Positivity for a noncooperative system of elliptic equations in $\mathbb{R}^{\mathbb{N}}$.

Nikos Stavrakakis*
Department of Mathematics
National Technical University
Zografou Campus
157 80 Athens
Greece

Guido Sweers†
Dept. of Pure Mathematics
Delft University of Technology
PObox 5031
2600 GA Delft
The Netherlands

(appeared in Advances in Diff. Eq. 4 (1999), pp 115-136)

Abstract

Elliptic equations such as $-\Delta u = f$ and elliptic systems with cooperative coupling are known to preserve positivity under appropriate boundary conditions or growth conditions near ∞. Here it is shown that some elliptic systems on $\mathbb{R}^N$ with small noncooperative coupling still have a restricted uniform positivity preserving property similar as in bounded domains. The proofs rely on optimal estimates for the Newtonian potential with weigths and on corresponding $3G$-type theorems.

1 Introduction

Consider the system

$$\begin{cases} -\Delta u &= f - \mu\ a\ v & \text{on } \mathbb{R}^N, \\ -\Delta v &= b\ u & \text{on } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) &= \lim_{|x|\to\infty} v(x) & = 0, \end{cases} \tag{1}$$

with $\mu \in \mathbb{R}^+$ and $N \geq 3$. With appropriate conditions on a, b and f we will prove the following:

There exists $\mu_c > 0$, depending only on a and b, such that for all $\mu \in [0, \mu_c]$ we have

$$f \geq 0 \Longrightarrow u \geq 0.$$

*stavraka@math.ntua.gr
†sweers@twi.tudelft.nl

For $\mu > 0$ and for example $a, b \geq 0$ the system is noncooperative and does not satisfy the conditions in [3]. Note that the system in (1) is cooperative (or quasimonotone) for $0 \neq a, b \geq 0$ if and only if $\mu \leq 0$. (Cooperative is also known as quasimonotone.)

On smooth bounded domains in $\mathbb{R}^N$ it has been shown in [9] and [12], using the so-called $3G$-theorem of Cranston, Fabes and Zhao [2], that small noncooperative coupling does not destroy the uniform positivity preserving property whenever $f > 0$. A positivity preserving property that depends on f, both for (1) and for the problem on bounded domains, can be shown by a perturbation argument in μ. But for such a result the critical number μ_c will depend on f. In this paper we will show that there is a bound $\mu_c > 0$ up to where positivity is preserved which does not depend on f. Such a uniform type of result can be used in semilinear systems.

As a corollary of our result we find a positivity preserving property for a nonhomogeneous biharmonic equation which seems to be of independent interest.

We finish with some remarks on the proofs. As in [12] we use two-sided pointwise estimate for the solution operator $G(x, y)$ (here the Newtonian potential). Instead of separately estimating the behaviour for $G(x, y)$ for $|x - y| \to 0$ and $x, y \to \partial\Omega$, we have to estimate $G(x, y)$ for $|x - y| \to 0$ and $|x|, |y| \to \infty$. In general we need, next to smallness in an appropriate sense, growth conditions on a and b near ∞.

The spaces that we use come from a paper of Kozono and Sohr ([7]) and also appear in [1]. The existence follows through the solution operator for the equation that is used by Weinberger in [14]. Positivity of the solution is a result from the approximation of the solution with the Green operator, for which it is possible to prove the positivity preserving property by pointwise estimates. Positivity of this Green operator is induced by the positivity preserving property of an auxiliary system.

Using Pinchover's ([10]) equivalence relations for Green functions, one can extend the results obtained here to a more general class of elliptic operators.

2 Main result

First we will fix the appropriate spaces. As in [1] we define the space $\mathcal{D}^{1,2}$, which will contain the solutions u, in the following way.

Definition 1 $\mathcal{D}^{1,2} = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_E}$, *where the closure is with respect to the energy norm defined by*

$$\|u\|_E^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

Remark 1. In [7, Prop. 2.4] it is proven that

$$\mathcal{D}^{1,2} = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N); |\nabla u| \in L^2(\mathbb{R}^N) \right\}.$$

Hence $\mathcal{D}^{1,2}$ is a reflexive Banach space and is continuously embedded in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$.

The weight functions a and b will be in the following type spaces.

Definition 2 *Let $\ell \geq 0$. We set*

$$L^\infty_\ell(\mathbb{R}^N) := \left\{a(\cdot)\,;\left(x \mapsto (1+|x|)^\ell \ \ a(x)\right) \in L^\infty(\mathbb{R}^N)\right\},$$

and

$$c_\ell(a) := \left\|(1+|\cdot|)^\ell \ \ a_+(\cdot)\right\|_\infty + \left\|(1+|\cdot|)^\ell \ \ a_-(\cdot)\right\|_\infty, \tag{2}$$

where $a_+ = \max(a,0)$ and $a = a_+ - a_-$.

Remark 2. Note that $L^\infty_\ell(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ whenever $p\ell > N$.

The main result of the paper is the next theorem.

Theorem 3 *Let $a \in L^\infty_{\ell_a}(\mathbb{R}^N)$ and $b \in L^\infty_{\ell_b}(\mathbb{R}^N)$ with ℓ_a, ℓ_b satisfying*

$$\ell_a + \ell_b > 4 \text{ and } \ell_a, \ell_b \geq 0 \text{ and } \ell_a, \ell_b > 4 - N. \tag{3}$$

Then there exists $\mu_{N,\ell_a,\ell_b} > 0$ such that for all $f \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and μ with

$$|\mu| \leq \mu_{N,\ell_a,\ell_b} \ \left(c_{\ell_a}(a)\ c_{\ell_b}(b)\right)^{-1}$$

the system

$$\left\{\begin{array}{rcll} -\Delta u &=& f - \mu\ a\ v & \text{on } \mathbb{R}^N, \\ -\Delta v &=& b\ u & \text{on } \mathbb{R}^N, \end{array}\right. \tag{4}$$

has a unique (weak) solution $u, v \in \mathcal{D}^{1,2}$. Moreover,

$$f \geq 0 \text{ implies } u \geq 0.$$

Corollary 4 *Suppose that $N \geq 5$ and that $a \in L^\infty_{\ell_a}(\mathbb{R}^N)$ with $\ell_a > 4$. Then for all $f \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and μ with $|\mu| \leq \mu_{N,\ell_a,0}\left(c_{\ell_a}(a)\right)^{-1}$ the biharmonic equation*

$$(-\Delta)^2 v + \mu\ a\ v = f$$

is positivity preserving.

Theorem 3 will be proven in the last section. The corollary follows by taking $b = 1$ and hence $\ell_b = 0$ and $c_{\ell_b} = 1$.

In the rest of the paper we will write $L^p = L^p(\mathbb{R}^N)$, $C_0^\infty = C_0^\infty(\mathbb{R}^N)$ et cetera. We will also denote $\|u\|_p = \|u\|_{L^p}$ and $\|u\|_{L^{q_1}\cap L^{q_2}} = \|u\|_{L^{q_1}} + \|u\|_{L^{q_2}}$.

3 The equation

In this section we will be concerned with solving

$$\left\{\begin{array}{c} -\Delta u = f \text{ in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}, \end{array}\right. \tag{5}$$

for appropriate functions f. As usual $u \in \mathcal{D}^{1,2}$ is called a weak solution of (5) if $\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v - f\ v)\, dx = 0$ for all $v \in \mathcal{D}^{1,2}$. The integral is well defined for $f \in L^{\frac{2N}{N+2}}$.

