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ON THE CONSTRUCTION OF SUPER AND SUBSOLUTIONS FOR ELLIPTIC EQUATIONS ON ALL OF $\mathbb{R}^N$

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1. INTRODUCTION

In this paper we shall prove the existence of solutions of the reaction–diffusion equation

$$-\Delta u(x) = \lambda g(x) f(u(x)) \qquad \text{for } x \in \mathbb{R}^N \tag{1.1}$$

$$0 \le u(x) \le 1 \tag{1.2}$$

arising in population genetics (see, e.g. [1]). In (1.1) the unknown function u represents the relative frequency of an allele population A_1 which is in competition with another allele population A_2 and so we are interested in solutions such that $0 \le u \le 1$. The real parameter $\lambda > 0$ is the reciprocal of the diffusion coefficient.

We shall assume throughout that $g: \mathbb{R}^N \to \mathbb{R}$ is a smooth function which takes on both positive and negative values, the sign of the function g indicating whether the A_1 allele is advantaged (where $g(x) > 0$) or disadvantaged (where $g(x) < 0$) over a rival allele A_2 at the point x.

We also assume throughout that f: $[0, 1] \to \mathbb{R}$ satisfies

$$f \in C^1[0, 1]; \quad f(0) = f(1) = 0; \quad f(u) > 0 \quad \text{for } 0 < u < 1; \quad f'(0) > 0.$$

These properties of f correspond to the fact that the fitness of the gene A_1A_2 is intermediate between those of the genes A_1A_1 and A_2A_2. Although we are interested only in solutions u satisfying $0 \le u \le 1$, our arguments will occasionally involve $f(u)$ with $u \notin [0, 1]$. We shall assume throughout that $f(u) \equiv 0$ whenever $u \notin [0, 1]$; with its domain of definition so extended f is a nonnegative Lipschitz continuous function on $\mathbb{R}$.

It is clear that $u \equiv 0$ and $u \equiv 1$ are solutions of (1.1), (1.2). These solutions correspond to the nonoccurrence of either the A_1 or the A_2 allele. Nonconstant solutions, on the other hand, correspond to the occurrence of both alleles in the population. If either allele has a significant overall advantage (which might be measured in terms of integrals involving g or g_+) and diffusion is large, then the dominant allele will diffuse rapidly and wipe out its competitor; thus, if one allele has a marked advantage, we would expect nonconstant solutions only when diffusion is small, i.e. λ is large. On the other hand, if neither allele has a marked overall advantage, then we may expect nonconstant solutions no matter how large the diffusion. Our main results, Theorems 4.5 and 4.6, are consistent with the preceding remarks.

The existence of solutions of (1.1) is well understood in the case where the domain is bounded and u satisfies Dirichlet or Neumann boundary conditions (see, e.g. [1, 2]). The situation is more complex in the case of the whole space $\mathbb{R}^N$ (see [2–8]). Results were obtain in [2] under the hypothesis that g is negative at infinity and it was shown that the nature of the existence results for the case $N \geq 3$ was different from the case $N = 1, 2$. Some of the solutions obtained correspond to bifurcation from the trivial solution $u = 0$ and such bifurcations occur at principal eigenvalues, i.e. eigenvalues corresponding to a positive eigenfunction of the linearized problem

$$-\Delta u(x) = \lambda g(x) f'(0) u(x) \qquad \text{for } x \in \mathbb{R}^N. \tag{1.3}$$

The existence of positive principal eigenvalues of (1.3) was proved in [4] under the hypotheses that

(i) g is negative and bounded away from zero at infinity, or

(ii) $N \geq 3$ and $|g(x)| \leq K(1 + |x|^2)^\alpha$ for some constants $K > 0$ and $\alpha > 1$.

These results were generalized and improved by Allegretto in [3] under the hypothesis that $N \geq 3$ and $g_+ \in L^{N/2}(\mathbb{R}^N)$, where $g_+(x) = \max\{g(x), 0\}$.

In this paper we shall prove results about the existence of solutions of (1.1) in the case where $N \geq 3$ and either g or g_+ is small at infinity, e.g. in $L^{N/2}(\mathbb{R}^N)$. Our proofs are based on the construction of appropriate sub and supersolutions some of which are suggested by the ideas introduced by Gamez in [7]. Our results, however, improve those of Gamez and also of [6] for the case of $\lim_{|x| \to +\infty} u(x) = 0$ and extend those Brown and Stavrakakis [5] for the case of $\lim_{|x| \to +\infty} u(x) = c$ where $0 < c < 1$.

