Global Bifurcation Results for a Semilinear Elliptic Equation on all of $\mathbb{R}^N$

K. J. BROWN and N. STAVRAKAKIS*
Department of Mathematics, Heriot-Watt University
Riccarton, Edinburgh EH14 4AS, UK
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Abstract

We prove the existence of positive solutions for the semilinear elliptic equation $-\Delta u(x) = \lambda g(x) f(u(x))$, $0 < u < 1$ for $x \in \mathbb{R}$, $\lim_{|x|\to+\infty} u(x) = 0$ which arises in population genetics, under the hypotheses that $N = 3, 4, 5$ and g lies in $L^{N/2}(\mathbb{R}^N)$. We establish the existence of a principal eigenvalue λ_1 for the corresponding linearized problem and, making use of the asymptotic properties of solutions and local and global bifurcation theory, prove the existence of a continuum of solutions lying in the space $\mathcal{D}^{1,2}$ extending from $\lambda = \lambda_1$ to $\lambda = \infty$.

1 Introduction

In this paper we shall discuss the existence of positive solutions of the equation

$$-\Delta u(x) = \lambda g(x) f(u(x)), \qquad x \in \mathbb{R}^N, \qquad (1.1)_\lambda$$

*on leave during 1993-1994 from the Department of Mathematics, National Technical University, Zografos Campus, 157 80 Athens, GREECE.

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$$0 < u < 1, \quad x \in \mathbb{R}^N, \quad \lim_{|x| \to +\infty} u(x) = 0, \tag{1.2}$$

by using methods of bifurcation theory. The equation arises in population genetics (see [6]) where the function g is assumed to change sign and $f : [0,1] \to \mathbb{R}^+$, with $f(0) = f(1) = 0$. The unknown function u corresponds to the relative frequency of an allele and is hence constrained to have values between 0 and 1. The real parameter $\lambda > 0$ corresponds to the reciprocal of a diffusion coefficient.

The problem is well understood on bounded domains where a fairly complete bifurcation analysis can be given (see [2]). The situation is more complicated in the case of unbounded domains as, in general, the equation does not give rise to compact operators and so it is unclear that there exist eigenvalues from which bifurcation can occur. It is also unclear *apriori* in which function spaces solutions of $(1.1)_\lambda$ might lie.

In order to discuss bifurcation from the zero solution of $(1.1)_\lambda$ it is first necessary to study the eigenvalues of the corresponding linear problem

$$\begin{array}{c} -\Delta u(x) = \lambda g(x) f'(0)\, u(x) \qquad \text{for} \quad x \in \mathbb{R}^N \\ \lim_{|x| \to +\infty} u(x) = 0. \end{array} \tag{1.3}$$

The existence of a positive principal eigenvalue (i.e., an eigenvalue corresponding to a positive eigenfunction and so a point at which positive solutions of $(1.1)_\lambda$ may bifurcate from the zero branch) for the above problem has been proved in ([3, 4]) under the hypotheses that $\int_{\mathbb{R}^N} g(x)dx < 0$ and $g(x) < 0$ for $|x|$ large and in ([1]) under the hypothesis that $N \geq 3$ and $g_+ \in L^{N/2}(\mathbb{R}^N)$.

We shall discuss the local and global bifurcation of solutions of $(1.1)_\lambda$ in the case where $N \geq 3$ and g lies in $L^{N/2}(\mathbb{R}^N)$. The proof of the existence of a principal eigenvalue in ([4]) involves constructing an appropriate function space $\mathcal{V}$ with inner product

$$< u, v >= \int_{\mathbb{R}^N} \bigtriangledown u \bigtriangledown v dx - \frac{\alpha}{2} \int_{\mathbb{R}^N} guvdx,$$

for an appropriate positive constant α.

In section 2, using the methods of ([4]), we show that, when $N \geq 3$ and g lies in $L^{N/2}(\mathbb{R}^N)$, a positive principal eigenvalue exists and that the space $\mathcal{V}$ coincides with the standard space $\mathcal{D}^{1,2}$, i.e., the closure of the C_0^∞ functions with respect to the norm $\int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx$. Bifurcation results are subsequently obtained in later sections in the setting of the function space $\mathcal{V}$.

In section 3 we discuss the asymptotic properties of the solutions of $(1.1)_\lambda$ and of (1.3) in $\mathcal{D}^{1,2}$. Our main tools are *apriori* estimates of ([8]) for the L^∞ norm of the solution of a linear elliptic problem in terms of its L^p norm over a larger set. We use our results to establish the uniqueness of the principal eigenfunction of (1.3).

In section 4 we prove our main bifurcation and existence results. We show that under the hypothesis $g \in L^{N/2}(\mathbb{R}^N)$ the local and global bifurcation theorems of ([5, 10]) hold to give the existence of bifurcating continua of solutions in $\mathcal{D}^{1,2}$. Since $\mathcal{D}^{1,2}$ does not embed continuously in L^∞, it is not immediately obvious that any solution u on these continua will satisfy $0 < u < 1$ (and so be of biological significance and satisfy the original problem $(1.1)_\lambda$). In order to prove that solutions are positive we must assume that $g < 0$ whenever $|x|$ is sufficiently large. We also discuss the extent of the continua in the (λ, u) plane; we show that the continuum of positive solutions bifurcating from $(\lambda_1, 0)$, where λ_1 is the principal eigenvalue, cannot cross $\lambda = 0$ and, under the additional hypothesis that $g \in L^p(\mathbb{R}^N)$ where $p < N/2$, that the continuum must extend to $\lambda = \infty$.

Finally in this introduction, we state the hypotheses which will be assumed throughout the paper.
(i) $N = 3, 4, 5$.
(ii) g is a smooth bounded function such that $g \in L^{N/2}(\mathbb{R}^N)$ and $g(x_0) > 0$, for some $x_0 \in \mathbb{R}^N$.
(iii) $f : [0, 1] \to \mathbb{R}^+$ is a smooth function such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(u) > 0$ for all $0 < u < 1$.

It is clear from the results in ([3]) that the cases $N \geq 3$ and $N < 3$, will be significantly different; we impose the restriction $N \leq 5$ to ensure the Frechet differentiability of certain operators and the applicability of *apriori* estimates.

Since u represents a relative frequency in the population genetics model, it is natural that restrictions should be placed on f on the interval $[0,1]$ only. The domain of definition of f, however, can be extended to all of $\mathbb{R}$ in such a way that $f(u) < 0$, whenever $u < 0$ or $u > 1$ and f, f' and f'' are uniformly bounded in $\mathbb{R}$. We shall assume throughout that f has been so extended but shall eventually prove the existence of solutions u with $0 \leq u \leq 1$.