3.1 Solution operator by Weinberger

Define the bilinear form $\mathcal{A}(\cdot,\cdot)$ for $u,v \in C_0^\infty$ by

$$\mathcal{A}(u,v) = (u, -\Delta v), \tag{6}$$

where $(u,v) = \int_{\mathbb{R}^N} u(x)\, v(x)\, dx$. Let us also define the bilinear functional

$$\mathcal{B}_f(u,v) = (u,f)(f,v),$$

with u and v as above and $f \in L^1_{loc}$. By an application of Theorem 4.1 in [14, page 54] we find the following result.

Lemma 5 *Let $f \in L^{\frac{2N}{N+2}}$. Then the value*

$$\mu_1 = \sup_{\substack{v \in C_0^\infty \\ \mathcal{A}(v,v)=1}} \mathcal{B}_f(v,v)$$

is attained for some $v = \phi_f$ in $\mathcal{D}^{1,2}$. Moreover, the operator $\mathcal{H} : L^{\frac{2N}{N+2}} \to \mathcal{D}^{1,2}$ is well defined by

$$\mathcal{H}f = (\phi_f, f)\, \phi_f \tag{7}$$

and one finds that $u = \mathcal{H}f$ is a weak solution of

$$-\Delta u = f \text{ on } \mathbb{R}^N. \tag{8}$$

Remark 3. Since $\mathcal{D}^{1,2} \subset L^{\frac{2N}{N-2}}$ the function $u = \mathcal{H}f$ goes to zero near ∞ in a weak sense. Hence, by Liouville's Theorem, such a solution is unique.

Proof. In order to apply Theorem 4.1 of Weinberger we have to check the conditions in Hypothesis 4.1 of [14]. Set $V = C_0^\infty$ and let $\mathcal{A}$ be defined by (6). Then $V_{\mathcal{A}} = \mathcal{D}^{1,2}$. The conditions are as follows.

(a) Let $w \in V$. Then $(v,w) = 0$ for all $v \in V$ implies $w = 0$.

(b) For $v,w \in C_0^\infty$ we find $\mathcal{A}(v,w) = \mathcal{A}(w,v)$ and $\mathcal{A}(v,v) = \|v\|_E^2 > 0$ for $v \neq 0$.

(c) The functional $v \to (v,f)$ for $v \in V$ is bounded in the following way:

$$|(v,f)| = \left|\textstyle\int_{\mathbb{R}^N} vf\ dx\right| \le \|v\|_{L^{\frac{2N}{N-2}}} \|f\|_{L^{\frac{2N}{N+2}}} \le$$
$$\le c\,\|v\|_E \|f\|_{L^{\frac{2N}{N+2}}} = c\,(A(v,v))^{\frac{1}{2}} \|f\|_{L^{\frac{2N}{N+2}}}.$$

Here we used the continuous embedding of $\mathcal{D}^{1,2}$ in $L^{\frac{2N}{N-2}}$.

By Weinberger's Theorem μ_1 is attained for some $\phi_f \in V_{\mathcal{A}} = \mathcal{D}^{1,2}$ and $u = \mathcal{H}f$ satisfies (8) in the weak sense. □

Lemma 6 *One finds that* $\mathcal{H} : L^{\frac{2N}{N+2}} \to \mathcal{D}^{1,2}$*, defined in* (7)*, is a bounded linear operator.*

Proof. For any $f \in L^{\frac{2N}{N+2}}$ we have

$$\|\mathcal{H}f\|_E = |(\phi_f, f)| \left(\int_{\mathbb{R}^N} |\nabla \phi_f|^2 \, dx \right)^{1/2} \leq$$

$$\leq \|f\|_{\frac{2N}{N+2}} \ \|\phi_f\|_{\frac{2N}{N-2}} \ \|\phi_f\|_E \leq c_N \ \|\phi_f\|_E^2 \ \|f\|_{\frac{2N}{N+2}} .$$

Since $\|\phi_f\|_E = 1$ the claim follows. □

Corollary 7 *If* $a \in L^{N/2}$*, then with* $\mathcal{H}_a f = \mathcal{H}(af)$ *we have that* $\mathcal{H}_a$ *is a bounded linear operator from* $L^{\frac{2N}{N-2}}$ *to* $\mathcal{D}^{1,2}$ *(and hence on* $\mathcal{D}^{1,2}$*).*

Proof. Since $\mathcal{D}^{1,2} \subset L^{\frac{2N}{N-2}}$ and $a \in L^{N/2}$ we find that $af \in L^{\frac{2N}{N+2}}$ and hence that we may use Lemma 5. □

Lemma 8 *If* $f \in L^{\frac{2N}{N+2}} \cap L^{N/2+\varepsilon}$ *for some* $\varepsilon > 0$ *then* $\mathcal{H}f \in L^\infty$*. If moreover* $f \in C^\gamma_{loc}$ *for some* $\gamma > 0$*, then* $\mathcal{H}f(x) \in C^{2,\gamma}_{loc}$*.*

Proof. Since $f \in L^{N/2+\varepsilon}$ we get by [1, Theorem 3.1] that $\mathcal{H}f \in L^\infty$. The regularity result is a direct consequence of [4, Theorem 9.19]. □

3.2 Solution operator by a Newtonian potential

The Newtonian potential (see [4]), or Green's function on $\mathbb{R}^N$,

$$G(x,y) = \frac{1}{N(N-2)\,\omega_N} |x-y|^{2-N} \text{ with } \omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

satisfies $-\Delta_x G(x,y) = \delta_y(x)$. We define the operator $\mathcal{G}_a : C_0^\infty \to L^1_{loc}$ by

$$(\mathcal{G}_a f)(x) = \int_{\mathbb{R}^N} G(x,y)\, a(y)\, f(y)\, dy. \tag{9}$$

We will prove that for appropriate a and p this operator can be extended such that $u = \mathcal{G}_a f$ with $f \in L^p$ solves

$$\begin{cases} -\Delta u = a\ f & \text{on } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{10}$$

Lemma 9 *For all $\varepsilon > 0$ we have that $\mathcal{G}_1 : \left(L^{\frac{N}{2}+\varepsilon} \cap L^{\frac{N}{2}-\varepsilon}\right) \to L^\infty$ is a well defined bounded linear operator.*

Proof. Set $p_1 = \frac{N}{2} - \varepsilon$, $p_2 = \frac{N}{2} + \varepsilon$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Then

$$\int\limits_{w\in\mathbb{R}^N} G(x,y)\,|f(y)|\ dy = c_N \left(\int\limits_{|x-y|>1} + \int\limits_{|x-y|<1} \right) |x-y|^{2-N}\,|f(y)|\ dy \le$$

$$\le c_N \left(\int\limits_{|x-y|>1} |x-y|^{q_1(2-N)}\ dy \right)^{\frac{1}{q_1}} \left(\int\limits_{|x-y|>1} |f(y)|^{p_1}\ dy \right)^{\frac{1}{p_1}} +$$

$$+ c_N \left(\int\limits_{|x-y|<1} |x-y|^{q_2(2-N)}\ dy \right)^{\frac{1}{q_2}} \left(\int\limits_{|x-y|<1} |f(y)|^{p_2}\ dy \right)^{\frac{1}{p_2}} \le$$

$$\le c_N\,\gamma_N \left(\int\limits_{r>1} r^{q_1(2-N)+N-1}\ dr \right)^{\frac{1}{q_1}} \|f\|_{L^{\frac{N}{2}+\varepsilon}} + c_N\,\gamma_N \left(\int\limits_{r<1} r^{q_2(2-N)+N-1}\ dr \right)^{\frac{1}{q_2}} \|f\|_{L^{\frac{N}{2}-\varepsilon}} \le$$

$$\le c_{N,\varepsilon} \left(\|f\|_{L^{\frac{N}{2}+\varepsilon}} + \|f\|_{L^{\frac{N}{2}-\varepsilon}} \right).$$