The plan of the paper is as follows: in Section 2 we describe the results we require on principal eigenvalues and eigenfunctions when g_+ is in $L^{N/2}(\mathbb{R}^N)$; in Section 3 we discuss how sub- (super-) solutions can be combined to produce new sub- (super-) solutions and in Section 4 we prove our existence results by constructing appropriate sub and supersolutions.

Notation: For simplicity we use the symbol $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$.

2. PRINCIPAL EIGENVALUES AND EIGENFUNCTIONS

Throughout this section we shall assume only that g is a bounded measurable function. We do so as, although g is assumed smooth in all the other sections, we require in Section 4 results on the principal eigenvalues and eigenfunctions of the discontinuous function g_+^R.

First, we consider the case where Ω is a bounded region with smooth boundary. It was proved by Manes and Micheletti in [9], using a variational approach, that the problem

$$-\Delta u(x) = \lambda g(x) u(x) \quad \text{for } x \in \Omega; \qquad u(x) = 0 \quad \text{for } x \in \partial\Omega$$

has a positive principal eigenvalue $\lambda_1(\Omega)$ and a positive principal eigenfunction $\phi_1(\Omega)$. In general $\phi_1(\Omega)$ satisfies the equation only in the weak sense but, if g is smooth (i.e. $g \in C^\alpha(\Omega)$), then $\phi_1(\Omega)$ is a classical solution. Also $\lambda_1(\Omega)$ has the variational characterisation

$$\lambda_1(\Omega) = \inf\left\{\int_\Omega |\nabla u|^2 \, \mathrm{d}x : u \in H_0^1(\Omega), \int_\Omega g u^2 \, \mathrm{d}x = 1\right\}.$$

If $u \in H_0^1(\Omega)$,

$$\int_\Omega gu^2\,dx \le \int_\Omega g_+u^2\,dx \le \left\{\int_\Omega g_+^{N/2}\,dx\right\}^{2/N}\left\{\int_\Omega u^{2N/N-2}\,dx\right\}^{(N-2)/N}$$
$$\le k\|g_+\|_{L^{N/2}(\Omega)}\|\nabla u\|^2_{L^2(\Omega)},$$

where k is the embedding constant of $H_0^1(\Omega)$ into $L^{2N/N-2}(\Omega)$ and is independent of Ω (see [10, p. 443). Hence $\lambda_1(\Omega) \ge 1/k\|g_+\|_{L^{N/2}(\Omega)}$.

If $B_R = \{x \in \mathbb{R}^N : |x| \le R\}$, we shall denote $\lambda_1(B_R)$ by $\lambda_1(R)$. It follows from the variational characterisation that $\lambda_1(R)$ is a decreasing function of R. Moreover, if $g_+ \in L^{N/2}(\mathbb{R}^N)$, it follows that $\lim_{R\to\infty}\lambda_1(R) \ge 1/k\|g_+\|_{N/2} > 0$.

In [3] Allegretto proves that, if $g_+ \in L^{N/2}(\mathbb{R}^N)$, then the equation

$$-\Delta u(x) = \lambda g(x)u(x) \quad \text{for } x \in \mathbb{R}^N; \qquad \lim_{|x|\to+\infty} u(x) = 0$$

has a positive principal eigenvalue λ^* with corresponding eigenfunction ϕ such that $\lim_{|x|\to\infty}\phi(x) = 0$. In fact Allegretto assumes that g is smooth but it is straightforward to check that all his arguments apply in the case where g is only bounded and measurable except that ϕ will now be only a weak solution instead of a classical solution of the equation. In addition we have the variational characterisation of λ^*, viz.,

$$\lambda^* = \inf\left\{\int_{\mathbb{R}^N}|\nabla u|^2\,dx : u \in C_0^\infty(\mathbb{R}^N),\ \int_{\mathbb{R}^N} gu^2\,dx = 1\right\}$$

and it follows easily that $\lambda^* = \lim_{R\to+\infty}\lambda_1(R)$.

