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Principal Eigenvalues and Anti – Maximum Principle for Some Quasilinear Elliptic Equations on $\mathbb{R}^N$

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Abstract. We improve some previous existence and nonexistence results for positive principal eigenvalues of the problem

$$\begin{aligned} -\Delta_p u &= \lambda g(x)\psi_p(u)\,, \quad x \in \mathbb{R}^N\,, \\ \lim_{|x|\to+\infty} u(x) &= 0\,. \end{aligned}$$

Also we discuss existence, nonexistence and antimaximum principle questions concerning the perturbed problem

$$-\Delta_p u \;=\; \lambda g(x)\psi_p(u) + f(x)\,, \quad x \in \mathbb{R}^N\,.$$

1. Introduction

In this paper we shall deal with existence, nonexistence and properties of the "first eigenpair" in $\mathbb{R}^N$, for some quasilinear elliptic eigenvalue problem, containing the p – Laplacian operator $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big)$, of the form

$$-\Delta_p u \;=\; \lambda g(x)\psi_p(u)\,, \quad x \in \mathbb{R}^N\,, \tag{1.1}$$

$$\lim_{|x|\to+\infty} u(x) \;=\; 0\,, \tag{1.2}$$

where $\psi_p(u) = |u|^{p-2}u$. Also we discuss existence, nonexistence and antimaximum principle questions concerning the perturbed problem

$$-\Delta_p u \;=\; \lambda g(x)\psi_p(u) + f(x)\,, \quad x \in \mathbb{R}^N\,. \tag{1.3}$$

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Throughout this work we will assume that $1 < p < N$, and that g in (1.1) satisfies the following hypothesis

($\mathcal{H}1$) g is a smooth function, at least $C^{0,\gamma}_{loc}\big(\mathbb{R}^N\big)$ for some $\gamma \in (0,1)$, such that $g \in L^\infty\big(\mathbb{R}^N\big)$ and $meas\big\{x \in \mathbb{R}^N : g(x) > 0\big\} > 0$;

($\mathcal{H}2$) there exist a regular open set $\Omega \subset \mathbb{R}^N$ and functions g_1, g_2, g_3, g_4 in $L^\infty\big(\mathbb{R}^N\big)$ such that $g = g_1 + g_2 - g_3 - g_4$ satisfying
(1) $g_1 \in L^{N/p}\big(\mathbb{R}^N\big)$,
(2) $g_2 \geq 0$, $\operatorname{Supp} g_2 \subset \Omega$ and for some $0 < \alpha < p$

$$\lim_{R\to+\infty} \sup_{|x|\geq R,\, x\in\Omega} \int_{|y|<1} g_2(x-y)\,|y|^{\alpha-N}\,dy \;=\; 0\,,$$

(3) $g_3 \geq 0$ and there is $\epsilon > 0$, $R > 0$ such that for any $x \in \Omega$ with $|x| \geq R$ we have $g_3 \geq \epsilon$,
(4) $g_4 \geq 0$.

Remark 1.1. Let $\tilde{g}$ be a function satisfying ($\mathcal{H}1$) and $\tilde{g} \leq g$. Then if g satisfies ($\mathcal{H}2$) it is easy to see that $\tilde{g}$ satisfies ($\mathcal{H}2$) also. If $\Omega = \emptyset$ then we can take $g_2 \equiv g_3 \equiv 0$.

Example 1.2. To be more clear about the kind of (nonradial) functions g satisfying conditions ($\mathcal{H}1$) and ($\mathcal{H}2$) we give the following example. For simplicity reasons we restrict ourselves to the case $N = 2$. So we consider $p \in (1,2)$. Also let $\Omega = \mathbb{R}^2_+$, i.e., $\Omega := \big\{(x_1,x_2) : x_2 > 0\big\}$. Consider a function $\Theta \in C^\infty\big(\mathbb{R}^2\big)$, such that $0 \leq \Theta \leq 1$ and

$$\Theta(x_1,x_2) \;:=\; \begin{cases} 1\,, & \text{for} \quad x_2 \;\geq\; 0\,,\\ 0\,, & \text{for} \quad x_2 \;\leq\; -1\,. \end{cases}$$

We set

$$h_1(x_1,x_2) \;:=\; \begin{cases} \dfrac{1}{(1+x_1^2)^{m_1}(1+x_2^2)^{m_2}}\,, & \text{for} \quad x_2 \;\leq\; 0\,,\\ 0\,, & \text{for} \quad x_2 \;>\; 0\,, \end{cases}$$

with $m_1 > \frac{p}{4}$, $m_2 > \frac{p}{4}$ and $g_1(x_1,x_2) := [1-\Theta(x_1,x_2)]h_1(x_1,x_2)$. It is easy to prove that $g_1 \in L^\infty\big(\mathbb{R}^2\big) \cap L^{2/p}\big(\mathbb{R}^2\big)$, g_1 is smooth and $g_1(x_1,x_2) > 0$ for $x_2 \leq -1$. Next we set

$$h_2(x_1,x_2) \;:=\; \begin{cases} \dfrac{1}{1+\log(1+|(x_1,x_2)|)}\,, & \text{for} \quad x_2 \;>\; 0\,,\\ 0\,, & \text{for} \quad x_2 \;\leq\; 0\,, \end{cases}$$

and $g_2(x_1,x_2) := \Theta(x_1,x_2-2)h_2(x_1,x_2)$. Then we have that g_2 is smooth, $g_2 \in L^\infty\big(\mathbb{R}^2\big)$ and $\operatorname{Supp} g_2 \subset \Omega$. Moreover, we have for $|x| \geq R$ and $|y| < 1$ that $\log(1+|x-y|) \geq \log R$. Therefore

$$\sup_{|x|\geq R} \int_{|y|<1} g_2(x-y)|y|^{1-2}\,dy \;\leq\; \frac{2\pi}{1+\log R}\,.$$

So the hypothesis $(\mathcal{H}2)\,(2)$ is satisfied for $\alpha = 1 < p$. Furthermore, we put

$$h_3(x_1,x_2) \;:=\; \begin{cases} \epsilon \;>\; 0\,, & \text{for} \quad x_2 \;>\; -1 \;\text{ and }\; |x| \;\geq\; R\,, \\ 0\,, & \text{elsewhere}\,, \end{cases}$$

and $g_3(x_1,x_2) \;:=\; \Theta(x_1,x_2)h_3(x_1,x_2)$, where we have that g_3 is smooth and $g_3(x_1,x_2) \geq \epsilon$ for $|x| \geq R$ and $x \in \Omega$. Finally, we put

$$g_4(x_1,x_2) \;:=\; 0\,.$$

Now we take as $g := g_1 + g_2 - g_3 - g_4$. Then g satisfies all hypothesis in conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$.