Indeed, since we have that $q_1(2-N) + N - 1 < -1$ and $q_2(2-N) + N - 1 > -1$, the last step holds. By γ_N we denoted the surface of the unit ball in $\mathbb{R}^N$. □

Lemma 10 *Suppose that $f \in L^p$ with $N/2 < p < \infty$ and assume that we have $a \in L^\infty_\ell$ for some $\ell > 2 - N/p$.*
Then $u = \mathcal{G}_a f$, defined in (9), is the unique solution in $W^{2,p}_{\mathrm{loc}}$ of (10). Moreover, there is $C_{N,\ell,p}$ such that

$$|u(x)| \le C_{N,\ell,p}\ c_\ell(a)\ \ (1+|x|)^{-m_{N,\ell,p}}\ \ \|f\|_p \quad \text{for all } x \in \mathbb{R}^N, \tag{11}$$

where

$$m_{N,\ell,p} = \min\left(\ell - 2 + \frac{N}{p}, N-2\right).$$

Remark 4. If a is bounded and has a compact support, then one obtains for $p > \frac{1}{2}N$:

$$|u(x)| \le C_a\ \ (1+|x|)^{2-N}\ \ \|f\|_p \quad \text{for all } x \in \mathbb{R}^N. \tag{12}$$

Proof. By a Liouville Theorem (see [4, p. 29]) (10) has at most one solution. If the integral in (9) converges then this u satisfies the differential equation in (10) and is in $W^{2,p}_{loc}$, see [4, Th. 9.19]. Hence it is sufficient to show (11). Since for $\frac{1}{p} + \frac{1}{q} = 1$ one has

$$\left| \int\limits_{\mathbb{R}^N} G(x,y)\,a(y)\,f(y)\,dy \right| \le \|f\|_p\ \ \|G(x,\cdot)\,a(\cdot)\|_q\,,$$

we will estimate $\|G(x,\cdot)\,a(\cdot)\|_q$.

For $|x| \leq 1$ we find that $|x-y| \geq 2$ implies $|y| \leq |y-x| + |x| \leq 2\,|y-x|$ and $|y| \geq |x-y| - |x| \geq 1$ and hence

$$\|G(x,\cdot)\,a(\cdot)\|_q^q = \int_{\mathbb{R}^N} |G(x,y)\,a(y)|^q\,dy \leq$$

$$\leq c_N\; c_\ell(a) \left(\int_{|x-y|\leq 2} + \int_{|x-y|\geq 2} \right) |x-y|^{(2-N)q}\,(1+|y|)^{-\ell q}\,dy \leq$$

$$\leq c_N\; c_\ell(a) \left(\int_{|x-y|\leq 2} |x-y|^{(2-N)q}\,dy + 2^{(N-2)q} \int_{|y|\geq 1} |y|^{(2-N-\ell)q}\,dy \right) \leq$$

$$\leq c_N\; c_\ell(a)\;\; M_{N,\ell,q}, \tag{13}$$

for some constant $M_{N,\ell,q}$ whenever

$$(2-N)\,q + N > 0 \text{ and } (2-N-\ell)\,q + N < 0. \tag{14}$$

The first inequality is satisfied since $p > \frac{1}{2}N$; the second one follows from $\ell > 2 - \frac{N}{p}$.

For $|x| \geq 1$ we find

$$\|G(x,\cdot)\,a(\cdot)\|_q^q = \int_{\mathbb{R}^N} |G(x,y)\,a(y)|^q\,dy \leq$$

$$\leq c_N\; c_\ell(a) \left(\int_{|y|\leq\frac{1}{2}} + \int_{\substack{|y-x|\leq\frac{1}{2}|x| \\ |y|\geq\frac{1}{2}}} + \int_{\substack{\frac{1}{2}|x|\leq|y-x|\leq 2|x| \\ |y|\geq\frac{1}{2}}} + \int_{\substack{|y-x|\geq 2|x| \\ |y|\geq\frac{1}{2}}} \right) |x-y|^{(2-N)q}\,(1+|y|)^{-\ell q}\,dy \leq$$

$$\leq c_N'\; c_\ell(a) \left(\int_{|y|\leq\frac{1}{2}} |x|^{(2-N)q}\,dy + \int_{\substack{|y-x|\leq\frac{1}{2}|x| \\ |y|\geq\frac{1}{2}}} |x-y|^{(2-N)q}\,|x|^{-\ell q}\,dy + \right.$$

$$\left. + \int_{\substack{\frac{1}{2}|x|\leq|y-x|\leq 2|x| \\ |y|\geq\frac{1}{2}}} |x|^{(2-N)q}\,|y|^{-\ell q}\,dy + \int_{\substack{|y-x|\geq 2|x| \\ |y|\geq\frac{1}{2}}} |x-y|^{(2-N-\ell)q}\,dy \right) \leq$$

$$\leq c_N''\; c_\ell(a) \left(\omega_N\,|x|^{(2-N)q} + \gamma_N\,|x|^{-\ell q} \int_{r=0}^{\frac{1}{2}|x|} r^{(2-N)q+N-1}\;dr + \right.$$

$$\left. + \gamma_N\,|x|^{(2-N)q} \int_{r=\frac{1}{2}}^{3|x|} r^{-\ell q+N-1}\;dr + \gamma_N \int_{r=2|x|}^{\infty} r^{(2-N-\ell)q+N-1}\;dr \right) \leq$$

(here we use (14))

$$\leq c_N''' \, c_\ell(a) \, \left(|x|^{(2-N)q} + |x|^{(2-N-\ell)q+N} + |x|^{(2-N)q} \left(1 + |x|^{N-\ell q}\right) + |x|^{(2-N-\ell)q+N} \right) \leq$$

$$\leq c_\ell(a) \; M'_{N,\ell,q} \; \left(|x|^{(2-N-\ell)q+N} + |x|^{(2-N)q} \right), \tag{15}$$

for some $M'_{N,\ell,q}$.

Together (13)-(15) imply

$$|u(x)| \leq c_\ell(a) \; M''_{N,\ell,q} \; \|f\|_p \; (1+|x|)^{\min\left(2-\ell-\frac{N}{p},2-N\right)} \quad \text{for all } x \in \mathbb{R}^N.$$

□

Corollary 11 *Let* $p_1 \in \left(\frac{1}{2}N, \infty\right]$, $p_2 \in \left(\frac{N}{N-2}, \infty\right]$ *and suppose that* $a \in L^\infty_\ell$ *for some* $\ell > 2 - \frac{N}{p_1} + \frac{N}{p_2}$. *Then the operator* $\mathcal{G}_a : L^{p_1} \to L^{p_2}$ *is bounded, i.e.: there is* c_{N,ℓ,p_1,p_2} *such that*

$$\|\mathcal{G}_a f\|_{p_2} \leq c_{N,\ell,p_1,p_2} \; \|f\|_{p_1}.$$

.