3. CONSTRUCTION OF SUB- (SUPER-) SOLUTIONS

Let $\Omega \subset \mathbb{R}^N$ be a bounded region with smooth boundary and consider the boundary value problem

$$-\Delta u(x) = \lambda g(x)f(u(x)) \quad \text{for } x \in \Omega; \qquad u(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{3.1}$$

If we fix $\lambda > 0$, we may choose $c > 0$ such that $u \to \lambda g(x)f(u) + cu$ is an increasing function for $0 \le u \le 1$ for every $x \in \Omega$. Let $h(x, u) = \lambda g(x)f(u) + cu$; then $h(x, 0) \equiv 0$ and $h(x, 1) \equiv c$.

We may rewrite (3.1) as

$$-\Delta u(x) + cu(x) = h(x, u(x)) \quad \text{for } x \in \Omega; \qquad u(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{3.2}$$

It is well known that u is a solution of (3.2) if and only if u is a solution of the integral equation

$$u(x) = \int_\Omega G(x, y)h(y, u(y))\,dy, \tag{3.3}$$

where G is the Green's function of $-\Delta + c$ with Dirichlet boundary conditions. Moreover equation (3.3) may be written in operator form as

$$u = KN(u),$$

where $K: C(\bar{\Omega}) \to C^{\alpha}(\bar{\Omega})$ is the compact linear integral operator with kernel G (see Amann [11]) and $N: C(\bar{\Omega}) \to C(\bar{\Omega})$ is the Nemytskii operator associated with h. Since $h(x, \cdot)$ is increasing, it follows easily that N is an increasing operator, i.e. $Nu_1 \geq Nu_2$ whenever $u_1 \geq u_2$.

We say that $u \in C^2(\bar{\Omega})$ is a subsolution of (3.2) (or equivalently of (3.1)) if

$$-\Delta u(x) + cu(x) \leq h(x, u(x)) \quad \text{for } x \in \Omega; \qquad u(x) \leq 0 \quad \text{for } x \in \partial\Omega$$

and that $u \in C(\bar{\Omega})$ is a subsolution of (3.3) if

$$u(x) \leq \int_{\Omega} G(x, y)h(y, u(y))\, \mathrm{d}y \qquad \text{for all } x \in \Omega, \text{ i.e. } u \leq KNu.$$

Supersolutions are defined in a similar manner.

It is well known, for both (3.2) and (3.3), that, if $\underline{u}$ is a subsolution and $\bar{u}$ is a supersolution such that $\underline{u} \leq \bar{u}$, then there exists a solution u such that $\underline{u} \leq u \leq \bar{u}$. In the next section we shall make use of the ideas that the maximum of subsolutions is a subsolution and that the minimum of supersolutions is a supersolution. Although this is "well known", we could not find what is required for our purposes in the literature and so we now prove the necessary results.

LEMMA 3.1. Suppose that $\underline{u}_1$ and $\underline{u}_2$ are subsolutions of (3.2). Then $\underline{u}_1$, $\underline{u}_2$ and $\max\{\underline{u}_1, \underline{u}_2\}$ are subsolutions of (3.3).

Proof. We have that

$$-\Delta \underline{u}_1(x) + c\underline{u}_1(x) \leq h(x, \underline{u}_1(x)) \quad \text{for } x \in \Omega; \qquad \underline{u}_1(x) \leq 0 \quad \text{for } x \in \partial\Omega.$$

Let $v(x) = \int_{\Omega} G(x, y)h(y, \underline{u}_1(y))\, \mathrm{d}y$. Then

$$-\Delta v(x) + cv(x) = h(x, \underline{u}_1(x)) \quad \text{for } x \in \Omega; \qquad v(x) = 0 \quad \text{for } x \in \partial\Omega.$$

Thus, by the maximum principle, $\underline{u}_1(x) \leq v(x)$ for $x \in \bar{\Omega}$, i.e.

$$\underline{u}_1(x) \leq \int_{\Omega} G(x, y)h(y, \underline{u}_1(y))\, \mathrm{d}y \qquad \text{for } x \in \bar{\Omega}$$

and so $\underline{u}_1$ is a subsolution of (3.3).

Let $\underline{u} = \max\{\underline{u}_1, \underline{u}_2\}$. Then $\underline{u} \geq \underline{u}_1$ and so $KN\underline{u} \geq KN\underline{u}_1$. But, since $\underline{u}_1$ is a subsolution of (3.3), $\underline{u}_1 \leq KN\underline{u}_1$ and so $\underline{u}_1 \leq KN\underline{u}$. A similar argument shows that $\underline{u}_2 \leq KN\underline{u}$. Hence $\underline{u} = \max\{\underline{u}_1, \underline{u}_2\} \leq KN\underline{u}$ and so $\underline{u}$ is a subsolution of (3.3). ■

It is easy to see that an analogous result holds for supersolutions, so that the minimum of two supersolutions is a supersolution.

