Sub and Supersolutions for Semilinear Elliptic Equations on all of R^n

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This paper is dedicated to the memory of Peter Hess.

Abstract

Sub and supersolutions are constructed for the semilinear elliptic equation $-\Delta u = \lambda g(x) f(u)$ on all of $\mathcal{R}^n$ which arises in population genetics. It is shown that the theory of existence of solutions is very different in the case $n = 1$ or 2 and in the case $n \geq 3$.

1 Introduction

In this paper we shall discuss the construction of sub- and supersolutions as well as the existence and nonexistence of solutions of the equation

$$-\Delta u = \lambda g(x) f(u), \quad 0 < u < 1, \quad x \in \mathcal{R}^n, \qquad (1.1)_\lambda$$

which arises in population genetics (see [1, 3]). The unknown function u corresponds to the relative frequency of an allele and is hence constrained to have values between 0 and 1. The real parameter $\lambda > 0$ corresponds to the reciprocal of a diffusion coefficient so that the case of small λ considered in

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this paper corresponds to diffusion being large. We assume throughout that g satisfies

$$(\mathbf{G}) \quad \begin{array}{l} g : \mathcal{R}^n \to \mathcal{R} \quad \text{is smooth such that } g(x_0) > 0 \quad \text{for some } \ x_0 \in \mathcal{R}^n \\ \text{and there exists } \ R_0 > 0 \ \text{ such that } \ g(x) < 0 \text{ whenever } \ |x| > R_0. \end{array}$$

This assumption corresponds to the fact that an allele has an advantage at some points x in $\mathcal{R}^n$ (where $g(x) > 0$), but is disadvantaged for $|x| > R_0$. In the population genetics model f is considered to be the cubic function $f(u) = u(1-u)[h(1-u) + (1-h)u]$, for some constant h, $0 < h < 1$. We shall assume throughout that f satisfies the condition

$$(\mathbf{F}) \quad \begin{array}{l} f : [0,1] \to \mathcal{R} \ \text{ is a smooth function such that } \ f(0) = f(1) = 0, \\ f'(0) > 0, f'(1) < 0, \quad \text{and } \ f(u) > 0 \ \text{ for all } \quad 0 < u < 1. \end{array}$$

We shall extend and unify existing results on solutions of $(1.1)_\lambda$ (see [1, 2, 3, 6, 7]) by constructing some new sub- and supersolutions for the problem. The results obtained show that the existence theory for solutions of $(1.1)_\lambda$ is very different in the case $n = 1, 2$ and the case $n \geq 3$. It is easy to show (see [1]) that sub- and supersolutions always exist for $(1.1)_\lambda$ when λ is sufficiently large. We shall be interested in constructing such solutions when λ is small and as a means of doing so we first construct numbers $lambda^\star, \lambda_\star \geq 0$ such that a supersolution (respectively subsolution) of $(1.1)_\lambda$ exists provided that $\lambda > \lambda^\star$ (respectively $\lambda > \lambda_\star$). First we recall how a class of subsolutions is constructed in [1]. Let $R > 0$ and let $\underline{u}$ be defined by

$$\underline{u}(x) = \begin{cases} \varphi(x), & \text{if} \quad |x| \leq R \\ 0, & \text{if} \quad |x| > R \end{cases},$$

where φ is a positive solution of

$$\begin{aligned} -\Delta\varphi(x) = &\ \lambda g(x) f(\varphi(x)), \qquad \text{if} \ \ |x| < R \\ \varphi(x) = &\ 0, \qquad \text{if} \ \ |x| = R. \end{aligned}$$

Such a solution φ exists provided $\lambda > \lambda_1(R)$ where $\lambda_1(R)$ is the principal eigenvalue of the problem

$$\begin{aligned} -\Delta\psi(x) = &\ \lambda g(x) f'(0)\psi(x), \qquad \text{if} \ \ |x| < R \\ \psi(x) = &\ 0, \qquad \text{if} \ \ |x| = R. \end{aligned}$$