2 Existence of positive principal eigenvalues

In this section we shall discuss the existence of a positive principal eigenvalue for the problem

$$-\Delta u = \lambda g(x) u, \qquad x \in \mathbb{R}^N, \tag{2.1$_\lambda$}$$

$$\lim_{|x|\to+\infty} u(x) = 0. \tag{2.2}$$

As we shall see below the corresponding principal eigenfunction will be in the space $\mathcal{D}^{1,2}$, i.e., the closure of the C_0^∞ functions with respect to the "energy" norm $\int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx$. It can be shown that

$$\mathcal{D}^{1,2} = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \bigtriangledown u \in L^2(\mathbb{R}^N) \right\}$$

and that $\mathcal{D}^{1,2}$ can be embedded continuously in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. For more information on the properties of this space see [9] and the references therein. Our approach is based on the following inequality.

Lemma 2.1 *Suppose $g \in L^{N/2}(\mathbb{R}^N)$. Then there exists $\alpha > 0$ such that*

$$\int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx \geq \alpha \int_{\mathbb{R}^N} |g| u^2 dx. \tag{2.3}$$

for all $u \in C_0^\infty$.

Proof Since $\mathcal{D}^{1,2}$ can be embedded continuously in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, there exists $k > 0$ such that, for all $u \in C_0^\infty$,

$$||u||_{\frac{2N}{N-2}} \leq k ||u||_{\mathcal{D}^{1,2}}.$$

Thus, if $u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we have

$$\begin{aligned} \textstyle\int_{\mathbb{R}^N} |g|u^2 dx \ &\leq \{\textstyle\int_{\mathbb{R}^N} |g|^{N/2} dx\}^{2/N} \ \{\textstyle\int_{\mathbb{R}^N} u^{\frac{2N}{N-2}} dx\}^{\frac{N-2}{N}} \\ &= ||g||_{N/2} \ ||u||^2_{\frac{2N}{N-2}} \end{aligned}$$

and so

$$\int_{\mathbb{R}^N} |g|u^2 dx \leq k^2 ||g||_{N/2} \ ||u||^2_{\mathcal{D}^{1,2}}$$

which completes the proof.$\diamond$

Thus, if $g \in L^{N/2}(\mathbb{R}^N)$ and $\alpha > 0$ is as in Lemma 2.1, then we can define an inner product on $C_0^\infty(\mathbb{R}^N)$ by

$$< u, v >= \int_{\mathbb{R}^N} \nabla u \ \nabla v \, dx - \frac{\alpha}{2} \int_{\mathbb{R}^N} guv \, dx.$$

As in ([4]) we define $\mathcal{V}$ to be the completion of C_0^∞ with respect to the above inner product. The space $\mathcal{V}$ would seem to depend on the function g; we might expect $\mathcal{V}$ to grow as $|g|$ becomes smaller at infinity. In fact we have

Lemma 2.2 *Suppose* $g \in L^{N/2}(\mathbb{R}^N)$. *Then* $\mathcal{V} = \mathcal{D}^{1,2}$.

Proof By standard density arguments it suffices to compare the $\mathcal{V}$ and $\mathcal{D}^{1,2}$ norms on $C_0^\infty(\mathbb{R}^N)$. For all $u \in C_0^\infty(\mathbb{R}^N)$ we have

$$||u||^2_{\mathcal{V}} = \int_{\mathbb{R}^N} | \nabla u|^2 dx - \frac{\alpha}{2} \int_{\mathbb{R}^N} gu^2 dx.$$

By Lemma 2.1

$$|\frac{\alpha}{2} \int_{\mathbb{R}^N} g(x)u^2 dx| \leq \frac{1}{2} \int_{\mathbb{R}^N} | \nabla u|^2 dx,$$

and so we must have

$$\frac{1}{2} \int_{\mathbb{R}^N} | \nabla u|^2 dx \ \leq \ ||u||^2_{\mathcal{V}} \ \leq \ \frac{3}{2} \int_{\mathbb{R}^N} | \nabla u|^2 dx$$

i.e.

$$\frac{1}{2} ||u||^2_{\mathcal{D}^{1,2}} \ \leq \ ||u||^2_{\mathcal{V}} \ \leq \ \frac{3}{2} ||u||^2_{\mathcal{D}^{1,2}}.$$

Hence it follows that $\mathcal{V} = \mathcal{D}^{1,2}$.$\diamond$

Since we assume throughout that $g \in L^{N/2}(\mathbb{R}^N)$, we shall have that $\mathcal{V} = \mathcal{D}^{1,2}$. Thus we may henceforth suppose that $||.||_{\mathcal{V}}$, the norm in $\mathcal{V}$, coincides with the norm in $\mathcal{D}^{1,2}$ and that the inner product in $\mathcal{V}$ is given by

$$< u, v >= \int_{\mathbb{R}^N} \nabla u \nabla v dx.$$

Proceeding as in ([4]) we define a bilinear form by

$$\beta(u, v) = \int_{\mathbb{R}^N} guvdx$$

for all $u, v \in \mathcal{V}$. Since $\mathcal{V} \subseteq L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we have

$$\begin{aligned} \beta(u, v) &= \textstyle\int_{\mathbb{R}^N} guvdx \\ &\leq ||g||_{N/2} \, ||u||_{\frac{2N}{N-2}} \, ||v||_{\frac{2N}{N-2}} \\ &\leq k^2 ||g||_{N/2} \, ||u||_{\mathcal{V}} \, ||v||_{\mathcal{V}} \end{aligned}$$

for all $u, v \in \mathcal{V}$ and so β is bounded. Hence by the Riesz Representation Theorem we can define a bounded linear operator L such that

$$\beta(u, v) =< Lu, v > \text{ for all } u, v \in \mathcal{V}.$$

It is easy to check that L is selfadjoint.

Lemma 2.3 *L is compact.*

Proof Let $\{u_n\}$ be a bounded sequence in $\mathcal{V}$. Then for all positive integers m and n we have

$$\begin{aligned} ||Lu_n - Lu_m||^2_{\mathcal{V}} &= \beta(u_n - u_m, Lu_n - Lu_m) \\ &= \int_{\mathbb{R}^N} g(u_n - u_m)(Lu_n - Lu_m)dx \\ &\leq \; ||g(u_n - u_m)||_{\frac{2N}{N+2}} \; ||Lu_n - Lu_m||_{\frac{2N}{N-2}} \\ &\leq \; k \; ||g(u_n - u_m)||_{\frac{2N}{N+2}} \; ||Lu_n - Lu_m||_{\mathcal{V}}, \end{aligned}$$

and so

$$||Lu_n - Lu_m||_{\mathcal{V}} \; \leq \; k \; ||g(u_n - u_m)||_{\frac{2N}{N+2}}. \tag{2.4}$$

