0, T; L^2_g(\mathbb{R}^N)]$, $u_{tt} \in L^2[0, T; \mathscr{D}^{-1,2}(\mathbb{R}^N)]$,

(ii) for all $v \in C^\infty_0([0, T] \times \mathbb{R}^N)$, satisfies the generalized formula

$$\begin{aligned} &\int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g}\, d\tau + \delta \int_0^T (u_t(\tau), v(\tau))_{L^2_g}\, d\tau \\ &\qquad + \int_0^T \int_{\mathbb{R}^N} \nabla u(\tau)\, \nabla v(\tau)\, dx\, d\tau + \lambda \int_0^T (f(u(\tau)), v(\tau))_{L^2_g}\, d\tau \\ &= \int_0^T (\eta, v)_{L^2_g(\mathbb{R}^N)}\, d\tau, \end{aligned} \tag{2.12}$$

(iii) satisfies the initial conditions

$$u(x,0)=u_0(x)\in\mathscr{D}^{1,2}(\mathbb{R}^N),\qquad u_t(x,0)=u_1(x)\in L_g^2(\mathbb{R}^N).$$

Remark 2.4. We may see by using a density argument, that the generalized formula (2.12) is satisfied for every $v\in L^2[0,T;\mathscr{D}^{1,2}(\mathbb{R}^N)]$. By the compactness and density of the embeddings in the evolution triple (2.3) we have that, for all $p\in(1,\infty)$, the embedding

$$\{u\in L^p(0,T;\mathscr{D}^{1,2}(\mathbb{R}^N)),\ u_t\in L^{p'}(0,T;\mathscr{D}^{-1,2}(\mathbb{R}^N))\}\subset C(0,T;L_g^2(\mathbb{R}^N))$$

is continuous (see, for example, [26, Lemma 2.45]). Therefore the above Definition 2.3 of the weak solution implies that

$$u\in C[0,T;L_g^2(\mathbb{R}^N)]\qquad\text{and}\qquad u_t\in C[0,T;\mathscr{D}^{-1,2}(\mathbb{R}^N)].$$

3. EXISTENCE AND UNIQUENESS OF SOLUTION

In this section we give existence and uniqueness results for the problem (1.1)–(1.2) in the space setting established in the previous section.

LEMMA 3.1. *Let f, g and η satisfy conditions $(\mathscr{F})$, $(\mathscr{G})$ and $(\mathscr{H})$ respectively. Suppose that the constants $T>0$, $R>0$, $\delta>0$ and the initial conditions*

$$u_0\in\mathscr{D}^{1,2}(B_R)\qquad\textit{and}\qquad u_1\in L_g^2(B_R),\tag{3.1}$$

are given. Then for the problem (1.1), (1.2), *restricted on $B_R\times(0,T)$ satisfying the boundary condition $u=0$ in $\partial B_R\times(0,T)$, there exists a unique (weak) solution such that*

$$u\in C[0,T;\mathscr{D}^{1,2}(B_R)]\qquad\textit{and}\qquad u_t\in C[0,T;L_g^2(B_R)].$$

Proof. We shall prove existence by means of the classical energy method (Faedo–Galerkin approximation). We consider the basis of $\mathscr{D}^{1,2}(B_R)$ generated by the eigenfunctions of A and we construct an approximating sequence of solutions

$$u^n(t,x)=\sum_{i=1}^{n}b_{in}(t)\,w_i,$$

solving the Galerkin system

$$(u^n_{tt}, w_j)_{L^2_g(B_R)} + \delta(u^n_t, w_j)_{L^2_g(B_R)} + \int_{B_R} \nabla u^n \nabla w_j \, dx + \lambda(f(u^n), w_j)_{L^2_g(B_R)}$$

$$= (\eta, w_j)_{L^2_g(B_R)}, \tag{3.2}$$

$$u^n(x, 0) = \mathscr{P}_n u_0(x), \qquad u^n_t(x, 0) = \mathscr{P}_n u_1(x), \tag{3.3}$$

where $\mathscr{P}_n$ is the continuous orthogonal projector operator of $\mathscr{D}^{1,2}(B_R) \to \text{span}\{w_i: i = 1, 2, ..., n\}$ and of $L^2_g(B_R) \to span\{w_i: i = 1, 2, ..., n\}$. Multiplying (3.2) by $\dot{b}_{in}(t)$ and adding from 1 to n, we obtain

$$\frac{1}{2}\frac{d}{dt} \|u^n_t\|^2_{L^2_g(B_R)} + \delta \|u^n_t\|^2_{L^2_g(B_R)} + \frac{1}{2}\frac{d}{dt} \|u^n\|^2_{\mathscr{D}^{1,2}(B_R)} + \lambda(f(u^n), u^n_t)_{L^2_g(B_R)}$$

$$= (\eta, u^n_t)_{L^2_g(B_R)}. \tag{3.4}$$

Using hypothesis $(\mathscr{F})$, we have the estimate

$$\left| \int_{B_R} g f(u^n) \, u^n_t \, dx \right| \leqslant c^* \int_{B_R} g^{1/2} g^{1/2} \, |u^n| \, |u^n_t| \, dx$$

$$\leqslant c \int_{B_R} g \, |u^n|^2 \, dx + c \int_{B_R} g \, |u^n_t|^2 \, dx$$

$$\leqslant c \int_{B_R} |\nabla u^n|^2 \, dx + c \int_{B_R} g \, |u^n_t|^2 \, dx, \tag{3.5}$$

$$\int_{B_R} g \, |\eta| \, |u^n_t| \, dx \leqslant c \, \|\eta\|^2_{L^2_g(B_R)} + c \, \|u^n_t\|^2_{L^2_g(B_R)}. \tag{3.6}$$

So, by relations (3.4)–(3.6) we get the inequality

$$\frac{d}{dt} (\|u^n_t\|^2_{L^2_g(B_R)} + \|u^n\|^2_{\mathscr{D}^{1,2}(B_R)})$$

$$\leqslant c \, \|\eta\|^2_{L^2_g(\mathbb{R}^N)} + C(\|u^n_t\|^2_{L^2_g(B_R)} + \|u^n\|^2_{\mathscr{D}^{1,2}(B_R)}). \tag{3.7}$$

