Principal Eigenvalues for some Quasilinear Elliptic Equations on $\mathbb{R}^N$

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Abstract

We improve some previous results for the principal eigenvalue of the p-laplacian defined on $\mathbb{R}^N$, study regularity of the corresponding eigenfunctions and give an existence result of the type below the first eigenvalue (or between the first eigenvalues) for a certain perturbed problem. Based in our approach for the equation we deduce existence,

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uniqueness and simplicity of positive principal eigenvalues for the p-Laplacian system

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1}, \quad x \in \mathbb{R}^N,$$

$$-\Delta_q v = \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v + \lambda d(x)|v|^{q-2}v, \quad x \in \mathbb{R}^N,$$

$$0 < u, 0 < v, \text{ in } \mathbb{R}^N, \lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0.$$

We also establish the regularity of the corresponding eigenfunctions.

1 Introduction

In this paper we shall deal with existence and properties of the "first eigenpair" in $\mathbb{R}^N$, for some quasilinear elliptic eigenvalue problems containing the $p-$Laplacian operator $\Delta_p u = div(| \bigtriangledown u|^{p-2} \bigtriangledown u)$.

Thus we shall first consider the scalar case

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \qquad (1.1)_\lambda$$

$$0 < u, \text{ in } \mathbb{R}^N, \lim_{|x|\to+\infty} u(x) = 0, \qquad (1.2)_\lambda$$

under certain conditions on p, N, and g, and then the system

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1}, \quad x \in \mathbb{R}^N, \qquad (1.3)_\lambda$$

$$-\Delta_q v = \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v + \lambda d(x)|v|^{q-2}v, \quad x \in \mathbb{R}^N, \qquad (1.4)_\lambda$$

$$0 < u, \ 0 < v, \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0. \qquad (1.5)$$

This system, under certain conditions on α, β, p, q, N and on the functions a, b and d, is an eigenvalue problem; for the bounded domain case see, for example [17, 29].

Problems where the operator $-\Delta_p$ is present arise both from pure mathematics, like in the theory of quasiregular and quasiconformal mappings (see [30] and the references therein), as well as from a variety of applications, e.g. non-Newtonian fluids, reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaseology, petroleum extraction, astronomy, etc (see [3],[4],[11]).

In the case of the eigenvalue problem for bounded domains, under various boundary conditions, there is quite an extensive literature and the picture for "the principal eigenpair" seems to be fairly complete. We mention among others, [2], [15], [19], [23], [28] for the case of the equation and [17], [29], for the case of a system.

The eigenvalue problem for unbounded domains becomes more complicate since, in general, the equation does not give rise to compact operators. Also it is unclear, a priori, the function spaces where the eigenfunctions might lie.

In the last few years several works dealing with the eigenvalue problem in unbounded domains have appeared, see [1], [7], [13], [14], [18] and [20]. Furthermore in [14] bifurcations technics are used to prove existence results for the p-Laplacian equation in $\mathbb{R}^N$.

Our paper is organized in two parts, Part I (sections 2 to 4) dedicated to the case of the equation and Part II (sections 5 and 6) to the system.

Our study in Part I, which also forms the basis for our treatment of the system in Part II, was originally motivated by a wish to understand the results on existence, simplicity and isolation for the case of the equation quoted in [1], [13], [14], [20]. From our point of view the simplicity result in [20] seems to be not correct due to an improper use of a Diaz and Saa's inequality and thus some of those results based on this property appear rather incomplete. (See section 3 for further comments).

Indeed our main intention in this paper was to extend some of the results in [1], [13], [14], [20] to the case of a system, as we do in Part 2. Nevertheless in doing so we realized that we needed some lemmas for the case of the equation fully justified and this became an additional reason for reviewing some results for the equation from our point of view.

Thus in Section 2, which can be considered as a review section, by first establishing some basic properties of the homogeneous space $\mathcal{D}^{1,p}(\mathbb{R}^N)$, we obtain the existence of a principal eigenvalue (i.e., an eigenvalue corresponding to a positive eigenfunction) in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, by standard variational methods. Using *a priori* estimates from [26], we determine the regularity as well as the asymptotic behavior of solutions of $(1.1)_\lambda$. Positivity of the eigenfunctions is then shown to be a consequence of Vasquez' Maximum Principle [33].

In section 3 we study the properties of the corresponding "principal eigenspace" and prove that the principal eigenvalue (for any $1 < p < N$) is simple by extending to our situation a uniqueness method developed for $p = 2$ in [10] (see also [27] for a more restrictive case).

In section 4 we prove solvability below the first eigenfunction (respectively between first eigenvalues) for a certain perturbation of $(1.1)_\lambda$. For this perturbed problem we then show a connection between the existence of a first eigenvalue and some sufficient conditions for the validity of a maximum principle, these results being interesting in their own.

In Part II, section 5, we show existence of positive principal eigenvalues for the system and establish the regularity as well as the asymptotic behavior of the corresponding eigenfunctions, the proof of those results, being rather technical, is done in the appendix (section 7) at the end of the paper. Then, in section 6, we prove the simplicity of the principal eigenvalues.

Notation. For simplicity we use the symbol $||.||_p$ for the norm $||.||_{L^p(\mathbb{R}^N)}$ and $\mathcal{D}^{1,p}$ for the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$. B_R and $B_R(c)$ will denote the balls in $\mathbb{R}^N$, center zero and center c respectively, and radius R. Also the Lebesgue measure of a set $\Omega \subset \mathbb{R}^N$ will be denoted by $|\Omega|$. The end of a proof is marked with a $\diamondsuit$

PART I. THE EQUATION

Throughout Part I we will assume that $1 < p < N$, and that g in $(1.1)_\lambda$ satisfies

$(\mathcal{G})$ g is a smooth function, at least $C^{0,\gamma}_{loc}(\mathbb{R}^N)$ for some $\gamma \in (0,1)$, such that $g \in L^{N/p}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ and $g(x) > 0$, in Ω^+, with $|\Omega^+| > 0$.

Also to differentiate the case when g changes sign (a.e.) with the case when it does not, we will say that g satisfies:

$(\mathcal{G}^+)$ if g satisfies $(\mathcal{G})$ and $g(x) \geq 0$, almost everywhere in $\mathbb{R}^N$, and

$(\mathcal{G}^-)$ if g satisfies $(\mathcal{G})$ and $g(x) < 0$, for $x \in \Omega^-$, with $|\Omega^-| > 0$.

2 Existence of a principal eigenvalue

In this section we shall first prove the existence of a positive principal eigenvalue for the problem $(1.1)_\lambda$. The natural setting for this problem is the space

$\mathcal{D}^{1,p}$, i.e., the closure of $C_0^\infty(\mathrm{I\!R}^N)$ with respect to the norm

$$||u||_{\mathcal{D}^{1,p}} = \left(\int_{\mathrm{I\!R}^N} | \bigtriangledown u|^p dx \right)^{1/p}.$$

It can be shown (see [22], Proposition 2.4) that

$$\mathcal{D}^{1,p} = \left\{ u \in L^{\frac{Np}{N-p}}(\mathrm{I\!R}^N) : \bigtriangledown u \in \left(L^p(\mathrm{I\!R}^N) \right)^N \right\}$$

and that there exists $K > 0$ such that for all $u \in \mathcal{D}^{1,p}$

$$||u||_{L^{\frac{Np}{N-p}}} \leq K \, ||u||_{\mathcal{D}^{1,p}}. \tag{2.1}$$

Clearly the space $\mathcal{D}^{1,p}$ is a *reflexive Banach space.* Our approach is based on the following inequality.

Lemma 2.1 *Suppose $g \in L^{N/p}(\mathrm{I\!R}^N)$. Then there exists $\alpha > 0$ such that*

$$\int_{\mathrm{I\!R}^N} | \bigtriangledown u|^p dx \geq \alpha \int_{\mathrm{I\!R}^N} |g||u|^p dx, \tag{2.2}$$

for all $u \in \mathcal{D}^{1,p}$.

Proof. The proof follows the lines of Lemma 2.1 in [10]. $\diamondsuit$

Let us define now $A : \mathcal{D}^{1,p} \to \mathrm{I\!R}$, by

$$A(u) = ||u||^p_{\mathcal{D}^{1,p}},$$

and $B : \mathcal{D}^{1,p} \to \mathrm{I\!R}$, by

$$B(u) = \int_{\mathrm{I\!R}^N} g(x)|u|^p dx.$$

It is well known that A is weakly lower semicontinuous and that A and B are of class C^1. Furthermore the functional B satisfies.

Lemma 2.2 *(i) if $\{u_n\}$ is a sequence in $\mathcal{D}^{1,p}$, with $u_n \rightharpoonup u$ weakly, then there is a subsequence, denoted again by $\{u_n\}$, such that $B(u_n) \to B(u)$; (ii) if $B'(u) = 0$, then $B(u) = 0$.*

Proof. The proof follows the lines of Lemma 2.1 in [10] (see also [6]). $\diamondsuit$

In the remaining part of this section we shall prove the existence of nonzero principal eigenvalues for $(1.1)_\lambda$ and the $C^{1,\alpha}_{loc}$ regularity as well as the asymptotic behavior of the associated eigenfunctions.