We can also obtain sub and supersolutions of (3.3) by pasting together sub and supersolutions of (3.2). We will denote by $(3.2)_R$ and $(3.3)_R$ the equations (3.2) and (3.3) when $\Omega = B_R$.

LEMMA 3.2. Suppose $R_1 < R_2$.

(i) If $\underline{u} \geq 0$ is a subsolution of $(3.2)_{R_1}$ with $\underline{u}(x) = 0$ when $|x| = R_1$ and $\underline{u}(x) \equiv 0$ when $R_1 \leq |x| \leq R_2$, then $\underline{u}$ is a subsolution of $(3.3)_{R_2}$.

(ii) If $0 \leq \bar{u} \leq 1$ on B_{R_2}, $\bar{u} \equiv 1$ when $|x| \leq R_1$ and $\bar{u}$ is a supersolution to (3.2) on $R_1 < |x| < R_2$ with $\bar{u}(x) = 1$ when $|x| = R_1$, then $\bar{u}$ is a supersolution of $(3.3)_{R_2}$.

Proof. (i) Let $G(x, y)$ denote the Green's function of $-\Delta + c$ with Dirichlet boundary conditions on B_{R_2} and define

$$v(y) = \int_{B_{R_2}} G(x, y)h(y, \underline{u}(y))\, dy.$$

Clearly $v(x) \geq 0$ for all $x \in B_{R_2}$ and so $\underline{u}(x) \leq v(x)$ for $R_1 \leq |x| \leq R_2$.

Since $\underline{u}$ is Lipschitz continuous, it follows that $y \to h(y, \underline{u}(y))$ lies in $C^{\alpha}(B_{R_2})$. Hence $v \in C^{2+\alpha}(B_{R_2})$ and we have

$$-\Delta v(x) + cv(x) = h(x, \underline{u}(x)) \quad \text{for } x \in B_{R_2}; \qquad v(x) = 0 \quad \text{for } |x| = R_2.$$

But

$$-\Delta \underline{u}(x) + c\underline{u}(x) \leq h(x, \underline{u}(x)) \quad \text{for } x \in B_{R_1}; \qquad \underline{u}(x) = 0 \quad \text{for } |x| = R_1$$

and so it follows from the maximum principle that $\underline{u}(x) \leq v(x)$ for $|x| \leq R_1$. Hence $\underline{u}$ is a subsolution for $(3.3)_{R_2}$.

(ii) Let $w(y) = \int_{B_{R_2}} G(x, y)h(y, \bar{u}(y))\, dy$.

Then $-\Delta w(x) + cw(x) = h(x, \bar{u}(x)) \leq h(x, 1) \leq c$ for $x \in B_{R_2}$ and $w(x) = 0$ for $|x| = R_2$.

It follows easily from the maximum principle that $w(x) \leq 1$ for all $x \in B_{R_2}$ and so $\bar{u}(x) \geq w(x)$ for $|x| \leq R_1$.

By using the maximum principle as in (i), it can also be shown that $\bar{u}(x) \geq w(x)$ for $R_1 \leq |x| \leq R_2$.

Hence $\bar{u}$ is a supersolution of $(3.3)_{R_2}$. ■

Finally in this section we show how sub and supersolutions on arbitrary large balls give rise to solutions on all of $\mathbb{R}^N$.

LEMMA 3.3. Suppose that $\underline{u}, \bar{u}: \mathbb{R}^N \to \mathbb{R}$ are continuous functions such that $\underline{u}(x) \leq \bar{u}(x)$ for all $x \in \mathbb{R}^N$ and $\underline{u}|_{B_R}$, $\bar{u}|_{B_R}$ are, respectively, a subsolution and a supersolution of $(3.3)_R$ for all large R. Then there exists a solution of

$$-\Delta u(x) = h(x, u(x)) \qquad \text{for } x \in \mathbb{R}^N \tag{3.4}$$

such that $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$ for all $x \in \mathbb{R}^n$.