Since $\{u_n\}$ is a bounded sequence in $\mathcal{V}$, $\{u_n\}$ is bounded in $H^1(B_R)$ for every ball $B_R = \{x \in \mathbb{R}^N : ||x|| \leq R\}$. Hence $\{u_n\}$ has a convergent subsequence in $L^2(B_R)$ and so in $L^{\frac{2N}{N+2}}(B_R)$. Thus by a diagonalization procedure we can find a subsequence, for convenience again denoted by $\{u_n\}$, which converges in $L^{\frac{2N}{N+2}}(B_R)$ for all $R > 0$. By using (2.4) we shall show that $\{Lu_n\}$ is a Cauchy sequence in $\mathcal{V}$.
Let $\varepsilon > 0$. Now

$$\begin{aligned} ||g(u_n - u_m)||_{\frac{2N}{N+2}} &= \left\{ \int_{|x| \leq R} + \int_{|x|>R} \; |g(u_n - u_m)|^{\frac{2N}{N+2}} \; dx \right\}^{\frac{N+2}{2N}} \\ &\leq ||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(B_R)} + ||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(\mathbb{R}^N - B_R)}. \end{aligned}$$

We have that

$$||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(\mathbb{R}^N - B_R)} \leq ||g||_{L^{\frac{N}{2}}(\mathbb{R}^N - B_R)} \; ||(u_n - u_m)||_{L^{\frac{2N}{N-2}}(\mathbb{R}^N - B_R)}.$$

Since $\{u_n\}$ is a bounded sequence in $\mathcal{V}$ and so in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, $||(u_n - u_m)||_{L^{\frac{2N}{N-2}}(\mathbb{R}^N - B_R)}$ is uniformly bounded. Also, as $g \in L^{N/2}(\mathbb{R}^N)$, we can make $||g||_{L^{N/2}(\mathbb{R}^N - B_R)}$ as small as we please by choosing R sufficiently large. Thus there exists $R_0 > 0$ such that

$$||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(\mathbb{R}^N - B_R)} < \varepsilon \quad \text{for all} \quad m, n \text{ if } \quad R \geq R_0.$$

But g is bounded on B_{R_0} and $\{u_n\}$ is convergent on $L^2(B_{R_0})$ and so in $L^{\frac{2N}{N+2}}(B_{R_0})$. Since

$$||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(B_{R_0})} \leq ||g||_{L^\infty(B_{R_0})} \, ||(u_n - u_m)||_{L^{\frac{2N}{N+2}}(B_{R_0})},$$

we have that $||g(u_n - u_m)||_{L^{\frac{2N}{N+2}}(B_{R_0})} < \varepsilon$ provided that m and n are sufficiently large.
Thus $\{Lu_n\}$ is a Cauchy sequence in $\mathcal{V}$ and the proof is complete.$\diamond$

We can now prove the existence of a positive principal eigenvalue.

Theorem 2.4 *Suppose that* $g \in L^{\frac{N}{2}}(\mathbb{R}^N)$ *and there exists* $x_0 \in \mathbb{R}^N$ *such that* $g(x_0) > 0$. *Then* L *has a positive principal eigenvalue.*

Proof Since L is compact and selfadjoint, the largest eigenvalue μ_1 of L is given by

$$\mu_1 = sup_{u\in\mathcal{V}} \frac{< Lu, u >}{< u, u >} = sup_{u\in\mathcal{V}} \frac{\int_{\mathbb{R}^N} g u^2 dx}{\int_{\mathbb{R}^N} | \bigtriangledown u|^2 dx}.$$

Suppose g is positive on an open set G containing x_0. Then we can find $\psi \in \mathcal{V}$ with support in G. Hence

$$\mu_1 \geq \frac{\int_{\mathbb{R}^N} g \psi^2 dx}{\int_{\mathbb{R}^N} | \bigtriangledown \psi|^2 dx} > 0.$$

A positive eigenfunction ϕ corresponding to μ_1 can be constructed as in ([3]) and so μ_1 is a positive principal eigenvalue of L.$\diamond$

Clearly ϕ is a weak solution of $(2.1)_\lambda$ with $\lambda = \lambda_1 = 1/\mu_1$ and standard regularity results show that ϕ is also a classical solution of $(2.1)_{\lambda_1}$. The results of the next section will show that ϕ also satisfies (2.2).

3 Asymptotic properties of solutions

In this section we shall derive asymptotic properties of the solutions in $\mathcal{D}^{1,2}$ of the nonlinear problem $(1.1)_\lambda$ - (1.2) and the linear eigenvalue problem $(2.1)_\lambda$ - (2.2).

We note first that it follows from standard regularity results that any weak solution in $\mathcal{D}^{1,2}$ of $(1.1)_\lambda$ or $(2.1)_\lambda$ is also a classical solution. Our results will be based on the following *apriori* estimates which are simple consequences of Theorem 8.17 in ([8]).

Theorem 3.1 *Let $q : \mathbb{R}^N \to \mathbb{R}$ be a smooth bounded function and let $h \in L^s(\mathbb{R}^N)$ be a smooth function with $s > N$. Suppose $R > 0$ and $p > 1$. Then there exists $C > 0$ (C depends only on N, R, p and $||q||_\infty$) such that for all solutions $u \in W^{1,2}_{loc}$ of*

$$-\Delta u(x) + q(x)u(x) = h(x), \qquad x \in \mathbb{R}^N, \tag{3.1}$$

we have

$$\sup_{y \in B_R(x)} |u(y)| \leq C\{R^{-N/p}||u||_{L_p(B_{2R}(x))} + R^{2\delta}||h||_{s/2}\},$$

for all $y \in \mathbb{R}^N$ where $\delta = 1 - (N/s)$.

First we prove that solutions must tend to zero as $|x| \to \infty$.

Theorem 3.2 *Suppose that $u \in \mathcal{D}^{1,2}$ is a solution of $(1.1)_\lambda$ or $(2.1)_\lambda$. Then* $\lim_{|x|\to\infty} |u(x)| = 0$.

Proof Suppose that $u \in \mathcal{D}^{1,2}$ is a solution of $(1.1)_\lambda$ - a similar, simpler proof holds when u is a solution of $(2.1)_\lambda$. Then u is a solution of (3.1) with $h(x) = 0$ and $q(x) = \lambda g(x)\frac{f(u(x))}{u(x)}$ so that q is smooth and bounded. Hence choosing $p = \frac{2N}{N-2}$ and $R = 1$ in Theorem 3.1, there exists $C > 0$ such that

$$|u(x)| \leq \sup_{y \in B_1(x)} |u(y)| \leq C||u||_{L_p(B_2(x))}, \tag{3.2}$$

where C depends on N and $||q||_\infty$. Since $u \in \mathcal{D}^{1,2}$, $u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and so $\lim_{|x|\to\infty} |u(x)| = 0$.$\diamond$

The next lemma is needed to deal with the boundary terms which arise when we integrate by parts.

Lemma 3.3 *Suppose that* $u \in \mathcal{D}^{1,2}$ *is a solution of (1.1)*$_\lambda$ *or (2.1)*$_\lambda$. *Then* $\lim_{R\to\infty} \int_{\partial B_R} u \frac{\partial u}{\partial n} dS = 0$.