Applying Gronwall's Lemma to the differential inequality (3.7) we get that

$$\|u^n_t\|^2_{L^2_g(B_R)} + \|u^n\|^2_{\mathscr{D}^{1,2}(B_R)} \leqslant K, \tag{3.8}$$

where K is indepedent of R, n and depends only on the initial conditions and T, λ, C, $\|\eta\|^2_{L^2_g(\mathbb{R}^N)}$. Now for all $v \in C^\infty_0([0, T] \times B_R)$ we have the inequality

$$\left|\int_0^T (u_{tt}^n(\tau), v(\tau))_{L_g^2(B_R)}\, d\tau\right| \leqslant \delta \left|\int_0^T (u_t^n(\tau), v(\tau))_{L_g^2(B_R)}\, d\tau\right| + \left|\int_0^T \int_{B_R} \nabla u^n(\tau)\, \nabla v(\tau)\, dx\, d\tau\right| + \lambda \left|\int_0^T (f(u^n(\tau)), v(\tau))_{L_g^2(B_R)}\, d\tau\right| + \left|\int_0^T (\eta, v(\tau))_{L_g^2(B_R)}\, d\tau\right|. \tag{3.9}$$

Using (3.8) and (3.9) we get the estimate

$$\left|\int_0^T (u_{tt}^n, v(\tau))_{L_g^2(B_R)}\, d\tau\right| \leqslant K_1 \left(\int_0^T \|v\|^2_{L_g^2(B_R)}\, d\tau + \int_0^T \|v\|^2_{D^{1,2}(B_R)}\, d\tau\right). \tag{3.10}$$

From estimates (3.8) and (3.10), we may extract a subsequence, still denoted by u^n, such that as $n \to \infty$, we get

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u, && \text{in } L^\infty[0, T; \mathscr{D}^{1,2}(B_R)], \\ u_t^n &\overset{*}{\rightharpoonup} z, && \text{in } L^\infty[0, T; L_g^2(B_R)], \\ u_{tt}^n &\rightharpoonup \omega, && \text{in } L^2[0, T; \mathscr{D}^{-1,2}(B_R)]. \end{aligned}$$

Note that, for all $v \in C_0^\infty([0, T] \times B_R)$, integration by parts implies

$$\begin{aligned} \int_0^T (u_t^n(\tau), v(\tau))_{L_g^2(B_R)}\, d\tau &= -\int_0^T (u^n(\tau), v_t(\tau))_{L_g^2(B_R)}\, d\tau, \\ \int_0^T (u_{tt}^n(\tau), v(\tau))_{L_g^2(B_R)}\, d\tau &= \int_0^T (u^n(\tau), v_{tt}(\tau))_{L_g^2(B_R)}\, d\tau. \end{aligned} \tag{3.11}$$

Then, as $n \to \infty$, we get

$$\begin{aligned} \int_0^T (z(\tau), v(\tau))_{L_g^2(B_R)}\, d\tau &= -\int_0^T (u(\tau), v_t(\tau))_{L_g^2(B_R)}\, d\tau, \\ \int_0^T (\omega(\tau), v(\tau))_{L_g^2(B_R)}\, d\tau &= \int_0^T (u(\tau), v_{tt}(\tau))_{L_g^2(B_R)}\, d\tau, \end{aligned} \tag{3.12}$$

which implies that $u_t = z$ and $u_{tt} = \omega$. By the compactness of the embeddings in the evolution triple (2.3) and the results in [33, Lemma 4(ii)] we have that

$$u^n \to u \qquad \text{in } L^2[0, T; L^2_g(B_R)].$$

Also the continuity of f implies that

$$f(u^n) \rightharpoonup f(u) \qquad \text{in } L^2[0, T; L^2_g(B_R)].$$

Summarizing all the above estimates, for all $v \in C_0^\infty([0, T] \times B_R)$, as $n \to \infty$, we have

$$\int_0^T (u^n_{tt}(\tau), v(\tau))_{L^2_g(B_R)}\, d\tau \to \int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g(B_R)}\, d\tau,$$

$$\delta \int_0^T (u^n_t(\tau), v(\tau))_{L^2_g(B_R)}\, d\tau \to \delta \int_0^T (u_t(\tau), v(\tau))_{L^2_g(B_R)}\, d\tau,$$

$$\int_0^T \int_{B_R} \nabla u^n(\tau)\, \nabla v(\tau)\, dx\, d\tau \to \int_0^T \int_{B_R} \nabla u(\tau)\, \nabla v(\tau)\, dx\, d\tau,$$

$$\lambda \int_0^T (f(u^n(\tau)), v(\tau))_{L^2_g(B_R)}\, d\tau \to \lambda \int_0^T (f(u(\tau)), v(\tau))_{L^2_g(B_R)}\, d\tau.$$

Therefore u is the weak solution of the problem (1.1)–(1.2) restricted to the ball B_R according to the Definition 2.3. The continuity and uniqueness properties stated in this lemma can be proved as in the following proposition. ∎

PROPOSITION 3.2. *Let f, g and η satisfy conditions $(\mathscr{F})$, $(\mathscr{G})$ and $(\mathscr{H})$, respectively. Suppose that the constants $T > 0$, $\delta > 0$ and the initial conditions*

$$u_0 \in C_0^\infty(\mathbb{R}^N) \qquad \textit{and} \qquad u_1 \in C_0^\infty(\mathbb{R}^N) \tag{3.11}$$

are given. Then for the problem (1.1)–(1.2) *there exists a (weak) solution such that*

$$u \in C[0, T; \mathscr{D}^{1,2}(\mathbb{R}^N)] \qquad \textit{and} \qquad u_t \in C[0, T; L^2_g(\mathbb{R}^N)].$$

Furthermore, the (weak) solution is unique if (i) $N = 3, 4$ *or* (ii) f' *satisfies* $(\mathscr{F}_\infty)$ *and* $N \geqslant 3$.