Proof. Since $\underline{u}$ and $\bar{u}$ provide sub and supersolutions for $(3.3)_R$, there exists a solution u of $(3.3)_R$ and hence of $(3.2)_R$ such that $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$ for all $x \in B_R$. Let $\{R_n\}$ be a sequence such that $\lim_{n\to+\infty} R_n = \infty$ and let u_n denote the solution corresponding to $(3.2)_{R_n}$. Then standard a priori estimates and a diagonalization argument show that there exists a subsequence of $\{u_n\}$ which converges to a solution u of (3.4) in $C^{2+\alpha}$ on every bounded subset of $\mathbb{R}^N$. Moreover, since $\underline{u}(x) \leq u_n(x) \leq \bar{u}(x)$ for all $x \in B_{R_n}$, it follows that $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$ for all $x \in \mathbb{R}^N$. ■

4. EXISTENCE OF SOLUTIONS

Throughout this section we shall assume that $g_+ \in L^{N/2}(\mathbb{R}^N)$ so that

$$\lambda^* = \lim_{R \to +\infty} \lambda_1(R) > 0.$$

We first prove the existence of nonconstant solutions of (1.1), (1.2) satisfying

$$\lim_{x \to \infty} u(x) = 0$$

by constructing appropriate sub and supersolutions. We begin by considering subsolutions.

LEMMA 4.1. If $\lambda > \lambda^*$, then there exists $\underline{u} \geq 0$ ($\underline{u} \not\equiv 0$) with compact support such that $\underline{u}$ is a subsolution of

$$-\Delta u(x) = \lambda g(x) f(u(x)) \quad \text{for } x \in B_R; \qquad u(x) = 0 \quad \text{for } x \in \partial B_R \qquad (4.1)_R$$

for all sufficiently large R; moreover, $\underline{u}$ may be chosen arbitrarily small.

Proof. We may choose R_0 such that $\lambda_1(R_0) < \lambda$. Consider the eigenvalue problem

$$-\Delta u(x) - \lambda g(x) f'(0) u(x) = \mu u(x) \qquad \text{for } x \in B_{R_0},$$

$$u(x) = 0 \qquad \text{for } |x| = R_0.$$

Since $\lambda_1(R_0) < \lambda$, the above problem has a principal eigenvalue $\mu < 0$; we denote the corresponding positive principal eigenfunction by ψ where $\sup_{x \in B_{R_0}} \psi(x) = 1$.

We now show that there exists $\varepsilon_0 > 0$ such that $\varepsilon\psi$ is a subsolution of $(4.1)_{R_0}$ whenever $0 < \varepsilon < \varepsilon_0$.

Since

$$-\Delta(\varepsilon\psi(x)) = \lambda g(x) f'(0) \varepsilon\psi(x) + \mu\varepsilon\psi(x)$$

and

$$\lambda g(x) f(\varepsilon\psi(x)) = \lambda g(x) \varepsilon\psi(x) f'(\xi(x))$$

where $0 < \xi(x) < \varepsilon$, it follows that $\varepsilon\psi(x)$ is a subsolution of $(4.1)_{R_0}$ provided that $\mu < \lambda g(x)(f'(\xi(x)) - f'(0))$ for all $x \in B_{R_2}$. Since $\mu < 0$ and f' is continuous at 0, clearly there exists $\varepsilon_0 > 0$ such that $\varepsilon\psi$ is a subsolution of $(4.1)_{R_0}$ whenever $0 < \varepsilon < \varepsilon_0$.

If we define

$$\underline{u}(x) = \begin{cases} \varepsilon\psi(x) & \text{for } |x| < R_0 \\ 0 & \text{for } |x| \geq R_0, \end{cases}$$

then by Lemma 3.2 $\underline{u}$ is a subsolution of $(4.1)_R$ for all $R > R_0$ provided that $0 < \varepsilon < \varepsilon_0$. ■

The next three lemmas show how supersolutions can be constructed under various hypotheses. The first is an immediate consequence of Lemma 3.2 and the fact that $1/r^{N-2}$ is a harmonic function for $N \geq 3$.

LEMMA 4.2. Suppose that g is negative at infinity, i.e. there exists $R_0 > 0$ such that $g(x) < 0$ whenever $|x| > R_0$. Then the function

$$\bar{u}(x) = \begin{cases} 1 & \text{for } |x| \le R_0 \\ \dfrac{R_0^{N-2}}{|x|^{N-2}} & \text{for } |x| \ge R_0 \end{cases}$$

is a supersolution of $(3.3)_R$ for all $R \ge R_0$.