Proof Suppose that u is a solution of $(2.1)_\lambda$. Multiplying both sides of $(2.1)_\lambda$ by u and integrating over B_R, we obtain

$$\int_{B_R} |\bigtriangledown u|^2 dx - \int_{\partial B_R} u \frac{\partial u}{\partial n}\, dS = \lambda \int_{B_R} g u^2\, dx.$$

Since $|\bigtriangledown u| \in L^2(\mathbb{R}^N)$ and $gu^2 \in L^1((\mathbb{R}^N)$, it follows that $\lim_{R\to\infty} \int_{\partial B_R} u \frac{\partial u}{\partial n}\, dS$ exists. Also

$$|\int_{\partial B_R} u \frac{\partial u}{\partial n}\, dS|^2 \le (\frac{1}{R}\int_{\partial B_R} u^2 dS)\ (R \int_{\partial B_R} |\bigtriangledown u|^2\, dS). \tag{3.3}$$

Since $|\bigtriangledown u| \in L^2(\mathbb{R}^N)$ and $u \in L^{2N/(N-2)}((\mathbb{R}^N)$, the integral

$$\int_0^\infty \{\int_{\partial B_R} [|\bigtriangledown u|^2 + u^{2N/(N-2)}] dS\}\, dR \ \text{ converges}$$

and so we can find a sequence $\{R_n\}$, with $R_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} R_n \int_{\partial B_{R_n}} |\bigtriangledown u|^2\, dS = 0 = \lim_{n\to\infty} R_n \int_{\partial B_{R_n}} u^{2N/(N-2)}\, dS.$$

Then

$$\frac{1}{R_n}\int_{\partial B_{R_n}} u^2 dS \le \frac{1}{R_n}\{\int_{\partial B_{R_n}} u^{2N/(N-2)}\, dS\}^{\frac{N-2}{N}}\ \{\int_{\partial B_{R_n}} dS\}^{\frac{2}{N}}$$

$$\le k R_n^{-1} (R_n^{N-1})^{\frac{2}{N}}\ \{\int_{\partial B_{R_n}} u^{2N/(N-2)} dS\}^{\frac{N-2}{N}} \ \text{ (for some constant } k)$$

$$= k\{R_n \int_{\partial B_{R_n}} u^{2N/(N-2)} dS\}^{1-\frac{2}{N}} \quad \to \quad 0 \quad \text{as} \quad n \to \infty.$$

Hence by (3.3), $\lim_{n\to\infty} \int_{\partial B_{R_n}} u \frac{\partial u}{\partial n} dS = 0$ and so $\lim_{R\to\infty} \int_{\partial B_R} u \frac{\partial u}{\partial n} dS = 0$. A similar argument applies if u satisfies $(1.1)_\lambda$.$\diamond$

Finally in this section we show that the principal eigenvalue of $(2.1)_\lambda$ and (2.2) is unique and is of multiplicity 1.

Theorem 3.4 *Suppose that ϕ is a positive eigenfunction of (2.1)$_{\lambda_1}$ corresponding to the principal eigenvalue λ_1 and that $u \in \mathcal{D}^{1,2}$ is also a positive eigenfunction of (2.1)$_\lambda$ corresponding to an eigenvalue $\lambda > 0$. Then $\lambda = \lambda_1$ and $u = c\phi$, for some constant $c > 0$.*

Proof Multiplying both sides of $(2.1)_{\lambda_1}$ by ϕ and integrating over B_R, we obtain

$$\int_{B_R} |\bigtriangledown \phi|^2\,dx - \int_{\partial B_R} \phi \frac{\partial \phi}{\partial n}\,dS = \lambda_1 \int_{B_R} g\phi^2\,dx. \tag{3.4}$$

Letting $R \to \infty$ and using Lemma 3.3 we get

$$\int_{\mathbb{R}^N} |\bigtriangledown \phi|^2\,dx = \lambda_1 \int_{\mathbb{R}^N} g\phi^2\,dx > 0. \tag{3.5}$$

Multiplying by $\frac{\phi^2}{u}$ the equation $(2.1)_\lambda$ satisfied by u and integrating over B_R, we obtain

$$2\int_{B_R} \frac{\phi}{u} \bigtriangledown\phi \bigtriangledown u\,dx - \int_{B_R} \frac{\phi^2}{u^2} |\bigtriangledown u|^2\,dx - \int_{\partial B_R} \frac{\phi^2}{u}\frac{\partial u}{\partial n}\,dS = \lambda \int_{B_R} g\phi^2\,dx. \tag{3.6}$$

Subtracting (3.6) from (3.4) we get

$$\begin{aligned} &\textstyle\int_{B_R} |\bigtriangledown \phi - \frac{\phi}{u} \bigtriangledown u|^2\,dx - \int_{\partial B_R} \phi \frac{\partial \phi}{\partial n}\,dS \\ &\qquad + \textstyle\int_{\partial B_R} \frac{\phi^2}{u}\frac{\partial u}{\partial n}\,dS = (\lambda_1 - \lambda) \int_{B_R} g\phi^2\,dx. \end{aligned} \tag{3.7}$$

Letting $R \to \infty$ in (3.7), using (3.5) and the fact that λ_1 is the smallest positive eigenvalue of $(2.1)_\lambda$ so that $\lambda_1 \le \lambda$, we obtain that

$$\lim_{R\to\infty} \{\int_{B_R} |\bigtriangledown \phi - \frac{\phi}{u} \bigtriangledown u|^2\,dx + \int_{\partial B_R} \frac{\phi^2}{u}\frac{\partial u}{\partial n}\,dS\}$$

exists and is nonpositive.

We shall prove that $\lim_{R\to\infty}\{\int_{\partial B_R} \frac{\phi^2}{u}\frac{\partial u}{\partial n}\,dS\} = 0$. Let

$$p(R) = \int_{\partial B_R} \frac{\phi^2}{u}\frac{\partial u}{\partial n}\,dS.$$

Since $\int_{B_R} | \bigtriangledown \phi - \frac{\phi}{u} \bigtriangledown u|^2 \, dx$ is a nondecreasing function of R, it follows that either $\lim_{R\to\infty} p(R)$ exists and $\lim_{R\to\infty} p(R) < 0$ or that $p(R) \to -\infty$ as $R \to \infty$.