Proof. Let $p_2 < \infty$. Notice that $x \mapsto (1+|x|)^{-m} \in L^{p_2}$ if and only if $mp_2 > N$. Hence it is sufficient that $\left(\ell - 2 + \frac{N}{p_1}\right) p_2 > N$ and $(N-2)\, p_2 > N$. For $p_2 = \infty$ it is sufficient that $\ell > 2 - \frac{N}{p_1}$. □

4 A formal decoupling

4.1 Nonnegative weights

First we will do a formal decoupling by using a solution operator that we denote by $\mathcal{G}$. Using this operator $\mathcal{G}$ we can replace the system in (1) by

$$\begin{cases} u = \mathcal{G}f - \mu \, \mathcal{G}(a\, v), \\ v = \mathcal{G}(b\, u). \end{cases} \tag{16}$$

The function u satisfies

$$u = \mathcal{G}f - \mu \, \mathcal{G}(a \, \mathcal{G}(b\, u)), \tag{17}$$

or equally

$$(\mathcal{I} + \mu \mathcal{G} \; a \; \mathcal{G} \; b)\, u = \mathcal{G}f.$$

If the power series converges we can use the expression

$$u = (\mathcal{I} + \mu\mathcal{G} \; a \; \mathcal{G} \; b)^{inv} \mathcal{G}f = \sum_{k=0}^{\infty} (-\mu\mathcal{G} \; a \; \mathcal{G} \; b)^k \mathcal{G}f =$$

$$= \left(\sum_{k=0}^{\infty} (\mu\mathcal{G} \; a \; \mathcal{G} \; b)^{2k} \right) (\mathcal{I} - \mu\mathcal{G} \; a \; \mathcal{G} \; b)\, \mathcal{G}f.$$

By $(\mathcal{I}+\mu\mathcal{G}\ a\ \mathcal{G}\ b)^{inv}$ we denote the (formal) inverse operator. If $a, b \geq 0$ the operator

$$\sum_{k=0}^{\infty} (\mu\mathcal{G}\ a\ \mathcal{G}\ b)^{2k} \tag{18}$$

is positive whenever it is well defined. Hence, it will be sufficient that

$$\mathcal{T}_{\mu}^{G} = (\mathcal{I} - \mu\mathcal{G}\ a\ \mathcal{G}\ b)\,\mathcal{G} \tag{19}$$

is a positive operator. This operator corresponds with the following system

$$\begin{cases} -\Delta u &= f - \mu a v \\ -\Delta v &= b w \qquad\qquad \text{in } \mathbb{R}^N, \\ -\Delta w &= f \\ \quad u, v, w \to 0 & \qquad \text{for } |x| \to \infty. \end{cases} \tag{20}$$

If $\mathcal{T}_{\mu}^{G}$ is well defined we have that the function u in (20) satisfies $u = \mathcal{T}_{\mu}^{G} f$.

Assuming still that all is well defined we get the following integral representation

$$\mathcal{T}_{\mu}^{G} f\,(x) = ((\mathcal{I} - \mu\mathcal{G}\ a\ \mathcal{G}\ b)\,\mathcal{G} f)\,(x) =$$

$$= \int\limits_{y\in\mathbb{R}^N} G(x,y) \left(1 - \mu \frac{\iint\limits_{v,w\in\mathbb{R}^N} G(x,w)\ a(w)\ G(w,v)\ b(v)\ G(v,y)\ dvdw}{G(x,y)} \right) f(y)\,dy.$$

The operator $(\mathcal{I} - \mu\mathcal{G}\ a\ \mathcal{G}\ b)\,\mathcal{G}$ is positive if and only if $\mu \leq M^{-1}$, where

$$M = \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^N} \frac{\iint\limits_{v,w\in\mathbb{R}^N} G(x,w)\ \ a(w)\ \ G(w,v)\ \ b(v)\ \ G(v,y)\ \ dvdw}{G(x,y)}.$$

Next to showing that this operator is well defined we have to show that $M < \infty$.

4.2 Indefinite weights

When a or b changes sign the operator inside (18), $(\mathcal{G}\ a\ \mathcal{G}\ b)^2$, although having an even power, is not positivity preserving (compare with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^2$). We will have to make an adaptation. First we split a and b in its positive and negative parts. Define

$$\begin{aligned} \mathcal{S}_c &= \mathcal{G}a_+\mathcal{G}b_- + \mathcal{G}a_-\mathcal{G}b_+, \\ \mathcal{S}_n &= \mathcal{G}a_+\mathcal{G}b_+ + \mathcal{G}a_-\mathcal{G}b_-, \end{aligned} \tag{21}$$

and note that $\mathcal{S}_n - \mathcal{S}_c = \mathcal{G}a\mathcal{G}b$. We replace (17) by

$$(I - \mu\mathcal{S}_c)\,u = \mathcal{G}f - \mu\mathcal{S}_n u. \tag{22}$$

Note that for $\mu\,\nu\left(\mathcal{S}_c\right) < 1$, where $\nu\left(\mathcal{S}_c\right)$ denotes the spectral radius of $\mathcal{S}_c$, the inverse of $\left(I-\mu\mathcal{S}_c\right)$ is well defined and positive. By similar arguments as before and assuming convergence of the power series we find that it will be sufficient to have positivity of

$$\left(\mathcal{I}-\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)\left(I-\mu\mathcal{S}_c\right)^{inv}\mathcal{G}$$

or, due to the fact that $\left(I-\mu\mathcal{S}_c\right)^{inv}\mathcal{G}=\mathcal{G}\left(I-\mu\widetilde{\mathcal{S}}_c\right)^{inv}$ with

$$\begin{aligned}\widetilde{\mathcal{S}}_c &= a_+\mathcal{G}b_-\mathcal{G}+a_-\mathcal{G}b_+\mathcal{G},\\ \widetilde{\mathcal{S}}_n &= a_+\mathcal{G}b_+\mathcal{G}+a_-\mathcal{G}b_-\mathcal{G},\end{aligned} \tag{23}$$

positivity of

$$\mathcal{T}_\mu^G=\left(\mathcal{I}-\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)\mathcal{G}. \tag{24}$$

Note that when $a_-=b_-=0$, hence $\mathcal{S}_c=0$ and $\mathcal{S}_n=\mathcal{G}a\mathcal{G}b$, the definition of $\mathcal{T}_\mu^G$ in (19) and (24) coincide.

The relation between u and f is as follows:

$$u=\left(\mathcal{I}+\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)^{inv}\ \left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{G}f. \tag{25}$$

We recapitulate the needed results. Let $\|\cdot\|$ denote an appropriate operator norm.

- $\|\mu\mathcal{S}_c\|<1$ in order that $\left(I-\mu\mathcal{S}_c\right)^{inv}$ is well defined by a series expansion; positivity of $\left(I-\mu\mathcal{S}_c\right)^{inv}$ follows from the Neumann series;
- $\left\|\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right\|<1$ in order that $\left(\mathcal{I}+\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)^{inv}$ is well defined by a series expansion;
- Positivity of $\mathcal{T}_\mu^G$ in (24) in order that positivity of $\left(\mathcal{I}+\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)^{inv}$ follows from the corresponding Neumann series.

The bounds for the operator norms above are related with the first eigenvalues. The bound to get positivity depends on a 3G-type theorem.

5 3G-type theorems

In this section we will show the result that is related to the 3G-Theorem of Cranston, Fabes and Zhao on bounded domains ([2]).

Theorem 12 *Suppose that $a\in L^q$ with $q>\frac{1}{2}N$ and*

$$\begin{aligned}&q<3 && \text{for } N=3,\\ &q<\infty && \text{for } N=4,\\ &q\le\infty && \text{for } N\ge 5.\end{aligned} \tag{26}$$

Then there exists $\alpha_{N,q} < \infty$ such that

$$\int\limits_{w \in \mathbb{R}^N} \frac{G(x,w)\ \ |a(w)|\ \ G(w,y)}{G(x,y)}\ dw \leq \alpha_{N,q}\ \ \|a\|_{L^q}\ \ |x-y|^{2-N/q}\ \ . \tag{27}$$

If $a \in L^{q_1} \cap L^{q_2}$, with $q_1 < \frac{1}{2}N$ and $q_2 > \frac{1}{2}N$, we find for some $\beta_{N,q_1,q_2} > 0$ that

$$\int\limits_{w \in \mathbb{R}^N} \frac{G(x,w)\ \ |a(w)|\ \ G(w,y)}{G(x,y)}\ dw \leq \beta_{N,q_1,q_2}\ \ \|a\|_{L^{q_1} \cap L^{q_2}}\ \ . \tag{28}$$

Remark 5. These results imply that for $a \in L^\infty \cap L^q$ with $q \in \left[1, \frac{1}{2}N\right)$ one finds

$$\int\limits_{w \in \mathbb{R}^N} \frac{G(x,w)\ \ |a(w)|\ \ G(w,y)}{G(x,y)}\ dw \leq \gamma_{N,q,\vartheta}\ \ \|a\|_{L^q \cap L^\infty}\ \ \min\left(1, |x-y|^{\vartheta}\right) \tag{29}$$

with $0 \leq \vartheta \leq 2$ and $\vartheta < N-2$.

