Global Bifurcation Results for Semilinear Elliptic Equations on $\mathbb{R}^N$:The Fredholm Case.*

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Abstract

We prove the existence of a continuum of positive solutions for the semilinear elliptic equation $-\Delta u(x) = \lambda g(x) f(u(x))$, $0 < u < 1$ for $x \in \mathbb{R}^N, \lim_{|x| \to +\infty} u(x) = 0$, which arises in population genetics, under the hypotheses that $N \geq 3$ and the weight g changes sign been negative and away from zero at ∞. After establishing the existence of a simple positive principal eigenvalue λ_1 for the corresponding linearized problem, we prove the existence of a continuum of solutions lying in the space $\mathbb{R} \times H^2$ extended from λ_1 to ∞. To complete this task we state a new version of the global bifurcation theory for nonlinear Fredholm (noncompact) operators and prove the compactness of the solution set of the problem.

1 Introduction

In this paper by using local and global bifurcation theory we prove the existence of a continuum of positive solutions of the following semilinear eigenvalue problem

$$-\Delta u(x) = \lambda g(x) f(u(x)), \quad x \in \mathbb{R}^N, \tag{1.1$_\lambda$}$$

$$0 < u < 1, \quad x \in \mathbb{R}^N, \quad \lim_{|x| \to +\infty} u(x) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ and $N \geq 3$. Here we state the general hypothesis which will be assumed throughout the paper:

$(\mathcal{G})$ *g is a smooth function, at least $C^{1,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, such that $g \in L^\infty(\mathbb{R}^N)$ and $g(x) > 0$, on Ω^+, with measure of Ω^+, $|\Omega^+| > 0$. Also there exist R_0 large and $k > 0$ such that $g(x) < -k$ for all $|x| > R_0$.*

$(\mathcal{F})$ *$f : [0,1] \longmapsto \mathbb{R}^+$ is a smooth function such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(u) > 0$ for all $0 < u < 1$.*

The equation arises in population genetics and ecology (see [18]). The unknown function u corresponds to the relative frequency of an allele and is hence constrained to have values between 0 and 1. The real parameter $\lambda > 0$ corresponds to the reciprocal of a diffusion coefficient.

It is well known that problems where an indefinite weight function is present arise both from pure mathematics (oscillatory integrals), as well as from a large variety of applications, e.g. ecology, transport theory, fluid dynamics, reaction-diffusion processes, electron scattering (see [5, 11, 23] and the references their in).

There is quite an extensive literature for the problem in the bounded domain case under various general boundary conditions and a fairly complete bifurcation analysis can be given. For the equation we mention, among others, the papers [9, 23].

The problem becomes more complicate in the case of unbounded domains as, in general, the equation does not give rise to compact operators and so it is not known if the classical spectral theory is applicable. It is also unclear *a priori* in which function spaces eigenfunctions of $(1.1)_\lambda$ might lie.

In the special case treated here, we have a *noncompact nonlinear operator*. A similar problem is treated by Drabeck and Huang in [16], but there the case is different because only the linear part is allowed to be noncompact, while the weight of the nonlinear part is essentially in $L^{\frac{N}{2}}$ (which induces compactness). H Matano in [27], for essentially the same problem, proves the existence and the L^∞-stability of solutions lying between "strict sub- and supersolutions", whenever they exist. To our knowledge the only paper were a case, quite similar to our, is studied is the work by N Dancer in [15]. There he studies the one dimensional problem and does the very crucial remark that one could restrict the noncompact nonlinear operator to a compact solution-set and get the same results as Rabinowitz's theorem does. So here although (maybe) the spectrum is mixed (discrete and continuous) we get a continuum of solutions for all $\lambda \geq \lambda^* > 0$, i.e. we can go through the (posible) continuous spectrum. Finally, we must notice that operators studied by Volpert and Volpert [38] are of different nature.

The complementary case, i.e. when g is going "weekly" to zero (in the sense that $g \in L^{N/2}(\mathbb{R}^N)$), is studied in several recent papers, see among others [10, 11, 12, 13, 20, 37] for the nonlinear Laplacian, in [16, 17, 19] for the p-Laplacian, and in [34, 35, 36] for the polyharmonic analog.

In order to discuss bifurcation from the zero solution of $(1.1)_\lambda$ it is first necessary to study the eigenvalues of the corresponding linearized problem

$$\begin{array}{c} -\Delta u(x) = \lambda g(x) f'(0)\, u(x) \qquad \text{for} \quad x \in \mathbb{R}^N \\ \lim_{|x|\to+\infty} u(x) = 0. \end{array} \tag{1.3}$$

The existence of a positive principal eigenvalue (i.e., an eigenvalue, to which corresponds a positive eigenfunction, so a point at which positive solutions of $(1.1)_\lambda$ may bifurcate from the zero branch) for the above problem has been proved in ([8, 13]) under the hypotheses that $\int_{\mathbb{R}^N} g(x)dx < 0$ and $g(x) < 0$ for $|x|$ large and in ([3]) under the hypothesis that $N \geq 3$ and $g_+ \in L^{N/2}(\mathbb{R}^N)$.

In Section 2, we study the space setting of the problem and give some equivalent norm results to be used later. A generalised version of Poincare's inequality plays a crucial role. Some of the ideas developed here appeared also in a different context in [11]. In Section 3 we discuss the linearised problem and basic characteristics, as the compactness of the operator, the simplicity of the positive principal eigenvalue and the H^2 nature of the eigenfunctions, are described. In Section 4 we prove the existence of a local continuum of positive solutions of the semilinear problem emanating from the principal eigenvalue λ_1 by applying the Crandall and Rabinowitz local bifurcation theory.

In order to study the global bifurcation behavior of our problem we have to overcome the lack of compactness of the nonlinear operator associated to the problem and the fact that the dependence on λ of the linear part of the operator is quite complicated. To complete this goal, it is necessary to reformulate some of the standard theorems of bifurcation theory for Fredholm (noncompact) operators. This is done in Section 5. To apply this global bifurcation theory for Fredholm operators developed in the previews section it is necessary to study the solution set of the problem and especially the compactness of the positive branch, which is the subject of Section 6.

Finally, in Section 7 we complete the study of the problem $(1.1)_\lambda$, (1.2) by showing that the continuum of positive solutions bifurcating from $(\lambda_1, 0)$ cannot cross $\lambda = 0$, the solutions are in the interval $(0, 1)$ in the L^∞ sense, and that the continuum must extend to $\lambda = \infty$.

Notation: We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R and $B^* =: \mathbb{R}^N \setminus B$. For simplicity reasons we use the symbols L^p, H^p, $1 \leq p \leq \infty$, respectively for the spaces $L^p(\mathbb{R}^N)$, $H^p(\mathbb{R}^N)$, respectively; $||.||_p$ for the norm $||.||_{L^p(\mathbb{R}^N)}$. Sometimes when the domain of integration is not stated it is assumed to be all of $\mathbb{R}^N$. Equalities introducing definitions are denoted by "=:". Embeddings are denoted by $'' \hookrightarrow''$. Denote by $g_\pm =: \max\{\pm g, 0\}$. The end of the proofs is marked by "$\lhd$".

2 Space Setting

In this section we are going to characterize the space $\mathcal{V}_g$ (introduced below) in terms of classical Sobolev spaces, in the case of g been negative and away from zero at infinity. Since g satisfies $(\mathcal{G})$, by applying Poincare's inequality in the ball B_{R_0}, it is easy to see that there exists a constant $a > 0$ such that

$$\int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx \geq \alpha \int_{\mathbb{R}^N} |g| u^2 dx. \tag{2.1}$$

for all $u \in C_0^\infty$. So we may introduce a real inner product on $C_0^\infty(\mathbb{R}^N)$ by

$$< u, v >=: \int_{\mathbb{R}^N} \bigtriangledown u \bigtriangledown v \, dx - \frac{\alpha}{2} \int_{\mathbb{R}^N} guv \, dx. \tag{2.2}$$

As in ([11]) we define $\mathcal{V}_g$ to be the completion of C_0^∞ with respect to the above inner product and let $||.||_g$ denote the corresponding norm. Although the space $\mathcal{V}_g$ would seem to depend on g, in fact we have the following Sobolev space characterization of it

Lemma 2.1 *Suppose that g satisfies $(\mathcal{G})$. Then $\mathcal{V}_g = H^1$.*

Proof Because of density we only compare $\mathcal{V}_g$ and H^1 norms on $C_0^\infty(\mathbb{R}^N)$.
(i) For all $u \in C_0^\infty(\mathbb{R}^N)$ we have

$$||u||_g^2 \leq \int |\bigtriangledown u|^2 dx + \frac{\alpha}{2} ||g||_\infty \int u^2 dx \leq C(\alpha, ||g||_\infty) ||u||_{H^1}^2$$

where $C(\alpha, ||g||_\infty) = \max\{1, \frac{\alpha ||g||_\infty}{2}\}$. Hence we have that $H^1 \subset \mathcal{V}_g$.
(ii) Let $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a Cauchy sequence in $\mathcal{V}_g$ converging in some $u \in \mathcal{V}_g$. Let also B be a ball centered at the origin in $\mathbb{R}^N$ such that $\int_B g(x) dx < 0$ and $g(x) \leq -k$, for all $x \notin B$. Then we have

$$\int_B |\bigtriangledown (u_n - u)|^2 dx \longrightarrow 0, \text{ and } \int_B g(x)(u_n - u)^2 dx \longrightarrow 0 \text{ as } n \to \infty.$$