Problems where the operator $-\Delta_p$ is present arise both from pure mathematics, like in the theory of quasiregular and quasiconformal mappings (see [19] and the references therein), as well as from a variety of applications, e. g. non – Newtonian fluids, reaction – diffusion problems, flow through porous media, nonlinear elasticity, glaciology, petroleum extraction, astronomy, etc (see [4], [5], [11]).

In the case of the eigenvalue problem for bounded domains, under various boundary conditions, there is quite an extensive literature and the picture for "the principal eigenpair" seems to be fairly complete. We mention among others, [3], [14], [17], [18].

The eigenvalue problem for unbounded domains in general becomes more complicate. In the last few years several works dealing with the eigenvalue problem in unbounded domains have been completed, see [2], [16] and the references therein; see also [12], where the existence of nontrivial solutions is proved for nonhomogeneous right – hand sides. Furthermore, in [13] bifurcation technics are used to prove existence results for the p – Laplacian equation in $\mathbb{R}^N$.

In Section 2, we shall prove the existence of a positive principal eigenvalue for the problem (1.1) and establish the natural space setting for this problem, which is the space $\mathcal{V}_\mu$, the completion of $C_0^\infty\big(\mathbb{R}^N\big)$ with respect to the norm $||u||_{\mathcal{V}}^p$. We generalize here some previous results concerning the case $p = 2$ [1], [7], [8] or the case $p \neq 2$ [16]. In Section 3 we give a necessary condition for existence of positive principal eigenvalues for the problem (1.1), (1.2). In Section 4, under certain conditions on f, we prove that there exists $\epsilon > 0$ such that the equation (1.3) has a solution for any $\lambda \in (0,\lambda_1) \cup (\lambda_1, \lambda_1 + \epsilon)$. This is done by applying a certain form of the Fredholm Alternative, i. e., Theorem 4.1. Finally, in Section 5 we discuss nonexistence results for (1.3) and a weak formulation of the antimaximum principle for unbounded domains, which seems to be not yet discussed (see [10]).

Notation 1.3. For simplicity we use the symbol $||\cdot||_p$ for the norm $||\cdot||_{L^p(\mathbb{R}^N)}$ and $\mathcal{D}^{1,p}$ for the space $\mathcal{D}^{1,p}\big(\mathbb{R}^N\big)$, see (2.2). B_R and $B_R(a)$ will denote the balls in $\mathbb{R}^N$, centered at zero and a respectively, and radius R. $B_R^c =: \big\{x \in \mathbb{R}^N : |x| > R\big\}$. Also the Lebesgue measure of a set $\Omega \subset \mathbb{R}^N$ will be denoted by $|\Omega|$ or by $meas\,\Omega$. $\mathrm{Supp}\,g$ is the support of the function g. The end of a proof is marked with a $\square$.

2. Existence of a principal eigenvalue

In this section we shall first prove the existence of a positive principal eigenvalue for the problem (1.1). The natural setting for this problem is the space $\mathcal{V}_\mu$, which is defined to be the completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$||u||_{\mathcal{V}_\mu}^p = \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \mu \int_{\mathbb{R}^N} (g_3 + g_4) \, |v|^p \, dx .$$

Since for all $\mu > 0$ the norms $||\cdot||_{\mathcal{V}_\mu}$ are equivalent, we denote these spaces simply by $\mathcal{V}$ and the common norm by $||\cdot||_{\mathcal{V}}$. By hypothesis $(\mathcal{H}2)\,(3)$ it is easy to see that for any $v \in \mathcal{V}$ and $\Omega \subset \mathbb{R}^N$ bounded, we have that $v|_\Omega \in W^{1,p}(\Omega)$ and

$$(2.1) \qquad ||v||_{W^{1,p}(\Omega)}^p \le \max\left\{1, \frac{\mu}{\epsilon}\right\} ||v||_{\mathcal{V}}^p .$$

Moreover, $\mathcal{V} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ and for any $v \in \mathcal{V}$ and $\mu > 0$ we have

$$(2.2) \qquad ||v||_{\mathcal{D}^{1,p}(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |\nabla u|^p \, dx \le ||v||_{\mathcal{V}}^p ,$$

where the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$||u||_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p} .$$

It is known that $\mathcal{D}^{1,p}(\mathbb{R}^N) = \left\{u \in L^{\frac{Np}{N-p}}(\mathbb{R}^N) : \nabla u \in \left(L^p(\mathbb{R}^N)\right)^N\right\}$ and that there exists $K > 0$ such that for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$

$$(2.3) \qquad ||u||_{L^{\frac{Np}{N-p}}} \le K \, ||u||_{\mathcal{D}^{1,p}} .$$

For more details we refer to [16]. To prove the existence of the principal eigenvalue we need the following lemmas.

Lemma 2.1. *For any $\mu > 0$, the mapping $u \mapsto g_2^{1/p} u$ is compact from $\mathcal{V}$ to $L^p(\mathbb{R}^N)$.*

Proof. By Theorem 2.3 of Berger and Schechter [6], for any $u \in \mathcal{V}$ we have

$$\begin{aligned} \left\|g_2^{1/p} u\right\|_{L^p\left(\Omega \cap B_R^c\right)} &\le C M_{\alpha,p}\left(g_2^{1/p}, \Omega \cap B_R^c\right) ||u||_{W^{1,p}\left(\Omega \cap B_R^c\right)} \\ &\le \sup_{|x| \ge R-1} \int_{|y|<1} g_2(x-y) \, |y|^{\alpha - N} \, dy \end{aligned}$$

where

$$(2.4) \qquad M_{\alpha,p}\left(g_2^{1/p}, \Omega \cap B_R^c\right) = \sup_{x \in \mathbb{R}^N} \int_{\left\{x-y \in \Omega \cap B_R^c,\, |y|<1\right\}} g_2(x-y) \, |y|^{\alpha - N} \, dy .$$

Since the limit of the last term is zero as $R \to +\infty$, it follows from [6, Theorem 2.4] that the mapping $u \mapsto g_2^{1/p} u$ is compact from $W^{1,p}(\Omega)$ to $L^p(\Omega \cap B_R^c)$. Since it is also compact from $\mathcal{D}^{1,p}(\mathbb{R}^N)$ to $L^p(B_R)$, it follows from (2.1) that it is compact from $\mathcal{V}$ to $L^p(\mathbb{R}^N)$. □

Lemma 2.2. *For any $\mu > 0$, the mapping $u \mapsto g_1^{1/p} u$ is compact from $\mathcal{V}$ to $L^p(\mathbb{R}^N)$.*

Proof. By [16, Lemma 2.2] this mapping is compact from $\mathcal{D}^{1,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$. Then the proof follows from (2.2). □

Lemma 2.3. *For any $\mu > 0$, the problem*

$$-\Delta_p u + \mu(g_3 + g_4)\,\psi_p(u) \;=\; k(\mu)(g_1 + g_2)\,\psi_p(u)\,, \quad x \in \mathbb{R}^N\,, \tag{2.5}$$

$$\lim_{|x| \to +\infty} u(x) \;=\; 0\,, \tag{2.6}$$

has a principal eigenvalue.