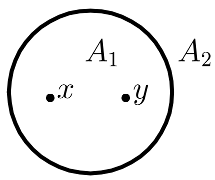
Proof. As before we estimate

$$G(x,y)^{-1} \int\limits_{w \in \mathbb{R}^N} G(x,w)\ \ |a(w)|\ \ G(w,y)\ \ dw\ =$$

$$=\ c_N\ \ |x-y|^{N-2} \int\limits_{w \in \mathbb{R}^N} |x-w|^{2-N}\ \ |a(w)|\ \ |w-y|^{2-N}\ \ dw.$$

We distinguish the areas

$$A_1 = \left\{w \in \mathbb{R}^N; \left|w - \tfrac{1}{2}(x+y)\right| \leq |x-y|\right\},$$
$$A_2 = \left\{w \in \mathbb{R}^N; \left|w - \tfrac{1}{2}(x+y)\right| \geq |x-y|\right\}.$$



For $w \in A_1$ we find that $|w-y| \leq 2|x-y|$ and $|w-x| \leq 2|x-y|$ holds. And also for all those w we have either $|w-y| \geq \frac{1}{2}|x-y|$ or $|w-x| \geq \frac{1}{2}|x-y|$. Hence the following estimate holds and is optimal in order:

$$c_N\ \ |x-y|^{N-2} \int\limits_{w \in A_1} |x-w|^{2-N}\ \ |a(w)|\ \ |w-y|^{2-N}\ \ dw \leq$$

$$\leq c_N' \left(\int\limits_{|w-x| \leq 2|x-y|} |w-x|^{2-N}\ \ |a(w)|\ \ dw + \int\limits_{|w-y| \leq 2|x-y|} |w-y|^{2-N}\ \ |a(w)|\ \ dw \right) \tag{30}$$

We find that

$$\int\limits_{|w-x| \leq 2|x-y|} |w-x|^{2-N}\ \ |a(w)|\ \ dw \leq$$

$$\leq \left(\int_{|w-x|\leq 2|x-y|} |w-x|^{(2-N)p} \ dw \right)^{1/p} \left(\int_{|w-x|\leq 2|x-y|} |a(w)|^q \ dw \right)^{1/q} \leq$$

$$\leq c_{N,q} \ |x-y|^{2-N/q} \ \|a\|_{L^q} \tag{31}$$

whenever $q > N/2$. By symmetry we obtain the same bound for the second integral in (30).

We may obtain an estimate independent of $|x-y|$ by proceeding as in Lemma 9. For $q_1 > N/2$ and $q_2 < N/2$ we find:

$$|x-y|^{N-2} \int_{w\in A_1} |x-w|^{2-N} \ |a(w)| \ |w-y|^{2-N} \ dw \leq$$

$$\leq c_N \left(\int_{|w-x|\leq 2|x-y|} |w-x|^{2-N} \ |a(w)| \ dw + \int_{|w-y|\leq 2|x-y|} |w-y|^{2-N} \ |a(w)| \ dw \right) \leq$$

$$\leq c_{N,q_1} \ \|a\|_{L^{q_1}} + c'_{N,q_2} \ \|a\|_{L^{q_2}} . \tag{32}$$

Note that (32) may be improved for $|x-y| < 1$ to

$$\int_{|w-x|\leq 2|x-y|} |w-x|^{2-N} \ |a(w)| \ dw \leq c_{N,q_1} \ \|a\|_{L^{q_1}} . \tag{33}$$

Now we consider the case $w \in A_2$. For $w \in A_2$ we find

$$\begin{aligned} \tfrac{1}{2}|w-x| &\leq \left|w - \tfrac{1}{2}(x+y)\right| \leq 2|w-x|, \\ \tfrac{1}{2}|w-y| &\leq \left|w - \tfrac{1}{2}(x+y)\right| \leq 2|w-y|. \end{aligned}$$

Hence

$$|x-y|^{N-2} \int_{w\in A_1} |x-w|^{2-N} \ |a(w)| \ |w-y|^{2-N} \ dw \leq$$

$$\leq c \ |x-y|^{N-2} \int_{|u|\geq |x-y|} |u|^{4-2N} \ \left| a\left(u + \tfrac{1}{2}(x+y)\right) \right| \ du \leq$$

$$\leq c \ |x-y|^{N-2} \left(\int_{|u|\geq|x-y|} |u|^{(4-2N)p} \ du \right)^{1/p} \left(\int_{w\in\mathbb{R}^N} |a(w)|^q \ dw \right)^{1/q} \leq$$

$$\leq c_{N,q} \ |x-y|^{2-N/q} \ \|a\|_{L^q} \tag{34}$$

whenever $(4-2N)\,p+N<0$. For $N\geq 5$ this bound is satisfied for all $p\geq 1$; for $N=3$, resp. 4 we need $p>\frac{3}{2}$, resp. $p>1$. The restrictions for q by (34) are then respectively

$$\begin{array}{ll} q\in[1,3) & \text{for } N=3,\\ q\in[1,\infty) & \text{for } N=4,\\ q\in[1,\infty] & \text{for } N\geq 5,\end{array}$$

which shows that for $q=N/2$ we have

$$|x-y|^{N-2}\int\limits_{w\in A_1}|x-w|^{2-N}\ \ |a(w)|\ \ |w-y|^{2-N}\ \ dw\leq c_N\ \ \|a\|_{L^{N/2}}\,.$$

Hence the restrictions for (28) follow from (32).

The restrictions on q by (31)-(34) become the ones that are stated in the theorem, yielding the result in (27). □

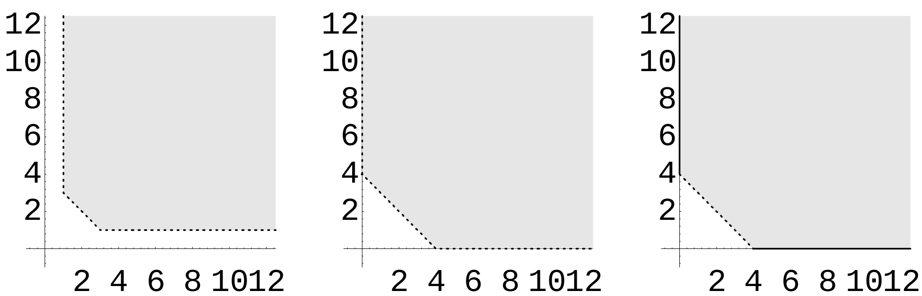
As a consequence of the 3G-result we obtain the following result which could be called a 4G-Theorem.