Suppose that $\int_B (u_n - u)^2 dx \not\to 0$. Then if $v_n =: \frac{u_n - u}{||u_n - u||_B}$, where by $||.||_B$ we denote the norm in $L^2(B)$, we have that $\lim_{n\to\infty} \int_B |\bigtriangledown v_n|^2 dx =$

$\lim\limits_{n\to\infty}\int_B g(x)v_n^2dx = 0$. Hence $\{v_n\}$ is a bounded sequence in $H^1(B)$. So there is a subsequence, denoted again by $\{v_n\}$, such that $\{v_n\}$ converges in $L^2(B)$. Since $\{\bigtriangledown v_n\}$ converges (to zero) in $L^2(B)$, $\{v_n\}$ is a Cauchy sequence in $H^1(B)$. Hence there exists $v \in H^1(B)$ such that $v_n \to v$ in $H^1(B)$. On the other hand since $\bigtriangledown v_n \to \bigtriangledown v$ in $(L^2(B))^N$, it implies that $\bigtriangledown v = 0$ or $v = c$. But $\int_B v^2 dx = 1$ hence $c \neq 0$. However,

$$0 = \lim_{n\to\infty}\int_B g(x)v_n^2dx = c\int_B g(x)dx \neq 0,$$

which is a contradiction. Hence we have

$$\int_B (u_n - u)^2 dx \to 0, \text{ as } n \to \infty. \tag{2.3}$$

Denote by $D_1 =: \{x \in B : g(x) > 0\}$, $D_2 =: \{x \in B : g(x) \leq 0\}$ and

$$\bar{g}(x) =: \begin{cases} g_+(x), & x \in D_1 \\ -g_-(x), & x \in D_2. \end{cases}$$

Then it is not difficult to prove that there exist constants K_0, K_1 that

$$\int g_+(x)(u_n - u)^2 dx \leq K_0||u||_g^2, \text{ and } -\int g_-(x)(u_n - u)^2 dx \leq K_1||u||_g^2.$$

By adding the two inequalities we get

$$\int_B \bar{g}(x)(u_n - u)^2 dx + \int_{B^*}(-g_-)(x)(u_n - u)^2 dx \leq (K_0 + K_1)||u||_g^2$$

But as $n \to \infty$ we have $\int_B \bar{g}(x)(u_n - u)^2 dx = M_n \int_B (u_n - u)^2 dx \to 0$, where M_n, given by the intermediate value theorem for integrals, is positive and finite for all $n \in \mathbb{N}$ (g been in L^∞). Also we have that

$$k\int_{B^*}(u_n - u)^2 dx \leq \int_{B^*}(-g_-)(x)(u_n - u)^2 dx \leq (K_0 + K_1)||u_n - u||_g^2,$$

which implies that as $n \to \infty$

$$\int_{B^*}(u_n - u)^2 dx \to 0. \tag{2.4}$$

Therefore by relations (2.3) and (2.4) we get $\int_{B^*}(u_n-u)^2dx \to 0,$ as $n \to \infty$. Summarizing we have that $u_n \to u,$ in H^1, that is $V_g \subset H^1$, for every g satisfying hypothesis $(\mathcal{G})$, and the proof is completed. $\triangleleft$

For any r_0 large enough ($r_0 \geq R_0$), there exists $\sigma_0 > 1$ such that $g(x) \leq -\frac{k}{\sigma_0}$, for all $|x| \geq r_0$. Then we introduce a new smooth function

$$g_2(x) =: \begin{cases} g(x), & \text{for } |x| \geq r_0, \\ \tilde{g}(x), & \text{for } |x| < r_0 \end{cases}$$

where $-k \leq \tilde{g}(x) \leq -\frac{k}{\sigma_0}$ and $g_1(x) =: g(x) - g_2(x)$. Let $\lambda > 0$ be chosen arbitrarily. Then define $V_{2,\lambda}$ to be the completion of C_0^∞ with respect to the norm

$$||u||^2_{2,\lambda} =: \int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx - \lambda \int_{\mathbb{R}^N} g_2 u^2 dx. \tag{2.5}$$

Remark 2.2 *It is easy to prove that the norms* $||u||_{2,\lambda}$ *and* $||u||_{2,\mu}$ *are equivalent for any positive* λ, μ.

Furthermore we have

Theorem 2.3 *For all* $\lambda > 0$ *the norms* $||.||_{H^1}$, $||.||_g$, $||.||_{2,\lambda}$ *are equivalent.*

Proof First we prove that $||.||_g$, $||.||_{2,\frac{\alpha}{2}}$ are equivalent in $C_0^\infty(\mathbb{R}^N)$. Indeed

$$||u||_g = \int \{|\bigtriangledown u|^2 - \frac{\alpha}{2} g u^2\} dx \leq \int \{|\bigtriangledown u|^2 - \frac{\alpha}{2} g_2 u^2\} dx = ||u||_{2,\frac{\alpha}{2}} \tag{2.6}$$

On the other hand, we have

$$\int g_1 u^2 dx \leq k \int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx - \frac{k\alpha}{2} \int g_1 u^2 dx - \frac{k\alpha}{2} \int g_2 u^2 dx,$$

or

$$(1 + \frac{k\alpha}{2}) \int g_1 u^2 dx \leq k \int |\bigtriangledown u|^2 dx - \frac{k\alpha}{2} \int g_2 u^2 dx = ||u||_{2,\frac{\alpha}{2}},$$

or

$$-\frac{\alpha}{2} \int g_1 u^2 dx \geq -\frac{k\alpha}{2 + k\alpha} ||u||_{2,\frac{\alpha}{2}}.$$

Hence

$$(1-\frac{k\alpha}{2+k\alpha})||u||_{2,\frac{\alpha}{2}} \leq \int |\bigtriangledown u|^2 dx - \frac{\alpha}{2}\int g_1 u^2 dx - \frac{\alpha}{2}\int g_2 u^2 dx = ||u||_g.$$

From relation (2.6) and the fact that $1-\frac{k\alpha}{2+k\alpha} > 0$ for any α, k positive, we get that the norms $||.||_g$, $||.||_{2,\frac{\alpha}{2}}$ are equivalent. Finally, the proof is completed by Lemma 2.1 and Remark 2.2. $\lhd$

3 The Linearized Problem

In this section we shall discuss the spectral and uniqueness properties for the linearization of the problem $(1.1)_\lambda$, close to the trivial solution $u \equiv 0$

$$\begin{aligned} -\Delta u &= \lambda g u, \quad x \in \mathbb{R}^N, \\ \lim_{|x|\to+\infty} u(x) &= 0, \end{aligned} \tag{3.1}$$

where without lose of generality we may assume that $f'(0) = 1$. Fix $\lambda_0 > 0$ arbitrary. Since $-\lambda g_2(x) > 0$ for all $x \in \mathbb{R}^N$ and $\lambda > \lambda_0$, the symmetric operator $-\Delta - \lambda g_2 : C_0^\infty(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$ is essentially selfadjoint (see [30, Vol II, Theorem X.28]). So the closure $L(\lambda)$ of this operator, where $L(\lambda) : D(L(\lambda)) \subset L^2(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$ is selfadjoint. It follows that $D(L(\lambda)) \subset H^1(\mathbb{R}^N)$ and

$$(L(\lambda)u, v) =: \int_{\mathbb{R}^N} (\bigtriangledown u \bigtriangledown v - \lambda g_2 uv)\, dx,$$

for all $\lambda > \lambda_0, u, v \in D(L(\lambda))$. Define the bilinear symmetric mapping

$$a_\lambda : L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \longmapsto \mathbb{R} \text{ by } a_\lambda(u,v) =: (L(\lambda)u, v).$$

Then a_λ is bounded in H^1, since $|a_\lambda(u,v)| \leq c||u||_{H^1}||v||_{H^1}$, for all $u, v \in D(L(\lambda))$ and $\lambda > \lambda_0$. Also a_λ is coersive. Indeed we have

$$\begin{aligned} a_\lambda(u,u) = (L(\lambda)u, u) &= \int_{\mathbb{R}^N} (|\bigtriangledown u|^2 - \lambda g_2 u^2)\, dx \\ &\geq \int_{\mathbb{R}^N} (|\bigtriangledown u|^2 + \frac{\lambda k}{\sigma_0} u^2)\, dx \geq \frac{\lambda k}{\sigma_0}||u||_2^2. \end{aligned}$$

Next we introduce an other bilinear form $b(u,v)$ by

$$b(u,v) = \int_{\mathbb{R}^N} g_1 uv\, dx, \quad \text{for all } u, v \in H^1(\mathbb{R}^N).$$