Proof. (a) *Existence.* Let $R_0 > 0$ such that $\operatorname{supp}(u_0) \subset B_{R_0}$ and $\operatorname{supp}(u_1) \subset B_{R_0}$. Then, for $R \geqslant R_0$, $R \in \mathbb{N}$, we consider the approximating problem

$$
\begin{aligned}
u^R_{tt}+\delta u^R_t-\phi(x)\Delta u^R+\lambda f(u^R)=\eta(x), \qquad & (x,t)\in B_R\times(0,T)\\
u^R(x,t)=0, \qquad & (x,t)\in\partial B_R\times(0,T)\\
u^R(\cdot,0)=u_0\in C_0^\infty(B_R), \qquad & u^R_t(\cdot,0)=u_1\in C_0^\infty(B_R).
\end{aligned}
\tag{3.12}
$$

By Lemma 3.1, problem (3.12) has a unique (weak) solution u^R such that

$$u^R\in C[0,T;\mathscr{D}^{1,2}(B_R)] \qquad \text{and} \qquad u^R_t\in C[0,T;L^2_g(B_R)].$$

We extend the solution of the problem (3.12) as

$$\tilde{u}^R(x,t)=:\begin{cases} u^R(x,t), & \text{if} \quad |x|\leqslant R,\\ 0, & \text{otherwise.}\end{cases}$$

Since $f(0)=0$, the solution $\tilde{u}^R$ satisfies the estimates

$$
\begin{aligned}
\|\tilde{u}^R\|_{L^\infty[0,T;\mathscr{D}^{1,2}(\mathbb{R}^N)]}\leqslant C, \qquad & \|f(\tilde{u}^R)\|_{L^\infty[0,T;\mathscr{D}^{1,2}(\mathbb{R}^N)]}\leqslant C,\\
\|\tilde{u}^R_t\|_{L^\infty[0,T;\mathscr{D}^{-1,2}(\mathbb{R}^N]}\leqslant C, \qquad & \|\tilde{u}^R_{tt}\|_{L^2[0,T;L^2_g(\mathbb{R}^N)]}\leqslant C,
\end{aligned}
\tag{3.13}
$$

where the constant C is independent of R. Lemma 2.2 and the estimates (3.13) applied to [33, Lemma 4(ii)] imply that

$$\tilde{u}^R \text{ is relatively compact in } C[0,T;L^2_g(\mathbb{R}^N)]. \tag{3.14}$$

Next using relations (3.13) and (3.14), the continuity of the embedding $C[0,T;L^2_g(\mathbb{R}^N)]\subset L^2[0,T;L^2_g(\mathbb{R}^N)]$, and the continuity of f we may extract a subsequence of $\tilde{u}^R$, denoted by $\tilde{u}^{R_m}$, such that as $R_m\to\infty$ we get

$$
\begin{aligned}
\tilde{u}^{R_m}\overset{*}{\rightharpoonup}\tilde{u}, \qquad & \text{in } L^\infty[0,T;\mathscr{D}^{1,2}(\mathbb{R}^N)],\\
\tilde{u}^{R_m}_t\overset{*}{\rightharpoonup} z, \qquad & \text{in } L^\infty[0,T;L^2_g(\mathbb{R}^N)],\\
\tilde{u}^{R_m}_{tt}\rightharpoonup \omega, \qquad & \text{in } L^2[0,T;\mathscr{D}^{-1,2}(\mathbb{R}^N)],\\
f(\tilde{u}^{R_m})\rightharpoonup f(\tilde{u}), \qquad & \text{in } L^2[0,T;L^2_g(\mathbb{R}^N)].
\end{aligned}
\tag{3.15}
$$

For the rest of the proof we proceed as in [4, Theorem 1.3]. For fixed $R=R_m$, let L_m denote the operator of restriction

$$L_m\colon [0,T]\times\mathbb{R}^N\to[0,T]\times B_R.$$

It is clear that the restricted subsequence $L_m\tilde{u}^{R_m}$ satisfies the estimates obtained in Lemma 3.1 (see also (3.13)). Therefore there exists a subsequence $\tilde{u}^{R_{m_j}}\equiv\tilde{u}^j$, for which it can be shown by following the procedure of Lemma 3.1, that $L_m\tilde{u}^j$ converges weakly to a (weak) solution $\tilde{u}_m$. We have that

$$\int_0^T (L_m\tilde{u}^j_{tt}, v)_{L^2_g(B_R)}\, d\tau + \delta \int_0^T (L_m\tilde{u}^j_t, v)_{L^2_g(B_R)} d\tau +$$

$$+ \int_0^T \int_{B_R} \nabla L_m \tilde{u}^{R_{mj}} \nabla v\, dx\, d\tau + \lambda \int_0^T (f(L_m\tilde{u}^j), v)_{L^2_g(B_R)}\, d\tau$$

$$- \int_0^T (\eta, v)_{L^2_g(B_R)}\, d\tau$$

$$= \int_0^T (\tilde{u}^j_{tt}, v)_{L^2_g(\mathbb{R}^N)}\, d\tau + \delta \int_0^T (\tilde{u}^j_t, v)_{L^2_g(\mathbb{R}^N)} d\tau +$$

$$+ \int_0^T \int_{\mathbb{R}^N} \nabla \tilde{u}^j \nabla v\, dx\, d\tau + \lambda \int_0^T (f(\tilde{u}^j), v)_{L^2_g(\mathbb{R}^N)}\, d\tau$$

$$- \int_0^T (\eta, v)_{L^2_g(\mathbb{R}^N)}\, d\tau, \tag{3.16}$$

for every $v \in C_0^\infty([0, T] \times B_R)$. Passing to the limit in (3.16) as $j \to \infty$, we obtain that $L_m\tilde{u} = \tilde{u}_m$. The equality (3.16) holds for any $v \in C_0^\infty([0, T] \times \mathbb{R}^N)$ since the radius R is arbitrarily chosen. Therefore $\tilde{u}$ is the weak solution of the problem (1.1)–(1.2).