Theorem 13 *Suppose that $a\in L^{\infty}_{\ell_a}$ and $b\in L^{\infty}_{\ell_b}$, with*

$$\ell_a+\ell_b>4 \text{ and } \ell_a,\ell_b\geq 0 \text{ and } \ell_a,\ell_b>4-N. \tag{35}$$

Then there exists $C_{N,\ell_a,\ell_b}>0$ such that

$$\iint\limits_{z,w\in\mathbb{R}^N}\frac{G(x,z)\,|a(z)|\,G(z,w)\,|b(w)|\,G(w,y)}{G(x,y)}dzdw\leq C_{N,\ell_a,\ell_b}\ c_{\ell_a}(a)\ \ c_{\ell_b}(b)\,. \tag{36}$$



Plot of feasible ℓ_a,ℓ_b for respectively $N=3$, $N=4$ and $N\geq 5$.

Remark 6. In fact it will be sufficient that $a\in L^q$, with q as in Theorem 12, and $b\in L^{\infty}_{\ell_b}$ with $\ell_b>4-N/q$. Recalling Remark 2 we have $L^{\infty}_{\ell}\subset L^q$ for $\ell>N/q$. In this case the right hand side of (36) will be replaced by $c_{N,q,\ell_b}\ \ \|a\|_{L^q}\ \ c_{\ell_b}(b)$.

Proof. Fix $q>N/2$. Since $L^{\infty}_{\ell}\subset L^q$ for $\ell>N/q$ and $L^{\infty}_0=L^{\infty}$ we find that $\|a\|_{L^q}\leq c_{N,\ell_a}\ c_{\ell_a}(a)$ whenever

$$\begin{array}{lll} \ell_a & > & N/q \quad \text{if } q<\infty,\\ \ell_a & \geq & 0 \qquad \text{if } q=\infty.\end{array}$$

Using (27) of the previous theorem we find

$$\iint\limits_{z,w\in\mathbb{R}^N} \frac{G(x,z)\,|a(z)|\,G(z,w)\,|b(w)|\,G(w,y)}{G(x,y)}dzdw =$$

$$= \int\limits_{w\in\mathbb{R}^N} \left(\int\limits_{z\in\mathbb{R}^N} \frac{G(x,z)\,|a(z)|\,G(z,w)}{G(x,w)}dz \right) \frac{G(x,w)\,|b(w)|\,G(w,y)}{G(x,y)}dw \leq$$

$$\leq \alpha_{N,q}\ \|a\|_{L^q} \int\limits_{w\in\mathbb{R}^N} \frac{G(x,w)\ |x-w|^{2-N/q}\ |b(w)|\ G(w,y)}{G(x,y)}dw \tag{37}$$

for all $q > N/2$.

Next we have to find the L^p-character of $w \mapsto |x-w|^{2-N/q}\ |b(w)|$. We read $N/q = 0$ if $q = \infty$.

By an inequality of Hardy-Littlewood, see [13], we have for $f, g \geq 0$ that

$$\int\limits_{\mathbb{R}^N} f(y)\ g(y)\ dy \leq \int\limits_0^\infty f^*(t)\ g^*(t)\ dt,$$

where f^* is the decreasing rearrangement of f :

$$\begin{aligned} \lambda_f(t) &= \lambda\left\{x \in \mathbb{R}^N; f(x) > t\right\}, \\ f^*(s) &= \sup\left\{t \geq 0; \lambda_f(t) > s\right\}, \end{aligned}$$

with λ the Lebesgue measure. Applied to positive decreasing functions $f, g : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ we find that

$$\int\limits_{\mathbb{R}^N} f(|x-y|)\ g(|y|)\,dy \leq \int\limits_0^\infty \Big(f(|x-\cdot|)\Big)^*(t)\ \Big(g(|\cdot|)\Big)^*(t)\ dt.$$

Since the rearrangement $\Big(f(|x-\cdot|)\Big)^*$ does not depend on x, we have

$$\int\limits_0^\infty \Big(f(|x-\cdot|)\Big)^*(t)\ \Big(g(|\cdot|)\Big)^*(t)\ dt = \int\limits_0^\infty \Big(f(|0-\cdot|)\Big)^*(t)\ \Big(g(|\cdot|)\Big)^*(t)\ dt =$$

$$= \int\limits_{\mathbb{R}^N} f(|0-y|)\ g(|y|)\,dy.$$

We will use this inequality for

$$f(|x-y|) = |x-y|^{(2-N/q)q_1} \text{ and } g(|y|) = (1+|y|)^{-\ell_b q_1}$$

to find that

$$\left\| |x-\cdot|^{2-N/q} \; b(\cdot) \right\|_{L^{q_1}} \leq \left\| |x-\cdot|^{2-N/q} \; c_{\ell_b}(b) \; (1+|\cdot|)^{-\ell_b} \right\|_{L^{q_1}} \leq$$

$$\leq \left\| |0-\cdot|^{2-N/q} \; c_{\ell_b}(b) \; (1+|\cdot|)^{-\ell_b} \right\|_{L^{q_1}} \leq$$

$$\leq \left(\gamma_N \int\limits_{r\geq 0} \left(r^{2-N/q} \; c_{\ell_b}(b) \; (1+r)^{-\ell_b} \right)^{q_1} \; r^{N-1} \; dr \right)^{\frac{1}{q_1}} \leq c_{N,\ell_b,q_1,q} \; c_{\ell_b}(b) \tag{38}$$

whenever $(2-N/q-\ell_b)\,q_1+N<0$. Since $q>N/2$ there is no singularity in (38) for $r=0$. In order to apply (28) we need $\ell_b>2-N/q$ and the existence of a number q_1 such that

$$\frac{N}{\ell_b-2+N/q}<q_1<N/2. \tag{39}$$

The L^{q_2} bound with $q_2>N/2$ is obtained without an extra condition. From (39) it follows that we need

$$\frac{N}{\ell_b-2+N/q}<N/2 \tag{40}$$

which is $\ell_b>4-N/q$. Note that $q>N/2$ implies that $4-N/q>2$.

We obtain the following conditions. For $q=\infty$ we have

$$\ell_a\geq 0,\;\; \ell_b>4. \tag{41}$$

For $q<\infty$ we have

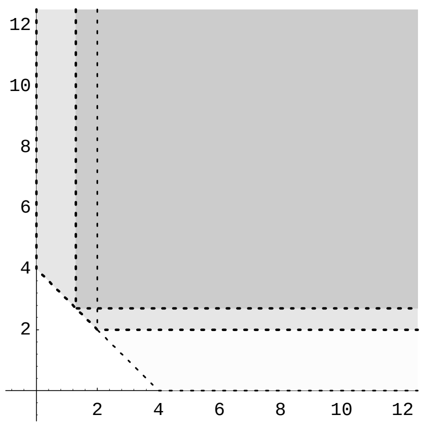
$$\ell_a>N/q,\;\; \ell_b>4-N/q. \tag{42}$$

Let us denote

$$K_{s,t}:=\left\{(x,y)\in\mathbb{R}^2; x>s \text{ and } y>t\right\}.$$

If $N\geq 4$ we find that we may take $N/q\in(0,2)$ to obtain that (42) is equivalent with

$$(\ell_a,\ell_b)\in\bigcup_{0<\vartheta<2}K_{\vartheta,4-\vartheta}. \tag{43}$$



$K_{1.3,2.7}$ (dark grey) and $\bigcup_{0<\vartheta<2}K_{\vartheta,4-\vartheta}$ (medium grey), with its reflection (light grey) in the diagonal