Suppose $\lim_{R\to\infty} p(R)$ exists and $\lim_{R\to\infty} p(R) < 0$. It follows that $\lim_{R\to\infty} \int_{B_R} | \bigtriangledown \phi - \frac{\phi}{u} \bigtriangledown u|^2 \, dx$ must exist and so, since $\bigtriangledown\phi \in L^2(\mathbb{R}^N)$, that $\lim_{R\to\infty} \int_{B_R} \frac{\phi^2}{u^2} | \bigtriangledown u|^2 dx$ exists. Hence, if

$$q(R) = \int_{B_R} \frac{\phi^2}{u^2} | \bigtriangledown u|^2 dx$$

there exists $R_1 > 0$ and $\theta \in (0,1)$ such that

$$-p(R) \geq \theta q(R), \quad \text{for all} \quad R \geq R_1. \tag{3.8}$$

Suppose now that $p(R) \to -\infty$ as $R \to \infty$. Then $\lim_{R\to\infty} \int_{B_R} |\bigtriangledown\phi - \frac{\phi}{u} \bigtriangledown u|^2 dx$ does not exist and so we must have that $q(R) \to \infty$ as $R \to \infty$. Hence (3.8) also holds in this case.
Suppose now that $\lim_{R\to\infty} p(R) \neq 0$ so that (3.8) holds; we shall obtain a contradiction. Let

$$H(R) = \int_{\partial B_R} \phi^2 dS.$$

Then we have the following

$$\begin{aligned} -p(R) &\leq \int_{\partial B_R} \frac{\phi^2}{u} | \bigtriangledown u| \, dS \\ &\leq \{\int_{\partial B_R} \phi^2 \, dS\}^{1/2} \{\int_{\partial B_R} \frac{\phi^2}{u^2} | \bigtriangledown u|^2 \, dS\}^{1/2} \end{aligned}$$

and so

$$-p(R) \leq \{H(R)\}^{1/2} \{\int_{\partial B_R} \frac{\phi^2}{u^2} | \bigtriangledown u|^2 \, dS\}^{1/2}. \tag{3.9}$$

Now

$$q(R) = \int_0^R \{\int_{\partial B_r} \frac{\phi^2}{u^2} | \bigtriangledown u|^2 \, dS\} \, dr$$

and so by (3.9)

$$\begin{aligned} q'(R) &= \int_{\partial B_R} \frac{\phi^2}{u^2} | \bigtriangledown u|^2 \, dS \\ &\geq \frac{[p(R)]^2}{H(R)} \geq \theta^2 \frac{[q(R)]^2}{H(R)} \ \text{ for all } \ R \geq R_1. \end{aligned}$$

Hence

$$\frac{d}{dR}\{\frac{1}{q(R)}\} + \frac{\theta^2}{H(R)} \leq 0$$

which implies

$$\frac{1}{q(R)} - \frac{1}{q(R_1)} + \theta^2 \int_{R_1}^{R} \frac{dr}{H(r)} \leq 0 \quad \text{for all} \quad R \geq R_1.$$

Thus we have a contradiction provided we can prove that $\int_{R_1}^{\infty} \frac{dr}{H(r)}$ is divergent. A straightforward argument used in proof of Lemma 3.3 shows that there exists a constant $k > 0$ such that for any smooth function u we have

$$\int_{\partial B_R} u^2 \, dS \leq kR^{2-\frac{2}{N}} \{\int_{\partial B_R} u^{\frac{2N}{N-2}} \, dS\}^{1-\frac{2}{N}}.$$

Since $\phi \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, $\int^{\infty}\{\int_{\partial B_r} \phi^{\frac{2N}{N-2}} \, dS\} \, dr$ converges and so

$$H(r) \leq kr^{2-\frac{2}{N}} \, \{I(r)\}^{1-\frac{2}{N}}.$$

where $I(r) = \int_{\partial B_r} \phi^{2N/(N-2)} dS$, i.e. $\int^{\infty} I(r) dr < \infty$. Thus

$$\frac{1}{H(r)} \geq \frac{1}{k} \, r^{\frac{2}{N}-2} \, \{I(r)\}^{\frac{2}{N}-1}.$$

But

$$\infty = \int^{\infty} \frac{dr}{r} = \int^{\infty} I(r)^{\alpha} \, \frac{1}{r} \, I(r)^{-\alpha} \, dr$$

$$\leq \{\int^{\infty} I(r)^{\alpha p} \, dr\}^{\frac{1}{p}} \, \{\int^{\infty} r^{-q} \, I(r)^{-\alpha q} \, dr\}^{\frac{1}{q}}$$

for any α, p, q such that $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now by choosing $q = 2 - \frac{2}{N}$, $p = \frac{2N-2}{N-2}$ and $\alpha = \frac{N-2}{2N-2}$ we obtain

$$\{\int^{\infty} I(r) \, dr\}^{\frac{1}{p}} \, \{\int^{\infty} r^{\frac{2}{N}-2} \, [I(r)]^{\frac{2}{N}-1} \, dr\}^{\frac{1}{q}} = \infty.$$

Hence $\int^{\infty} r^{\frac{2}{N}-2}[I(r)]^{\frac{2}{N}-1} \, dr = \infty$ and so $\int_{R_1}^{\infty} \frac{dr}{H(r)}$ is divergent. Thus we have obtained a contradiction.

Therefore $\lim_{R\to\infty} p(R) = 0$, i.e., $\lim_{R\to\infty}\{\int_{\partial B_R} \frac{\phi^2}{u} \frac{\partial u}{\partial n} \, dS\} = 0$. Hence $\lim_{R\to\infty} \int_{B_R} | \bigtriangledown \phi - \frac{\phi}{u} \bigtriangledown u|^2 \, dx = 0$ and so $\bigtriangledown\phi - \frac{\phi}{u} \bigtriangledown u = 0$ on $\mathbb{R}^N$, i.e., $u \bigtriangledown \phi - \phi \bigtriangledown u = 0$ on $\mathbb{R}^N$. Thus $\bigtriangledown(\frac{\phi}{u}) \equiv 0$ and so $u = c\phi$ for some constant c. Finally, letting $R \to \infty$ in (3.7), we see that $\lambda = \lambda_1$ and the proof is complete.$\diamond$

4 Bifurcation Results

In this section we shall obtain results on the existence of solutions for the nonlinear problem $(1.1)_\lambda$, (1.2), by considering bifurcation of solutions from the zero solution.
We define the nonlinear operator $T : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$ through the relation

$$< T(\lambda, u), \phi >= \int_{\mathbb{R}^N} \nabla u \nabla \phi \, dx - \lambda \int_{\mathbb{R}^N} g f(u) \phi \, dx, \tag{4.1}$$

for all $\phi \in \mathcal{V}$, where $<,>$ denotes the inner product in $\mathcal{V}$, i.e. in $\mathcal{D}^{1,2}$.