We see that

$$|b(u,v)| \leq c||u||_{H^1}||v||_{H^1},$$

for some $c > 0$ and all $u, v \in H^1(\mathbb{R}^N)$. Hence b is a bilinear bounded form and by Riesz theory we can define a linear operator $L_1(\lambda) : D(L_1(\lambda)) \subset L^2 \longmapsto L^2$ such that $(L_1(\lambda)u, v) = b(u,v)$, for all $u, v \in D(L_1(\lambda))$ and $\lambda > 0$. It is easy to see that $D(L_1(\lambda)) \subset H^1(\mathbb{R}^N)$. Furthermore we have

Lemma 3.1 *(i) The operator* $L_1(\lambda)$ *is compact, self-adjoint and there exists* $k_1(\lambda) > 0$ *and* $\psi_\lambda \in L^2$ *such that* $L_1(\lambda)\psi_\lambda = k_1(\lambda)\psi_\lambda$*,*
(ii) the problem

$$(L_1(\lambda)\psi_\lambda, \psi_\lambda) = \frac{1}{k_1(\lambda)} b(\psi_\lambda, \psi_\lambda)$$

admits a positive principal eigenvalue at some $\lambda = \lambda_1$ *such that* $\frac{1}{k_1(\lambda_1)} = \lambda_1$*. The corresponting principal eigenfunction* $\phi = \psi_{\lambda_1}$ *is a positive classical solution of equation (3.1),*
(iii) the eigenpair (λ_1, ϕ) *is unique, i.e. if* (λ, u) *is an other solution of (3.1) with* $u(x) > 0$ *for all* $x \in \mathbb{R}^N$*, then* $\lambda = \lambda_1$ *and there is* $c > 0$ *such that* $u(x) = c\phi(x)$ *for all* $x \in \mathbb{R}^N$*.*

Proof (i) The compactness of $L_1(\lambda)$ is a straitforward consequence of the fact that g_1 has compact support, so the imbedding of $H^1(B)$ into $L^p(B)$ is compact for any $p \in [1, \frac{2N}{N-2})$ and any ball B. Note that the compactness of the operator $L_1(\lambda)$ is related to the norm $||.||_{2,\lambda}$. Since $L_1(\lambda)$ is symmetric and defined on the whole space $D(L_1(\lambda))$ it is self-adjoint. Hence $L_1(\lambda)$ has a principal eigenvalue $k_1(\lambda) > 0$ and a (positive) eigenfunction $\psi_\lambda \in L^2$ such that $L_1(\lambda)\psi_\lambda = k_1(\lambda)\psi_\lambda$ (for a similar reasoning see [3]).
(ii) This is proved in [8, Theorem 2.1]. The smoothness of the eigenfunction

ϕ is implied by Weyl's lemma [30, Vol II, p.53] and the strict positivity by [30, Vol IV, Theorem XII.48].
(iii) The proof follows the same steps as in the paper [11]. Actually, it is simpler (see also [37]). $\lhd$

Theorem 3.2 *For the operator* $L(\lambda) : D(L(\lambda)) \subset L^2 \longmapsto L^2$, *we have that* $D(L(\lambda)) = H^2(\mathbb{R}^N)$ *for all* $\lambda > \lambda_0$.

Proof Let $u \in D(L(\lambda))$ i.e. there exists $w \in L^2$ such that $L(\lambda)u = w$. Then by [24, Chapt VI, Theorem 4.6] $u \in H^1(\mathbb{R}^N)$. The proof that $u \in H^2(\mathbb{R}^N)$ follows the spirit of [7, Theorem IX.25]. Denote by

$$D_h u(x) = \frac{u(x+h) - u(x)}{|h|}.$$

For all $u \in H^1(\mathbb{R}^N)$ we have

$$\int (\bigtriangledown u \bigtriangledown v - \lambda g_2 uv)\, dx = \int w\, v\, dx.$$

Let $v = D_{-h}(D_h u)$. Since $u \in H^1(\mathbb{R}^N)$ then $v \in H^1(\mathbb{R}^N)$ too. So we get

$$\int | \bigtriangledown D_h u|^2 - \lambda g_2 |D_h u|^2\, dx = \int w\, D_{-h}(D_h u)\, dx.$$

Hence

$$\begin{aligned} &\min \quad \{1, \tfrac{\lambda_0 k}{\sigma_0}\}\{\int | \bigtriangledown D_h u|^2 + \int | \bigtriangledown D_h u|^2\} \\ \leq \quad &\int | \bigtriangledown D_h u|^2 - \lambda \int g_2 |D_h u|^2\, dx = \int w\, D_{-h}(D_h u)\, dx. \end{aligned}$$

and

$$\min\{1, \tfrac{\lambda_0 k}{\sigma_0}\} ||D_h u||^2_{H^1} \leq ||w||_2 ||D_{-h}(D_h u)||_2.$$

But in general $||D_{-h} u||_2 \leq || \bigtriangledown u||_2$ for all $u \in H^1(\mathbb{R}^N)$. So that

$$||D_h u||^2_{H^1} \leq ||w||_2 ||(D_h u)||_{H^1} \quad \text{or} \quad ||D_h u||_{H^1} \leq ||w||_2.$$

In particular we get $||D_h \frac{\partial u}{\partial x_i}||_2 \leq ||w||_2, \ \ i = 1, 2, ..., N$. The last inequality implies that $\frac{\partial u}{\partial x_i} \in H^1(\mathbb{R}^N)$ (see [7, Prop. IX.3(ii)]). Hence $u \in H^2(\mathbb{R}^N)$.

Conversely, let $u \in H^2(\mathbb{R}^N)$. Setting $-\Delta u - \lambda g_2 u = w$, we get that $w \in L^2(\mathbb{R}^N)$. Multiplying by $v \in C_0^\infty(\mathbb{R}^N)$ and integrating we get

$$\int (\bigtriangledown u \ \bigtriangledown v - \lambda g_2 uv)\, dx, = \int w\, v\, dx$$

or

$$(u, L(\lambda)v) = (w, v), \quad \text{for all } \ v \in C_0^\infty(\mathbb{R}^N)$$

Since $L(\lambda)$ is self-adjoint we obtain $L(\lambda)u = w$, i.e. $u \in D(L(\lambda))$. $\lhd$

Lemma 3.3 *The inverse operator* $L^{-1}(\lambda) : L^2(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$ *exists, is linear, continuous and self-adjoint. For each* $w \in L^2(\mathbb{R}^N)$ *the equation*

$$-\Delta u - \lambda g_2 u = w,$$

has a unique solution $u \in H^2$.

Proof The operator $L(\lambda)$ satisfies all necessary hypothesis for the application of Friedrichs' Theorem (see, for example [41, pg 126]). $\lhd$

4 The Local Bifurcation Theory

For the proof of the existence of a continuum of positive solutions of the problem $(1.1)_\lambda$ for λ close to the principal eigenvalue λ_1 we shall apply the local bifurcation theory developed by Crandall and Rabinowitz in [14]. For the rest of the paper we assume that f satisfies the following hypothesis

$(\mathcal{F}_1)$ f *is extended to a new function, denoted again by* $f : \mathbb{R} \mapsto \mathbb{R}$, $f \in C^2(\mathbb{R})$, $f, f', f'' \in L^\infty(\mathbb{R})$, $f(t) < 0$ *for all* $t \notin [0,1]$, *and there exists* $k^* > 0$ *such that, for all* $t \in \mathbb{R}$, $|f(t)| \leq k^*|t|$.

We introduce the nonlinear operator $T : \mathbb{R} \times H^2(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$ by

$$(T(\lambda, u), v) = \int_{\mathbb{R}^N} \{\bigtriangledown u \ \bigtriangledown v - \lambda g f(u) v\}\, dx, \tag{4.1}$$

Then by standard procedure we can prove the following preliminary results

Lemma 4.1 *Let* $N = 3, 4, ..., 8$ *and* f, g *satisfy hypothesis* $(\mathcal{F})$, $(\mathcal{F}_1)$ *and* $(\mathcal{G})$ *respectively. Then*
(i) the operator T *is well defined and continuous,*
(ii) for each $\lambda \in \mathbb{R}$ *the operator* $T(\lambda, .) : H^2(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$ *is Fréchet differentiable and its Fréchet derivative is defined by*

$$(T_u(\lambda, u)h, \phi) = \int_{\mathbb{R}^N} \{\nabla h \ \nabla \phi - \lambda g f'(u) h\phi\} \, dx, \tag{4.2}$$

(iii) the derivative $T_u : \mathbb{R} \times H^2(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$ *is continuous,*
(iv) the derivatives $T_\lambda, T_{u\lambda}$ *exist and are continuous.*