The following theorem is a direct consequence of the properties of the functionals A and B, lemma 2.3, and Theorem (6.3.2) in [5] for nonlinear eigenvalue problems.

Theorem 2.3 *(i) Let g satisfies $(\mathcal{G}^+)$. Then equation $(1.1)_\lambda$ admits a positive first eigenvalue given by*

$$\lambda_1 = \inf_{B(u)=1} ||u||^p_{\mathcal{D}^{1,p}}. \tag{2.3}$$

(ii) Let g satisfies $(\mathcal{G}^-)$. Then problem $(1.1)_\lambda$ admits two first eigenvalues of opposite sign given by

$$\lambda_1^+ = \inf_{B(u)=1} ||u||^p_{\mathcal{D}^{1,p}}, \tag{2.4}$$

$$\lambda_1^- = -\inf_{B(u)=-1} ||u||^p_{\mathcal{D}^{1,p}}. \tag{2.5}$$

In both cases the associated eigenfunctions ϕ (respectively ϕ^+, ϕ^-) belong to $\mathcal{D}^{1,p}$.

In our next theorem we study the L^σ character and asymptotic behavior of the $\mathcal{D}^{1,p}$ solutions of $(1.1)_\lambda$.

Theorem 2.4 *Suppose that $u \in \mathcal{D}^{1,p}$ is a solution of $(1.1)_\lambda$. Then $u \in L^\sigma$ for all $\sigma \in [\frac{Np}{N-p}, +\infty]$. Moreover, the solutions $u(x)$ decay uniformly to zero as $|x| \to +\infty$.*

Proof. Let $\gamma = \frac{N}{N-p}$, $\sigma_n = p\gamma^n$ and $s_n = (\gamma^n - 1)p$. Assume that $u \in L^{\sigma_1}(\mathbb{R}^N)$, then we shall prove by induction that $u \in L^{\sigma_n}(\mathbb{R}^N)$, for all $n \geq 1$.

Let $u \in L^{\sigma_n}(\mathbb{R}^N)$, for some fixed n. Consider $T_k(u) = max(-k, min(k,u))$, for $k > 0$ and $w = |T_k(u)|^{s_n} T_k(u)$. Since $w \in \mathcal{D}^{1,p}$, we multiply (1.1) by w and from (2.1) we obtain

$$\begin{aligned} |||T_k(u)|^{\gamma^n}||^p_{\gamma p} &\leq K^p || \bigtriangledown \{|T_k(u)|^{\gamma^n - 1} T_k(u)\}||^p_p \\ &\leq K^p \gamma^{np} \textstyle\int_{\mathbb{R}^N} | \bigtriangledown u|^{p-2} \bigtriangledown u \cdot \bigtriangledown T_k(u) |T_k(u)|^{s_n}| dx \\ &\leq K^p \gamma^{n(p-1)} \textstyle\int_{\mathbb{R}^N} |\lambda g| |u|^{p-1} |w| dx \\ &\leq K_0 \gamma^{n(p-1)} ||u||^{\sigma_n}_{\sigma_n}, \end{aligned}$$

where $K_0 = K^p |\lambda| ||g||_\infty$. Letting $k \to +\infty$, by the dominated convergence theorem,

$$||u||^{\sigma_{n+1}/\gamma}_{\sigma_{n+1}} \leq K_0 \gamma^{n(p-1)} ||u||^{\sigma_n}_{\sigma_n},$$

and thus $u \in L^{\sigma_{n+1}}(\mathbb{R}^N)$. As in [31], we deduce from the above inequality that $u \in L^\infty(\mathbb{R}^N)$. Moreover, since $u \in L^{\sigma_1}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$, we obtain that $u \in L^\sigma(\mathbb{R}^N)$ for all $\sigma \in [\sigma_1, +\infty]$.

By Theorem 1 of Serrin in [26], for any ball $B_r(x)$ of radius r centered at any $x \in \mathbb{R}^N$ and some constant $C(N, \sigma_2)$, the solution $u \in \mathcal{D}^{1,p}$ of the equation

$$-\Delta_p u = f$$

satisfies the estimate

$$\sup_{y \in B_1(x)} |u(y)| \leq C \left\{ ||u||_{L^p(B_2(x))} + ||f||_{L^{\sigma_2}(B_2(x))} \right\}.$$

For $q = \frac{\sigma_n}{p-1} \geq \sigma_2$ we obtain for the solution of (1.1)

$$\begin{aligned} \sup_{y \in B_1(x)} |u(y)| &\leq C_1 \Big\{ ||u||_{L^{\sigma_1}(B_2(x))} \\ &\quad + |\lambda| ||g||_\infty |||u|^{p-1}||^{\frac{1}{p-1}}_{L^q(B_2(x))} \Big\}. \end{aligned}$$

By the preceding results $|u|^{p-1}$ belongs to $L^q(\mathbb{R}^N)$, so the uniform vanishing of u is implied. $\diamondsuit$

The next regularity characterization of the solutions of $(1.1)_\lambda$ is a direct consequence of the previous theorem and an argument of Tolksdorf [30].

Corollary 2.5 *For any $r > 0$, the solutions of $(1.1)_\lambda$ belongs to $C^{1,\alpha}(B_r)$, where $\alpha = \alpha(r) \in (0,1)$.*

Now we are ready to discuss the sign of the eigenfunctions corresponding to the first eigenvalues.

Theorem 2.6 *(i) Let g satisfy either $(\mathcal{G}^+)$ or $(\mathcal{G}^-)$. Then, there is an eigenfunction which is strictly positive everywhere in $\mathbb{R}^N$.*
(ii) Let g satisfy $(\mathcal{G}^+)$ (respectively $(\mathcal{G}^-)$). Then all eigenfunctions associated to λ_1 (respectively λ_1^+, λ_1^-) are of constant sign, i.e. λ_1 (respectively λ_1^+, λ_1^-) are principal eigenvalues.

Proof. (i) Since $A(|u|) = A(u)$ and $B(|u|) = B(u)$, if u_λ achieves the infimum in one of (2.3), (2.4) or (2.5), then $|u_\lambda|$ does the same. So we can consider that $u_\lambda \geq 0$. Also since

$$-\Delta_p u_\lambda = \lambda g(x)|u_\lambda|^{p-2}u_\lambda, \quad \text{almost everywhere in } \mathbb{R}^N,$$

we have that

$$\Delta_p u_\lambda \leq |\lambda| ||g||_\infty |u_\lambda|^{p-1}, \quad \text{almost everywhere in } \mathbb{R}^N.$$

The conclusion is implied by Vasquez' Maximum Principle [33].

(ii) Assume that g satisfies $(\mathcal{G}^+)$ and let ϕ be an eigenfunction corresponding to λ_1. Let $\phi_+ \geq 0$, $\phi_- \leq 0$ denote respectively the positive and negative parts of ϕ, i.e. $\phi = \phi_+ + \phi_-$. Then $\phi_+, \phi_- \in \mathcal{D}^{1,p}$ and

$$A(\phi) = A(\phi_+) + A(\phi_-); \quad B(\phi) = B(\phi_+) + B(\phi_-).$$

As usual we have that

$$\max\left\{\frac{B(\phi_+)}{A(\phi_+)}, \frac{B(\phi_-)}{A(\phi_-)}\right\} \geq \frac{B(\phi)}{A(\phi)} = \frac{1}{\lambda_1}.$$

Suppose now that $\frac{B(\phi_+)}{A(\phi_+)}$ correspond to the maximum (the other case being similar), then $\lambda_1 B(\phi_+) \geq A(\phi_+)$. Setting $v_+ = \frac{\phi_+}{\mu}$, where $\mu = B(\phi_+)^{1/p}$, we find that

$$B(v_+) = 1 \quad \text{and} \quad \lambda_1 = \lambda_1 B(v_+) \geq A(v_+).$$

Hence v_+ is an eigenfunction for λ_1. Also since $v_+ \geq 0$, Vasquez' Maximum Principle implies that $v_+ > 0$. Hence $v_- \equiv 0$ and finally $\phi_- \equiv 0$. Thus $\phi(x) > 0$ everywhere in $\mathbb{R}^N$. In the case that g satisfies $(\mathcal{G}^-)$ the proof follows the same lines. $\diamondsuit$

3 Simplicity of the Principal Eigenvalues

In this section we shall discuss first the dimension of the eigenspace associated to the principal eigenvalues of the quasilinear problem $(1.1)_\lambda$. A result in this direction was announced in [1] and a proof given in [20]. This proof nevertheless is not correct since it is based in Diaz and Saa's inequality [12]; this inequality holds only when dealing with two functions u and v whose ratio is bounded, this is far away from being obvious in unbounded domains.

The following preliminary lemmas will be useful for the proof of the main result of this section.