Lemma 4.1 *The operator T is well defined by (4.1).*

Proof For fixed $u \in \mathcal{D}^{1,2}$ consider the functional

$$F(\phi) = \int_{\mathbb{R}^N} \nabla u \nabla \phi \, dx - \lambda \int_{\mathbb{R}^N} g f(u) \phi \, dx$$

for $\phi \in \mathcal{D}^{1,2}$. Since $|f(u)| \leq K|u|$ for some constant K, $f(u) \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and so

$$\begin{aligned} |F(\phi)| &\leq || \nabla u||_2 \, || \nabla \phi||_2 + |\lambda| \, ||g||_{N/2} \, ||f(u)||_{\frac{2N}{N-2}} \, ||\phi||_{\frac{2N}{N-2}} \\ &\leq K_1(|| \nabla u||_2 + |\lambda| \, ||g||_{N/2} \, ||f(u)||_{\frac{2N}{N-2}}) \, ||\phi||_{\mathcal{V}} \end{aligned}$$

for some constant K_1 and so F is a bounded linear functional. Hence by the Riesz Representation Theorem we may define T as in (4.1). $\diamond$

It is straightforward to check that T is a continuous function and, using the fact that $N = 3, 4, 5$, that T is Frechet differentiable with continuous Frechet derivatives given by

$$< T_u(\lambda, u)\phi, \psi > = \int_{\mathbb{R}^N} \nabla \phi \nabla \psi \, dx - \lambda \int_{\mathbb{R}^N} g f'(u) \phi \psi \, dx,$$

$$< T_\lambda(\lambda, u), \phi > = - \int_{\mathbb{R}^N} g f(u) \phi \, dx,$$

$$< T_{\lambda u}(\lambda, u)\phi, \psi > = - \int_{\mathbb{R}^N} g f'(u) \phi \psi \, dx$$

for all $\phi, \psi \in \mathcal{D}^{1,2}$.

To simplify notation but without loss of generality we shall assume that $f'(0) = 1$ so that $(2.1)_\lambda$ becomes exactly the linearisation of $(1.1)_\lambda$. Consider the linear operator $T_u(\lambda_1, 0)$ where λ_1 is the principal eigenvalue of $(2.1)_\lambda$. It is easy to check that $T_u(\lambda_1, 0)$ is a bounded selfadjoint operator and that $T_u(\lambda_1, 0)\phi = 0$ if and only if $\phi \in \mathcal{V}$ is a solution of $(2.1)_{\lambda_1}$. Thus $N(T_u(\lambda_1, 0)) = [\phi]$ where ϕ is the principal eigenfunction of $(2.1)_{\lambda_1}$. Since $T_u(\lambda_1, 0)$ is selfadjoint, $R(T_u(\lambda_1, 0)) = [\phi]^\perp$, i.e., $\psi \in R(T_u(\lambda_1, 0))$ if and only if $< \psi, \phi >= 0$. Since, using (3.5),

$$< T_{\lambda u}(\lambda_1, 0)\phi, \phi >= - \int_{\mathbb{R}^N} g\, \phi^2 dx < 0,$$

we have that T satisfies all the hypotheses for the theorem on bifurcation from a simple eigenvalue (see [5]) and so the following result on local bifurcation holds.

Theorem 4.2 *There exists $\epsilon_0 > 0$ and continuous functions $\eta : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ and $\psi : (-\epsilon_0, \epsilon_0) \to [\phi]^\perp$ such that $\eta(0) = \lambda_1$, $\psi(0) = 0$ and every nontrivial solution of $T(\lambda, u) = 0$ in a small neighbourhood of $(\lambda_1, 0)$ is of the form $(\lambda_\epsilon, u_\epsilon) = (\eta(\epsilon), \epsilon\phi + \epsilon\psi(\epsilon))$.*

In order that a solution u of $T(\lambda, u) = 0$ is also a solution of $(1.1)_\lambda$, (1.2) it is necessary to ensure that $0 < u(x) < 1$ for all $x \in \mathbb{R}^N$. By (3.2) we have

$$|u_\epsilon(x)| \ \leq \ C \, ||u_\epsilon||_{L^{\frac{2N}{N-2}}(B_2(x))} \ \leq \ K \, ||u_\epsilon||_{\mathcal{V}}$$

for all $x \in \mathbb{R}^N$ where K is independent of ϵ. Thus, as $||u_\epsilon||_{\mathcal{V}} \to 0$ as $\epsilon \to 0$, it follows that $|u_\epsilon(x)| < 1$ for all $x \in \mathbb{R}^N$ provided ϵ is sufficiently small.

It seems harder to establish the positivity of solutions; in order to do so we must assume that g satisfies the condition

($\mathbf{G}^-$) $g(x_0) > 0$ for some $x_0 \in \mathbb{R}^n$ and there exists $R_0 > 0$ such that $g(x) < 0$ whenever $|x| > R_0$.

Theorem 4.3 *Suppose that g satisfies condition $(\mathbf{G}^-)$. Then there exists $\epsilon_1 > 0$ such that $u_\epsilon(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $0 < \epsilon < \epsilon_1$.*

Proof Since $u_\epsilon = \epsilon\phi + \epsilon\psi(\epsilon)$ satisfies $(1.1)_\lambda$ where $\lambda = \eta(\epsilon)$, we have

$$-\Delta\psi(\epsilon) = \eta(\epsilon) g \frac{f(\epsilon\phi + \epsilon\psi(\epsilon))}{\epsilon} - \lambda_1 g\phi.$$

A straightforward computation yields

$$-\Delta\psi(\epsilon) - q(x)\psi(\epsilon) = (\eta(\epsilon) - \lambda_1) g\phi + \frac{1}{2}\eta(\epsilon) g\phi f''(\xi(\epsilon, x)) u_\epsilon$$

where $q(x) = \eta(\epsilon) g[1 + \frac{1}{2} f''(\xi(\epsilon, x)) u_\epsilon]$ and $\xi(\epsilon, x)$ lies between 0 and $u_\epsilon(x)$. For sufficiently small $\epsilon > 0$ we have that $|u_\epsilon(x)| \leq 1$ for all x and so $||q||_\infty$ is uniformly bounded. Moreover

$$||g\phi||_{\frac{2N}{N-2}} \;\leq\; ||g||_\infty\, ||\phi||_{\frac{2N}{N-2}} \;\leq\; ||g||_\infty\, ||\phi||_{\mathcal{V}}$$

and

$$||g\phi u_\epsilon||_{\frac{2N}{N-2}} \;\leq\; ||g||_\infty\, ||\phi||_\infty\, ||u_\epsilon||_{\frac{2N}{N-2}} \;\leq\; ||g||_\infty\, ||\phi||_\infty\, ||u_\epsilon||_{\mathcal{V}}.$$

If $N = 3, 4, 5$ and $\frac{s}{2} = \frac{2N}{N-2}$, then $s > N$. Hence by Theorem 3.1 with $p = \frac{s}{2} = \frac{2N}{N-2}$, there exists a constant $K > 0$ (independent of ϵ) such that

$$\sup_{|x| \leq R_0} |\psi(\epsilon)(x)| \;\leq\; K\, ||g||_\infty\, \{(\eta(\epsilon) - \lambda_1)\, ||\phi||_{\mathcal{V}} + ||\phi||_\infty\, ||u_\epsilon||_{\mathcal{V}}\}.$$

Since $\phi(x) > 0$ for all x in the compact set $B_{R_0} = \{x \in \mathbb{R}^N : |x| \leq R_0\}$, it follows that there exists $\epsilon_1 > 0$ such that $\phi(x) + \psi(\epsilon)(x) > 0$ for all $x \leq R_0$ provided that $0 < \epsilon < \epsilon_1$.