(b) *Continuity*. Following Remark 2.4 we get that $u \in C[0, T; L^2_g(\mathbb{R}^N)]$ and $u_t \in C[0, T; \mathscr{D}^{-1,2}(\mathbb{R}^N)]$, so u and u_t are weakly continuous with values in $\mathscr{D}^{-1,2}(\mathbb{R}^N)$ and $L^2_g(\mathbb{R}^N)$ respectively (for example, see [30, Lemma 10.9]). Since the solution u is the limit of the sequence of solutions $\tilde{u}^j$ satisfying inequality (3.7), we integrate (3.7) with respect to time in the interval $(0, t)$ to obtain

$$\|\tilde{u}^j(t)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|\tilde{u}^j_t(t)\|^2_{L^2_g(\mathbb{R}^N)} - \|\tilde{u}^j(0)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} - \|u^j_t(0)\|^2_{L^2_g(\mathbb{R}^N)}$$

$$\leqslant C \int_0^t \{\|\eta\|^2_{L^2_g(\mathbb{R}^N)} + \|\tilde{u}^j_t(s)\|^2_{L^2_g(\mathbb{R}^N)} + \|\tilde{u}^j(s)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)}\}\, ds. \tag{3.17}$$

Consider any fixed $s \in (0, T]$. The quantity

$$\sup_{t \in [0, s]} \{\|u(t)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(t)\|^2_{L^2_g(\mathbb{R}^N)}\}$$

is equivalent to the square of the norm of the space $L^\infty[0, s; \mathscr{D}^{1,2}(\mathbb{R}^N)] \times L^\infty[0, s; L^2_g(\mathbb{R}^N)]$. But balls in this space are weak*-compact, therefore they are weak*-closed. So we conclude from estimate (3.17) that, at the limit $j \to \infty$, we obtain

$$\sup_{t\in[0,s]} \{\|u(t)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(t)\|^2_{L^2_g(\mathbb{R}^N)}\} \leqslant \|u(0)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(0)\|^2_{L^2_g(\mathbb{R}^N)}$$

$$+ \limsup_{j\to\infty} C\int_0^t \{\|\eta\|^2_{L^2_g(\mathbb{R}^N)} + \|u_t^j(s)\|^2_{L^2_g(\mathbb{R}^N)} + \|u^j(s)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)}\}\, ds.$$

Letting $s\to 0$, we have that

$$\limsup_{t\to 0^+} \{\|u(t)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(t)\|^2_{L^2_g(\mathbb{R}^N)}\} \leqslant \|u(0)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(0)\|^2_{L^2_g(\mathbb{R}^N)}.$$

On the other hand, by weak continuity of $u(t)$ and $u_t(t)$ we get

$$\limsup_{t\to 0^+} \{\|u(t)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(t)\|^2_{L^2_g(\mathbb{R}^N)}\} \geqslant \|u(0)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(0)\|^2_{L^2_g(\mathbb{R}^N)}.$$

So $\|u(t)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(t)\|^2_{L^2_g(\mathbb{R}^N)}$ is right continuous and by the solvability of the time-reversed problem we get the left continuity. Moreover, since

$$\begin{aligned}
&\|u(t)-u(s)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(t)-u_t(s)\|^2_{L^2_g(\mathbb{R}^N)}\\
&\quad = \|u(t)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} - 2(u(t), u(s))_{\mathscr{D}^{1,2}(\mathbb{R}^N)}\\
&\qquad + \|u(s)\|^2_{\mathscr{D}^{1,2}(\mathbb{R}^N)} + \|u_t(t)\|^2_{L^2_g(\mathbb{R}^N)} - 2(u_t(t), u_t(s))_{L^2_g(\mathbb{R}^N)} + \|u_t(s)\|^2_{L^2_g(\mathbb{R}^N)},
\end{aligned}$$

where the right-hand side of the equality tends to zero as $t\to s$, we complete the proof of the last part of the theorem.

(c) *Uniqueness.* Assume that u and v are two solutions of (1.1), (1.2) associated to the initial data u_0, u_1 and v_0, v_1, respectively. Let $w=u-v$. Then w is a solution of the equation

$$w_{tt} + dw_t - \phi(x)\,\Delta w + \lambda(f(u)-f(v)) = 0. \tag{3.18}$$

Following the lines of the proof of Lemma 3.1 and Proposition 3.2(a) we get that w satisfies the equality

$$\frac{1}{2}\frac{d}{dt}\|w_t\|^2_{L^2_g} + \delta\,\|w_t\|^2_{L^2_g} + \frac{1}{2}\frac{d}{dt}\|w\|^2_{\mathscr{D}^{1,2}} + \lambda\int_{\mathbb{R}^N} g(x)(f(u)-f(v))\,w_t\,dx = 0. \tag{3.19}$$

(i) For the last integral in Eq. (3.19), we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} g(x)(f(u)-f(v))\,w_t\,dx\right| &\leqslant \int_{\mathbb{R}^N} g^{1/2}\,g^{1/2}\,|f(u)-f(v)|\,|w_t|\,dx\\
&\leqslant C\int_{\mathbb{R}^N} g\,|f(u)-f(v)|^2\,dx\\
&\quad + C\int_{\mathbb{R}^N} g\,|w_t|^2\,dx.
\end{aligned} \tag{3.20}$$

For some $\vartheta \in [0, 1]$ we have that

$$\int_{\mathbb{R}^N} g\, |f(u)-f(v)|^2\, dx \leqslant c_2^2 \int_{\mathbb{R}^N} g\, |\vartheta u+(1-\vartheta)\, v|^2\, |u-v|^2\, dx$$

$$\leqslant C \int_{\mathbb{R}^N} (g^{1/2}|u+v|)^2\, |u-v|^2\, dx$$

$$\leqslant C\, \|g^{1/2}(u+v)\|_N^2\, \|u-v\|_{2N/(N-2)}^2. \tag{3.21}$$

Since $N=3, 4$, by interpolation we have the inequality

$$\|g^{1/2}(u+v)\|_N^2 \leqslant \|g^{1/2}(u+v)\|_2^{2\theta}\, \|g^{1/2}(u+v)\|_{2N/(N-2)}^{2(1-\theta)}$$

$$= \|u+v\|_{L_g^2}^{2\theta}\, \|g^{1/2}(u+v)\|_{2N/(N-2)}^{2(1-\theta)}. \tag{3.22}$$

Moreover, we have that

$$\|g^{1/2}(u+v)\|_{2N/(N-2)}^2 \leqslant \|g\|_\infty\, \|u+v\|_{2N/(N-2)}^2. \tag{3.23}$$

Therefore, by using (3.20)–(3.23) and relations (2.1), (2.2) we obtain that

$$\int_{\mathbb{R}^N} g\, |f(u)-f(v)|^2\, dx \leqslant C\, \|g\|_\infty^{1-\theta}\, \|u+v\|_{\mathscr{D}^{1,2}}^2\, \|u-v\|_{\mathscr{D}^{1,2}}^2. \tag{3.24}$$

Finally, by (3.19) and (3.24) we have the inequality

$$\frac{d}{dt}(\|w_t\|_{L_g^2}^2 + \|w\|_{\mathscr{D}^{1,2}}^2) \leqslant C(\|w_t\|_{L_g^2}^2 + \|w\|_{\mathscr{D}^{1,2}}^2). \tag{3.25}$$

Once more the application of Gronwall's Lemma gives the result.