Proof. We define

$$k(\mu) \;=:\; \inf_{u \in \mathcal{V}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \mu \int_{\mathbb{R}^N} (g_3 + g_4)|u|^p\,dx}{\int_{\mathbb{R}^N} (g_1 + g_2)\,|u|^p\,dx}\,.$$

By Lemmas 2.1 and 2.2, $k(\mu)$ is well defined and is attained for some $u_\mu \geq 0$; moreover, $u_\mu \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$-\Delta_p u_\mu \;\leq\; (|\mu| + |k(\mu)|)\,||g||_\infty\,|u_\mu|^{p-1}\,.$$

Hence $u_\mu > 0$ in $\mathbb{R}^N$ by Vázquez' Maximum Principle [20]. For a more detailed proof see Theorems 2.4 and 2.6 in [16]. □

Theorem 2.4. *If g satisfies $(\mathcal{H})$ then problem (1.1), (1.2) admits a positive principal eigenvalue.*

Proof. It is sufficient to show that for some μ_0 we have that $k(\mu_0) = \mu_0$. First, let $v \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{Supp} v \subset \{x \in \mathbb{R}^N : g(x) > 0\}$; then we have

$$k(\mu) - \mu \;\leq\; \frac{\int_{\mathbb{R}^N} |\nabla v|^p\,dx - \mu \int_{\mathbb{R}^N} g\,|v|^p\,dx}{\int_{\mathbb{R}^N} (g_1 + g_2)\,|v|^p\,dx}\,.$$

Obviously, $\int_{\mathbb{R}^N} (g_1 + g_2)\,|v|^p\,dx \;\geq\; \int_{\mathbb{R}^N} g\,|v|^p\,dx \;>\; 0$. So for $\mu \to +\infty$, we obtain that

$$\text{there exists } \mu_1 \;>\; 0 \text{ such that } k(\mu_1) - \mu_1 \;<\; 0\,. \tag{2.7}$$

Now for any $u \in \mathcal{V}$ and $\mu \in (0,1)$ we have that

$$\int_{\mathbb{R}^N} |g_1|\,|u|^p\,dx \;\leq\; ||g_1||_{N/p}\,||u||^p_{Np/(N-p)} \;\leq\; ||g_1||_{N/p}\,||u||^p_{\mathcal{V}}\,, \tag{2.8}$$

$$(2.9)\qquad \begin{aligned}\int_{B_R} |g_2|\,|u|^p\,dx &\le ||g_2||_\infty \int_{B_R} \left(|u|^{Np/(N-p)}\right)^{(N-p)/p}\,dx\,R^p\\ &\le CR^p\,||g_2||_\infty\,||u||^p_{\mathcal{D}^{1,p}}\\ &\le CR^p\,||g_2||_\infty\,||u||^p_{\mathcal{V}}\,,\end{aligned}$$

$$(2.10)\qquad \begin{aligned}\int_{\Omega\cap B_R^c} g_2\,|u|^p &\le CM^p_{\alpha,p}\left(g_2^{1/p},\Omega\cap B^c_R\right)\,||u||^p_{W^{1,p}(\Omega)}\\ &\le \frac{C}{\mu}\,M^p_{\alpha,p}\left(g_2^{1/p},\Omega\cap B^c_R\right)\,||u||^p_{\mathcal{V}}\,.\end{aligned}$$

We fix some $R > 0$ such that

$$CM^p_{\alpha,p}\left(g_2^{1/p},\Omega\cap B^c_R\right) \le \frac{1}{2}\,.$$

So inequalities (2.8), (2.9) and (2.10) for $\mu \in (0,1)$ imply that

$$\int_{\mathbb{R}^N} |g_1+g_2|\,|u|^p\,dx \le \frac{1}{\mu}\left\{\frac{1}{2} + \mu C(||g_1||_{N/p} + R^p\,||g_2||_\infty)\right\}\,||u||^p_{\mathcal{V}}\,.$$

So there exists some μ_2 small enough, that

$$\int_{\mathbb{R}^N} |g_1+g_2|\,|u|^p\,dx \le \frac{1}{\mu_2}\,||u||^p_{\mathcal{V}}\,,$$

therefore

$$(2.11)\qquad k(\mu_2) - \mu_2 > 0\,.$$

As in Allegretto [1], we observe also that $k(\mu)$ is locally Lipschitz on $\mathbb{R}^+$, since for $h > 0$ we have that

$$\begin{aligned}k(\mu+h) &= \inf_{u\in\mathcal{V}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx + (\mu+h)\int_{\mathbb{R}^N}(g_3+g_4)|u|^p\,dx}{\int_{\mathbb{R}^N}(g_1+g_2)\,|u|^p\,dx}\\ &\le \frac{\mu+h}{\mu}\cdot\frac{\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \mu\int_{\mathbb{R}^N}(g_3+g_4)\,|u|^p\,dx}{\int_{\mathbb{R}^N}(g_1+g_2)\,|u|^p\,dx}\\ &= \left(1+\frac{h}{\mu}\right)k(\mu)\,.\end{aligned}$$

Since $k(\mu+h) \ge k(\mu)$ we obtain

$$|k(\mu+h)-k(\mu)| \le |h|\,\frac{k(\mu)}{\mu} \le \max_{\mu\in[\mu_1,mu_2]} \frac{k(\mu)}{\mu}\,,$$

so k is continuous. Therefore by (2.7) and (2.11) there exists some μ_0 such that $k(\mu_0) = \mu_0$. $\square$

3. A necessary condition for non – existence of positive principal eigenvalues

We follow here the ideas developped in [7]. To prove the necessary condition, we exhibit some particular functions in $\mathcal{V}$ and we need the following lemma. First, for $R' > R > 0$ we introduce the following notations

$$\Sigma_{R,R'} \;=:\; \left\{x \in \mathbb{R}^N : R \le |x| < R'\right\}, \quad \delta_1(R) \;=:\; \inf_{u \in W_0^{1,p}(B_R)} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\int_{\mathbb{R}^N} g\, |u|^p \, dx}.$$

Lemma 3.1. *Let $N > p$ and let $g \in L^\infty\big(\mathbb{R}^N\big)$ satisfy hypothesis $(\mathcal{H}1)$. We assume that there is $R_0 > 0$ such that $g(x) \ge 0$ for all $|x| > R_0$ and*

$$\limsup_{R\to\infty} \frac{1}{R^{N-p}} \int_{\Sigma_{R,2R}} g(x)\, dx \;=\; +\infty\,.$$

Then we have $\lim_{R\to\infty} \delta_1(R) = 0$.