Then the local bifurcation result can be phrased as follows

Theorem 4.2 (Local Bifurcation) *Let* $N = 3, 4, ..., 8$ *and* f, g *satisfy hypothesis* $(\mathcal{F})$, $(\mathcal{F}_1)$ *and* $(\mathcal{G})$ *respectively. Then there is a neighborhood* U *of* $(\lambda_1, 0)$ *in* $\mathbb{R} \times H^2(\mathbb{R}^N)$, *an interval* $(0, a)$, *and continuous functions*

$$\begin{array}{lll} \eta : (0, a) \longmapsto \mathbb{R}, & \textit{with} & \eta(0) = \lambda_1, \\ \psi : (0, a) \longmapsto H^2(\mathbb{R}^N), & \textit{with} & \psi(0) = 0, \end{array}$$

such that

$$T^{-1}(0) \cap U \supset \{(\eta(\epsilon), \epsilon\phi + \epsilon\psi(\epsilon)) : 0 < \epsilon < a\} \cup \{(t, 0) : (t, 0) \in U\}.$$

Proof By Lemma 3.1 (iii) the null space of $T_u(\lambda_1, 0)$ is spanned by the corresponding eigenfunction ϕ. The range $R(T_u(\lambda_1, 0)) = \{v \in H^2 : \int_{\mathbb{R}^N} g v \phi \, dx = 0\}$ is of codimension 1. Also Lemma 4.1 implies the necessary differentiability conditions. Finally, for the transversality condition we observe that

$$(T_{u\lambda}(\lambda_1, 0)\phi, \phi) = \int (-g(x))\phi^2 \, dx = -\int g\phi^2 \, dx = -\frac{1}{\lambda_1} \int |\nabla \phi|^2 \, dx < 0,$$

and the proof is completed. $\lhd$

Remark 4.3 *By Theorem 4.2 we get a weak solution* $u_\epsilon = \epsilon\phi + \epsilon\psi_\epsilon$ *of*

$$-\Delta u_\epsilon - \eta_\epsilon g(x) f(u_\epsilon) = 0, \quad x \in \mathbb{R}^N$$

where we denote $\psi(\epsilon) = \psi_\epsilon$. *By standard regularity arguments for elliptic problems (e.g. see [22]) we get that* $\epsilon\phi + \epsilon\psi_\epsilon \in C^{2+\alpha}(\mathbb{R}^N)$. *So for all* $\epsilon \in (0, a)$, $\psi_\epsilon \in C^{2+\alpha}(\mathbb{R}^N)$.

Finally we prove the positivity of the local branch of nontrivial solutions

Theorem 4.4 *Let* $N = 3, 4, ..., 7$ *and* f, g *as in Theorem 4.2. Then there exists* $a^* > 0$ *small enough such that for all* $\epsilon \in (0, a^*)$ *the function* $u_\epsilon = \epsilon\phi + \epsilon\psi(\epsilon)$ *defined above is a positive solution of the problem*

$$-\Delta u_\epsilon - \eta(\epsilon) g(x) f(u_\epsilon) = 0, \quad x \in \mathbb{R}^N$$

Proof For all $y \in \mathbb{R}^N$, the solutions of the equation

$$-\Delta u - \lambda g(x) f(u) = 0, \quad x \in \mathbb{R}^N$$

satisfies the following Serrin estimates from [33] (also [22, Theorem 8.17])

$$\begin{aligned} sup_{B_R(y)}|u(x)| &\leq c^*\{R^{N/p}||u||_{L^p(B_{2R}(y))} + R^{\frac{2(q-N)}{q}}||\lambda g f(u)||_{q/2}\} \\ &\leq c^*\{R^{N/2}||u||_{L^2(B_{2R}(y))} + R^{\frac{2(q-N)}{q}} k^* \lambda ||g||_\infty ||u||_{q/2}\} \quad (4.3) \\ &\leq c^*\{R^{N/2} + R^{\frac{2(q-N)}{q}} k^* \lambda ||g||_\infty\}||u||_{H^2} \end{aligned}$$

where the constant c^* depends only on N and q. By Sobolev embedding and [22, Theorem 8.17] q must satisfy: $q \geq 2$, $q > N$ and $\frac{2}{q} \geq \frac{N-4}{2N}$. All these conditions are true for $3 \leq N \leq 7$. On the other hand, for $\epsilon\phi + \epsilon\psi_\epsilon$ have

$$\epsilon\Delta\phi + \epsilon\Delta\psi_\epsilon = \eta(\epsilon) g(x) f(\epsilon\phi + \epsilon\psi_\epsilon)$$

or

$$\epsilon\Delta\psi_\epsilon = \frac{\eta(\epsilon)}{\epsilon} g(x) f(\epsilon\phi + \epsilon\psi_\epsilon) - \lambda_1 g(x)\phi.$$

Applying [22, Theorem 8.17] on the last equation, for any $y \in \mathbb{R}^N$ we get

$$\begin{aligned}
sup_{B_R(y)}|\psi_\epsilon(x)| &\leq c_* R^{N/2}||\psi_\epsilon||_{L^2(B_{2R}(y))} \\
&+ c_* R^{\frac{2(q-N)}{q}}||g\{\tfrac{\eta(\epsilon)}{\epsilon}f(\epsilon\phi+\epsilon\psi_\epsilon)-\lambda_1\phi\}||_{q/2} \\
&+ \eta(\epsilon)\psi_\epsilon+\epsilon^2\int_0^1(1-\tau)f''(\tau\epsilon(\phi+\psi_\epsilon))(\phi+\psi_\epsilon)^2\,d\tau\}||_{q/2} \\
&\leq c_* R^{N/2}||\psi_\epsilon||_{H^2}+c_* R^{\frac{2(q-N)}{q}}\{||g||_\infty|\eta(\epsilon)-\lambda_1|||\phi||_{q/2} \\
&+ \eta(\epsilon)||g||_\infty||\psi_\epsilon||_{q/2}+\epsilon sup_{\tau\in(0,1)}||f''(\tau\epsilon(\phi+\psi_\epsilon))||_\infty \\
&\times ||g||_\infty||\epsilon(\phi+\psi_\epsilon)||_\infty||\phi+\psi_\epsilon||_{q/2}\} \\
&\leq c_* R^{N/2}||\psi_\epsilon||_{H^2}+c_* R^{\frac{2(q-N)}{q}}\{||g||_\infty|\eta(\epsilon)-\lambda_1|||\phi||_{H^2} \\
&+ |\eta(\epsilon)|||g||_\infty||\psi_\epsilon||_{H^2}+c^*\epsilon sup_{\tau\in(0,1)}||f''(\tau\epsilon(\phi+\psi_\epsilon))||_\infty||g||_\infty \\
&\times ||\phi+\psi_\epsilon||_{H^2}[R^{-N/2}+k^* R^{\frac{2(q-N)}{q}}\eta(\epsilon)||g||_\infty]||\epsilon(\phi+\psi_\epsilon)||_{H^2}\}
\end{aligned}$$

by relation (4.3), where again $3 \leq N \leq 7$ and c_* depends only on N, q. Since $\phi(x) > 0$ for all x in the compact set B_{R_0}, it follows that there exists $\epsilon_1 > 0$ such that $\phi(x)+\psi_\epsilon(x) > 0$, for all $|x| \leq R_0$ provided that $0 < \epsilon < \epsilon_1$. For the global positivity we assume that for $0 < \epsilon < \epsilon_1$ there exists some $x_0 \in \mathbb{R}^N$ such that $u_\epsilon < 0$. Since $u_\epsilon(x) \to 0$ as $x \to \infty$ it follows that there must be some $x_1, |x_1| > R_0$ such that u_ϵ attains a negative minimum at x_1. But then $-\Delta u_\epsilon(x_1) = \lambda g(x_1) f(u_\epsilon(x_1)) > 0$, which is imposible, and the proof is completed. $\lhd$

5 Abstract Global Bifurcation Theory

In order to study the global bifurcation behavior of our problem we have to handle two kinds of difficulties. The first one is related to the lack of compactness of the operator associated to the problem and the second with the fact that the dependence on λ of the linear part of the operator is quite complicated. To overcome these difficulties, it is necessary to reformulate some of the standard theorems of bifurcation theory. For the sake of completeness we recall the notions of admissible (noncompact) operators and their degree.

Let X be a Banach space and $B : X \longmapsto X$ be a continuous linear operator. Then B is said to be *admissible* if $\lambda I - B$ is a Fredholm operator for all $\lambda \geq 1$. Let Ω be an open subset of X and $F : \bar{\Omega} \longmapsto X$ a (nonlinear) operator. F is said to be *admissible* if (i) F is twice continuously (Fréchet-) differentiable on $\bar{\Omega}$, (ii) $I - F$ is proper (that is $(I-F)^{-1}(D)$ is compact if D is compact), and (iii) $\lambda I - F'(x)$ is a Fredholm operator of index zero for all $\lambda \geq 1$ and $x \in \Omega$.

A *homotopy* $H : [a,b] \times \bar{\Omega} \longmapsto X$ is said to be *admissible* if (i) it is twice continuously differentiable on $[a,b] \times \bar{\Omega}$, (ii) $I - H$ is proper on $[a,b] \times \bar{\Omega}$, and (iii) $H_u(\lambda, u)$ (the partial derivative with respect to u) is admissible of all $(\lambda, u) \in [a,b] \times \bar{\Omega}$.