If we assume that g_+ has appropriate power decay at infinity, we can construct a supersolution with power decay.

LEMMA 4.3. Suppose there exists $K > 0$ and $\alpha > 1$ such that $g_+(x) \le K/(1 + |x|^2)^\alpha$ for all $x \in \mathbb{R}^N$. Then, for all $\lambda > 0$, there exists $\bar{u} > 0$ such that

$$-\Delta\bar{u}(x) \ge \lambda g(x) f(\bar{u}(x)) \qquad \text{for all } x \in \mathbb{R}^N \tag{4.2}$$

and $|\bar{u}(x)| \le C|x|^{-\beta}$ where $C > 0$ is a constant and $\beta = N - 2$ if $N < 2\alpha$ and $\beta = 2\alpha - 2$ if $N > 2\alpha$.

Proof. Consider the equation

$$-\Delta u(x) = g_+(x) \qquad x \in \mathbb{R}^N. \tag{4.3}$$

It follows from [12], Lemma 2.3 that equation (4.3) has a solution ϕ such that

$$\phi(x) = C_N \int_{\mathbb{R}^N} \frac{g_+(y)}{|x - y|^{N-2}} \, dy,$$

where $C_N = [N(N-2)\omega_N]^{-1}$ and ω_N is the volume of the unit ball in $\mathbb{R}^N$. Since $|g_+(y)| \le K/|y|^{2\alpha}$ at ∞, we have $|\varphi(x)| \le C|x|^{-\beta}$ where $\beta = N - 2$ if $N \le 2\alpha$ and $\beta = 2\alpha - 2$ if $N > 2\alpha$.

We now show that $\bar{u} = k\phi$ satisfies (4.2) provided that k is chosen sufficiently large. Since, for all $x \in \mathbb{R}^N$, $-\Delta(k\phi(x)) = kg_+(x)$ and $\lambda g(x) f(k\phi(x)) \le \lambda g_+(x) M$ where $M = \sup_{u \ge 0} f(u(x))$, it follows that $\bar{u} = k\phi$ satisfies (4.2) provided that $k > \lambda M$. ■

Finally, using an idea introduced by Gamez in [7], we construct a supersolution in the case where $g_+ \in L^{N/2}$.

LEMMA 4.4. Suppose $g_+ \in L^{N/2}$. Then, for all $\lambda > 0$, there exists $\bar{u} > 0$ such that (4.2) holds and $\lim_{x \to \infty} \bar{u}(x) = 0$.

Proof. We define

$$g_+^R(x) = \begin{cases} g_+(x) & \text{for } |x| > R \\ 0 & \text{for } |x| \le R. \end{cases}$$

Because of Lemma 4.2, we need consider only the case where g_+^R is not identically equal to zero. Thus g_+^R is a bounded measurable function such that $\|g_+^R\|_{N/2} \to 0$ as $R \to \infty$ and so, by the results of Section 2, there exists a positive principal eigenvalue $\gamma(R)$ corresponding to g_+^R and $\lim_{R\to+\infty} \gamma(R) = \infty$. We denote by ϕ_R the positive principal eigenfunction corresponding to $\gamma(R)$ such that $\max_{x\in\mathbb{R}^N}\phi_R = 1$. We shall show that $k\phi_R$ may be taken as the required function $\bar{u}$ provided R and k are chosen sufficiently large.

Let $L = \max_{u>0}(f(u)/u) > 0$. We can choose $R > 0$ such that $\gamma_R > \lambda L$. Since ϕ_R is a positive function on B_R, we can choose $k > 0$ such that $k\phi_R(x) > 1$ for all $x \in B_R$. Hence $f(k\phi_R(x)) \equiv 0$ on B_R. Thus, if $|x| \le R$,

$$-\Delta(k\phi_R)(x) = k\gamma_R g_+^R(x)\phi_R(x) = 0 = \lambda g(x) f(k\phi_R(x)).$$

Moreover, if $|x| \ge R$, then

$$\begin{aligned}-\Delta(k\phi_R)(x) &= k\gamma_R g_+^R(x)\phi_R(x) \ge \gamma_R g_+^R(x)\frac{f(k\phi_R(x))}{L}\\ &\ge \lambda g_+^R(x) f(k\phi_R(x)) \ge \lambda g(x) f(k\phi_R(x)).\end{aligned}$$

Thus we may take $\bar{u} = k\phi_R$ and the proof is complete. ■

Using the sub and supersolutions constructed above, we obtain the following existence result.