Proof. We introduce the following auxiliary function

$$\Theta_{R,2R}(x) \;=\; \begin{cases} 0\,, & \text{for} \quad |x| \;\le\; \dfrac{R}{2}\,, \\ c_N^* \left[\left(\dfrac{R}{|x|}\right)^{N-p} - 2^{N-p}\right], & \text{for} \quad \dfrac{R}{2} \;\le\; |x| \;\le\; R\,, \\ 1\,, & \text{for} \quad R \;\le\; |x| \;\le\; 2R\,, \\ c_N \left[\left(\dfrac{2R}{|x|}\right)^{N-p} - 2^{p-N}\right], & \text{for} \quad 2R \;\le\; |x| \;\le\; 4R\,, \\ 0\,, & \text{for} \quad |x| \;\ge\; 4R\,. \end{cases}$$

Then we can easily see that $\Theta_{R,2R}$ is continuous $\big($we can take $c_N^* = \frac{1}{1-2^{N-p}}$, $c_N = \frac{1}{1-2^{p-N}}\big)$ and that $\Theta_{R,2R} \in W^{1,p}\big(\mathbb{R}^N\big)$. Moreover, we get that

$$|\nabla\, \Theta_{R,2R}(x)| \;=\; \begin{cases} (N-p)\, c_N^* \, \dfrac{R^{N-p}}{|x|^{N-p+1}}\,, & \text{for} \quad \dfrac{R}{2} \;\le\; |x| \;\le\; R\,, \\ (N-p)\, c_N \, \dfrac{R^{N-p}}{|x|^{N-p+1}}\,, & \text{for} \quad 2R \;\le\; |x| \;\le\; 4R\,, \\ 0\,, & \text{elsewhere}\,, \end{cases}$$

and

$$\int_{\mathbb{R}^N} |\nabla\, \Theta_{R,2R}(x)|^p \, dx \;=\; \big(c_N^*\big)^p \int_{R/2}^{R} (N-p)^p \, R^{p(N-p)} \, r^{-p(N-p+1)+N-1} \, dr$$
$$+ \, (c_N)^p \int_{2R}^{4R} (N-p)^p \, R^{p(N-p)} \, r^{-p(N-p+1)+N-1} \, dr\,.$$

So there is a constant $C = C(N, R)$ such that

$$\int_{\mathbb{R}^N} |\nabla \Theta_{R,2R}(x)|^p \, dx \;\le\; C R^{N-p} .$$

On the other hand for any $R \ge 2R_0$ we get

$$\int_{B_{4R}} g\, |\Theta_{R,2R}(x)|^p \, dx \;\ge\; \int_{\Sigma_{R,2R}} g(x)\, dx .$$

Therefore, we finally obtain that

$$\delta_1(4R) \;\le\; \frac{C R^{N-p}}{\int_{\Sigma_{R,2R}} g(x)\, dx} \;\longrightarrow\; 0 , \quad \text{for} \quad R \;\to\; \infty . \qquad \square$$

Theorem 3.2. *Assume that g satisfies hypothesis of Lemma* 3.1 *then problem* (1.1), (1.2) *has no positive principal eigenvalue.*

Proof. Suppose that the theorem is false. So let $\lambda_1 > 0$ be a principal eigenvalue and u be the corresponding positive eigenfunction, i.e., we have

$$-\Delta_p u \;=\; \lambda_1 g(x) \psi_p(u) , \quad x \in \mathbb{R}^N . \tag{3.1}$$

By hypothesis $(\mathcal{H}1)$, it is easy to see that by choosing B_ϵ to be a ball sufficiently small, on which $g > 0$, we can make $\delta_1(\epsilon)$ as big as we need. Since the principal eigenvalue depends continuously on the domain, using Lemma 3.1, we can find some R such that $\delta_1(R) = \lambda_1$. If ϕ denotes the corresponding positive eigenfunction on B_R, we have

$$\begin{aligned} -\Delta_p \phi &= \lambda_1 g(x) \psi_p(\phi) , \quad x \;\text{ in }\; B_R , \\ \phi &= 0 \quad \text{on} \quad \partial B_R . \end{aligned} \tag{3.2}$$

Multiplying (3.2) by ϕ, (3.1) by $\frac{|\phi|^p}{u^{p-1}}$, integrating over B_R and taking the difference we obtain

$$\int_{B_R} \left\{ |\nabla \phi|^p + (p-1) \left(\frac{\phi}{u}\right)^p |\nabla u|^p - p \nabla\phi . \nabla u \; |\nabla u|^{p-2} \left(\frac{\phi}{u}\right)^{p-1} \right\} dx \;=\; 0 ,$$

which implies, by Lemma 3.2 in [16], that there exists a constant $c > 0$ such that $u = c\phi$. But this is impossible, since we have that $u > 0$ in $\mathbb{R}^N$. $\square$

Remark 3.3. This estimate is sharp in the following sense. We know by Theorem 2.4 that problem (1.1), (1.2) has a positive principal eigenvalue if g is in $L^{N/p}(\mathbb{R}^N)$; and in this case we have

$$\int_{\Sigma_{R,2R}} g(x)\, dx \;\le\; ||g||_{N/p} \left\{ \int_{\Sigma_{R,2R}} 1 \, dx \right\}^{\frac{N-p}{N}} \;\le\; C R^{N-p} .$$

On the other hand, if g is radially symmetric $g(x) = g(|x|) = g(r)$, $g(r) \geq 0$, decaying for $r \geq R_0$ and if we have for some $\epsilon > 0$

$$\limsup_{R\to+\infty} \frac{1}{R^{N-p-\epsilon}} \int_{\Sigma_{R,2R}} g(x)\,dx \;<\; +\infty\,,$$

we can deduce that for $r \geq R_0$ then $g(r) \leq \frac{C}{r^{p+\epsilon}}$. Therefore g is in $L^{N/p}(\mathbb{R}^N)$.

Example 3.4. For $g(x) = \frac{1}{(1+|x|^2)^{m/2}}$, Theorem 3.2 proves the nonexistence of a positive principal eigenvalue for $m < p$; on the other hand for $m > p$, g is in $L^{N/p}(\mathbb{R}^N)$ and the principal eigenvalue exists.

Remark 3.5. If K is some cone of summit 0 and of infinite volume, then the condition

$$\limsup_{R\to\infty} \frac{1}{R^{N-p}} \int_{K\cap\Sigma_{R,2R}} g(x)\,dx \;=\; +\infty\,,$$

also implies the non existence of a positive principal eigenvalue. The details of the construction of $\Theta_{R,2R}(x)$ are left to the reader.