Suppose $0 < \epsilon < \epsilon_1$ and that $u_\epsilon(x_0) < 0$ for some $x_0 \in \mathbb{R}^N$. Since $\lim_{|x|\to\infty} u_\epsilon(x) = 0$, it follows that there must exist x_1, $|x_1| > R$, such that u_ϵ attains a negative minimum at x_1. But then

$$-\Delta u_\epsilon(x_1) = \lambda g(x_1) f(u_\epsilon(x_1)) > 0$$

which is impossible. Hence $u_\epsilon(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $0 < \epsilon < \epsilon_1$.$\diamond$

We now discuss the global nature of the continuum of solutions bifurcating from $(\lambda_1, 0)$. It is easy to see that we can write the operator T as $T(\lambda, u) = u - \lambda S(u)$ where

$$< S(u), \phi >= \int_{\mathbb{R}^N} g(x) f(u(x)) \phi(x)\, dx \quad \text{for all } \phi \in \mathcal{V}.$$

Also we have that

$$S(u) = Lu + \mathcal{H}(u)$$

where L denotes the same linear operator as in section 2, i.e.,

$$< Lu, v >= \int_{\mathbb{R}^N} guv\, dx \text{ for all } u, v \in \mathcal{V}$$

and $\mathcal{H}(u) = O(||u||^2_{\mathcal{V}})$ as $||u||_{\mathcal{V}} \to 0$. We showed in section 2 that λ_1 is an eigenvalue of L and by Theorem 3.4 the eigenspace associated with λ_1 has algebraic multiplicity 1. Also, as f is a Lipschitz function, it can be proved by modifying slightly the proof of Lemma 2.3 that S is a compact operator.

Thus we can apply the classical result of Rabinowitz (see [10]) on global bifurcation to obtain the existence of a continuum $\mathcal{C}$ of nonzero solutions of $(1.1)_\lambda$, (1.2) bifurcating from $(\lambda_1, 0)$ which is either unbounded or contains a point $(\lambda, 0)$, where $\lambda \neq \lambda_1$ is an eigenvalue of L, i.e., λ is an eigenvalue of $(2.1)_\lambda$, (2.2). In addition $\mathcal{C}$ has a connected subset $\mathcal{C}^+ \subset \mathcal{C} - \{(\eta(\epsilon), u_\epsilon) : -\epsilon_0 \leq \epsilon \leq 0\}$ for some $\epsilon_0 > 0$ such that $\mathcal{C}^+$ also satisfies one of the above alternatives. Clearly, close to the bifurcation point $(\lambda_1, 0)$, $\mathcal{C}^+$ consists of the curve $\epsilon \to (\eta(\epsilon), u_\epsilon)$, $0 < \epsilon \leq \epsilon_0$. We now investigate the nature of solutions lying on $\mathcal{C}^+$. First we show that $\mathcal{C}^+$ is bounded below in λ.

Theorem 4.4 *There exists $\lambda_* > 0$ such that $\lambda > \lambda_*$ whenever $(\lambda, u) \in \mathcal{C}^+$.*

Proof Suppose $u \in \mathcal{V}$ is a solution of $(1.1)_\lambda$, (1.2). Multiplying equation $(1.1)_\lambda$ by u, integrating over $\mathbb{R}^N$ and using Lemma 3.3 gives

$$\begin{aligned} ||u||^2_{\mathcal{V}} &= \textstyle\int_{\mathbb{R}^N} |\bigtriangledown u|^2 dx = \lambda \int_{\mathbb{R}^N} g f(u) u dx \\ &\leq \lambda K ||g||_{N/2}\, ||u||^2_{\frac{2N}{N-2}}, \quad \text{where } |f(u)| \leq Ku \text{ for all } u, \\ &\leq \lambda K_1 ||g||_{N/2}\, ||u||^2_{\mathcal{V}} \end{aligned}$$

where K_1 is a constant and the result follows.$\diamond$

In order to proceed further we must first prove that solutions which are close in $\mathbb{R} \times \mathcal{V}$ are also close in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$; since $\mathcal{V}$ does not embed in $L^\infty(\mathbb{R}^N)$, this is not immediately obvious.

Lemma 4.5 *Suppose that $u_\lambda \in \mathcal{V}$ is a solution of (1.1)$_\lambda$, (1.2). Then there exist constants K_1 and K_2 such that*

$$|u_\lambda(x) - u_\mu(x)| \leq K_1 \, |\lambda - \mu| + K_2 \, ||u_\lambda - u_\mu||_{\mathcal{V}} \quad \textit{for all} \quad x \in \mathbb{R}^N$$

whenever μ is close to λ and $u_\mu \in \mathcal{V}$ is a solution of (1.1)$_\mu$.

Proof It is easy to see that

$$-\Delta(u_\lambda - u_\mu) = g\{\lambda f(u_\lambda) - \mu f(u_\mu)\}.$$

Hence by Theorem 3.1 there exists $C > 0$ such that

$$|u_\lambda(x) - u_\mu(x)| \;\leq\; \sup_{y \in B_1(x)} |u_\lambda(y) - u_\mu(y)|$$

$$\leq C \; \{||u_\lambda - u_\mu||_{L_p(B_2(x))} + ||g\,[\lambda f(u_\lambda) - \mu f(u_\mu)\,]||_p\}$$

(where $p = \frac{2N}{N-2}$ and so $2p = \frac{4N}{N-2} > N$ for $N = 3, 4, 5$)

$$\leq C_1 \; ||u_\lambda - u_\mu||_{\mathcal{V}} + C \; ||g||_\infty \; \{||(\lambda - \mu) f(u_\lambda)||_p + ||\mu[f(u_\lambda) - f(u_\mu)]||_p\}$$

$$\leq C_1 \; ||u_\lambda - u_\mu||_{\mathcal{V}} + C_2 \; |\lambda - \mu| \; ||g||_\infty \; ||u_\lambda||_{\mathcal{V}} + C_3 \; |\mu| \; ||g||_\infty \; ||u_\lambda - u_\mu||_{\mathcal{V}}$$

where C_1, C_2 and C_3 are constants and the result follows.$\diamond$

Theorem 4.6 *Suppose g satisfies condition* $(\mathbf{G}^-)$. *Then $0 < u(x) < 1$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+$.*

Proof Suppose that there exists $(\lambda, u) \in \mathcal{C}^+$ such that $u(x_0) < 0$ for some $x_0 \in \mathbb{R}^N$. By Theorem 4.3, $u(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+$ is close to $(\lambda_1, 0)$. Moreover by Lemma 4.5 points in $\mathcal{C}^+$ which are close in $\mathbb{R} \times \mathcal{V}$ must also be close in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. Hence there must exist $(\lambda_0, u_0) \in \mathcal{C}^+$ such that $u_0(x) \geq 0$ for all $x \in \mathbb{R}^N$ but $u_0(x_0) = 0$

for some $x_0 \in \mathbb{R}^N$ and in any neighbourhood of (λ_0, u_0) we can find a point $(\hat{\lambda}, \hat{u}) \in \mathcal{C}^+$ with $\hat{u}(x) < 0$ for some $x \in \mathbb{R}^N$. Let B denote any open ball containing x_0. Then

$$-\Delta u_0(x) - \lambda g(x)\frac{f(u_0(x))}{u_0(x)}u_0(x) = 0 \text{ on } B \text{ and } u_0(x) \geq 0 \text{ on } \partial B$$

It follows from the Serrin Maximum principle (see [7]) that $u_0 \equiv 0$ on B. Hence $u_0 \equiv 0$ on $\mathbb{R}^N$.