Lemma 3.1 *Suppose that* $u \in \mathcal{D}^{1,p}$ *is a solution of* $(1.1)_\lambda$. *Then*

$$\lim_{R\to+\infty} \int_{\partial B_R} u| \bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS = 0. \tag{3.1}$$

Proof. Let u satisfies $(1.1)_\lambda$. Multiplying both sides of $(1.1)_\lambda$ by u and integrating over B_R, we obtain

$$\int_{B_R} | \bigtriangledown u|^p \, dx - \int_{\partial B_R} u| \bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS = \lambda \int_{B_R} g|u|^p \, dx. \tag{3.2}$$

Since $| \bigtriangledown u| \in L^p(\mathbb{R}^N)$, and $g|u|^p \in L^1(\mathbb{R}^N)$, it follows that

$$\lim_{R\to\infty} \int_{\partial B_R} u| \bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS = L,$$

exists and it is finite.

We claim that $L = 0$. For this to hold it is sufficient that there is a sequence $\{R_n\}$ with $R_n \to +\infty$, such that

$$\lim_{n\to\infty} \int_{\partial B_{R_n}} u| \bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS = 0. \tag{3.3}$$

Indeed we have that

$$\int_{\partial B_R} u| \bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS \leq \left(R^{-\frac{p}{p'}} \int_{\partial B_R} |u|^p dS \right)^{\frac{1}{p}} \left(R \int_{\partial B_R} | \bigtriangledown u|^p dS \right)^{\frac{1}{p'}}, \tag{3.4}$$

where from now on $p' = \frac{p}{p-1}$. Since $\bigtriangledown u \in \left(L^p(\mathbb{R}^N) \right)^N$, and $u \in L^{\frac{Np}{N-p}}(\mathbb{R}^N)$ it follows that the integral

$$\int_0^\infty \left\{ \int_{\partial B_R} \left(| \bigtriangledown u|^p + |u|^{\frac{Np}{N-p}} \right) dS \right\} dR,$$

is bounded, and so

$$\lim_{R\to\infty} \int_R^{2R} \int_{\partial B_R} \left(| \bigtriangledown u|^p + |u|^{\frac{Np}{N-p}} \right) dS dr = 0.$$

By the mean value theorem of the integral calculus, we can find a sequence $\{R_n\}$, with $R_n \to \infty$ as $n \to \infty$, such that

$$\lim_{R_n\to\infty} R_n \int_{\partial B_{R_n}} | \bigtriangledown u|^p dS = 0 = \lim_{R_n\to\infty} R_n \int_{\partial B_{R_n}} |u|^{\frac{Np}{N-p}} dS. \tag{3.5}$$

Furthermore we have

$$\begin{aligned} \textstyle\int_{\partial B_{R_n}} |u|^p dS &\leq \left(\textstyle\int_{\partial B_{R_n}} |u|^{\frac{Np}{N-p}} dS \right)^{\frac{N-p}{N}} \left(\textstyle\int_{\partial B_{R_n}} 1 dS \right)^{\frac{p}{N}} \\ &= K {R_n}^{\frac{(N-1)p}{N}} \left(\textstyle\int_{\partial B_{R_n}} |u|^{\frac{Np}{N-p}} dS \right)^{\frac{N-p}{N}}. \end{aligned}$$

Hence

$${R_n}^{-\frac{p}{p'}} \int_{\partial B_{R_n}} |u|^p dS \leq K \left(R_n \int_{\partial B_{R_n}} |u|^{\frac{Np}{N-p}} dS \right)^{\frac{N-p}{N}} {R_n}^{\frac{(N-1)p}{N} + \frac{p-N}{N} - \frac{p}{p'}}.$$

So we get

$$R_n{}^{-\frac{p}{p'}} \int_{\partial B_{R_n}} |u|^p dS \le K \left(R_n \int_{\partial B_{R_n}} |u|^{\frac{Np}{N-p}} dS \right)^{\frac{N-p}{N}},$$

and by (3.5) $\lim_{R_n \to \infty} R_n{}^{-\frac{p}{p'}} \int_{\partial B_{R_n}} |u|^{\frac{Np}{N-p}} dS = 0$. Thus inequality (3.4) implies (3.3) and the claim is proved. $\diamondsuit$

Lemma 3.2 *Suppose that* $u, \phi \in \mathcal{D}^{1,p} \cap C^{1,\alpha}$ *are two solutions of* $(1.1)_\lambda$, *and that* $u(x) > 0$ *in* $\mathbb{R}^N$. *Then for all* $R > 0$ *the function*

$$\Theta(R) = \int_{B_R} \Big\{ | \bigtriangledown \phi|^p + (p-1) \left(\frac{|\phi|}{u} \right)^p | \bigtriangledown u|^p - p \bigtriangledown \phi . \bigtriangledown u \, |\bigtriangledown u|^{p-2} \left(\frac{|\phi|^{p-2}\phi}{u^{p-1}} \right) \Big\} dx, \tag{3.6}$$

is non decreasing and satisfies $\Theta(R) \ge 0$. *Moreover if* $\lim_{R\to\infty} \Theta(R) = 0$, *then there exists a constant* $c > 0$ *such that* $u = c\phi$.

Proof. For any $\mu > 0$ we have

$$\begin{aligned} \bigtriangledown \phi . \bigtriangledown u \, |\bigtriangledown u|^{p-2} \left(\frac{\phi|\phi|^{p-2}}{u^{p-1}} \right) &\le | \bigtriangledown \phi| \, |\bigtriangledown u|^{p-1} \left| \frac{\phi}{u} \right|^{p-1} \\ &\le \frac{\mu^p}{p} | \bigtriangledown \phi|^p + \frac{p-1}{p\mu^{p'}} |\bigtriangledown u|^p \left| \frac{\phi}{u} \right|^p . \end{aligned} \tag{3.7}$$

For $\mu = 1$, integrating (3.7) on B_R we find that $\Theta(R)$ is non decreasing and that $\Theta(R) \ge 0$.

Now suppose that $\lim_{R\to\infty} \Theta(R) = 0$, then by (3.7), we obtain that for any $R > 0$

$$\int_{B_R} \Big\{ | \bigtriangledown \phi| \, |\bigtriangledown u|^{p-1} \left| \frac{\phi}{u} \right|^{p-1} - \frac{1}{p} | \bigtriangledown \phi|^p - \frac{(p-1)}{p} |\bigtriangledown u|^p \left| \frac{\phi}{u} \right|^p \Big\} dx = 0, \tag{3.8}$$

and

$$\int_{B_R} \Big\{ \bigtriangledown \phi . \bigtriangledown u |\bigtriangledown u|^{p-2} \left(\frac{\phi|\phi|^{p-2}}{u^{p-1}} \right) - | \bigtriangledown \phi| \, |\bigtriangledown u|^{p-1} \left| \frac{\phi}{u} \right|^{p-1} \Big\} dx = 0. \tag{3.9}$$

From (3.8) we first find that $|\bigtriangledown \phi(x)| = \left|\frac{\phi(x)}{u(x)} \bigtriangledown u(x)\right|$, then from (3.9) it follows that $\bigtriangledown\phi(x) = \varepsilon\frac{\phi(x)}{u(x)} \bigtriangledown u(x)$, where $\varepsilon = \pm1$, and finally from $\lim_{R\to\infty}\Theta(R) = 0$, we obtain that $\varepsilon = 1$, and hence that $\bigtriangledown\left(\frac{\phi(x)}{u(x)}\right) = 0$. Thus $u = C\phi$, where C is a constant, and the proof is ended. $\diamondsuit$

Lemma 3.3 *Suppose that* $\phi \in \mathcal{D}^{1,p}$ *is such that* $\phi(x) > 0$ *on* $\mathbb{R}^N$. *Then, for any fixed* $R_1 > 0$, *we have that*

$$\int_{R_1}^{+\infty} \frac{dr}{(H(r))^{p'/p}} = +\infty,$$

where

$$H(r) := \int_{\partial B_r} (\phi(S))^p dS. \tag{3.10}$$

Proof. We have

$$H(r) \le \left(\int_{\partial B_r} (\phi(S))^{\frac{Np}{N-p}} dS\right)^{\frac{N-p}{N}} \left(\int_{\partial B_r} 1 dS\right)^{\frac{p}{N}} = Kr^{\frac{p(N-1)}{N}} (I(r))^{\frac{N-p}{N}}, \tag{3.11}$$

where $I(r) := \int_{\partial B_r} (\phi(S))^{\frac{Np}{N-p}} dS$.

We note that since $\phi \in L^{\frac{Np}{N-p}}(\mathbb{R}^N)$, then for any fixed $R_1 > 0$, we have that $\int_{R_1}^{+\infty} I(r) dr < +\infty$.