(ii) If $(\mathscr{F}_\infty)$ is satisfied, instead of the estimate (3.21) we have that

$$\int_{\mathbb{R}^N} g\, |f(u)-f(v)|^2\, dx \leqslant C \int_{\mathbb{R}^N} g\, |u-v|^2\, dx, \tag{3.26}$$

which is valid for any $N \geqslant 3$. From (3.19), (3.20), and (3.26) we again obtain (3.25) and the proof is completed. ∎

We associate with the problem (1.1), (1.2) the mapping $\mathscr{T}(t)\colon C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N) \mapsto \mathscr{X}_0$ by

$$\mathscr{T}(t)\colon \{u_0, u_1\} \mapsto \{u(t), u_t(t)\}.$$

Then Proposition 3.2 has an immediate consequence

THEOREM 3.3. *We may associate to the problem* (1.1), (1.2) *a nonlinear Lipschitz continuous semigroup* $\mathcal{S}(t)\colon \mathcal{X} \mapsto \mathcal{X}_0$, $t \geqslant 0$, *such that for* $\varphi =: \{u_0, u_1\} \in \mathcal{X}_0$, $\mathcal{S}(t)\,\varphi = \{u(t), u_t(t)\}$ *is the weak solution of the problem* (1.1), (1.2).

Proof. It is clear from Proposition 3.2(c), that the mapping $\mathcal{T}(t)$ is Lipschitz continuous from $C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ endowed with the norm of $\mathcal{X}_0$ into $C[0, T; \mathcal{X}_0]$. By the density of $C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ into $\mathcal{X}_0$ there exists a unique Lipschitz continuous extension $\tilde{\mathcal{T}}(t)$ from $\mathcal{X}_0$ into $C[0, T; \mathcal{X}_0]$ such that

$$\tilde{\mathcal{T}}(t)\,\varphi = \mathcal{T}(t)\,\varphi, \qquad \text{for every} \quad \varphi \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N).$$

Then, we define the mapping

$$\mathcal{S}(t)\colon \mathcal{X}_0 \mapsto \mathcal{X}_0, \qquad t \geqslant 0 \quad \text{by} \quad \mathcal{S}(t)\,\varphi := \tilde{\mathcal{T}}(t)\,\varphi.$$

The semigroup $\mathcal{S}(t)$, $t \geqslant 0$ defines a dynamical system on $\mathcal{X}_0$. From inequality (3.25) we have that for φ, $\tilde{\varphi}$ in $\mathcal{X}_0$

$$\|\mathcal{S}(t)\,\varphi - \mathcal{S}(t)\,\tilde{\varphi}\|_{\mathcal{X}_0} \leqslant C\,\|\varphi - \tilde{\varphi}\|_{\mathcal{X}_0}.$$

i.e., it is clear that S is a Lipschitz continuous semigroup. ∎

4. EXISTENCE OF A GLOBAL ATTRACTOR

In this section we shall prove that the dynamical system generated by the semigroup $\mathcal{S}(t)$ posseses a global attractor. In order to obtain this result we need a series of lemmas. The first lemma is related to the existence of an absorbing set in $\mathcal{X}_0$.

LEMMA 4.1. *Let* f, g *and* η *satisfy conditions* $(\mathcal{F})$, $(\mathcal{G})$, *and* $(\mathcal{H})$, *respectively. Then for*

$$\lambda < \min\left(\frac{\alpha^{1/2}\delta}{4c^*}, \left(\frac{\alpha\mu_1}{8}\right)^{1/2} \frac{1}{c^*}\right) \tag{4.1}$$

there exists an absorbing set for the semigroup $\mathcal{S}$ *associated to the problem* (1.1)–(1.2).

Proof. Let $0 \leqslant \varepsilon \leqslant \varepsilon_0$, where $\varepsilon_0 = \min(\delta/4, \mu_1/2\delta)$. Note that

$$\mu_1 = \inf\left\{\frac{\|u\|^2_{\mathcal{D}^{1,2}}}{\|u\|^2_{L^2_g}} : u \in \mathcal{D}^{1,2}\right\} = \inf\left\{\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\int_{\mathbb{R}^N} g u^2\,dx} : u \in \mathcal{D}^{1,2}\right\}.$$

We set $v=u_t+\varepsilon u$ and multiply the Galerkin Eqs. (3.2) by $\dot{b}_j^n(t)+\varepsilon b_j^n(t)$. By following the same arguments as in Proposition 3.2 and Theorem 3.3 we get that u, v satisfy the "energy relation"

$$\frac{1}{2}\frac{d}{dt}\{\|u\|^2_{\mathscr{D}^{1,2}}+\|v\|^2_{L^2_g}\}+\varepsilon\|u\|^2_{\mathscr{D}^{1,2}}$$

$$+(\delta-\varepsilon)\|v\|^2_{L^2_g}-\varepsilon(\delta-\varepsilon)\int_{\mathbb{R}^N} guv\,dx+\lambda\int_{\mathbb{R}^N} gf(u)\,v\,dx$$

$$=\int_{\mathbb{R}^N} g\eta v\,dx. \tag{4.2}$$

We observe that

$$\begin{aligned}(u,v)_{L^2_g}&=\int_{\mathbb{R}^N} guv\,dx\\ &\leqslant\left(\int_{\mathbb{R}^N} gu^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^N} gv^2\,dx\right)^{1/2}\\ &\leqslant\frac{1}{\mu_1^{1/2}}\|u\|_{\mathscr{D}^{1,2}}\|v\|_{L^2_g}.\end{aligned}$$

With the assumptions on ε and the above inequality, we get that

$$\varepsilon\|u\|^2_{\mathscr{D}^{1,2}}+(\delta-\varepsilon)\|v\|^2_{L^2_g}-\varepsilon(\delta-\varepsilon)(u,v)_{L^2_g}\geqslant\frac{\varepsilon}{2}\|u\|_{\mathscr{D}^{1,2}}+\frac{\delta}{2}\|v\|_{L^2_g}. \tag{4.3}$$