The set of all admissible operators on Ω is denoted by $\mathcal{F}(\Omega)$. A point $p \in X$ is a *regular value* of F if $F'(u)$ is surjective for all $u \in F^{-1}(p)$.

Under these assumptions N. Dancer [15] was able to extend the Leray-Schauder degree to the class $\mathcal{F}(\Omega)$ as follows

Definition 5.1 *Suppose that $\Phi = I - F \in \mathcal{F}(\Omega)$ and $p \notin \Phi(\partial\Omega)$. If p is a regular value of Φ, define the* degree *of F at u to be*

$$deg(\Phi, \Omega, u) = \sum_{u \in \Phi^{-1}(u)} (-1)^{v(u)}$$

where $v(u)$ is the sum of the multiplicities of the eigenvalues of $F'(u)$ in $(1, +\infty)$. If p is not a regular value of Φ, choose a sequence of regular values $\{p_n\}$ such that $p_n \to p$ in X and defined $deg(\Phi, \Omega, p) = \lim_{n\to\infty} deg(\Phi, \Omega, p_n)$.

It is proved that the above introduced generalized degree satisfies all the properties of Leray-Schauder degree. Moreover, the new definition agrees with the old one when they both are defined. We refer to [15, 26].

Definition 5.2 *Let $F \in \mathcal{F}(\Omega)$ and u_0 be an isolated fixed point of F. Then we define the* index *of F at u_0 as $i(F, u_0) = deg(I - F, B, u_0)$, where B is a ball centered at u_0, with u_0 the only fixed point of F in B.*

If u_0 is a fixed point of F such that $I - F'(u_0)$ is invertible, then u_0 is an isolated fixed point of F and

$$i(F, u_0) = deg(I - F, B, u_0) = deg(I - F'(u_0), \hat{B}, 0),$$

where $B, \hat{B}$ are sufficiently small balls centered at u_0, 0 respectively.

Using the above degree we can state a generalized version of Rabinowitz' global bifurcation theorem, which unifies two earlier adaptations appearing in [6, 15].

Theorem 5.3 *Let X be a Banach space and $E = \mathrm{I\!R} \times X$. Assume that U is an open subset of E and $G : U \longmapsto X$ is a twice continuously differentiable mapping such that (a) $G(\lambda, 0) = 0$, for all $(\lambda, 0) \in U$, (b) the partial derivative $G_u(\lambda, 0)$ is a linear compact operator with positive principal eigenvalue λ_1, such that the operator $I - G_u(\lambda, 0)$ is invertible for all $0 < |\lambda - \lambda_1| < \epsilon$, (c) for any $(\lambda, u) \in U$ the linear operator $G_u(\lambda, u)$ is admissible, (d) $i(G(\lambda, .))$ is constant on $(\lambda_1 - \epsilon, \lambda_1)$ and $(\lambda_1, \lambda_1 + \epsilon)$, such that if $\lambda_1 - \epsilon < \underline{\lambda} < \lambda_1 < \bar{\lambda} < \lambda_1 + \epsilon$, then $i(G(\underline{\lambda}, .), 0) \neq i(G(\bar{\lambda}, .), 0)$. Then there exists a continuum $\mathcal{C}_{\lambda_1}$, in the $\lambda - u$ plane of solutions of $u = G(\lambda, u)$, such that either*
(i) $\mathcal{C}_{\lambda_1}$ joins $(\lambda_1, 0)$ to $(\mu, 0)$, where $I - G_u(\mu, 0)$ is not invertible, or
(ii) $\mathcal{C}_{\lambda_1}$ is not a compact set in E.

Proof Suppose by way of contradiction that $\mathcal{C}_{\lambda_1}$ is a compact set in E and $\mathcal{C}_{\lambda_1} \cup \mathrm{I\!R} \times \{0\} = \{(\lambda_1, 0)\}$. Since $G \in C^2(U)$ by Sard-Smale theorem there is an open neighborhood V of $\mathcal{C}_{\lambda_1}$ (in E) such that $\bar{V} \subset U$ and $I - G|\bar{V}$ is proper. By [29, lemma1.1] there exist disjoint compact sets K_1, K_2 in E such that $K_1 \cap \mathrm{I\!R} \times \{0\} = \{(\lambda_1, 0)\}$, $\mathcal{C}_{\lambda_1} \subset K_1$, $\mathcal{L}_U \cap \partial V \subset K_2$ and $K_1 \cup K_2 = \mathcal{L}_U \cap \bar{V}$, where $\mathcal{L}_U$ stands for the closure (in U) of the set $\{(\lambda, u) \in U : u \neq 0, \ u = G(\lambda, u)\}$. For the rest of the proof we follow [29], except that we use the degree defined above and for the last part we use condition (d). $\lhd$

Remark 5.4 *Here the linearized operator $G(\lambda, 0)$ can be of the form $\lambda K(\lambda)$, where $K(\lambda)$ is a linear bounded operator of λ for $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$.*

6 Compactness of the Solution Set

To apply the abstract bifurcation theory developed in section 5 it is necessary to characterise the solution set of the equation $(1.1)_\lambda$

$$S_\Lambda =: \{(\lambda, u) \in [\lambda_0, \Lambda) \times H^2 : u \text{ solution of } (1.1)_\lambda, \text{with } u \in (0,1),\ \lambda_0 > 0\}.$$

Some of the ideas applied here were inspired from the papers [4, 20]. To complete the aim of this section we need the following results

Lemma 6.1 *Let* $\rho \in L^2(\mathbb{R}^N)$. *Then the equation*

$$-\Delta\psi - g^-\psi = \rho, \quad in\ \mathbb{R}^n, \tag{6.1}$$

admits a unique solution $\psi \in H^1$. *Moreover, if* $\rho \geq 0$, $\rho \not\equiv 0$ *in* $\mathbb{R}^N$, *then* $\psi(x) > 0$ *for all* $x \in \mathbb{R}^N$.

Proof Introduce the functional $\Psi : H^1 \mapsto \mathbb{R}$ defined by $\Psi(u) = \int_{\mathbb{R}^N} \rho u dx$ for all $u \in H^1$. Since it is continuous and linear we can apply Riesz' theory to get the existence of a unique function $\psi \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla\psi \nabla u dx - \int_{\mathbb{R}^N} g^-\psi u dx = \int_{\mathbb{R}^N} \rho u dx, \text{ for all } u \in H^1.$$

The positivity of $\psi(x)$ is implied by substituting u by ψ^- in the last relation and then applying the strong maximum principle. $\lhd$

Let $M =: \max_{t\in[0,1]} f(t)$. If we set $\rho = Mg^+$, then (g^+ been of bounded support, can be considered as an L^p-function, $p \geq 1$) we get a unique solution $\psi_0 \in H^1(\mathbb{R}^N)$ of the equation (6.1). Since $g^+ \not\equiv 0$ then $\psi_0 > 0$ in $\mathbb{R}^N$. Next lemma gives a "uniform" upper bound for the solutions of the equation $(1.1)_\lambda$ in $(0,1)$; note that $g^+ \equiv g_1^+$.

Lemma 6.2 *For every solution* $u \in (0,1)$ *of (1.1)*$_\lambda$ *we have* $u < \lambda(\psi_0 + C)$, *for some positive constant* C.

Proof By [25, lemma 2.3] we get the Newtonian potential formulation of the solutions of $(1.1)_\lambda$

$$\begin{aligned} u(x) &= \lambda \int \frac{g(y)f(u)}{|x-y|^{N-2}}dy \\ &= \lambda \int \frac{g_1(y)f(u)}{|x-y|^{N-2}}dy + \lambda \int \frac{g_2(y)f(u)}{|x-y|^{N-2}}dy \qquad (6.2) \\ &\leq \lambda M \int \frac{g^+(y)}{|x-y|^{N-2}}dy + \lambda \int \frac{g_2(y)f(u)}{|x-y|^{N-2}}dy \end{aligned}$$

The same formulation applied to equation (6.1), where $\rho = Mg^+$ implies

$$\psi_0(x) = \int \frac{g^+(y)M + g^-(y)\psi_0(y)}{|x-y|^{N-2}}dy.$$

from where we obtain

$$\begin{aligned} 0 < \psi_0(x) &\leq \int_{\mathbb{R}^N} \frac{g^+(y)M}{|x-y|^{N-2}}dy \\ &\leq M||g^+||_\infty \int_{B_{R_0}} \frac{r^{N-1}}{r^{N-2}}dr = \frac{1}{2}M||g^+||_\infty R_0^2. \end{aligned} \qquad (6.3)$$

and

$$0 \leq -\int \frac{g^-(y)\psi_0(y)}{|x-y|^{N-2}}dy \leq \int \frac{g^+(y)M}{|x-y|^{N-2}}dy \leq \frac{1}{2}M||g^+||_\infty R_0^2.$$

Hence from equation (6.2), (6.3) we get the estimate

$$\begin{aligned} u(x) &\leq \lambda\psi_0(x) - \lambda \int \frac{g^-(y)\psi_0(y)}{|x-y|^{N-2}}dy + \lambda \int \frac{g_2(y)f(u)}{|x-y|^{N-2}}dy \\ &\leq \lambda\psi_0(x) - \lambda \int \frac{g^-(y)\psi_0(y)}{|x-y|^{N-2}}dy \\ &\leq \lambda\psi_0(x) + \tfrac{1}{2}\lambda M||g^+||_\infty R_0^2 \equiv \lambda(\psi_0(x) + C). \qquad \lhd \end{aligned}$$

We also have the following *a priori* estimate for the H^2-norm of all solutions of $(1.1)_\lambda$ lying in $(0, 1)$.