Thus we can construct a sequence $\{(\lambda_n, u_n)\} \subseteq \mathcal{C}^+$ such that $u_n(x) > 0$ for all $n \in \mathcal{N}$ and $x \in \mathbb{R}^N$, $u_n \to 0$ in $\mathcal{V}$ and $\lambda_n \to \lambda_0$. Let $v_n = \frac{u_n}{||u_n||_{\mathcal{V}}}$. Since

$$u_n = \lambda_n L(u_n) + \lambda_n \mathcal{H}(u_n)$$

we have

$$v_n = \lambda_n L(v_n) + \lambda_n \frac{\mathcal{H}(u_n)}{||u_n||_{\mathcal{V}}}.$$

Since L is compact, there exists a subsequence of $\{v_n\}$ (which we again denote by $\{v_n\}$) such that $\{L(v_n)\}$ is convergent. Since $\lim_{n\to\infty} \frac{\mathcal{H}(u_n)}{||u_n||_{\mathcal{V}}} = 0$, $\{v_n\}$ is convergent to v_0, say, and $v_0 = \lambda_0 L(v_0)$. Since $v_n \geq 0$ for all $n \in \mathcal{N}$, $v_0 \geq 0$. Since by Theorem 3.4 λ_1 is the only positive eigenvalue corresponding to a positive eigenfunction, it follows that $\lambda_1 = \lambda_0$. Thus $(\lambda_0, u_0) = (\lambda_1, 0)$ and this contradicts the fact that every neighbourhood of (λ_0, u_0) must contain a solution $(\hat{\lambda}, \hat{u}) \in \mathcal{C}^+$ with $\hat{u}(x) < 0$, for some $x \in \mathbb{R}^N$. Hence $u(x) > 0$ for all $x \in \mathbb{R}^N$ whenever $(\lambda, u) \in \mathcal{C}^+$.

Suppose that there exists $(\lambda, u) \in \mathcal{C}^+$ with $u(x_1) > 1$ for some $x_1 \in \mathbb{R}^N$. Then there must exist $(\lambda_0, u_0) \in \mathcal{C}^+$ such that $u_0(x) \leq 1$ for all $x \in \mathbb{R}^N$ and $u_0(x_0) = 1$ for some $x_0 \in \mathbb{R}^N$. If $v_0 = 1 - u_0$, then $v_0(x) \geq 0$ for all $x \in \mathbb{R}^N$, $v_0(x_0) = 0$ and

$$-\Delta v_0 = \lambda \hat{g}(x)\hat{f}(v_0(x))$$

where $\hat{g}(x) = -g(x)$ and $\hat{f}(v) = f(1-v)$. The same maximum principle argument as used above shows that $v_0 \equiv 0$ and so $u_0 \equiv 1$ on $\mathbb{R}^N$. But as $u_0 \in \mathcal{V}$ this is impossible and so the proof is complete.$\diamond$

The following result is a consequence of an argument very similar to that used in the first part of the proof above.

Corollary 4.7 *$\mathcal{C}^+$ contains no points of the form $(\lambda, 0)$, where $\lambda \neq \lambda_1$.*

Thus $\mathcal{C}^+$ must connect $(\lambda_1, 0)$ to ∞ in $\mathbb{R} \times \mathcal{V}$. The next theorem shows that $\mathcal{C}^+$ cannot become unbounded at a finite value of λ; in order to prove the result we must strengthen slightly the hypothesis on g.

Theorem 4.8 *Suppose g satisfies the hypotheses of Theorem 4.6 and $g \in L^p(\mathbb{R}^N)$, where $p < \frac{N}{2}$. Then there exists a continuous function $\mathcal{K} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $||u||_{\mathcal{V}} \leq \mathcal{K}(\lambda)$ whenever $(\lambda, u) \in \mathcal{C}^+$.*

Proof As in the proof of Theorem 4.4 we obtain that if u satisfies $(1.1)_\lambda$, (1.2), then

$$\begin{aligned} ||u||_{\mathcal{V}}^2 &= \lambda \textstyle\int_{\mathbb{R}^N} g f(u) u dx \\ &\leq \lambda K \textstyle\int_{\mathbb{R}^N} |g| u^2 dx. \end{aligned}$$

Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then as $p < \frac{N}{2}, q > \frac{N}{N-2}$ and so we can choose β, $0 < \beta < 2$, such that $\beta q = \frac{2N}{N-2}$. Hence

$$\begin{aligned} ||u||_{\mathcal{V}}^2 &\leq \lambda\, K\, ||u||_\infty^{2-\beta} \textstyle\int_{\mathbb{R}^N} |g| u^\beta dx \\ &\leq \lambda\, K\, ||u||_\infty^{2-\beta}\, ||g||_p \,\{\textstyle\int_{\mathbb{R}^N} u^{\beta q} dx\}^{1/q} \\ &\leq \lambda\, K\, ||g||_p\, ||u||_{\frac{2N}{N-2}}^{\frac{2N}{q(N-2)}} \leq \lambda\, K_1\, ||g||_p\, ||u||_{\mathcal{V}}^\beta. \end{aligned}$$

for some constant K_1 where we have used the fact that $|u(x)| < 1$ for all $x \in \mathbb{R}^N$ and so the proof is complete.◇

As an immediate consequence of the previous results we can give the following description of the continuum $\mathcal{C}^+$.

Theorem 4.9 *Suppose that $g \in L^p(\mathbb{R}^N)$, where $1 < p < \frac{N}{2}$ and satisfies condition $(\mathbf{G}^-)$. Then there exists a continuum $\mathcal{C}^+ \subseteq \mathbb{R} \times \mathcal{V}$ of solutions bifurcating from the zero solution at $(\lambda_1, 0)$ such that*

(i) if $(\lambda, u) \in \mathcal{C}^+$ *then* $\lambda > 0$ *and* $0 < u(x) < 1$ *for all* $x \in \mathbb{R}^N$*,*
(ii) $\{\lambda : (\lambda, u) \in \mathcal{C}^+$ *for some* $u \in \mathcal{V}\} \supseteq (\lambda_1, \infty]$*.*
In particular (1.1)$_\lambda$*, (1.2) has a nontrivial solution* $u \in \mathcal{V}$ *such that* $0 < u(x) < 1$ *for all* $x \in \mathbb{R}^N$ *whenever* $\lambda > \lambda_1$*.*

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