By hypothesis $\mathscr{F}$ and the assumption for $\eta(x)$ we get that

$$\begin{aligned}2\lambda\int_{\mathbb{R}^N} g\,|f(u)|\,|v|\,dx&\leqslant 2\lambda c^*\int_{\mathbb{R}^N} g\,|u|\,|v|\,dx\\ &\leqslant 2\lambda c^*\|u\|_{L^2_g}\|v\|_{L^2_g}\\ &\leqslant\frac{2\lambda c^*}{\alpha^{1/2}}\|u\|_{\mathscr{D}^{1,2}}\|v\|_{L^2_g}\\ &\leqslant\frac{\varepsilon}{2}\|u\|^2_{\mathscr{D}^{1,2}}+\gamma\|v\|^2_{L^2_g},\end{aligned} \tag{4.4}$$

$$2\int_{\mathbb{R}^N} g\eta v\,dx\leqslant\frac{\delta}{2}\|v\|^2_{L^2_g}+\frac{2}{\delta}\|\eta\|^2_{L^2_g}, \tag{4.5}$$

where $\gamma =: 2\lambda^2 c^{*2}/\alpha\varepsilon$. The requirement $\delta > 2\gamma$ justifies assumption (4.1). Setting $\rho =: \min(\varepsilon/2, (\delta - 2\gamma)/2)$ we get from (4.2)–(4.5), that

$$\frac{d}{dt} H(t) + \rho H(t) \leqslant B,$$

where $B =: 2\delta^{-1} \|\eta\|^2_{L^2_g}$ and $H(t) = \|u(t)\|^2_{\mathscr{D}^{1,2}} + \|v(t)\|^2_{L^2_g}$. By application of Gronwall's lemma we get

$$H(t) \leqslant H(0)\, e^{-\rho t} + \frac{1 - e^{-\rho t}}{\rho} B.$$

Clearly, $\lim_{t \to \infty} H(t) \leqslant \mu_0^2$, where $\mu_0^2 =: B/\rho$. We get $\mu_0^* > \mu_0$ fixed and we assume that $H(0) \leqslant K$. Then there exists time $t \geqslant t_0(K, \mu_0')$ such that $H(t) \leqslant \mu_0^*$. Moreover, we have the inequality

$$\begin{aligned} \|u(t)\|^2_{\mathscr{D}^{1,2}} + \|u_t(t)\|^2_{L^2_g} &\leqslant L(\varepsilon, \lambda)(\|u(t)\|^2_{\mathscr{D}^{1,2}} + \|v(t)\|^2_{L^2_g}) \\ &\leqslant LH(t) \leqslant L\mu_0^*. \end{aligned}$$

Therefore summarizing we see that for any $\mathscr{B}$ bounded subset of $\mathscr{X}_0 = \mathscr{D}^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$ we obtain

$$K = \sup_{\tilde{\phi} \in \mathscr{B}} \{ \|\phi_0\|^2_{\mathscr{D}^{1,2}} + \|\phi_0 + \varepsilon\phi_1\|^2_{L^2_g} \} < \infty,$$

where $\tilde{\phi} = \{\phi_0, \phi_1\}$. Setting $\sigma_0 =: L\mu_0^*$ we easily see that the ball $B_0 = B(0, \sigma_0)$ is an absorbing set in X_0 for the semigroup $\mathscr{S}(t)$, i.e., for any bounded set $\mathscr{B}$ of $\mathscr{X}_0$ we have that $\mathscr{S}(t)\,\mathscr{B} \subset B_0$, for $t \geqslant t_0$. ∎

Remark 4.2 (Global Existence). From Lemma 4.1 we may see that solutions of problem (1.1), (1.2) (given by Theorem 3.2) belong to the space $C_b(\mathbb{R}_+, \mathscr{X}_0)$ of bounded continuous functions from $\mathbb{R}_+$ to $\mathscr{X}_0$, that is, it is proved that if λ, α, δ, $\|g\|_{N/2}$, c^*, μ_1, satisfy condition (4.1), solutions exist *globally in time*.

Remark 4.3 (Pseudocoercivity Hypothesis). In the absence of an external force $\eta(x)$, the existence of an absorbing set in $\mathscr{X}_0$ may be shown for all $\lambda > 0$, if the functions g, f satisfy the following *pseudocoercivity hypothesis*

$$\liminf_{\|\phi\|_{\mathscr{D}^{1,2}} \to \infty} \frac{\int_{\mathbb{R}^N} g(x)\, F(\phi)\, dx}{\|\phi\|^2_{\mathscr{D}^{1,2}}} \geqslant 0,$$

$$\liminf_{\|\phi\|_{\mathscr{D}^{1,2}} \to \infty} \frac{\int_{\mathbb{R}^N} g(x)\, f(\phi)\, \phi\, dx - C_0 \int_{\mathbb{R}^N} g(x)\, F(\phi)\, dx}{\|\phi\|^2_{\mathscr{D}^{1,2}}} \geqslant 0,$$

for some $C_0 > 0$, where $F(s) = \int_0^s f(s)\, ds$.

In the rest of the paper we show that the ω-limit set of the absorbing set is a compact attractor. To this end, we need to decompose the semigroup $\mathcal{S}(t)$ in the form $\mathcal{S}(t)=\mathcal{S}_1(t)+\mathcal{S}_2(t)$, where for any bounded set $\mathcal{B}\subset\mathcal{X}_0$, the semigroups $\mathcal{S}_1(t)$, $\mathcal{S}_2(t)$ satisfy the following properties,

(S1) *$\mathcal{S}_1(t)$ is uniformly compact for t large, i.e., $\bigcup_{t\geqslant t_0}\mathcal{S}_1(t)\,\mathcal{B}$ is relatively compact in $\mathcal{X}_0$,*

(S2) $\sup_{\phi\in\mathcal{B}}\|\mathcal{S}_2(t)\,\phi\|_{\mathcal{X}_0}\to 0$, *as $t\to\infty$.*

For this, we need some additional results concerning the linear equation, given in the following lemmas.

LEMMA 4.4. *The linear homogeneous initial value problem*

$$\begin{aligned} &u_{tt}+\delta u_t-\phi(x)\,\Delta u=0, \qquad x\in\mathbb{R}^N, \quad t\in[0,T],\\ &u(\cdot,0)=u_0\in\mathcal{D}^{1,2}(\mathbb{R}^N),\\ &u_t(\cdot,0)=u_1\in L^2_g(\mathbb{R}^N), \end{aligned} \tag{4.6}$$

admits a unique solution such that

$$u\in C_b[\mathbb{R}_+,\mathcal{D}^{1,2}(\mathbb{R}^N)] \qquad \text{and} \qquad u_t\in C_b[\mathbb{R}_+,L^2_g(\mathbb{R}^N)].$$

Moreover, this solution decays exponentially as $t\to\infty$.