Lemma 6.3 *There exist* $\Lambda_0 > 0$ *such that for all solutions* (λ, u) *of (1.1)*$_\lambda$ *lying in* $(0,1)$ *with* $\lambda > \Lambda_0$*, we have*

$$||u||_{H^2} \leq \{\lambda^2 (k^*)^2 ||g||_\infty^2 + 1\}\lambda^2 M \int_{\mathbb{R}^N} g^+(x)(\psi_0(x) + C)dx. \qquad (6.4)$$

Proof First we prove that there exists $l < 1$ such that $0 < u(x) < l$ for all $x \in \mathbb{R}^N$. Indeed, by $(\mathcal{G})$ we have that $-\Delta u = \lambda g(x) f(u) \leq 0$ for all $|x| > R_0$, i.e. u is subharmonic for $|x| > R_0$. The Hadamard's three circles theorem [28, pg 131] implies that $\sup\{u(y) : |y| = r\} \leq \sup\{u(y) : |y| = R_0\}$ for all $r > R_0$. Hence $u(x) \leq \sup\{u(y) : |y| = R_0\} =: l$ for all $x \in \mathbb{R}^N$. Therefore there exists Λ_0 such that for all (λ, u) solutions of $(1.1)_\lambda$ with $0 < u(x) < 1$, f satisfies the condition $u \leq \Lambda_0 \frac{k}{\sigma_0} f(u)$ i.e. $u \leq -\lambda g_2 f(u)$, where $\lambda > \Lambda_0$ Then we have that

$$\begin{aligned} ||u||_{H^1} &= \lambda \int g_1 f(u) u dx + \int \{\lambda g_2 f(u) + u\} u dx \\ &\leq \lambda^2 M \int g^+(x)(\psi_0(x) + C) dx. \end{aligned}$$

Therefore for the H^2-norm we get

$$\begin{aligned} ||u||_{H^2} &\leq \lambda^2 \int g^2 f^2(u) dx + ||u||_{H^1} \\ &\leq \lambda^2 (k^*)^2 ||g||_\infty^2 \int |u|^2 dx + ||u||_{H^1} \leq \{\lambda^2 (k^*)^2 ||g||_\infty^2 + 1\} ||u||_{H^1} \\ &\leq \{\lambda^2 (k^*)^2 ||g||_\infty^2 + 1\} \lambda^2 M \int g^+(x)(\psi_0(x) + C) dx. \qquad \triangleleft \end{aligned}$$

Theorem 6.4 S_Λ *is a compact subset of* $[\lambda_0, \Lambda] \times H^2(\mathbb{R}^N)$.

Proof Relation (6.4) implies that for all $(\lambda, u) \in S_\Lambda$

$$||u||_{H^2} \leq \{\Lambda^2 (k^*)^2 ||g||_\infty^2 + 1\}\Lambda^2 M \int_{\mathbb{R}^N} g^+(x)(\psi_0(x) + C)dx =: K^*. \qquad (6.5)$$

So S_Λ is bounded in $[\lambda_0, \Lambda] \times H^2(\mathbb{R}^N)$. Hence for any sequence $\{\lambda_n, u_n\}$

of S_Λ, there is a subsequence, denoted again by $\{\lambda_n, u_n\}$, such that $u_n \rightharpoonup u$, in $H^2(\mathbb{R}^N)$ and $\lambda_n \to \lambda$ in $\mathbb{R}$. Sobolev embeddings imply that

$$u_n \rightharpoonup u, \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{for all} \ p \geq 1 \tag{6.6}$$

Also the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^p(B_{r_0}),\ p \geq 1$ is compact, so we have

$$||u_n - u||_{L^p(B_{r_0})} \to 0, \quad \text{as} \quad n \to \infty. \tag{6.7}$$

Then using Theorem 2.3 we obtain

$$\begin{aligned}
||u_n - u||_{H^2} &\leq ||\Delta u_n - \Delta u||_2 + c\,||u_n - u||_{2,\lambda} \\
&\leq |\lambda_n^2 - \lambda^2| \int g^2 f^2(u_n)dx + \lambda^2 \int g^2 |f^2(u_n) - f^2(u)|dx \\
&+ \int |g_1 u_n(\lambda_n f(u_n) - \lambda f(u))|dx \\
&+ \lambda \int |g_1 f(u)(u_n - u)|dx + |\lambda_n - \lambda| \int |g_2 f(u_n) u_n|dx \\
&+ \lambda \int |g_2 f(u)(u_n - u)|dx + \int |g_2 u_n(f(u_n) - f(u))|dx \\
&+ \lambda \int |g_2 u_n(u_n - u)|dx + \lambda \int |g_2 u(u_n - u)|dx
\end{aligned}$$

So relations (6.5), (6.6), (6.7) imply

$$\begin{aligned}
||u_n - u||_{H^2} &\leq 2\Lambda(k^*)^2 K^* ||g||_\infty^2 |\lambda_n - \lambda| + \Lambda^2 k^* ||g||_\infty \int |u_n - u|\psi_0 dx \\
&+ \Lambda^2 k^* ||g||_\infty C \int |u_n - u|dx \\
&+ ||g_1||_\infty \int_{B_{r_0}} (\psi_0 + C)|\lambda_n f(u_n) - \lambda f(u)|dx \\
&+ \Lambda^2 k^* ||g_1||_\infty \int_{B_{r_0}} (\psi_0 + C)|u_n - u|dx \\
&+ |\lambda_n - \lambda| k^* K^* ||g||_\infty + c_1 \Lambda^2 ||g||_\infty \int (\psi_0 + C)|u_n - u|dx \to 0
\end{aligned}$$

and the proof is completed. $\lhd$

7 The Global Continuation

In this section we prove that the branch of positive solutions of $(1.1)_\lambda$ obtained in Section 4 can be continued for all $\lambda > \lambda_1$. For the rest of this section we assume that f satisfies the additional hypothesis

$(\mathcal{F}_2)$ *there exists a smooth function* $f_1 : \mathbb{R} \longmapsto \mathbb{R}$ *such that* $f(u) = u + f_1(u)$, *where* f_1 *satisfies the following conditions* $f_1(0) = {f'}_1(0) = 0$, ${f'}_1, {f''}_1, {f'''}_1 \in L^\infty(\mathbb{R})$ *and there is* $k_1 > 0$ *such that* $|{f'}_1(u)| \leq k_1|u|$.

Define the operator $G : \mathcal{E} =: [\lambda_0, +\infty) \times H^2(\mathbb{R}^N) \longmapsto H^2(\mathbb{R}^N)$ by

$$G(\lambda, u) =: \lambda L^{-1}(\lambda) g_1(x) u + \lambda L^{-1}(\lambda) g(x) f_1(u)$$

Then equation $(1.1)_\lambda$ can be written

$$u = G(\lambda, u) \equiv K(\lambda)u + R(\lambda, u)$$

where $K(\lambda) =: \lambda L^{-1}(\lambda) g_1(x)$ and $R(\lambda, u) =: \lambda L^{-1}(\lambda) g(x) f_1(u)$ the linear and nonlinear part of the operator $G(\lambda, u)$ respectively. Next lemma describes the properties of the operator G.

Lemma 7.1 *Let* $N = 3, 4, 5$ *and* f, g *satisfy hypothesis* $(\mathcal{F})$, $(\mathcal{F}_1)$, $\mathcal{F}_2)$ *and* $(\mathcal{G})$ *respectively. Then (i)* G *is twice continuously differentiable with* $G(\lambda, 0) = 0$ *and* $R_u(\lambda, 0) = 0$, *for all* λ, *(ii) the partial derivative* $G_u(\lambda, 0)$ *is a linear compact operator with positive principal eigenvalue* λ_1 *and there exists* $\epsilon > 0$ *such that the operator* $I - G_u(\lambda, 0)$ *is invertible, if* $0 < |\lambda_1 - \lambda| < \epsilon$.

Proof (i) The proof follows a standard procedure which is omitted.