Now, for any $\delta > 0$, let us consider the identity

$$\log\frac{R}{R_1} = \int_{R_1}^{R} \frac{1}{r} I(r)^{\delta} I(r)^{-\delta} dr,$$

by using Hölder's inequality,

$$\log\frac{R}{R_1} \le \left(\int_{R_1}^{R} I(r)^{\delta q'} dr\right)^{\frac{1}{q'}} \left(\int_{R_1}^{R} r^{-q} I(r)^{-\delta q} dr\right)^{\frac{1}{q}},$$

for any $\delta > 0$ and any $q > 1$. Thus taking $\delta = \frac{N-p}{(N-1)p}$, $q = \frac{(N-1)p}{N(p-1)}$, and $q' = \frac{q}{q-1}$, we obtain

$$\log\frac{R}{R_1} \le \left(\int_{R_1}^{R} I(r)\, dr\right)^{\frac{1}{q'}} \int_{R_1}^{R} r^{-\frac{p'(N-1)}{N}} I(r)^{-\frac{p'(N-p)}{Np}}\, dr.$$

Letting $R \to \infty$ in this expression, we find

$$\int_{R_1}^{+\infty} r^{-\frac{p'(N-1)}{N}} I(r)^{-\frac{p'(N-p)}{Np}} \, dr = \infty.$$

By (3.11),

$$\int_{R_1}^{R} \frac{dr}{(H(r))^{p'/p}} \geq \int_{R_1}^{R} K^{-1} r^{-\frac{p'(N-1)}{N}} I(r)^{-\frac{p'(N-p)}{Np}} dr,$$

and thus by letting $R \to \infty$ in this last expression the proof of the lemma is ended . $\diamondsuit$

Now we prove the simplicity of the principal eigenvalue of $(1.1)_\lambda$, which is the main result of this section.

Theorem 3.4 *Let g satisfies $(\mathcal{G}^+)$ (respectively $(\mathcal{G}^-)$). Then*
(i) the eigenspace corresponding to the principal eigenvalue λ_1 (respectively λ_1^+, λ_1^-) has dimension 1.
(ii) λ_1 (respectively λ_1^+, λ_1^-) is the only eigenvalue of $(1.1)_\lambda$ which admits positive eigenfunctions.

Proof. We only consider the case when g satisfies $(\mathcal{G}^+)$. The case when g satisfies $(\mathcal{G}^-)$ can be treated in the same way.

Suppose that ϕ is any eigenfunction of $(1.1)_{\lambda_1}$ corresponding to the principal eigenvalue λ_1. Also suppose that $u \in \mathcal{D}^{1,p}$ is a positive eigenfunction of $(1.1)_{\lambda_1}$ corresponding to an eigenvalue $\lambda > 0$. Thus in this case $\lambda \geq \lambda_1$. Multiplying $(1.1)_{\lambda_1}$ by ϕ and integrating by parts over B_R, we obtain

$$\int_{B_R} | \bigtriangledown \phi|^p dx - \int_{\partial B_R} \phi | \bigtriangledown \phi|^{p-2} \frac{\partial \phi}{\partial n} dS = \lambda_1 \int_{B_R} g|\phi|^p dx. \tag{3.12}$$

Letting $R \to \infty$ and using lemma 3.1, we get

$$\int_{\mathbb{R}^N} | \bigtriangledown \phi|^p dx = \lambda_1 \int_{\mathbb{R}^N} g|\phi|^p dx > 0. \tag{3.13}$$

Multiplying $(1.1)_\lambda$ by $\frac{|\phi|^p}{u^{p-1}}$ and integrating by parts over B_R, we find

$$\begin{aligned} &p \textstyle\int_{B_R} \bigtriangledown \phi . \bigtriangledown u |\bigtriangledown u|^{p-2} \left(\frac{\phi|\phi|^{p-2}}{u^{p-1}} \right) dx - (p-1) \textstyle\int_{B_R} \left(\frac{|\phi|}{u} \right)^p | \bigtriangledown u|^p dx \\ &- \textstyle\int_{\partial B_R} \frac{|\phi|^p}{u^{p-1}} | \bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS = \lambda \textstyle\int_{B_R} g|\phi|^p dx. \end{aligned} \tag{3.14}$$

Subtracting equation (3.14) from (3.12), we obtain

$$\Theta(R) - \textstyle\int_{\partial B_R} \phi| \bigtriangledown \phi|^{p-2} \frac{\partial \phi}{\partial n} dS + \beta(R) = (\lambda_1 - \lambda) \textstyle\int_{B_R} g|\phi|^p dx, \tag{3.15}$$

where

$$\beta(R) := \int_{\partial B_R} \frac{|\phi|^p}{u^{p-1}} | \bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS. \tag{3.16}$$

By lemma 3.2, $\Theta(R)$ converges as $R \to \infty$, and $\lim_{R\to\infty} \Theta(R) \in \mathbb{R}_+ \cup \{+\infty\}$, then by lemma 3.1 and (3.15) $\beta(R)$ also converges and $\lim_{R\to\infty} \beta(R) \in \mathbb{R}_- \cup \{-\infty\}$.

(α) Suppose that $-\infty < \lim_{R\to\infty} \beta(R) < 0$, then $\lim_{R\to\infty} \Theta(R)$ is finite. Furthermore, for any $\mu > 1$, we have by (3.7)

$$(p-1)(1-\frac{1}{\mu^{p'}}) \int_{B_R} \left(\frac{|\phi|}{u} \right)^p | \bigtriangledown u|^p dx \leq \Theta(R) + (\mu^p - 1) \int_{B_R} | \bigtriangledown \phi|^p dx. \tag{3.17}$$

Let us set $\gamma(R) = \int_{\partial B_R} \left| \frac{\phi}{u} \right|^p | \bigtriangledown u|^p dS > 0$ and $\Gamma(R) = \int_0^R \gamma(r) dr$. We see that, since $\Gamma(R)$ is increasing and bounded, it is convergent. Thus we can find a constant $\sigma \in (0,1)$ and R_1 large enough so that

$$\sigma \Gamma(R) \leq -\beta(R), \quad \text{for all} \quad R \geq R_1. \tag{3.18}$$

(β) Suppose that $\lim_{R\to\infty} \beta(R) = -\infty$. Then from (3.15) it follows that $\lim_{R\to\infty} \Theta(R) = \infty$. Furthermore, (3.15), and (3.17) respectively imply that

$$\Theta(R) \leq -\beta(R) + \text{constant}, \quad \text{and} \quad \sigma\Gamma(R) \leq \Theta(R) + \text{constant},$$

and thus relation (3.18) is again satisfied.

Moreover, for any $R > 0$, we have

$$-\beta(R) \leq \textstyle\int_{\partial B_R} \frac{|\phi|^p}{u^{p-1}} | \bigtriangledown u|^{p-1} dS$$

$$\leq \left(\textstyle\int_{\partial B_R} |\phi|^p dS \right)^{\frac{1}{p}} \left(\textstyle\int_{\partial B_R} \left(\frac{|\phi|}{u} \right)^p | \bigtriangledown u|^p dS \right)^{\frac{1}{p'}} = (H(R))^{\frac{1}{p}} (\gamma(R))^{\frac{1}{p'}},$$

and thus

$$\Gamma'(R) \geq \frac{(-\beta(R))^{p'}}{(H(R))^{\frac{p'}{p}}} \geq \sigma^{p'} \frac{(\Gamma(R))^{p'}}{(H(R))^{\frac{p'}{p}}}, \text{ for all } R \geq R_1.$$

Hence

$$\frac{\Gamma'(R)}{(\Gamma(R))^{p'}} = -\frac{1}{p-1}\frac{d}{dR}\left(\frac{1}{(\Gamma(R))^{p'-1}}\right) \geq \sigma^{p'}\frac{1}{(H(R))^{\frac{p'}{p}}}, \text{ for all } R \geq R_1,$$

which by an integration implies

$$\frac{1}{(\Gamma(R))^{p'-1}} - \frac{1}{(\Gamma(R_1))^{p'-1}} + K\int_{R_1}^{R}\frac{dr}{(H(r))^{\frac{p'}{p}}} \leq 0.$$

By letting $R \to \infty$ in this last expression we obtain a contradiction to lemma 3.3 and thus we must have $\lim_{R\to\infty}\beta(R) = 0$. By (3.15) this in turn implies that $\lim_{R\to\infty}\Theta(R) = 0$.

Then lemma 3.2 implies that $u = c\phi$ for some positive constant c. Using again (3.15), we obtain that

$$0 = (\lambda_1 - \lambda)\int_{\mathbb{R}^N} g|\phi|^p dx.$$

Since $\int_{\mathbb{R}^N} g|\phi|^p dx > 0$ we must have that $\lambda_1 = \lambda$ and the theorem is proved. ♢

Finally we mention from [14] that if g satisfies $(\mathcal{G}^+)$ (respectively $(\mathcal{G}^-)$), then the principal eigenvalue λ_1 (respectively λ_1^+, λ_1^-) of $(1.1)_\lambda$ is isolated.

4 Existence Results for a Perturbation of the p-Laplacian

Consider the following perturbation of the p-Laplacian problem

$$-\Delta_p u = ag(x)|u|^{p-2}u + f(x), \qquad x \in \mathbb{R}^N, \tag{4.1}$$

where $a \in \mathbb{R}$. As far as we know this problem is not included in recent papers on the subject. Moreover, the theorem on a maximum principle we prove is of particular interest on its own. First we give the following existence result.