4. Existence results for a perturbation of the p−Laplacian

In this section, under certain conditions on f, we shall prove that there exists $\epsilon > 0$ such that the following perturbation of the p−Laplacian equation

$$-\Delta_p u \;=\; \lambda g(x)\psi_p(u) + f(x)\,, \quad x \in \mathbb{R}^N\,, \tag{4.1}$$

has a solution for any $\lambda \in (0,\lambda_1) \cup (\lambda_1, \lambda_1 + \epsilon)$. We need the following form of the Fredholm Alternative for nonlinear operators, obtained from the book of Fučik, Nečas, Souček and Souček [15, p. 61].

Theorem 4.1. *Let X and Y be two Banach spaces. Let T be an odd (K, L, a) – homeomorphism of X onto Y, which is an a – homogeneous operator. Let $S : X \mapsto Y$ be an odd completely continuous a – homogeneous operator. Then $\mu T - S$ is surjective from X onto Y if and only if μ is not an eigenvalue for the couple (T, S).*

The previous terminology needs clarification. The operator T is said to be a− homogeneous if $T(su) = s^a T(u)$, holds for any $s \geq 0$ and all $u \in X$. We call (K, L, a)− homeomorphism of X onto Y a homeomorphism, for which there exist real numbers $K > 0$, $L > 0$ and $a > 0$ such that

$$L\,||u||_X^a \;\leq\; ||T(u)||_Y \;\leq\; K\,||u||_X^a\,, \quad \text{for each} \quad u \in X\,.$$

μ is said to be an eigenvalue for the couple (T, S), if there exists $u_0 \in X$, $u_0 \neq 0$ such that $\mu T(u_0) - S(u_0) = 0$.

Lemma 4.2. *Let* $X = \mathcal{V}$, $Y = \mathcal{V}^*$, $\lambda \in \mathbb{R}^N$, $\lambda > 0$, $T : \mathcal{V} \mapsto \mathcal{V}^*$ *defined by* $T(u) = -\Delta_p u + \lambda\big(g_3 + g_4\big)|u|^{p-2}u$ *and* $f \in \mathcal{V}^*$. *Then* T *is an* $(1, 2, p-1)$ *– homeomorphism and* $(p-1)$ *– homogeneous of* $\mathcal{V}$ *onto* $\mathcal{V}^*$.

Proof. First we prove that $T : \mathcal{V} \mapsto \mathcal{V}^*$. For any $u, v_n \in V$ we have

$$\langle T(u), v_n\rangle \;=\; \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v_n\, dx + \lambda \int_{\mathbb{R}^N} \big(g_3 + g_4\big)\, |u|^{p-2} u v_n\, dx\,.$$

Let $v_n \to v$ in $\mathcal{V}$ then $|\nabla v_n|^{p-2} \nabla v_n \to |\nabla v|^{p-2} \nabla v$ in $(L^p)^N\big(\mathbb{R}^N\big)$. Moreover, we have that

$$\int_{\mathbb{R}^N} (g_3+g_4)\, |u|^{p-2} u (v_n - v) dx \le \left\{\int_{\mathbb{R}^N} (g_3 + g_4)\, |u|^{p'}\right\}^{\frac{1}{p'}} \left\{\int_{\mathbb{R}^N} (g_3 + g_4)\, |v_n - v|^p\right\}^{\frac{1}{p}}.$$

So $T(u) \in \mathcal{V}^*$. The functional

$$\Phi(u) \;=:\; \frac{1}{p}\, ||u||_{\mathcal{V}}^p - \langle f, u\rangle \;\ge\; \frac{1}{p}\, ||u||_{\mathcal{V}}^p - ||f||_{\mathcal{V}^*} ||u||_{\mathcal{V}}\,,$$

is weakly semicontinuous, strictly convex and coercive for any $f \in \mathcal{V}^*$; hence T is surjective. Similarly, we prove that T is continuous. Indeed, if $u_n \to u$ in $\mathcal{V}$ then

$$\begin{array}{rcll} |\nabla u_n|^{p-2} \nabla u_n & \longrightarrow & |\nabla u|^{p-2} \nabla u\,, & \text{in} \quad \big(L^{p'}\big(\mathbb{R}^N\big)\big)^N, \quad \text{and} \\ (g_3 + g_4)^{\frac{1}{p'}}\, |u_n|^{p-2} u_n & \longrightarrow & (g_3 + g_4)^{\frac{1}{p'}}\, |u|^{p-2} u & \text{in} \quad L^{p'}\big(\mathbb{R}^N\big)\,. \end{array}$$

Obviously T is $(p-1)$ – homogeneous. Moreover,

$$||T(u)||_{\mathcal{V}^*} \;=\; \sup_{v \in \mathcal{V},\, v \neq 0} \frac{\int_{\mathbb{R}^N} [|\nabla u|^{p-2} \nabla u \cdot \nabla v + \lambda (g_3 + g_4)\, |u|^{p-2} u v]\, dx}{||v||_{\mathcal{V}}}\,,$$

so

$$||u||_{\mathcal{V}}^{p-1} \;\le\; ||T(u)||_{\mathcal{V}^*} \;\le\; 2\, ||u||_{\mathcal{V}}^{p-1}\,. \tag{4.2}$$

T is $(1, 2, p-1)$ – homeomorphism if we can prove that the operator T^{-1} is continuous. Indeed, let $f_n \to f$ in $\mathcal{V}^*$. Setting $u_n = T^{-1}(f_n)$, $u = T^{-1}(f)$ we have

$$\begin{aligned} & \langle T(u_n) - T(u), u_n - u\rangle \\ = & \int_{\mathbb{R}^N} [|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u] \cdot (\nabla u_n - \nabla u)\, dx \\ & + \lambda \int_{\mathbb{R}^N} (g_3 + g_4) \big[|u_n|^{p-2} u_n - |u|^{p-2} u\big] (u_n - u)\, dx\,. \end{aligned} \tag{4.3}$$

Using the following well known algebraic relation

$$|\xi - \xi'|^p \;\le\; C\big\{\big[|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'\big](\xi - \xi')\big\}^{\frac{\alpha}{2}} \big\{\big[|\xi|^p + |\xi'|^p\big]\big\}^{\frac{2-\alpha}{2}}\,, \qquad \text{for all} \quad \xi\,, \xi' \in \mathbb{R}^N\,,$$

where $\alpha = 2$ if $p \geq 2$, $a = p$ if $1 < p \leq 2$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} [|\bigtriangledown u_n - \bigtriangledown u|^p]\, dx \\
&\leq\; C \left\{ \int_{\mathbb{R}^N} \left[|\bigtriangledown u_n|^{p-2} \bigtriangledown u_n - |\bigtriangledown u|^{p-2} \bigtriangledown u \right] \cdot (\bigtriangledown u_n - \bigtriangledown u)\, dx \right\}^{\frac{\alpha}{2}} \\
&\quad\times\; \left\{ \int_{\mathbb{R}^N} [|\bigtriangledown u_n|^p]\, dx + \int_{\mathbb{R}^N} [|\bigtriangledown u|^p]\, dx \right\}^{\frac{2-\alpha}{2}} .
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} (g_3 + g_4)\, |u_n - u|^p\, dx \\
&\leq\; C \left\{ \lambda \int_{\mathbb{R}^N} (g_3 + g_4) \left[|u_n|^{p-2} u_n - |u|^{p-2} u \right] (u_n - u)\, dx \right\}^{\frac{\alpha}{2}} \\
&\quad\times\; \left\{ \lambda \int_{\mathbb{R}^N} (g_3 + g_4) \left[|u_n|^p + |u|^p \right] dx \right\}^{\frac{2-\alpha}{2}} .
\end{aligned}$$