If $N=3$ we find that we may take $N/q\in(1,2)$ to obtain that (42) is equivalent with

$$(\ell_a,\ell_b)\in\bigcup_{1<\vartheta<2}K_{\vartheta,4-\vartheta}. \tag{44}$$

If $N\geq 5$ we may add to (43) the line

$$(\ell_a,\ell_b)\in\{(0,t)\,;t>4\}\,. \tag{45}$$

We may interchange the roles of a and b, using

$$\iint\limits_{z,w\in\mathbb{R}^N}\frac{G(x,z)\,|a(z)|\,G(z,w)\,|b(w)|\,G(w,y)}{G(x,y)}dzdw=$$

$$=\int\limits_{z\in\mathbb{R}^N}\left(\int\limits_{w\in\mathbb{R}^N}\frac{G(z,w)\,|b(w)|\,G(w,y)}{G(z,y)}dw\right)\frac{G(x,z)\,|a(z)|\,G(z,y)}{G(x,y)}dz$$

instead of (37). Hence we can add the pairs (ℓ_b,ℓ_a) (with ℓ_a and ℓ_b interchanged) that are in the areas defined in (43)-(44)-(45) above. Hence we find

$$\begin{array}{llll}\ell_a>1, & \ell_b>1, & \ell_a+\ell_b>4 & \text{if}\quad N=3,\\ \ell_a>0, & \ell_b>0, & \ell_a+\ell_b>4 & \text{if}\quad N=4,\\ \ell_a\geq 0, & \ell_b\geq 0, & \ell_a+\ell_b>4 & \text{if}\quad N\geq 5.\end{array}$$

These inequalities are summarized in the conditions of (35). □

6 The auxiliary system

In this section we will show that for appropriate a,b and μ sufficiently small, the operator $\mathcal{T}_\mu^G$, defined in (24), is positivity preserving for $f\in L^{N/2-\varepsilon}\cap L^{N/2+\varepsilon}$. We will proceed by several lemmas.

Lemma 14 *Let $\mathcal{S}_c$ and $\widetilde{\mathcal{S}}_c$ be as in* (21) *respectively* (23) *with $\mathcal{G}$ the Green operator. Assume that $a\in L^\infty_{\ell_a},b\in L^\infty_{\ell_b}$ with ℓ_a,ℓ_b as in* (3)*. Then for*

$$\mu^*:=\mu\;C_{N,\ell_a,\ell_b}\;\left(c_\ell(a_+)c_\ell(b_-)+c_\ell(a_-)c_\ell(b_+)\right)<1$$

the operator $\left(I-\mu\widetilde{\mathcal{S}}_c\right)^{inv}:L^p\rightarrow L^p$ is well defined for all $p\in(N/2,\infty]$. Moreover for all $0\leq f\in L^{N/2-\varepsilon}\cap L^{N/2+\varepsilon}$ we have

$$0\leq\left(I-\mu\mathcal{S}_c\right)^{inv}\mathcal{G}f\leq\frac{1}{1-\mu^*}\mathcal{G}f.$$

Remark 7. Note that $\mathcal{G}\left(I-\mu\widetilde{\mathcal{S}}_c\right)^{inv}=(I-\mu\mathcal{S}_c)^{inv}\mathcal{G}$.

Proof. From Theorem 13 we find that

$$\mu\mathcal{G}\widetilde{\mathcal{S}}_c=\mu\mathcal{S}_c\mathcal{G}=\mu\mathcal{G}a_+\mathcal{G}b_-\mathcal{G}+\mu\mathcal{G}a_-\mathcal{G}b_+\mathcal{G}\leq$$

$$\leq\mu\ C_{N,\ell_a,\ell_b}\ \left(c_{\ell_a}(a_+)c_{\ell_b}(b_-)+c_{\ell_a}(a_-)c_{\ell_b}(b_+)\right)\mathcal{G}\leq\mu^*\mathcal{G}.$$

Hence $\mathcal{S}_c\mathcal{G}$ is well defined for $f\in L^p$. By a Neumann series we have

$$\left(\mathcal{I}-\mu\widetilde{\mathcal{S}}_c\right)^{inv}=$$

$$=\mathcal{I}+\mu\sum_{k=1}^{\infty}\mathcal{G}a_+\left(\mu\mathcal{G}b_-\mathcal{G}a_++\mu\mathcal{G}b_+\mathcal{G}a_-\right)^{k-1}\mathcal{G}b_-+$$

$$+\mu\sum_{k=1}^{\infty}\mathcal{G}a_-\left(\mu\mathcal{G}b_-\mathcal{G}a_++\mu\mathcal{G}b_+\mathcal{G}a_-\right)^{k-1}\mathcal{G}b_+\leq$$

$$\leq\mathcal{I}+\mu\left(\sum_{k=1}^{\infty}(\mu^*)^{k-1}\right)\left(\mathcal{G}a_-\mathcal{G}b_++\mathcal{G}a_+\mathcal{G}b_-\right).$$

The series converges and is hence well defined when $\mu^*<1$. Since $\mathcal{I},\mathcal{G}a_+,\mathcal{G}a_-,\mathcal{G}b_+$ and $\mathcal{G}b_-$ are positive operators we obtain $(\mathcal{I}-\mu\mathcal{S}_c)^{inv}\mathcal{G}f\geq 0$. □

Lemma 15 *Let $\mathcal{S}_c,\mathcal{S}_n$ resp. $\mathcal{T}_\mu^G$ be as in (21) resp. (24) and $a\in L^\infty_{\ell_a},b\in L^\infty_{\ell_b}$ with ℓ_a,ℓ_b as in (3). Then we find that for all μ with*

$$0\leq\mu\leq\left(C_{N,\ell_a,\ell_b}\right)^{-1}\ \left(c_{\ell_a}(a)\right)^{-1}\left(c_{\ell_b}(b)\right)^{-1}\tag{46}$$

and $f\in L^{N/2-\varepsilon}\cap L^{N/2+\varepsilon}$ that

$$f\geq 0\Rightarrow\mathcal{T}_\mu^G f\geq 0.$$

Proof. From (46) it follows that μ^* defined in Lemma 14 satisfies $\mu^*<1$. Then this lemma implies, using $\mathcal{S}_n\mathcal{G}=\mathcal{G}\widetilde{\mathcal{S}}_n$, that

$$\left(\mathcal{I}-\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)\mathcal{G}f=$$

$$=\left(\mathcal{G}-\mu\left(I-\mu\mathcal{S}_c\right)^{inv}\ \mathcal{G}\widetilde{\mathcal{S}}_n\right)f\geq\left(\mathcal{G}-\mu\frac{1}{1-\mu^*}\ \mathcal{G}\widetilde{\mathcal{S}}_n\right)f=$$

$$=\left(\mathcal{I}-\mu\frac{1}{1-\mu^*}\ \mathcal{S}_n\right)\mathcal{G}f.$$

From Theorem 13 it follows that

$$\left(\mathcal{I}-\mu\frac{1}{1-\mu^*}\ \mathcal{S}_n\right)\mathcal{G}f\geq$$

$$\geq\left(\mathcal{I}-\frac{\mu\ C_{N,\ell_a,\ell_b}\ \left(c_{\ell_a}(a_+)c_{\ell_b}(b_+)+c_{\ell_a}(a_-)c_{\ell_b}(b_-)\right)}{1-\mu^*}\right)\mathcal{G}f=$$

$$=\frac{1-\mu\ C_{N,\ell_a,\ell_b}\ c_{\ell_a}(a)c_{\ell_b}(b)}{1-\mu\ C_{N,\ell_a,\ell_b}\ \left(c_{\ell_a}(a_+)c_{\ell_b}(b_-)+c_{\ell_a}(a_-)c_{\ell_b}(b_+)\right)}\ \mathcal{G}f\geq 0.\tag{47}$$

In the last step we used (46). □

7 The full system

First we show that (1) has at most one solution in $\mathcal{D}^{1,2}$.