Theorem 4.1 *Let $f \in \mathcal{D}^{-1,p'}(\mathbb{R}^N)$. If either (i) g satisfies hypothesis $(\mathcal{G}^+)$ and $a < \lambda_1$, or (ii) g satisfies hypothesis $(\mathcal{G}^-)$ and $\lambda_1^- < a < \lambda_1^+$, then the equation (4.1) admits a solution in $\mathcal{D}^{1,p}$.*

Proof. We split the set of possible $\mathcal{D}^{1,p}$ solutions of equation (4.1) in two parts depending on the sign of $\int_{\mathbb{R}^N} g|u|^p dx$. Let us set

$$V^+ = \left\{ u \in \mathcal{D}^{1,p} : 0 < \int_{\mathbb{R}^N} g|u|^p dx \right\},$$

$$V^- = \left\{ u \in \mathcal{D}^{1,p} : \int_{\mathbb{R}^N} g|u|^p dx < 0, \right\}.$$

Hence if $u \in V^+$ we have that $\int_{\mathbb{R}^N} g|u|^p dx \leq \frac{1}{\lambda_1} \int_{\mathbb{R}^N} |\bigtriangledown u|^p dx$, while if $u \in V^-$ then $-\int_{\mathbb{R}^N} g|u|^p dx \leq -\frac{1}{\lambda_1^-} \int_{\mathbb{R}^N} |\bigtriangledown u|^p dx$.

(a) Let $u \in V^+$. If $0 < a < \lambda_1$, then

$$-a \int_{\mathbb{R}^N} g|u|^p dx \geq \frac{-a}{\lambda_1} \int_{\mathbb{R}^N} |\bigtriangledown u|^p dx.$$

Since $a < \lambda_1$, there exists $\delta \in (0,1)$ such that

$$-a \int_{\mathbb{R}^N} g|u|^p dx > -\delta \int_{\mathbb{R}^N} |\bigtriangledown u|^p dx, \text{ for any } u \in V^+.$$

For $a \leq 0$ we get $-a \int_{\mathbb{R}^N} g|u|^p dx \geq 0$.

(b) Let $u \in V^-$. If $0 < a$, then $-a \int_{\mathbb{R}^N} g|u|^p dx \geq 0$. While if $0 > a > \lambda_1^-$, then

$$-a \int_{\mathbb{R}^N} g|u|^p dx \geq -\frac{a}{\lambda_1^-} \int_{\mathbb{R}^N} |\bigtriangledown u|^p dx.$$

Since $\frac{a}{\lambda_1^-} < 1$, there exists $\delta' \in (0,1)$ such that $\frac{a}{\lambda_1^-} < \delta' < 1$. So when $u \in V^-$ we get

$$-a \int_{\mathbb{R}^N} g|u|^p dx > -\delta' \int_{\mathbb{R}^N} |\bigtriangledown u|^p dx.$$

Hence in both cases we obtain

$$\begin{aligned} J(u) \quad &:= \tfrac{1}{p} \textstyle\int_{\mathbb{R}^N} |\bigtriangledown u|^p dx - \tfrac{a}{p} \int_{\mathbb{R}^N} g|u|^p dx - \int_{\mathbb{R}^N} fu dx \\ &\geq \tfrac{1}{p}(1-\delta) \textstyle\int_{\mathbb{R}^N} |\bigtriangledown u|^p dx - \int_{\mathbb{R}^N} fu dx \\ &\geq \tfrac{1}{p}(1-\delta)||u||^p_{\mathcal{D}^{1,p}} - ||f||_{\mathcal{D}^{-1,p'}} ||u||_{\mathcal{D}^{1,p}}. \end{aligned}$$

Therefore J is coercive in $\mathcal{D}^{1,p}$. It is easy to see that J is also weakly lower semicontinuous, so equation (4.1) admits a solution in $\mathcal{D}^{1,p}$ and the proof is complete. $\diamondsuit$

The next theorem gives conditions for the validity of the maximum principle.

Theorem 4.2 *Let* $f \in L^{\frac{Np}{Np-N+p}}(\mathbb{R}^N)$ *and* $f \geq 0$. *Then we have:*
(i) (Necessary and Sufficient Condition) if g *satisfies* $(\mathcal{G}^+)$ *then all solutions of the equation (4.1) are non negative if and only if* $a < \lambda_1$;
(ii) (Sufficient Condition) if g *satisfies* $(\mathcal{G}^-)$ *and* $\lambda_1^- < a < \lambda_1^+$, *then all solutions of the equation (4.1) are non negative.*

Proof. (i) (Sufficient Condition) Let u be a solution of the equation (4.1) with $u = u^+ + u^-$, u^+, u^- being the positive and negative parts of u. Multiplying (4.1) by u^- we find, after an integration, that

$$\int_{\mathbb{R}^N} |\bigtriangledown u^-|^p dx = a\int_{\mathbb{R}^N} g|u^-|^p dx + \int_{\mathbb{R}^N} fu^- dx. \tag{4.2}$$

We have $\int_{\mathbb{R}^N} g|u^-|^p dx \geq 0$. So if $a \leq 0$ then

$$\int_{\mathbb{R}^N} |\bigtriangledown u^-|^p dx \leq \int_{\mathbb{R}^N} fu^- dx \leq 0,$$

which implies that $u^- \equiv 0$. While if $0 < a < \lambda_1$, since $\int_{\mathbb{R}^N} |\bigtriangledown u^-|^p dx \geq \lambda_1 \int_{\mathbb{R}^N} g|u^-|^p dx$, we obtain that

$$a\int_{\mathbb{R}^N} g|u^-|^p dx \leq \frac{a}{\lambda_1}\int_{\mathbb{R}^N} |\bigtriangledown u^-|^p dx,$$

which implies that

$$(1 - \frac{a}{\lambda_1})\int_{\mathbb{R}^N} |\bigtriangledown u^-|^p dx \leq \int_{\mathbb{R}^N} fu^- dx \leq 0.$$

Hence again $u^- \equiv 0$.

(Necessary Condition) Now suppose that $a \geq \lambda_1$. If we set $u = -\phi_1 < 0$, then u is a negative eigenfunction of equation $(1.1)_{\lambda_1}$, that satisfy

$$
\begin{aligned}
-\Delta_p(u) \quad &= \lambda_1 g(x)|u|^{p-2}u \\
&= ag(x)|u|^{p-2}u + f(x),
\end{aligned}
$$

where

$$f(x) = (\lambda_1 - a)| - \phi_1|^{p-2}(-\phi_1) > 0.$$

Clearly this contradicts Vazquez's Maximum Principle and hence $a < \lambda_1$.

(ii) If $\int_{\mathbb{R}^N} g|u^-|^p dx \geq 0$ we argue as before. If $\int_{\mathbb{R}^N} g|u^-|^p dx < 0$ and $a \geq 0$ then from equation (4.2) we have

$$\int_{\mathbb{R}^N} | \nabla u^-|^p dx \leq \int_{\mathbb{R}^N} fu^- dx \leq 0,$$

which implies that $u^- \equiv 0$.
While if $\lambda_1^- < a < 0$, since $\int_{\mathbb{R}^N} | \nabla u^-|^p dx \geq \lambda_1^- \int_{\mathbb{R}^N} g|u^-|^p dx$, we get

$$a \int_{\mathbb{R}^N} g|u^-|^p dx \leq \frac{a}{\lambda_1^-} \int_{\mathbb{R}^N} | \nabla u^-|^p dx.$$

So again equation (4.2) implies that

$$(1 - \frac{a}{\lambda_1^-}) \int_{\mathbb{R}^N} | \nabla u^-|^p dx \leq \int_{\mathbb{R}^N} fu^- dx \leq 0.$$

Hence we have again $u^- \equiv 0$ and the proof of the theorem is complete. $\diamondsuit$

PART II. THE SYSTEM

In this part we will consider system $(1.3)_\lambda$, $(1.4)_\lambda$. Throughout this part we will suppose that the exponents p, q, α, β and the coefficient functions a, b, d which appear in that problem satisfy the following conditions:

$$p > 1,\ q > 1,\ \alpha \geq 0,\ \beta \geq 0,$$

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1 \quad \text{and} \quad \alpha + \beta + 2 < N. \tag{4.3}$$

$(\mathcal{H}_1)$ a is a smooth function, at least $C^{0,\gamma}_{loc}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$, such that $a \in L^{N/p}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ and $a(x) > 0$, in Ω^+, with $|\Omega^+| > 0$.

$(\mathcal{H}_2)$ d is a smooth function, at least $C^{0,\gamma}_{loc}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$, such that $d \in L^{N/q}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ and $d(x) > 0$, in Ω^+, with $|\Omega^+| > 0$.

$(\mathcal{H}_3)$ b is a smooth function, at least $C^{0,\gamma}_{loc}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$, such that $b \in L^{N/(\alpha+\beta+2)}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$, $b \geq 0$.

$(\mathcal{H}_4)$ a and d both satisfy either $(\mathcal{G}^+)$ or $(\mathcal{G}^-)$.