So from relations (4.2) and (4.3) and the fact that $||f_n||_{\mathcal{V}^*}$ is bounded, we deduce

$$\|u_n - u\|_{\mathcal{V}}^p \;\leq\; C' \left\{ \langle T(u_n) - T(u), u_n - u \rangle \right\}^{\frac{\alpha}{2}} \;\leq\; ||f_n - f||_{\mathcal{V}^*}^{\frac{\alpha}{2}}\, ||u_n - u||_{\mathcal{V}}^{\frac{\alpha}{2}} ,$$

i.e., $u_n \to u$ in $\mathcal{V}$ and T^{-1} is continuous. □

Lemma 4.3. *Assume that g satisfies (H_2) and let $S_i : \mathcal{V} \mapsto \mathcal{V}^*$ be defined by $S_i(u) = g_i(x)\, |u|^{p-2} u,\ i = 1, 2$. Then S_1, S_2 are completely continuous and $(p-1)$ – homogeneous.*

Proof. Let $\{u_n\}$ be a bounded sequence in $\mathcal{V}$. Without loss of generality, we may assume that $u_n \rightharpoonup u$ in this space. Moreover, $\{u_n\}$ is a bounded sequence in $L^{\frac{Np}{N-p}}(\mathbb{R}^N)$, because of the continuous embedding $\mathcal{V} \subset \mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{\frac{Np}{N-p}}(\mathbb{R}^N)$. For S_1 it is sufficient to prove that $S_1(u_n) \to S_1(u)$ in $L^q(\mathbb{R}^N)$ with $q = \frac{Np}{Np-N+p}$, since $L^q(\mathbb{R}^N) \subset \mathcal{D}^{-1,p'}(\mathbb{R}^N) \subset \mathcal{V}^*$. For any $\epsilon > 0$ there is R_0 large enough such that for any $R > R_0,\ n \in \mathbb{N}$ we have

$$\begin{aligned}
&\int_{|x|>R} \left| g_1(x) \left[|u_n|^{p-2} u_n - |u|^{p-2} u \right] \right|^q dx \\
&\leq\; 2^q \int_{|x|>R} |g_1|^q \left[|u_n|^{(p-1)q} + |u|^{(p-1)q} \right] dx \\
&\leq\; C \left\{ \int_{|x|>R} |g_1|^{N/p}\, dx \right\}^{\frac{p^2}{Np-N+p}} \times \left[\left\{ \int_{|x|>R} |u_n|^{\frac{Np}{N-p}}\, dx \right\}^{\frac{(p-1)(N-p)}{Np-N+p}} \right. \\
&\quad \left. +\; \left\{ \int_{|x|>R} |u|^{\frac{Np}{N-p}}\, dx \right\}^{\frac{(p-1)(N-p)}{Np-N+p}} \right] \\
&\leq\; \epsilon .
\end{aligned}$$

Let R be as above. Then $\mathcal{V}_{B_R}$ is compactly embedded in $L^p(B_R)$; so there is a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$ such that $u_n \to u$ in $L^p(B_R)$ and $|u_n|^{p-2}\,u_n \to |u|^{p-2}\,u$ strongly in $L^{\frac{p}{p-1}}(B_R)$. Since $\frac{p}{p-1} > q$ and $g \in L^\infty(\mathbb{R}^N)$, we deduce that

$$\lim_{n\to\infty} \int_{|x|<R} \bigl|g_1(x)\bigl[|u_n|^{p-2}\,u_n - |u|^{p-2}\,u\bigr]\bigr|^q\,dx \;=\; 0\,,$$

and therefore $S_1(u_n) \to S_1(u)$ in $L^q(\mathbb{R}^N)$. By uniqueness of the limit, we have proved that S_1 is completely continuous. Moreover, it is obviously $(p-1)-$homogeneous.

To prove that S_2 is completely continuous, we split it in two parts $S_2 = S_{2,B_R} + S_{2,B_R^c}$ where $S_{2,B_R}(u) =: g_2\chi_{B_R}|u|^{p-2}\,u$ and $S_{2,B_R^c}(u) =: g_2\chi_{B_R^c}|u|^{p-2}\,u$, where χ_A denotes the function which is equal to 1 in A and 0 elsewhere. Let ϕ in $\mathcal{V}$, then we have

$$\begin{aligned}\bigl|\bigl\langle S_{2,B_R^c}(u_n),\phi\bigr\rangle\bigr| \;&\le\; \int_{B_R^c} \bigl|g_2\,|u_n|^{p-1}\phi\bigr|\,dx\\ &\le\; \left\{\int_{B_R^c} |g_2|\,|u_n|^p\,dx\right\}^{\frac{1}{p'}} \left\{\int_{B_R^c} |g_2|\,|\phi|^p\,dx\right\}^{\frac{1}{p}}.\end{aligned}$$

Applying [6, Theorem 2.3] of Berger and Schechter, we have

$$\Bigl\|g_2^{1/p}u\Bigr\|_{L^p\left(\Omega\cap B_R^c\right)} \;\le\; C M_{\alpha,p}\Bigl(g_2^{1/p},\Omega\cap B_R^c\Bigr)\;||u||_{W^{1,p}\left(\Omega\cap B_R^c\right)}\,,$$

where $M_{\alpha,p}\Bigl(g_2^{1/p},\Omega\cap B_R^c\Bigr)$ is given by (2.4) and we have

$$M_{\alpha,p}\Bigl(g_2^{1/p},\Omega\cap B_R^c\Bigr) \;\le\; k_{R-1} \;=:\; \sup_{|x|\ge R-1}\int_{|y|<1} g_2(x-y)\,|y|^{\alpha-N}\,dy\,.$$

Hence

$$\bigl\|g_2^{1/p}u_n\bigr\|_{L^p\left(\Omega\cap B_R^c\right)} \;\le\; Ck_{R-1}\,\|u_n\|^p_{\mathcal{V}}\,.$$

Using the same estimates on $\phi\in\mathcal{V}$, we obtain

$$\bigl\|S_{2,B_R^c}(u_n)\bigr\|_{\mathcal{V}^*} \;=\; \sup_{\phi\in\mathcal{V},\,\phi\ne0} \frac{\bigl\langle S_{2,\,B_R^c}(u_n),\phi\bigr\rangle}{\|\phi\|_{\mathcal{V}}} \;\le\; Ck_{R-1}\,\|u_n\|^{p-1}_{\mathcal{V}}\,.$$