Proof. We proceed as in the Proposition 3.2 and the Lemma 4.1 to obtain the estimate

$$\|u(t)\|^2_{\mathcal{D}^{1,2}}+\|u_t(t)+\varepsilon u(t)\|^2_{L^2_g}\leqslant\{\|u_0\|^2_{\mathcal{D}^{1,2}}+\|u_1+\varepsilon u_0\|^2_{L^2_g}\}\,e^{-Ct}$$

with $C>0$. The last estimate apart of giving the existence and uniqueness results for problem (4.6) (as in Proposition 3.2), implies also the exponential decay of solutions by letting $t\to\infty$. ∎

This lemma implies that the semigroup associated with the problem (4.6), satisfy the property (S2). Concerning semigroups satisfying property (S1) we need to prove the following lemmas.

LEMMA 4.5. *Consider the linear nonhomogeneous initial value problem*

$$\begin{aligned} &\tilde{u}_{tt}+\delta\tilde{u}_t-\phi(x)\,\Delta\tilde{u}+\lambda f(u)=\eta(x), \qquad x\in\mathbb{R}^N, \quad t\in[0,T],\\ &\tilde{u}(x,0)=\tilde{u}_0\in\mathcal{D}^{1,2}(\mathbb{R}^N),\\ &\tilde{u}_t(x,0)=\tilde{u}_1\in L^2_g(\mathbb{R}^N), \end{aligned} \tag{4.7}$$

where u denotes the solution of the original problem given by Theorem 3.3. Then problem (4.7) possesses a unique solution such that

$$\tilde{u} \in C_b[\mathbb{R}_+, \mathcal{D}^{1,2}(\mathbb{R}^N)] \qquad \text{and} \qquad \tilde{u}_t \in C_b[\mathbb{R}_+, L_g^2(\mathbb{R}^N)].$$

Proof. Working as in Lemmas 4.1 and 4.4 we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\{\|\tilde{u}\|^2_{\mathcal{D}^{1,2}} + \|\tilde{v}\|^2_{L_g^2}\} + \frac{\varepsilon}{2}\|\tilde{u}\|_{\mathcal{D}^{1,2}} + \frac{\delta}{2}\|\tilde{v}\|_{L_g^2}$$

$$\leqslant \lambda \int_{\mathbb{R}^N} g\,|f(u)|\,|\tilde{v}|\,dx + \int_{\mathbb{R}^N} g\,|\eta|\,|\tilde{v}|\,dx.$$

Note that

$$\begin{aligned}\lambda \int_{\mathbb{R}^N} g\,|f(u)|\,|\tilde{v}|\,dx &\leqslant \lambda c^* \int_{\mathbb{R}^N} g\,|u|\,|\tilde{v}|\,dx \\ &\leqslant \lambda c^* \|u\|_{L_g^2}\|\tilde{v}\|_{L_g^2} \\ &\leqslant \frac{\lambda c^*}{\alpha^{1/2}}\|u\|_{\mathcal{D}^{1,2}}\|\tilde{v}\|_{L_g^2} \\ &\leqslant \frac{\rho_1}{4}\|\tilde{v}\|^2_{L_g^2} + \frac{1}{\rho_1}M^2\|u\|^2_{\mathcal{D}^{1,2}},\end{aligned} \tag{4.8}$$

$$\int_{\mathbb{R}^N} g\,|\eta|\,|\tilde{v}|\,dx \leqslant \frac{\rho_1}{4}\|\tilde{v}\|^2_{L_g^2} + \frac{1}{\rho_1}\|\eta\|^2_{L_g^2}, \tag{4.9}$$

where $M = \lambda c^*/\alpha^{1/2}$ and $\rho_1 = \min(\varepsilon/2, \delta/2)$. Since u is the solution of the original problem, the last term of the right-hand side of (4.8) is bounded. Finally we get the inequality

$$\frac{d}{dt}\{\|\tilde{u}\|^2_{\mathcal{D}^{1,2}} + \|\tilde{v}\|^2_{L_g^2}\} + \rho_1\{\|\tilde{u}\|^2_{\mathcal{D}^{1,2}} + \|\tilde{v}\|^2_{L_g^2}\} \leqslant \tilde{C}$$

and by Gronwall's lemma we get

$$\begin{aligned}&\|\tilde{u}(t)\|^2_{\mathcal{D}^{1,2}} + \|\tilde{u}_t(t) + \varepsilon\tilde{u}(t)\|^2_{L_g^2} \\ &\qquad \leqslant \{\|\tilde{u}_0\|^2_{\mathcal{D}^{1,2}} + \|\tilde{u}_1 + \varepsilon\tilde{u}_0\|^2_{L_g^2}\}\, e^{-\rho_1 t} + \tilde{C}(1 - e^{-\rho_1 t}).\end{aligned}$$

Leting $t \to \infty$ we obtain the result. ∎

This lemma gives the existence of the semigroup $\mathcal{S}_1(t)$. To prove uniform compactness for t large, i.e., property $(S1)$ we need the next two lemmas

LEMMA 4.6. *Let f satisfy* $(\mathcal{F}_\infty)$. *Then there exists* $\varepsilon>0$, *such that for every* ϕ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$ *the functional* $f'(\phi)\in\mathcal{L}(L^2_g, V_{\varepsilon-1})$, *and for every* $R>0$

$$\sup_{\|\phi\|_{\mathcal{D}^{1,2}}\leqslant R} |f'(\phi)|_{\mathcal{L}(L^2_g,\, V_{\varepsilon-1})}<\infty.$$

Proof. We define the operator $T\colon L^2_g\mapsto V_{\varepsilon-1}$, such that

$$T\theta=f'(\phi)\,\theta, \qquad \text{for every} \quad \theta\in L^2_g.$$

Since hypothesis $(\mathcal{G})$ and $(\mathcal{F})$ are satisfied $f'(\phi)\in L^\infty(\mathbb{R}^N)$, for every $\phi\in\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since for any $\varepsilon\in(0,1)$ the embedding $L^2_g(\mathbb{R}^N)\equiv V_0\subset V_{\varepsilon-1}$ is compact, we have

$$\|f'(\phi)\,\theta\|_{V_{\varepsilon-1}}\leqslant C\,\|f'(\phi)\,\theta\|_{L^2_g}\leqslant C\,\|f'(\phi)\|_{L^\infty}\,\|\theta\|_{L^2_g},$$

and the proof is completed. ∎

The last lemma shows that semigroup $\mathcal{S}_1(t)$ satisfies property (S1) and so the decomposition of the semigroup $\mathcal{S}(t)$ is achieved.