5 Existence of Principal Eigenvalues

As in the case of the equation we will work in $\mathcal{D}^{1,p}$. For any $(u,v) \in \mathcal{D}^{1,p} \times \mathcal{D}^{1,q}$ we define the functionals A and B by

$$A(u,v) = \frac{\alpha+1}{p} ||u||^p_{\mathcal{D}^{1,p}} + \frac{\beta+1}{q} ||v||^q_{\mathcal{D}^{1,q}}.$$

$$\begin{aligned} B(u,v) = \; & \tfrac{\alpha+1}{p} \textstyle\int_{\mathbb{R}^N} a(x)|u|^p dx + \tfrac{\beta+1}{q} \int_{\mathbb{R}^N} d(x)|v|^q dx \\ & + \textstyle\int_{\mathbb{R}^N} b(x)|u|^{\alpha+1}|v|^{\beta+1} dx. \end{aligned}$$

It is well known that $A, B \in C^1$ and that A is weakly lower semicontinuous. In order to prove existence of the first eigenvalue for the system of equations $(1.3)_\lambda$, $(1.4)_\lambda$, we need the following

Lemma 5.1 *(i) If (u_n, v_n) is a sequence in $\mathcal{D}^{1,p} \times \mathcal{D}^{1,q}$ with $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{1,p}$, $v_n \rightharpoonup v$ weakly in $\mathcal{D}^{1,q}$, then there is a subsequence, denoted again by (u_n, v_n), such that $B(u_n, v_n) \to B(u,v)$; (ii) if $B'(u,v) = 0$, then $B(u,v) = 0$.*

Proof. As in Lemma 2.2, we split $\mathbb{R}^N$ in the two parts $|x| > R$ and $|x| \leq R$, and then use Holder's inequality with equation (4.3) to estimate the terms involving u and v. $\diamondsuit$

In the remaining part of this section we shall prove the existence of nonzero principal eigenvalues. We first have

Theorem 5.2 *Let a, d, b satisfy hypotheses ($\mathcal{H}_1$), ($\mathcal{H}_2$) and ($\mathcal{H}_3$), respectively. Suppose that $(u,v) \in \mathcal{D}^{1,p} \times \mathcal{D}^{1,q}$ is a solution of (1.3)$_\lambda$, (1.4)$_\lambda$. Then for any $x \in \mathbb{R}^N$ and any $R > 0$, there is a constant K depending only of $p, q, N, \lambda, ||a||_\infty, ||b||_\infty$ and $||d||_\infty$ such that*

$$\begin{aligned} ||u||_{L^\infty(B_R(x))} \leq\ & K\left(1 + R^{\max(p,q)}\right)^{\frac{N}{p\min(p,q)}} \\ & \times \max\{R^{\frac{p-N}{p}}||u||_{L^{\frac{Np}{N-p}}(B_{2R}(x))},\ R^{\frac{q-N}{p}}||v||^{\frac{q}{p}}_{L^{\frac{Nq}{N-q}}(B_{2R}(x))}\}, \\ ||v||_{L^\infty(B_R(x))} \leq\ & K\left(1 + R^{\max(p,q)}\right)^{\frac{N}{q\min(p,q)}} \\ & \times \max\{R^{\frac{p-N}{q}}||u||^{\frac{p}{q}}_{L^{\frac{Np}{N-p}}(B_{2R}(x))},\ R^{\frac{q-N}{q}}||v||_{L^{\frac{Nq}{N-q}}(B_{2R}(x))}\}. \end{aligned}$$

Moreover, we have that

$$\lim_{|x|\to+\infty} u(x) = 0, \textit{and}, \lim_{|x|\to+\infty} v(x) = 0,$$

uniformly.

Proof Our proof consists in adapting Theorem 1 of Serrin in [26] to our system. For completeness we will give all the details in the Appendix, section 7. $\diamondsuit$

Remark 5.1 *In the case of one equation (theorem 2.4) we proved that the solution u is uniformly bounded on $\mathbb{R}^N$. We have not been able yet to produce a similar result for the system.*

Theorem 5.3 *(i) Let a, d satisfy ($\mathcal{G}^+$). Then the system (1.3)$_\lambda$, (1.4)$_\lambda$ admits a positive principal eigenvalue given by*

$$\lambda_1 = \inf_{B(u,v)=1} A(u,v). \tag{5.1}$$

(ii) Let a, b satisfy $(\mathcal{G}^-)$. Then the system $(1.3)_\lambda$, $(1.4)_\lambda$ admits two principal eigenvalues of opposite sign given by

$$\lambda_1^+ = \inf_{B(u,v)=1} A(u,v), \quad \lambda_1^- = -\inf_{B(u,v)=-1} A(u,v). \tag{5.2}$$

In both cases the associated eigenfunctions (ϕ, ψ) (respectively (ϕ^+, ψ^+), (ϕ^-, ψ^-)) belong to $\mathcal{D}^{1,p} \times \mathcal{D}^{1,q}$ and each component is of class $\mathcal{C}^{1,\alpha}(B_r)$, for any $r > 0$, where $\alpha = \alpha(r) \in (0,1)$. Moreover, there is an eigenfunction which is positive (componentwise) everywhere in $\mathrm{I\!R}^N$.

Proof. The existence and positivity (negativity, when corresponds) of the first eigenvalues in (i) and (ii) is a direct consequence of the properties of the functionals A and B, lemma 5.1 above and Theorem (6.3.2) in [5] for nonlinear eigenvalue problems. The $\mathcal{C}^{1,\alpha}$ regularity of the eigenfunctions follows directly from the previous theorem 5.2 and an argument of Tolksdorf [30]. Since $A(|u|,|v|) = A(u,v)$ and $B(|u|,|v|) = B(u,v)$, if (u_λ, v_λ) achieves the infimum in one of (5.1), (5.2), then $(|u_\lambda|, |v_\lambda|)$ does the same. So we can consider that $u_\lambda \geq 0, v_\lambda \geq 0$.
Hence theorem 5.2 and Vasquez' Maximum Principle [33] implies that $u_\lambda > 0, v_\lambda > 0$. $\diamondsuit$

6 Simplicity of the Principal Eigenvalues

We begin this section with a lemma which is the analog of lemma 3.1.

Lemma 6.1 *Suppose that $(u,v) \in \mathcal{D}^{1,p} \times \mathcal{D}^{1,q}$ is a solution of $(1.3)_\lambda$, $(1.4)_\lambda$. Then*

$$\lim_{R \to +\infty} \int_{\partial B_R} u |\bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS = 0 = \lim_{R \to +\infty} \int_{\partial B_R} v |\bigtriangledown v|^{q-2} \frac{\partial v}{\partial n} dS. \tag{6.1}$$

Proof. Let (u,v) satisfies $(1.3)_\lambda$, $(1.4)_\lambda$. Multiplying both sides of $(1.3)_\lambda$, by u and integrating over B_R, we obtain

$$\begin{aligned} \int_{B_R} |\bigtriangledown u|^p \, dx \quad &- \quad \int_{\partial B_R} u |\bigtriangledown u|^{p-2} \frac{\partial u}{\partial n} dS \\ &= \lambda \int_{B_R} a |u|^p \, dx + \lambda \int_{B_R} b |u|^{\alpha+1} |v|^{\beta+1} \, dx. \end{aligned} \tag{6.2}$$

Since $|\bigtriangledown u| \in L^p(\mathbb{R}^N)$, $a|u|^p \in L^1(\mathbb{R}^N)$, and $b|u|^{\alpha+1}|v|^{\beta+1} \in L^1(\mathbb{R}^N)$, it follows that

$$\lim_{R\to\infty}\int_{\partial B_R} u|\bigtriangledown u|^{p-2}\frac{\partial u}{\partial n}dS = L_1,$$

exists and it is finite. Similarly working with $(1.4)_\lambda$, we find

$$\lim_{R\to\infty}\int_{\partial B_R} v|\bigtriangledown v|^{q-2}\frac{\partial v}{\partial n}dS = L_2.$$

The lemma is ended if we prove that $L_1 = 0 = L_2$, but this follows identically to the corresponding argument in lemma 3.1. $\diamondsuit$

Suppose now that (ϕ,ψ) is any eigenfunction of $(1.3)_{\lambda_1}$, $(1.4)_{\lambda_1}$ corresponding to the principal eigenvalue λ_1. Also let $(u,v) \in \mathcal{D}^{1,p}\times\mathcal{D}^{1,q}$ be a positive eigenfunction of $(1.3)_\lambda$, $(1.4)_\lambda$ corresponding to an eigenvalue $\lambda > 0$.

In our next result we will use the functions Θ, H, and β defined respectively in (3.6), (3.11), and (3.16). Furthermore we define

$$\hat{\Theta}(R) := \int_{B_R}\{|\bigtriangledown \psi|^q + (q-1)\left|\frac{\psi}{u}\right|^q|\bigtriangledown v|^q dx - q\bigtriangledown\psi.\bigtriangledown v|\bigtriangledown v|^{q-2}\frac{|\psi|^{q-2}\psi}{v^{q-1}}\}dx,$$

and

$$\tilde{\Theta}(R) := \frac{\alpha+1}{p}\Theta(R) + \frac{\beta+1}{q}\hat{\Theta}(R).$$

Also let us set $\hat{H}(r) := \int_{\partial B_r}(\psi(x))^q dS$, and

$$\hat{\beta}(R) := \int_{\partial B_R}\frac{|\psi|^q}{v^{q-1}}|\bigtriangledown v|^{q-2}\frac{\partial v}{\partial n}dS,$$

and

$$\tilde{\beta}(R) := \frac{\alpha+1}{p}\beta(R) + \frac{\beta+1}{q}\hat{\beta}(R).$$

Now we are ready to state and prove our main simplicity result for the system. Since the proof follows the same lines as those of theorem 3.4 for the case of the equation, we just sketch it.