Therefore, by hypothesis $(\mathcal{H}2)\,(3)$ for any $\epsilon > 0$ there is some R_0 such that for any $R > R_0$ we have

(4.4) $$\bigl\|S_{2,\,B_R^c}(u_n) - S_{2,\,B_R^c}(u)\bigr\|_{\mathcal{V}^*} \;\le\; \epsilon\,.$$

For such an R and any $\phi\in V$ we have

$$\begin{aligned}&\int_{|x|<R} \bigl|g_2(x)\bigl[|u_n|^{p-2}\,u_n - |u|^{p-2}\,u\bigr]\bigr|\,\phi\\ \le\;& ||g_2||_\infty\,||\phi||_{L^p(B_R)}\,\bigl\||u_n|^{p-2}\,u_n - |u|^{p-2}\,u\bigr\|_{L^{p'}(B_R)}\,.\end{aligned}$$

Since $\mathcal{V}$ is compactly embedded in $L^p(B_R)$ then $|u_n|^{p-2}\,u_n$ converges strongly to $|u|^{p-2}\,u$ in $L^{p'}(B_R)$. So for some $n \geq n_0$ we have

$$\big\||u_n|^{p-2}\,u_n - |u|^{p-2}\,u\big\|_{L^{p'}(B_R)} \;\leq\; \frac{1}{||g_2||_\infty}\,\epsilon\,.$$

Hence using the same embedding for $\phi \in V$ we get

$$||S_{2,B_R}(u_n) - S_{2,B_R}(u)||_{\mathcal{V}^*} \;\leq\; \epsilon\,,$$

which together with relation (4.4) proves that S_2 is completely continuous. □

Theorem 4.4. *Let g satisfies hypothesis $(\mathcal{H}1)$ and $(\mathcal{H}2)$. Then there exists $\epsilon > 0$ such that Equation* (4.1) *has a solution $u \in \mathcal{V}$, for any $\lambda \in (0, \lambda_1) \cup (\lambda_1, \lambda_1 + \epsilon)$ and $f \in \mathcal{V}^*$.*

Proof. We use the above form of Fredholm Alternative, i. e., Theorem 4.1 and Lemmas 4.2, 4.3. Since $\lambda_1 > 0$ is the smallest positive eigenvalue of $(-\Delta_p)$ and is isolated [13], then 1 is not an eigenvalue for the couple $(T, S_1 + S_2)$. So $T - S$ is surjective and problem (4.1) has a nontrivial solutions for $\lambda \in (0, \lambda_1) \cup (\lambda_1, \lambda_1 + \epsilon)$ and the proof is complete. □

5. An anti – maximum principle

In this section, we consider the perturbation of the $p-$Laplacian equation

$$-\Delta_p u \;=\; \mu g(x)\psi_p(u) + f(x)\,, \quad x \in \mathbb{R}^N\,. \tag{5.1}$$

We have the following nonexistence result

Theorem 5.1. *Assume that $f \in L^\infty$, $f \geq 0$, $f \not\equiv 0$. Then*
(i) *Equation* (5.1) *has no solution in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ for $\mu = \lambda_1$.*
(ii) *Equation* (5.1) *has no solution $u \geq 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ for $\mu \geq \lambda_1$.*

Proof. (i) First we prove that any solution of (5.1) for $\mu = \lambda_1$ satisfies $u \geq 0$. Multiplying (5.1) by $u^- =: \max(-u, 0)$ we obtain

$$-\int_{\mathbb{R}^N} |\nabla u^-|^p\,dx \;=\; -\lambda_1 \int_{\mathbb{R}^N} g(x)\,|u^-|^p\,dx + \int_{\mathbb{R}^N} f(x)\,|u^-|\,dx\,.$$

Hence

$$-\lambda_1 \int_{\mathbb{R}^N} g(x)\,|u^-|^p\,dx \;\leq\; \int_{\mathbb{R}^N} |\nabla u^-|^p\,dx \;\leq\; \lambda_1 \int_{\mathbb{R}^N} g(x)\,|u^-|^p\,dx\,.$$

The functional $v \mapsto -\int_{\mathbb{R}^N} |\nabla v|^p\,dx - \lambda_1 \int_{\mathbb{R}^N} g(x)\,|v|^p\,dx$ reaches its minimum (say 0) at $v = u^-$; hence its gradient vanishes for $v = u^-$; which implies that u^-; is an eigenvalue associated to λ_1 and by simplicity of λ_1, $u^- = c\phi_1$. Then by (5.1) we get that $c = 0$.

(ii) Now we consider a solution u of equation (5.1) in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ for $\mu \geq \lambda_1$ satisfying $u \geq 0$. Since $f \in L^\infty$ then $u \in L^\infty$ and Tolksdorf' estimates imply that $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$. Moreover by Vazquez' Maximum Principle we obtain that $u > 0$ in all $\mathbb{R}^N$. Next, to prove the nonexistence we multiply by ϕ_1 the equation

$$-\Delta_p \phi_1 \; = \; \lambda_1 g(x) \psi_p(\phi_1)\,, \quad x \in \mathbb{R}^N\,,$$

and by $\frac{|\phi_1|^p}{\psi_p(u)}$ the equation (5.1); integrating on B_R and substracting, we obtain

$$\Theta(R) - \int_{\partial B_R} \phi_1 \, |\bigtriangledown \phi_1|^{p-2} \, \frac{\partial \phi_1}{\partial \eta} \, ds + \beta(R) \; = \; - \int_{B_R} f \, \frac{|\phi_1|^p}{\psi_p(u)} + (\lambda_1 - \mu) \int_{B_R} g \, |\phi_1|^p\,,$$

which is nonpositive for $R \geq R_0$ since $\int_{B_R} g \, |\phi_1|^p > 0$. Here

$$\Theta(R) \; =: \; \int_{B_R} \left\{ |\bigtriangledown \phi_1|^p + (p-1) \left(\frac{\phi_1}{u} \right)^p |\bigtriangledown u|^p - p \bigtriangledown \phi_1 \cdot \bigtriangledown u \, |\bigtriangledown u|^{p-2} \left(\frac{\phi_1}{u} \right)^{p-1} \right\} dx$$

and

$$\beta(R) \; =: \; \int_{\partial B_R} \frac{\phi_1^p}{u^{p-1}} \, |\bigtriangledown u|^{p-2} \, \frac{\partial u}{\partial \eta} \, ds\,.$$

Arguing as in the proof of [16, Theorem 3.4], we prove that $\lim_{R\to\infty} \beta(R) = 0$ and therefore $\lim_{R\to\infty} \Theta(R) = 0$. Then [16, Lemma 3.2] implies that $u = c\phi_1$ for some positive c. Substituting u by $c\phi_1$ in (5.1) we obtain a contradiction, since $f \not\equiv 0$. □

In the case of $p = 2$ we have the following weak formulation of the antimaximum principle for unbounded domains. This result can be considered as an extension of the antimaximum principle [9] to the unbounded domain.