LEMMA 4.7. *The semigroup* $\mathcal{S}_1(t)$ *satisfies the property* (S1).

Proof. We write the solution of the problem (1.1), (1.2) as $u=w+\tilde{u}$, where w is the solution of the problem (4.6) and $\tilde{u}=u-w$ is the solution of the problem (4.7), with initial conditions $\tilde{u}(x,0)=0$ and $\tilde{u}_t(x,0)=0$. The semigroup $\mathcal{S}_2(t)$ associated with solution w has the property (S2). We shall show that $\mathcal{S}_1(t)=\mathcal{S}(t)-\mathcal{S}_2(t)$ is uniformly compact. Let $\{u_0,u_1\}$ be in a bounded set $\mathcal{B}$ of $\mathcal{X}_0$, then Lemma 4.1 implies that for all $t\geqslant t_0$, $\{u,u_t\}$ is in $\mathcal{B}_0$ and

$$\|u(t)\|^2_{\mathcal{D}^{1,2}}+\|u_t(t)\|^2_{L^2_g}\leqslant\sigma_0^2, \qquad \text{for all} \quad t\geqslant t_0. \tag{4.10}$$

We differentiate Eq. (4.7) with respect to time. Then $U=\tilde{u}_t$, is the solution of the problem

$$\begin{aligned} U_{tt}+\delta U_t-\phi\,\Delta U&=-\lambda f'(u)\,u_t\\ U(x,0)&=0,\\ U_t(x,0)&=-\lambda f(u_0). \end{aligned} \tag{4.11}$$

For the rest of the proof we follow ideas developed in [15]. By Theorem 3.2 and Lemma 4.5, $U\in C_b(\mathbb{R}_+, V_0)$, $U_t\in C_b(\mathbb{R}_+, V_{-1})$ (see also Remark 2.4) and by Lemma 4.6, $f'(u)\,u_t\in C_b(\mathbb{R}_+, V_{\varepsilon-1})$. So applying the operator

$A^{(\varepsilon-1)/2}$ to the Eq. (4.11) and setting $\psi = A^{(\varepsilon-1)/2}U$ and $\xi = A^{(\varepsilon-1)/2}(-f'(u)\,u_t)$ we get

$$\psi_{tt} + \delta\psi_t + A\psi = \lambda\xi, \qquad t \in \mathbb{R}_+. \tag{4.12}$$

From the properties of the operators A^s and relation (2.11) we have that

$$A^{(\varepsilon-1)/2}\colon V_{\varepsilon-1} \mapsto V_0,$$
$$A^{(\varepsilon-1)/2}\colon V_0 \mapsto V_{1-\varepsilon},$$
$$A^{(\varepsilon-1)/2}\colon V_{-1} \mapsto V_{-\varepsilon},$$

are isomorphisms. Therefore $\{\psi, \psi_t\} \in C_b(\mathbb{R}^+, V_{1-\varepsilon} \times V_{-\varepsilon})$. Since $\xi \in C_b(\mathbb{R}^+, V_0)$, by Lemma 4.5 we obtain that $\{\psi, \psi_t\} \in C_b(\mathbb{R}^+, V_1 \times V_0)$ (see [15; 35, p. 182]). Furthermore the isomorphisms

$$A^{(1-\varepsilon)/2}\colon V_1 \mapsto V_\varepsilon,$$
$$A^{(1-\varepsilon)/2}\colon V_0 \mapsto V_{\varepsilon-1},$$

imply that the following relations are true

$$\{\tilde{u}_t, \tilde{u}_{tt}\} = \{U, U_t\} = A^{(1-\varepsilon)/2}\{\psi, \psi_t\} \in C_b(\mathbb{R}^+, V_\varepsilon \times V_{\varepsilon-1}). \tag{4.13}$$

But $f(u) \in V_{\varepsilon-1}$ so by (4.13) we obtain that $-\phi\Delta\tilde{u} = -\tilde{u}_{tt} - d\tilde{u}_t - \lambda g(x) f(u) \in V_{\varepsilon-1}$ and using again (2.11) we have the isomorphism

$$(-\phi\Delta)^{-1} = A^{-2/2}\colon V_{\varepsilon-1} \mapsto V_{\varepsilon+1}.$$

Therefore

$$\{\tilde{u}, \tilde{u}_t\} = \{A^{-1}\tilde{u}, \tilde{u}_t\} \in C_b(\mathbb{R}^+, V_{\varepsilon+1} \times V_\varepsilon),$$

that is, $\bigcup_{t \geqslant t_0} \mathcal{S}_1(t)\,\mathcal{B}$ is in a bounded set of $V_{\varepsilon+1} \times V_\varepsilon$. So the compact embeddings $V_{\varepsilon+1} \subset V_1$ and $V_\varepsilon \subset V_0$ imply that, the set $\bigcup_{t \geqslant t_0} \mathcal{S}_1(t)\,\mathcal{B}$ is relatively compact in $\mathcal{X}_0$. ∎

Summarizing the previous lemmas we may state the main result

THEOREM 4.8. *Let g satisfying* $(\mathcal{G})$, η *satisfies* $(\mathcal{H})$ *and f satisfying* $(\mathcal{F})$ *and* $(\mathcal{F}_\infty)$. *Then the dynamical system associated to the problem* (1.1), (1.2), *possesses a global attractor* $\mathcal{A} = \omega(\mathcal{B}_0)$, *which is compact, connected and maximal among the functional invariant sets in* $\mathcal{X}_0$.

ACKNOWLEDGMENTS

We thank the referee for his/her helpful suggestions which improved the original results. This work was partially financially supported by a grant from Papakyriakopoulos Legacy in the Department of Mathematics at NTU, Athens.

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