Theorem 6.2 *Let a, d satisfy $(\mathcal{G}^+)$ (respectively $(\mathcal{G}^-)$. Then (i) the eigenspace corresponding to the principal eigenvalue λ_1 (respectively λ_1^+, λ_1^-) is of dimension 1.*

(ii) λ_1 *(respectively* λ_1^+, λ_1^- *) is the only eigenvalue of* $(1.1)_\lambda$ *to which corresponds a positive eigenfunction.*

Proof. We consider only the case when g satisfies $(\mathcal{G}^+)$, then in this case $\lambda \geq \lambda_1$. The case $(\mathcal{G}^-)$ can be treated similarly. Multiplying $(1.3)_{\lambda_1}$ by ϕ and $(1.3)_\lambda$ by $\frac{|\phi|^p}{u^{p-1}}$, integrating by parts over B_R, and taking the difference of the resulting expressions, we obtain

$$\begin{aligned} &\Theta(R) - \textstyle\int_{\partial B_R} \phi| \bigtriangledown \phi|^{p-2} \frac{\partial \phi}{\partial n} dS + \beta(R) \\ &= (\lambda_1 - \lambda) \textstyle\int_{B_R} a|\phi|^p dx + \lambda_1 \int_{B_R} b|\phi|^{\alpha+1}|\psi|^{\beta+1} dx \\ &-\lambda \textstyle\int_{B_R} b|u|^{\alpha+1}|v|^{\beta+1} \left(\frac{|\phi|}{u}\right)^p dx. \end{aligned} \tag{6.3}$$

Similarly, by an analogous procedure, from $(1.4)_{\lambda_1}$ and $(1.4)_\lambda$, we find

$$\begin{aligned} \hat{\Theta}(R) \quad &- \textstyle\int_{\partial B_R} \psi| \bigtriangledown \psi|^{q-2} \frac{\partial \psi}{\partial n} dS + \hat{\beta}(R) = (\lambda_1 - \lambda) \int_{B_R} d|\psi|^q dx \\ &+\lambda_1 \textstyle\int_{B_R} b|\phi|^{\alpha+1}|\psi|^{\beta+1} dx - \lambda \int_{B_R} b|u|^{\alpha+1}|v|^{\beta+1} \left(\frac{|\psi|}{v}\right)^q dx. \end{aligned} \tag{6.4}$$

Multiplying (6.3) by $\frac{\alpha+1}{p}$, (6.4) by $\frac{\beta+1}{q}$, and adding, we obtain

$$\begin{aligned} &\tilde{\Theta}(R) + \tilde{\beta}(R) - \frac{\alpha+1}{p} \int_{\partial B_R} \phi| \bigtriangledown \phi|^{p-2} \frac{\partial \phi}{\partial n} dS \\ &- \frac{\beta+1}{q} \int_{\partial B_R} \psi| \bigtriangledown \psi|^{q-2} \frac{\partial \psi}{\partial n} dS = \int_{B_R} bE dx \\ &+(\lambda_1 - \lambda) \int_{B_R} \{\frac{a(\alpha+1)}{p} |\phi|^p + \frac{b(\beta+1)}{q} |\psi|^q\} dx, \end{aligned} \tag{6.5}$$

where

$$E = |u|^{\alpha+1}|v|^{\beta+1} \lambda_1 \left|\frac{\phi}{u}\right|^{\alpha+1} \left|\frac{\psi}{v}\right|^{\beta+1} - \lambda \left\{ \frac{\alpha+1}{p} \left|\frac{\phi}{u}\right|^p - \frac{\beta+1}{q} \left|\frac{\psi}{v}\right|^q \right\} \leq 0,$$

because $\lambda \geq \lambda_1$ and by Young's inequality. As in theorem 3.4 we set

$$\gamma(R) = \int_{\partial B_R} \left|\frac{\phi}{u}\right|^p | \bigtriangledown u|^p dS > 0, \ \text{ and } \ \Gamma(R) = \int_0^R \gamma(r) dr,$$

$$\hat{\gamma}(R)=\int_{\partial B_R}\left|\frac{\psi}{v}\right|^q|\bigtriangledown v|^q dS>0,\;\; \text{and}\;\; \hat{\Gamma}(R)=\int_0^R\hat{\gamma}(r)dr,$$

$$\tilde{\Gamma}(R)=\frac{\alpha+1}{p}\Gamma(R)+\frac{\beta+1}{q}\hat{\Gamma}(R).$$

We have from (6.3) and lemma 6.1 that $\Theta(R)$ and $|\beta(R)|$ converge to either a finite limit or to $+\infty$ at the same time; so $l:=\lim_{R\to+\infty}\beta(R)$ exists and is either finite or equal to $-\infty$. Similarly, it follows from (6.4) and lemma 6.1 that $\hat{l}:=\lim_{R\to+\infty}\hat{\beta}(R)$ exists and is either finite or equal to $-\infty$.
Claim: We have $\lim_{R\to+\infty}\beta(R)=\lim_{R\to+\infty}\hat{\beta}(R)=0$. To prove this claim we proceed by contradiction.

i) Suppose that l and $\hat{l}$ are finite . Since $\lim_{R\to+\infty}\hat{\Theta}(R)\geq 0$ by (6.5), we have that $l+\hat{l}\leq 0$; so we can assume that one of them, say $l<0$. Then arguying as in the proof of theorem 3.4 for the equation, and by using (3.17), we can find a constant $\sigma\in(0,1)$ and R_1 large enough such that

$$\sigma\Gamma(R)\leq-\beta(R),\quad \text{for all}\quad R\geq R_1, \tag{6.6}$$

and the argument follows like in theorem 3.4.
ii) Suppose now that l or $\hat{l}$ is $-\infty$, say $\hat{l}$ is $-\infty$. Then from (6.4),

$$\hat{\Theta}(R)\leq-\hat{\beta}(R)+Const.$$

and since similar to (3.17), for any $\mu>1$, we have now that

$$(q-1)(1-\frac{1}{\mu^{q'}})\int_{B_R}\left(\frac{|\psi|}{v}\right)^q|\bigtriangledown v|^q dx\leq\hat{\Theta}(R)+(\mu^q-1)\int_{B_R}|\bigtriangledown\psi|^q dx, \tag{6.7}$$

we can again follow the argument in theorem 3.4 to obtain a contradiction to the corresponding version of lemma 3.3. Thus the claim is proved.

Now we deduce from equation (6.5) that $\lim_{R\to+\infty}\tilde{\Theta}(R)=0$, which implies that $u=c\phi$ and $v=\hat{c}\psi$ for some constants $c,\hat{c}$, and also that $\lambda_1=\lambda$. Thus the theorem is proved.$\diamondsuit$

7 Appendix

Proof of theorem 5.2 Let $x \in \mathbb{R}^N$ and $R > 0$. For $y \in B_{2R}(x)$ and any function h defined on $B_{2R}(x)$ we define

$$\hat{h}(t) = h(x), \quad \text{where,} \quad t = \frac{y-x}{R}.$$

Let $(u, v) \in \mathcal{D}^{1,p} \times \mathcal{D}^{1,q}$ be a solution of $(1.3)_\lambda$, $(1.4)_\lambda$. Simple computations show that $(\hat{u}, \hat{v})$ satisfy the following system

$$-\Delta_p \hat{u} = \lambda R^p \hat{a}(x)|\hat{u}|^{p-2}\hat{u} + \lambda R^p \hat{b}(x)|\hat{u}|^{\alpha-1}\hat{u}|\hat{v}|^{\beta+1}, \tag{7.1}$$

$$-\Delta_q \hat{v} = \lambda R^q \hat{b}(x)|\hat{u}|^{\alpha+1}|\hat{v}|^{\beta-1}\hat{v} + \lambda R^q \hat{d}(x)|\hat{v}|^{q-2}\hat{v}. \tag{7.2}$$

Hereafter K, K', K'' denote any constants depending only on p, q, N, λ, $||a||_\infty$, $||b||_\infty$ and $||d||_\infty$.

Without loss of generality, we can assume $p > q$. For any ball $B \subset B_2(0)$, there is a constant $c = \frac{N}{N-q} > 1$ such that

$$\text{for any} \quad w \in W_0^{1,p}(B), \quad ||w||_{L^{cp}(B)} \leq K|| \bigtriangledown w||_{L^p(B)}$$
$$\text{for any} \quad w \in W_0^{1,q}(B), \quad ||w||_{L^{cq}(B)} \leq K|| \bigtriangledown w||_{L^q(B)}$$

We construct the following sequences

$$p_k = pc^k, \;\; q_k = qc^k, \quad \text{for any} \;\; k \geq 0, \;\; \text{and}$$

$$m_k = p(c^k - 1), \; t_k = q(c^k - 1);$$

$$\rho_0 = 2, \; \rho_k = 2 - \frac{1}{\sigma}\sum_{j=0}^{k-1} c^{\frac{-j}{p'}}, \;\; \text{for any} \;\; k \geq 1,$$

where $\sigma = \sum_{j=0}^{\infty} c^{\frac{-j}{p'}}$.