Theorem 5.2. *Assume that $p = 2$ and that $f \in L^\infty$, $f \geq 0$, $f \not\equiv 0$. Then for any $R > 0$ there exists $\delta = \delta(f, R)$ such that for any $\lambda_1 < \mu \leq \lambda_1 + \delta$, any solution u of*

$$-\Delta u \; = \; \mu g(x) u + f(x)\,, \quad x \in \mathbb{R}^N\,,$$

satisfies $u < 0$ in B_R.

Proof. By contradiction, we assume that there exists $\hat{R}$, $\alpha_k \searrow \lambda_1$, $x_k \in B_{\hat{R}}$ and u_k satisfying $u_k(x_k) \geq 0$ and

$$-\Delta u_k \; = \; \alpha_k g(x) u_k + f(x)\,, \quad x \in \mathbb{R}^N\,. \tag{5.2}$$

First, we remark that $\lim_{k\to\infty} ||u_k||_\infty = +\infty$. If not, i.e., $||u_k||_\infty \leq C$ for some positive C and we have uniform estimate of u_k in $\mathcal{D}^{1,2}(\mathbb{R}^N)$; passing to the limit in (5.2), we obtain a solution of (5.1) which is impossible by Theorem 5.1. Setting $v_k = \frac{u_k}{||u_k||_\infty}$, we see that v_k converges to some $v \neq 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and in all $C^1(B_R)$. Because of the uniform estimates v_k satisfies

$$-\Delta v_k \; = \; \alpha_k g(x) v_k + \frac{f(x)}{||u_k||_\infty}\,, \quad x \in \mathbb{R}^N\,. \tag{5.3}$$

So passing to the limit as $k \to +\infty$ we have

$$-\Delta v \;=\; \lambda_1 g(x) v\,, \quad x \in \mathbb{R}^N\,.$$

Hence $v = c\,\phi_1$ with $c \neq 0$. Multiplying (5.3) by ϕ_1 and integrating on $\mathbb{R}^N$ we obtain

$$\begin{aligned}\lambda_1 \int_{\mathbb{R}^N} g(x)\phi_1 v_k\,dx \;&=\; \langle -\Delta\phi_1, v_k\rangle \;=\; \langle -\Delta v_k, \phi_1\rangle \\ &=\; \alpha_k \int_{\mathbb{R}^N} g(x)\phi_1 v_k\,dx + \int_{\mathbb{R}^N} \frac{f(x)}{\|u_k\|_\infty}\,\phi_1\,dx\,.\end{aligned}$$

Hence

$$(\alpha_k - \lambda_1)\int_{\mathbb{R}^N} g(x)\phi_1 v_k\,dx \;=\; -\int_{\mathbb{R}^N} \frac{f(x)}{\|u_k\|_\infty}\,\phi_1\,dx \;\leq\; 0\,,$$

which implies that $\int_{\mathbb{R}^N} g(x)\phi_1 v_k\,dx \leq 0$. So passing to the limit we obtain that $\int_{\mathbb{R}^N} g(x) c\phi_1^2\,dx \leq 0$, i.e., $c < 0$. But then $u_k \to c\phi_1$ uniformly on B_R, which contradicts the existence of the sequence x_k. □

When $g(x) \leq 0$ at infinity then the preceding result can be improved.

Theorem 5.3. *Assume that there exists $R_0 > 0$ such that for all $|x| > R_0$ we have $g(x) \leq 0$. Then for any $1 < p < +\infty$, Theorem 5.2 remains valid for the equation*

$$-\Delta_p u \;=\; \mu g(x)\psi_p(u) + f(x)\,, \quad x \in \mathbb{R}^N\,.$$

P r o o f. As in Theorem 5.2, $v_k = \frac{u_k}{\|u_k\|_\infty}$ converges to some $v \neq 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and, because of Tolksdorf's estimates [19], in all $C^1(B_R)$. If $c > 0$, for any $k \geq k_0$ we have that $v_k(x) > 0$ for any $x \in B_{R_0}$. We multiply the equation

$$-\Delta_p v_k \;=\; \alpha_k g(x)\psi_p(v_k) + \frac{f(x)}{\|u_k\|_\infty^{p-1}}\,, \quad x \in \mathbb{R}^N\,, \tag{5.4}$$

by v_k^- and obtain

$$\int_{\mathbb{R}^N} |\nabla v_k^-|^p\,dx \;\leq\; \alpha_k \int_{\mathbb{R}^N} g(x)\,|v_k^-|^p\,dx \;=\; \alpha_k \int_{B_{R_0}^c} g(x)\,|v_k^-|^p\,dx \;\leq\; 0\,,$$

since $g(x) \leq 0$ on $B_{R_0}^c$. Hence $v_k \geq 0$ and Equation (5.4) can be written as

$$-\Delta_p v_k \;=\; \alpha_k g(x)\psi_p(v_k) + h_k(x)\,, \quad x \in \mathbb{R}^N\,,$$

with $h_k \geq 0$, which contradicts to the Theorem 5.2. Therefore, $c < 0$ and the conclusion follows. □

Corollary 5.4. *Assume that there exists $R_0 > 0$ such that for all $|x| > R_0$ we have $g(x) \leq 0$ and $1 < p < +\infty$. Then for any $f \in L^\infty$, $f \geq 0$, $f \not\equiv 0$ and such that*

$f(x) = 0$ for any $|x| \geq R_0$ the following conclusion holds: There exists $\delta = \delta(f)$ such that for any $\lambda_1 < \mu \leq \lambda_1 + \delta$, any solution u of

$$-\Delta_p u \;=\; \mu g(x)\psi_p(u) + f(x)\,, \quad x \in \mathbb{R}^N\,,$$

satisfies $u < 0$ in $\mathbb{R}^N$.

Proof. By Theorem 5.3, v_k converges to some $c\,\phi_1$ $c < 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and in all $C^1(B_R)$. So for any $k \geq k_0$ we have that $v_k(x) < 0$ for any $x \in B_{R_0}$. Multiplying Equation (5.4) by v_k^+ we obtain

$$\int_{\mathbb{R}^N} |\nabla v_k^+|^p \, dx \;=\; \alpha_k \int_{\mathbb{R}^N} g(x)\, |v_k^+|^p \, dx \;\leq\; 0\,.$$

Hence $v_k(x) \leq 0$ and applying Vazquez' Maximum Principle in $B_{R_0}^c$ we get that $v_k(x) < 0$ in $B_{R_0}^c$; and the proof is completed. □

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