(ii) We have that the operator

$$G_u(\lambda, 0) \equiv \lambda L^{-1}(\lambda) g_1 : H^2(\mathbb{R}^N) \longmapsto H^2(\mathbb{R}^N)$$

is linear with eigenvalue 1 at $\lambda = \lambda_1$ by lemma 3.1. Hence there exists $\epsilon > 0$ sufficiently small such that $G_u(\lambda, 0)$ has no other eigenvalue for

$0 < |\lambda_1 - \lambda| < \epsilon$ i.e. the operator $I - G_u(\lambda, 0)$ is invertible. As far as the compactness of $G_u(\lambda, 0)$ is concerned, let $\{u_n\}$ be a bounded sequence in $H^2(\mathbb{R}^N)$. By Rellich - Kontrachov Theorem (see [1]) for any $R > 0$ and any $N \geq 1$ the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^2(B_R)$ is compact. Hence there is a subsequence denoted again by $\{u_n\}$ converging in $L^2(B_R)$. Choose R large enough so that $supp(g_1) \subset B_R$. Then we have

$$\begin{aligned}
||G_u(\lambda, 0)u_n - G_u(\lambda, 0)u_m||_{H^2} &= \lambda||L^{-1}(\lambda)g_1 u_n - L^{-1}(\lambda)g_1 u_m||_{H^2} \\
&\leq \lambda||L^{-1}(\lambda)||\, ||g_1(u_n - u_m)||_{L^2(B_R)} \\
&\leq \lambda||L^{-1}(\lambda)||\, ||g_1||_\infty \, ||u_n - u_m||_{L^2(B_R)}
\end{aligned}$$

and the compactness of $G_u(\lambda, 0)$ is proved. $\lhd$

The following notation is going to be used next. Let $r^* > 0$ (to be fixed later). Then we define

$$g_*(x) =: \begin{cases} g(x), & \text{for} \quad |x| \leq r_* \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g_\infty =: g(x) - g_*(x).$$

Lemma 7.2 *Let* $N \geq 1$ *and* f, g *satisfy conditions of Lemma 7.1. Then* $\mu I - G_u(\lambda, u)$ *is a linear Fredholm operator of index zero for all* $(\lambda, u) \in \mathcal{E}$ *and* $\mu \geq 1$.

Proof Let $r^* > 0$ (to be defined later). Then we have the following operator decomposition

$$\begin{aligned}
\mu I - G_u(\lambda, u) &= \mu I - \lambda L^{-1}(\lambda)g_1(x) - \lambda L^{-1}(\lambda)g(x){f_1}'(u) \\
&= \mu I - \lambda L^{-1}(\lambda)g_1(x) - \lambda L^{-1}(\lambda)g_*(x){f_1}'(u) \\
&- \lambda L^{-1}(\lambda)g_\infty(x){f_1}'(u) \equiv F(\mu, \lambda, u) + N(\lambda, u)
\end{aligned}$$

where

$$F(\mu, \lambda, u) =: \mu I - \lambda L^{-1}(\lambda)g_1(x) - \lambda L^{-1}(\lambda)g_*(x){f_1}'(u),$$

and

$$N(\lambda, u) =: \lambda L^{-1}(\lambda) g_\infty(x) {f_1}'(u).$$

Following similar ideas as in Lemma 7.1 (ii) we can prove that the operator

$$F_1(\lambda, u) =: -\lambda L^{-1}(\lambda) g_*(x) {f_1}'(u),$$

is compact on U. Hence the operator $F(\mu, \lambda, u)$ is Fredholm of index zero for all $\mu \geq 1$ and $(\lambda, u) \in U$ (see [40, pg 368]). According to [32, Theorems 3.1,3.2], to complete the proof we must show that

$$||N(\lambda, u)||_{H^2} < 1.$$

Indeed, we have

$$\begin{aligned}
||N(\lambda, u)||_{H^2} &= \inf_{||v||_{H^2}=1} ||\lambda L^{-1}(\lambda) g_\infty(x) {f_1}'(u) v||_{H^2} \\
&\leq \inf_{||v||_{H^2}=1} \lambda ||L^{-1}(\lambda)|| \, ||g_\infty(x) {f_1}'(u) v||_{L^2} \\
&= \inf_{||v||_{H^2}=1} \lambda \, ||L^{-1}(\lambda)|| \, ||g_\infty(x) {f_1}'(u) v||_{L^2(B^*_{r^*})} \\
&\leq \inf_{||v||_{H^2}=1} \lambda \, k_1 \, ||L^{-1}(\lambda)|| \, ||g_\infty||_{L^\infty(B^*_{r^*})} \, ||uv||_{L^2(B^*_{r^*})}
\end{aligned}$$

By Hölder inequality we have that

$$||uv||_{L^2(B^*_{r^*})} \leq ||u||_{L^\sigma(B^*_{r^*})} \, ||v||_{L^\rho(B^*_{r^*})},$$

where $\sigma, \rho \geq 1$ and $\frac{1}{2} = \frac{1}{\sigma} + \frac{1}{\rho}$. The embedding $H^2(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, is continuous for $1 \leq q < +\infty$, if $N \leq 4$ and for $1 \leq q < \frac{2N}{N-4}$, if $N \geq 5$. Then by choosing $\rho \in [1, 2)$ and $\sigma \geq 1$ so that $\frac{1}{2} = \frac{1}{\sigma} + \frac{1}{\rho}$ we have that $H^2(\mathbb{R}^N) \hookrightarrow L^\sigma(\mathbb{R}^N)$, $H^2(\mathbb{R}^N) \hookrightarrow L^\rho(\mathbb{R}^N)$. Hence

$$||uv||_{L^2(B^*_{r^*})} \leq ||u||_{H^2(B^*_{r^*})} \, ||v||_{H^2(B^*_{r^*})}.$$

Therefore we get

$$||N(\lambda, u)||_{H^2} \leq \lambda \, k_1 \, ||L^{-1}(\lambda)|| \, ||g_\infty||_{L^\infty(B^*_{r^*})} \, ||u||_{H^2(B^*_{r^*})}$$

where for any $\lambda > 0$, $||L^{-1}(\lambda)||$ is bounded by Lemma 3.3. Hence for all $\lambda > 0$ we can find r^* large enough so that

$$||N(\lambda, u)||_{H^2} \leq 1$$

and the proof is completed. $\lhd$

Theorem 7.3 *Let* $N = 3, 4, 5$ *and* f, g *satisfy conditions of Lemma 7.1. Then there exists a continuum of solutions* $\mathcal{C}_{\lambda_1}$ *of (1.1)*$_\lambda$ *in the* (λ, u) *- plane emanating from* $(\lambda_1, 0)$ *such that either*
(i) $\mathcal{C}_{\lambda_1}$ *joins* $(\lambda_1, 0)$ *to* $(\mu, 0)$*, where* $I - G_u(\mu, 0)$ *is not invertible, or*
(ii) $\mathcal{C}_{\lambda_1}$ *is not a compact set in* $\mathcal{E}$*.*

Proof Since λ_1 is a principal eigenvalue of $K(\lambda) = G_u(\lambda, 0)$, there is an $\epsilon > 0$ such that λ_1 is the only eigenvalue of $K(\lambda)$ in the interval $(\lambda_1 - \epsilon, \lambda_1 + \epsilon)$ which in addition is simple. Hence for any $t_1 \in (\lambda_1 - \epsilon, \lambda_1 + \epsilon)$ and $t_2 \in (\lambda_1 - \epsilon, \lambda_1 + \epsilon)$ we can easily see that

$$1 \;=\; i(G(t_1, .), 0) \;\neq\; i(G(t_2, .), 0) \;=\; -1.$$

Combining this fact along with the results of lemmas 7.1, 7.2, we have that all hypothesis of the global bifurcation theorem 5.3 are satisfied. Thus there exists a continuum $\mathcal{C}_{\lambda_1}$ of nonzero solutions of $(1.1)_\lambda$ in $\mathcal{E}$ satisfying one of the alternatives (i) or (ii). $\lhd$

$\mathcal{C}_{\lambda_1}$ has a connected subset $\mathcal{C}^+_{\lambda_1} \subset \mathcal{C}_{\lambda_1} - \{(\eta(\epsilon), u_\epsilon) : -\epsilon_0 \leq \epsilon \leq 0\}$ for some $\epsilon_0 > 0$ such that $\mathcal{C}^+_{\lambda_1}$ also satisfies one of the above alternatives. Finally, it is clear that close to the bifurcation point $(\lambda_1, 0)$, $\mathcal{C}^+_{\lambda_1}$ consists of the curve $\epsilon \to (\eta(\epsilon), u_\epsilon)$, $0 < \epsilon \leq \epsilon_0$.

We now investigate the nature of solutions lying on $\mathcal{C}^+_{\lambda_1}$. First, we show that solutions u_λ of $(1.1)_\lambda$ on the branch $\mathcal{C}^+_{\lambda_1}$ remain strictly in the open interval $(0, 1)$ in the L^∞ - norm for all $\lambda > \lambda_1$. For this it is necessary to prove that solutions which are close in $\mathbb{R} \times H^2$ are also close in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$; since $H^2(\mathbb{R}^N)$ does not embed in $L^\infty(\mathbb{R}^N)$, for $N > 4$, this is not immediately obvious.

Lemma 7.4 *Let* $N = 3, 4, ..., 7$ *and* f, g *satisfy conditions of Lemma 7.1. Suppose that* $u_\lambda \in H^2$ *is a solution of (1.1)*$_\lambda$. *Then there exist constants* K_1 *and* K_2 *such that*

$$|u_\lambda(x) - u_\mu(x)| \leq K_1 \, ||u_\lambda - u_\mu||_{H^2} + K_2|\lambda - \mu| \quad \textit{for all} \quad x \in \mathbb{R}^N$$

whenever μ *is close to* λ *and* $u_\mu \in H^2(\mathbb{R}^N)$ *is a solution of (1.1)*$_\mu$.