Denoting by $D_k = B_{\rho_k}$, (we recall here we have agreed that B_d means the ball in $\mathbb{R}^N$, center zero and radius d), we consider a function $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, defined such that $0 \leq \eta \leq 1$, $\eta = 1$ on D_{k+1}, $\sup \eta \subset D_k$, and satisfying

$$| \bigtriangledown \eta(t)| \leq K \, c^{\frac{k}{p'}}, \;\; \text{for all} \;\; t \in D_k \tag{7.3}$$

Multiplying (7.1) by $|\hat{u}|^{m_k}\, \hat{u}\, \eta^p$, and integrating over D_k, we obtain

$$I_1 + I_2 = I_3 + I_4 \tag{7.4}$$

where

$$I_1 = (1+m_k) \int_{D_k} \eta^p |\hat{u}|^{m_k} | \bigtriangledown \hat{u}|^p \, dx,$$

$$I_2 = p \int_{D_k} \eta^{p-1} \bigtriangledown \eta \cdot \bigtriangledown \hat{u} | \bigtriangledown \hat{u}|^{p-2} \hat{u}^{m_k} \hat{u} \, dx,$$

$$I_3 = \lambda\, R^p \int_{D_k} \hat{a} \eta^p |\hat{u}|^{p+m_k} \, dx,$$

and

$$I_4 = \lambda\, R^p \int_{D_k} \hat{b} |\hat{u}|^{\alpha+1+m_k} |\hat{v}|^{\beta+1} \eta^p \, dx.$$

Defining next $E_k = \max\{||u^{c^k}||^p_{L^p(D_k)},\ ||v^{c^k}||^q_{L^q(D_k)}\}$, we obtain that

$$|I_3| \leq R^p\, ||a||_\infty \int_{D_k} \eta^p |\hat{u}|^{pc^k} dx \leq R^p\, K\, E_k. \tag{7.5}$$

Similarly, observing that $\frac{\alpha+1+m_k}{p_k} + \frac{\beta+1}{q_k} = 1$, we have that

$$|I_4| \leq R^p\, ||b||_\infty \{\int_{D_k} |\hat{u}|^{pc^k} dx\}^{\frac{\alpha+1+m_k}{p_k}}\ \{\int_{D_k} |\hat{v}|^{qc^k} dx\}^{\frac{\beta+1}{q_k}} \leq R^p\, K\, E_k. \tag{7.6}$$

On the other hand, since $(1+m_k) = (p-1)(c^k-1) + c^k$, for any $s > 0$, we get

$$|I_2| \leq \frac{ps^{p'} c^k}{p'} \int_{D_k} \eta^p | \bigtriangledown \hat{u}|^p |\hat{u}|^{m_k} dx + \frac{c^{\frac{-kp}{p'}}}{s^{p'}} \int_{D_k} | \bigtriangledown \eta|^p |\hat{u}|^{pc^k} dx.$$

Since $c^k \leq (1+m_k)$, by (7.3) and for $2ps^{p'} \leq p$, we obtain

$$|I_2| \leq \frac{1}{2} I_1 + K E_k \tag{7.7}$$

Now, by the imbedding theorem, we have

$$||\eta \hat{u}^{c_k}||^p_{L^{cp}(D_k)} \leq K\, || \bigtriangledown (\eta \hat{u}^{c_k})||^p_{L^p(D_k)} \leq K\,(I_5 + I_6),$$

where

$$0 \leq I_5 = \int_{D_k} |\bigtriangledown \eta|^p |\hat{u}|^{pc^k} dx \leq K\, c^{k(p-1)}\, E_k, \tag{7.8}$$

and

$$0 \leq I_6 = c^{kp} \int_{D_k} \eta^p | \bigtriangledown \hat{u}|^p |\hat{u}|^{p(c^k-1)} dx \leq c^{k(p-1)} I_1. \tag{7.9}$$

By (7.4), (7.5), (7.6), (7.7), (7.8), (7.9), we obtain

$$||\eta \hat{u}^{c_k}||^p_{L^{cp}(D_k)} \leq (1+R^p)\ K\ c^{k(p-1)}\ E_k. \tag{7.10}$$

Similarly, multiplying (7.2) by $|\hat{v}|^{t_k}\hat{v}\eta^q$, integrating over D_k, and using the fact that $p > q$, we find that

$$||\eta \hat{v}^{c^k}||^q_{L^{cq}(D_k)} \leq (1+R^p)\ K\ c^{k(p-1)}\ E_k. \tag{7.11}$$

Setting $\Theta_k = E_k^{\frac{1}{p_k}}$, by (7.10) and (7.11), we obtain

$$\Theta_{k+1} \leq \{(1+R^p)K\}^{\frac{1}{p_k}}\ c^{\frac{k(p-1)}{p_k}}\ \Theta_k, \quad \text{for all } k \geq 0.$$

Hence,

$$||\hat{u}||_{L^{p_k}(D_k)} \leq \Theta_k \leq \{(1+R^p)K\}^{\sum_{j=0}^{\infty}\frac{1}{pc^j}}\ c^{\sum_{j=0}^{\infty}\frac{j(p-1)}{pc^j}}\ \Theta_0,$$

where

$$\sum_{j=0}^{\infty}\frac{1}{p_j} = \sum_{j=0}^{\infty}\frac{1}{pc^j} = \frac{1}{p}\frac{c}{c-1} = \frac{N}{pq}$$

We therefore obtain

$$\begin{aligned} ||\hat{u}||_{L^\infty(B_1)} &\leq \limsup_{k\to+\infty} ||\hat{u}||_{L^{p_k}(D_k)} \\ &\leq K'\ (1+R^p)^{\frac{N}{pq}}\ \max\{||\hat{u}||_{L^p(B_2)}, ||\hat{v}||^{\frac{q}{p}}_{L^q(B_2)}\}. \end{aligned} \tag{7.12}$$

Similarly, defining $\Psi_k = E_k^{\frac{1}{q_k}}$, from (7.10), (7.11), we have

$$\Psi_{k+1} \leq \{(1+R^p)K\}^{\frac{1}{q_k}} c^{\frac{k(p-1)}{q_k}} \Psi_k, \ \text{ and,}$$

$$||\hat{v}||_{L^\infty(B_1)} \leq K'\ (1+R^p)^{\frac{N}{q^2}}\ \max\{||\hat{u}||^{\frac{p}{q}}_{L^p(B_2)},\ ||\hat{v}||_{L^q(B_2)}\}. \tag{7.13}$$

By the imbeddings

$$L^{\frac{Np}{N-p}}(B_2) \subset L^p(B_2) \ \text{ and } \ L^{\frac{Nq}{N-q}}(B_2) \subset L^q(B_2),$$

we obtain from (7.12) and (7.13) that

$$||\hat{u}||_{L^\infty(B_1)} \le K''\,(1+R^p)^{\frac{N}{pq}}\ \max\{||\hat{u}||_{L^{\frac{Np}{N-p}}(B_2)},\ ||\hat{v}||^{\frac{q}{p}}_{L^{\frac{Nq}{N-q}}(B_2)}\}, \tag{7.14}$$

and,

$$||\hat{v}||_{L^\infty(B_1)} \le K''\,(1+R^p)^{\frac{N}{q^2}}\ \max\{||\hat{u}||^{\frac{p}{q}}_{L^{\frac{Np}{N-p}}(B_2)},\ ||\hat{v}||_{L^{\frac{Nq}{N-q}}(B_2)}\}. \tag{7.15}$$

Coming back to (u, v), by a simple change of linear dimension, we find from (7.14) and (7.15)

$$\begin{aligned} ||u\ \ ||_{L^\infty(B_R(x))} &\le K\ (1+R^p)^{\frac{N}{pq}}\times \\ &\times\max\{R^{\frac{p-N}{p}}||u||_{L^{\frac{Np}{N-p}}(B_{2R}(x))}, R^{\frac{q-N}{p}}||v||^{\frac{q}{p}}_{L^{\frac{Nq}{N-q}}(B_{2R}(x))}\}, \end{aligned} \tag{7.16}$$

$$\begin{aligned} ||v\ \ ||_{L^\infty(B_R(x))} &\le K\ (1+R^p)^{\frac{N}{q^2}}\times \\ &\times\max\{R^{\frac{p-N}{q}}||u||^{\frac{p}{q}}_{L^{\frac{Np}{N-p}}(B_{2R}(x))}, R^{\frac{q-N}{q}}||v||_{L^{\frac{Nq}{N-q}}(B_{2R}(x)}\}, \end{aligned} \tag{7.17}$$

ending the proof of the theorem.

Finally, from the last estimates (7.16) and (7.17), we deduce that

$$\lim_{|x|\to+\infty} u(x) = 0, \text{and}, \lim_{|x|\to+\infty} v(x) = 0,$$

uniformly for $x \in \mathbb{R}^N$. $\diamondsuit$

Remark 7.1 *As in the proof of theorem 2.4, we can truncate the function* $\hat{u}$ *and justify the use of* $|\hat{u}|^{m_k}\,\hat{u}\,\eta^p$ *as a test function.*

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Abbreviated Version of the Title:

Quasilinear Elliptic Eigenvalue Equations on $\mathbb{R}^N$

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