THEOREM 4.5. Suppose that $g_+ \in L^{N/2}$ and $\lambda > \lim_{R\to\infty}\lambda_1(R)$.

(i) There exists a nonconstant solution u of (1.1), (1.2) such that $\lim_{|x|\to+\infty} u(x) = 0$.

(ii) If $g_+(x) \le K/(1 + |x|^2)^\alpha$ for some constants $K > 0$ and $\alpha > 1$, then there exists a nonconstant solution u of (1.1), (1.2) such that $|u(x)| \le C|x|^{-\beta}$ for $|x|$ large where C is a positive constant and $\beta = N - 2$ if $N \le 2\alpha$ and $\beta = 2\alpha - 2$ if $N > 2\alpha$.

Proof. Suppose that $\bar{u}$ and $\underline{u}$ are the sub and supersolutions constructed in Lemmas 4.1 and 4.4. Clearly $u \equiv 1$ is a supersolution for $(3.2)_R$ and so for $(3.3)_R$ for all $R > 0$. Hence, if $\bar{v} = \min\{1, \bar{u}\}$, $\bar{v}$ is also a supersolution of $(3.3)_R$ for all $R > 0$ with $0 < \bar{v}(x) \le 1$ for all $x \in \mathbb{R}^N$ and $\lim_{|x|\to+\infty}\bar{v}(x) = 0$. Since $\underline{u}$ has compact support and can be chosen with arbitrarily small sup-norm, we may choose $\underline{u}$ so that $\underline{u} \le \bar{v}$. Hence by Lemma 3.3 there exists a solution u of (1.1) such that $\underline{u} \le u \le \bar{v}$.

(ii) can be proved similarly by using Lemma 4.3. ■

The nonconstant solution whose existence is proved in the previous theorem occurs in the situation where the A_1 allele is disadvantaged at infinity (since $\lim_{|x|\to+\infty} u(x) = 0$) and has only limited advantage at other points (because of the smallness assumptions on g_+). Thus it can be expected that such solutions can occur only when diffusion is sufficiently small, i.e. only for λ sufficiently large. In fact it is proved in [5] that, if $|g(x)| \le K/(1 + |x|^2)^\alpha$, then there exists $\lambda_0 > 0$ such that (1.1), (1.2) has no nonconstant solutions u such that $\lim_{|x|\to+\infty} u(x) = 0$ when $0 < \lambda < \lambda_0$. Our final theorem shows that, if neither A_1 nor A_2 is completely dominant at infinity and have only limited advantage at other points, then nonconstant solutions can exist no matter how large the diffusion is.

THEOREM 4.6. Suppose that $|g(x)| \le K/(1 + |x|^2)^\alpha$ for constants $K > 0$ and $\alpha > 1$ and c is any constant such that $0 < c < 1$. Then there exists a nonconstant solution u of (1.1), (1.2) such that $\lim_{|x|\to+\infty} u(x) = c$.

Proof. Let $M = \max_{u \geq 0} f(u)$. As in the proof of Lemma 4.3, there exists a positive solution ϕ of

$$-\Delta u(x) = \lambda M g_+(x) \quad \text{for all } x \in \mathbb{R}^N, \qquad \lim_{|x| \to +\infty} u(x) = 0.$$

Let $\bar{u} = c + \phi$. Then

$$-\Delta \bar{u}(x) = -\Delta \phi(x) = \lambda M g_+(x) \geq \lambda g(x) f(\bar{u}) \qquad \text{for all } x \in \mathbb{R}^N.$$

Hence $\bar{u}$ is a supersolution of $(3.1)_R$ for all $r > 0$ and so $\bar{v} = \min\{1, \bar{u}\}$ is a supersolution of $(3.3)_R$ for all $R > 0$. Similarly, if $g_-(x) = \min\{g(x), 0\}$, ψ is the negative solution of

$$-\Delta u = \lambda M g_-(x) \quad \text{for all } x \in \mathbb{R}^N; \qquad \lim_{|x| \to +\infty} u(x) = 0$$

and $\underline{u} = c + \psi$, then $\underline{v} = \max\{0, \underline{u}\}$ is a subsolution of $(3.3)_R$ for all $R > 0$. The result now follows from Lemma 3.3. ■

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