Proof Introduce the operator $\Lambda : \mathbb{R} \longmapsto H^2(\mathbb{R}^N)$ by $\Lambda(\lambda) =: u_\lambda$. Then we have that

$$-\Delta(u_\lambda - u_\mu) = g\{\lambda f(u_\lambda) - \mu f(u_\mu)\} =: h.$$

By Sobolev imbeddings $h \in L^p$ for all $p \geq 1$. So by [22, Theorem 8.17] for any $x \in \mathbb{R}^N$ there exists $C = C(p, N, ||g||_\infty) > 0$ such that

$$\begin{aligned} &|\Lambda(\lambda) - \Lambda(\mu)| = |u_\lambda(x) - u_\mu(x)| \leq \sup_{y \in B_1(x)} |u_\lambda(y) - u_\mu(y)| \\ &\leq C \, \{||u_\lambda - u_\mu||_{L_q(B_2(x))} + ||g \, [\lambda f(u_\lambda) - \mu f(u_\mu) \,]||_q\}. \end{aligned} \tag{7.1}$$

where we take $p = q/2 > 1$ and $q > N$. For the last term of the relation (7.1) we have the following estimate

$$\begin{aligned} &||g \, [\lambda f(u_\lambda) - \mu f(u_\mu) \,]||_{q/2} \leq \\ &\leq |\lambda - \mu| ||g f(u_\mu)||_{q/2} + ||\lambda g[f(u_\lambda) - f(u_\mu)]||_{q/2} \leq \\ &\leq \lambda \, \sup_{\tau \in (0,1)} ||f'(\tau u_\lambda + (1 - \tau) u_\mu)||_\infty ||g||_\infty ||u_\lambda - u_\mu||_{H^2} + \\ &+ |\lambda - \mu| k^* ||g||_\infty ||u_\mu||_{H^2} \leq \\ &\leq K_1 \, ||u_\lambda - u_\mu||_{H^2} + K_2 |\lambda - \mu| \end{aligned} \tag{7.2}$$

where K_1 and K_2 depend on $k^*, f', \lambda, ||g||_\infty, ||u_\mu||_{H^2}$ This relation is true for $\frac{N-4}{2N} \leq \frac{2}{q}$ and $\frac{q}{2} \leq 1$. It is easy to see that all the above restrictions on q are satisfied for $N < 8$. So from relations (7.1), (7.2) we have

$$\sup_{y \in B_1(x)} |u_\lambda(y) - u_\mu(y)| \leq K_1 \, ||u_\lambda - u_\mu||_{H^2} + K_2 |\lambda - \mu| \tag{7.3}$$

Therefore the operator Λ is continuous in $L^\infty(\mathbb{R}^N)$, which completes the proof of the lemma. $\triangleleft$

Theorem 7.5 *Suppose f, g satisfy conditions of Lemma 7.1. Then $0 < u(x) < 1$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+_{\lambda_1}$.*

Proof For the proof we follow the ideas developed in [11, Theorem 4.6]. Suppose that there exists $(\lambda, u) \in \mathcal{C}^+_{\lambda_1}$ such that $u(x_0) < 0$ for some $x_0 \in \mathbb{R}^N$. By Theorem 4.4, $u(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+_{\lambda_1}$ is close to $(\lambda_1, 0)$. Moreover, by Lemma 7.4 points in $\mathcal{C}^+_{\lambda_1}$ which are close in $\mathbb{R} \times H^2$ must also be close in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. Hence there must exist $(\lambda_0, u_0) \in \mathcal{C}^+_{\lambda_1}$ such that $u_0(x) \geq 0$ for all $x \in \mathbb{R}^N$ but $u_0(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$ and in any neighbourhood of (λ_0, u_0) we can find a point $(\hat{\lambda}, \hat{u}) \in \mathcal{C}^+_{\lambda_1}$ with $\hat{u}(x) < 0$ for some $x \in \mathbb{R}^N$. Let B denote any open ball containing x_0. Then

$$-\Delta u_0(x) - \lambda g(x) \frac{f(u_0(x))}{u_0(x)} u_0(x) = 0 \text{ on } B \text{ and } u_0(x) \geq 0 \text{ on } \partial B$$

It follows from the Serrin Maximum principle (see [21]) that $u_0 \equiv 0$ on B. Hence $u_0 \equiv 0$ on $\mathbb{R}^N$. Thus we can construct a sequence $\{(\lambda_n, u_n)\} \subseteq \mathcal{C}^+_{\lambda_1}$ such that $u_n(x) > 0$ for all $n \in \mathcal{N}$ and $x \in \mathbb{R}^N$, $u_n \to 0$ in H^2 and $\lambda_n \to \lambda_0$. Let $v_n = \frac{u_n}{||u_n||_{H^2}}$. Since $u_n = K(\lambda_n)(u_n) + R(\lambda_n, u_n)$, then

$$v_n = K(\lambda_n)(v_n) + \frac{R(\lambda_n, u_n)}{||u_n||_{H^2}}.$$

Since $K(\lambda)$ is compact, there exists a subsequence of $\{v_n\}$, again denote by $\{v_n\}$, such that $\{K(\lambda_n)(v_n)\}$ is convergent. Since $\lim_{n\to\infty} \frac{R(\lambda_n, u_n)}{||u_n||_{\mathcal{V}}} = 0$, $\{v_n\}$ is convergent to v_0, say, and $v_0 = K(\lambda_0)(v_0)$. Since $v_n \geq 0$ for all $n \in \mathcal{N}$, $v_0 \geq 0$. Since by lemma 3.1 λ_1 is the only positive eigenvalue corresponding to a positive eigenfunction, it follows that $\lambda_1 = \lambda_0$. Thus $(\lambda_0, u_0) = (\lambda_1, 0)$ and this contradicts the fact that every neighbourhood of (λ_0, u_0) must contain a solution $(\hat{\lambda}, \hat{u}) \in \mathcal{C}^+_{\lambda_1}$ with $\hat{u}(x) < 0$, for some

$x \in \mathbb{R}^N$. Hence $u(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+_{\lambda_1}$.

In a similar way we can prove that $u(x) < 1$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+_{\lambda_1}$ (we work with the function $v_0 = 1 - u_0$) and so the proof is complete. $\lhd$

Following arguments very similar to the ones developed above we have the next result.

Corollary 7.6 $\mathcal{C}^+_{\lambda_1}$ *contains no points of the form* $(\lambda, 0)$*, where* $\lambda \neq \lambda_1$*, i.e.* $\mathcal{C}^+_{\lambda_1}$ *must connect* $(\lambda_1, 0)$ *to* ∞ *in* $\mathbb{R} \times H^2$.

Next we show that $\mathcal{C}^+_{\lambda_1}$ is bounded below in λ.

Lemma 7.7 *There exists* $\lambda_* > 0$ *such that* $\lambda > \lambda_*$ *whenever* $(\lambda, u) \in \mathcal{C}^+_{\lambda_1}$.

Proof Suppose $u \in H^2$ is a solution of $(1.1)_\lambda$, (1.2). Multiplying equation $(1.1)_\lambda$ by u, and integrating over $\mathbb{R}^N$ we get

$$\int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx = \lambda \int_{\mathbb{R}^N} gf(u)udx \leq |\lambda| k^* ||g||_\infty \; ||u||^2_2, \leq |\lambda| K_1 ||g||_\infty \; ||u||^2_{H^2},$$

where K_1 is a constant. If $\lambda = 0$ then $|| \bigtriangledown u||_2 = 0$ which combined with the fact that $\lim_{|x| \to \infty} u = 0$ implies that $u \equiv 0$, which contradicts with lemma 7.5. Negative λ are excluded because of lemma 7.4 and the proof is completed. $\lhd$

As an immediate consequence of the previous results we can give the following complete description of the continuum $\mathcal{C}^+_{\lambda_1}$.

Theorem 7.8 *Suppose that* $N = 3, 4, 5$ *and* g, f *as in Lemma 7.1. Then there exists a continuum* $\mathcal{C}^+_{\lambda_1} \subseteq \mathbb{R} \times H^2$ *of solutions of* $(1.1)_\lambda$*, (1.2) bifurcating at* $(\lambda_1, 0)$ *such that*
(i) if $(\lambda, u) \in \mathcal{C}^+_{\lambda_1}$ *then* $\lambda > 0$ *and* $0 < u(x) < 1$ *for all* $x \in \mathbb{R}^N$,
(ii) $\{\lambda : (\lambda, u) \in \mathcal{C}^+_{\lambda_1}$ *for some* $u \in H^2\} \supseteq (\lambda_1, \infty]$. *In particular,* $(1.1)_\lambda$*, (1.2) has a nontrivial solution* $u \in H^2$ *such that* $0 < u(x) < 1$ *for all* $x \in \mathbb{R}^N$ *whenever* $\lambda > \lambda_1$.

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