Lemma 16 *Let ℓ_a, ℓ_b satisfy* (3) *and let $f \in L^{\frac{2N}{N-2}}$. Then with C_{N,ℓ_a,ℓ_b} as in Theorem 13 we find that if*

$$|\mu| \leq \left(C_{N,\ell_a,\ell_b}\, c_{\ell_a}(a)\, c_{\ell_b}(b)\right)^{-1}$$

the system in (1) *has at most one solution in $\mathcal{D}^{1,2}$.*

Proof. Suppose that both (u_1, v_1) and (u_2, v_2) are solutions of (1). Then we have $u_1, u_2 \in L_{\frac{2N}{N-2}}$,

$$-\Delta\left(u_1 - u_2\right) = \mu a \left(v_1 - v_2\right)$$

and

$$-\Delta\left(v_1 - v_2\right) = b\left(u_1 - u_2\right).$$

We obtain by Theorem 13 that

$$\|u_1 - u_2\|_{\frac{2N}{N-2}} = \|\mathcal{G}\mu a \mathcal{G} b \mathcal{G}\mu a \left(v_1 - v_2\right)\|_{\frac{2N}{N-2}} \leq$$

$$\leq |\mu|\ C^* \ \|\mathcal{G}\mu a\left(v_1 - v_2\right)\|_{\frac{2N}{N-2}} = |\mu|\ C^* \ \|u_1 - u_2\|_{\frac{2N}{N-2}}.$$

For

$$|\mu|\ C^* = |\mu|\ C_{N,\ell_a,\ell_b}\ c_{\ell_a}(a)\, c_{\ell_b}(b) < 1$$

we find $u_1 = u_2$. □

Proof of Theorem 3: Let $\mu \geq 0$. First we assume that $0 \leq f \in L^{N/2-\varepsilon} \cap L^{N/2+\varepsilon}$. By Theorem 13 it follows that

$$\mu \mathcal{S}_c \mathcal{G} f \leq \mu\ C_{N,\ell_a,\ell_b}\left(c_{\ell_a}(a_+) c_{\ell_b}(b_-) + c_{\ell_a}(a_-) c_{\ell_b}(b_+)\right) \mathcal{G} f$$

When

$$\mu^* = \mu\ C_{N,\ell_a,\ell_b}\ \left(c_{\ell_a}(a_+) c_{\ell_b}(b_-) + c_{\ell_a}(a_-) c_{\ell_b}(b_+)\right) < 1 \tag{48}$$

we may use Lemma 14 to find that $\left(I - \mu \mathcal{S}_c\right)^{inv} \mathcal{G} f$ is well defined and that

$$0 \leq \left(I - \mu \mathcal{S}_c\right)^{inv} \mathcal{G} f \leq \frac{1}{1 - \mu^*} \mathcal{G} f. \tag{49}$$

For

$$\mu^{**} = \mu\ C_{N,\ell_a,\ell_b}\ \left(c_{\ell_a}(a_+) + c_{\ell_a}(a_-)\right)\left(c_{\ell_b}(b_-) + c_{\ell_b}(b_+)\right) < 1 \tag{50}$$

Lemma 15 implies that

$$\left(\mathcal{I} - \mu\left(I - \mu \mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right) \mathcal{G} g \geq 0.$$

We use $g = \left(I - \mu \widetilde{\mathcal{S}}_c\right)^{inv} f$, which is positive and well defined by Lemma 14 for $\mu^* < 1$. Hence we obtain

$$\left(\mathcal{I} - \mu\left(I - \mu \mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)\left(I - \mu \mathcal{S}_c\right)^{inv} \mathcal{G} f =$$

$$= \left(\mathcal{I} - \mu\left(I - \mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)\mathcal{G}\left(I - \mu\widetilde{\mathcal{S}}_c\right)^{inv} f \geq 0. \tag{51}$$

Note that $\mu^* \leq \mu^{**}$.

The estimates in (49)-(51) show that for $0 \leq f \in L^{N/2-\varepsilon} \cap L^{N/2+\varepsilon}$ and $\mu^{**} < 1$ the solution u of (1) is well defined by

$$u = \left(\mathcal{I} + \mu\left(I - \mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)^{inv}\ \left(I - \mu\mathcal{S}_c\right)^{inv}\ \mathcal{G}f \tag{52}$$

and satisfies $u \geq 0$. See also the remarks at the end of section 4,

Since $L^{\frac{2N}{N+2}} \cap L^{N/2+\varepsilon}$ is dense in $L^{\frac{2N}{N+2}}$ we can approximate $f \in L^{\frac{2N}{N+2}}$ by $f_n \in L^{\frac{2N}{N+2}} \cap L^{N/2+\varepsilon}$. Since we have for $f \geq 0$ that $\|f - f_n^+\|_{L^{\frac{2N}{N+2}}} \leq \|f - f_n\|_{L^{\frac{2N}{N+2}}}$, we may approximate $0 \leq f \in L^{\frac{2N}{N+2}}$ by $0 \leq f_n \in L^{\frac{2N}{N+2}} \cap L^{N/2+\varepsilon}$. Moreover, since for $|\mu|$ as in the previous Lemma there is at most one solution for (1), it is defined by (52). By the continuity of the operators involved it follows that $u_n \to u$ in $\mathcal{D}^{1,2}$ with the sequence $\{u_n\}$, defined by

$$u_n = \left(\mathcal{I} + \mu\left(I - \mu\mathcal{S}_c\right)^{inv}\ \mathcal{S}_n\right)^{inv}\ \left(I - \mu\mathcal{S}_c\right)^{inv}\ \mathcal{G}f_n.$$

Since $u_n > 0$ we find $u \geq 0$. □

Acknowledgement.

The authors gratefully acknowledge the support by the HCM-Programme of the E.U. to the network Reaction-Diffusion Equations (ERBCHRXCT930409)

References

[1] K. J. Brown and N. Stavrakakis, Global bifurcation results for a semilinear elliptic equation on all of $\mathbb{R}^N$, to appear in Duke Journ. of Math..

[2] M. Cranston, E. Fabes and Z. Zhao, Conditional gauge and potential theory for the Schrödinger operator, Transactions A.M.S. 307 (1988), 171-194.

[3] J. Fleckinger, J Hernandez and F. de Thélin, On maximum principles and existence of positive solutions for some cooperative elliptic systems, to appear in Differ. Int. Equations.

[4] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, second edition, Springer, Berlin Heidelberg New York, 1983.

[5] P. Hess, On the spectrum of elliptic operators with respect to indefinite weights, Linear Algebra and its Applications 84 (1986), 99-109.

[6] D. Kinderlehrer and G. Stampacchia, An introduction to variational inequalities and their applications, Academic Press, New York, 1980.

[7] H. Kozono and H. Sohr, New a priori estimates for the Stokes equation in exterior domains, Indiana Univ. Math. J. 40 (1991), 1-27.

[8] Yi Li and Wei-Ming Ni, On conformal scalar curvature equations in $\mathbb{R}^N$, Duke Math. J. 57 (1988), 895-924.

[9] E. Mitidieri and G. Sweers, Weakly coupled systems and positivity, Math. Nachr. 173 (1995), 259-286.

[10] Y. Pinchover, On the equivalence of Green functions of second order elliptic equations in $\mathbb{R}^n$, Diff. and Int. Equations 5 (1992), 481-493.

[11] B. Simon, Schrödinger semigroups, Bull.A.M.S. 7 (1982), 447-526.

[12] G. Sweers, Positivity for a strongly coupled system by Green function estimates, The Journal of Geometric Analysis 4 (1994), 121-142.

[13] G.G. Talenti, Rearrangements and P.D.E., in: Inequalities fifty years on from Hardy, Littlewood and Polya, ed. W. Norrie Everitt, Marcel Dekker Inc., New York, 1991.

[14] H.F. Weinberger, Variational methods for eigenvalue approximation, Regional Conference in Mathematics Series, SIAM, 1974.