Since

$$\lambda_1(R) = \inf \left\{ \frac{\int_{B_R} | \bigtriangledown u(x)|^2 dx}{\int_{B_R} g(x) f'(0) u^2(x) dx} : \quad u \in H_0^1(B_R), \quad \int_{B_R} g(x) u^2(x) dx > 0 \right\}$$

where $B_R = \{x \in \mathcal{R}^n : |x| \leq R\}$, $\lambda_1(R)$ is a decreasing function of R and we define $\lambda_* = \lim_{R\to\infty} \lambda_1(R)$. Thus if $\lambda > \lambda_\star$ we can choose R such that $\lambda > \lambda_1(R)$, and then construct a subsolution $\underline{u}(x)$ as described above. In particular when $\lambda_\star = 0$ we can construct such a subsolution for any $\lambda > 0$.

We now construct supersolutions which are identically equal to 1 on an exterior domain in a similar way. If we define

$$\bar{u}(x) = \begin{cases} 1 - \varphi(x) & \text{if} \quad |x| \leq R \\ 1 & \text{if} \quad |x| > R \end{cases} ,$$

then $\bar{u}(x)$ is a supersolution of $(1.1)_\lambda$ if φ is a positive solution of

$$\begin{aligned} -\Delta\varphi(x) = & \quad \lambda h(x) \hat{f}(\varphi(x)), & \text{if} \quad |x| < R \\ \varphi(x) = & \quad 0, & \text{if} \quad |x| = R, \end{aligned}$$

where $h(x) = -g(x)$ and $\hat{f}(u) = f(1-u)$. Note that $\hat{f}'(0) = -f'(1) > 0$. Using an argument very similar to that used above, it can be shown that such a supersolution can be constructed provided that $\lambda > \lambda^\star$ where $\lambda^\star = \lim_{R\to\infty} \hat{\lambda}_1(R)$, and $\hat{\lambda}_1(R)$ is the principal eigenvalue of the problem

$$\begin{aligned} -\Delta\psi(x) = & \quad \lambda h(x) \hat{f}'(0) \psi(x), & \text{if} \quad |x| < R \\ \psi(x) = & \quad 0, & \text{if} \quad |x| = R, \end{aligned}$$

and so

$$\hat{\lambda}_1(R) = \inf \left\{ \frac{\int_{B_R} | \bigtriangledown u(x)|^2 dx}{\int_{B_R} h(x) \hat{f}'(0) u^2(x) dx} : \quad u \in H_0^1(B_R), \quad \int_{B_R} h(x) u^2(x) dx > 0 \right\}.$$

If $\lambda > \lambda_\star$, it is easy to see that by an appropriate choice of R we can construct an arbitrarily small subsolution, and if $\lambda > \lambda^\star$ that we can construct a supersolution, which is arbitrarily close to 1. Thus we have

Theorem 1.1 *If $\lambda > \max\{\lambda^\star, \lambda_\star\}$, then there exists a solution of $(1.1)_\lambda$. In particular, if $\lambda^\star = \lambda_\star = 0$, then $(1.1)_\lambda$ has a solution for all $\lambda > 0$.*

Thus it is of interest to fi nd conditions under which $\lambda_\star = 0$ or $\lambda^\star = 0$. We investigate this problem in Section 2 and the results we obtain depend heavily on whether $n = 1, 2$ or $n \geq 3$, as well as on the sign of $\int_{\mathcal{R}^n} g(x)dx$.

In Section 3 we prove some existence and nonexistence results which are suggested by the results of Section 2. In particular we show that when $n = 1, 2$, then $(1.1)_\lambda$ has a solution for all $\lambda > 0$ when $\int_{\mathcal{R}^n} g(x)dx \geq 0$, but has no solution for sufficiently small λ when $\int_{\mathcal{R}^n} g(x)dx < 0$. In contrast when $n \geq 3$, g is small at infinity (viz. $|g(x)| \leq k(1 + |x|^2)^{-\alpha})$, for some $\alpha > 1$ and $k > 0$) and λ sufficiently small, then $(1.1)_\lambda$ has no solution u such that $\lim_{|x|\to\infty} u(x) = 0$, but has a solution u such that $\lim_{|x|\to\infty} u(x) = c$ for any constant c , $0 < c < 1$.

2 Sub and supersolutions for small λ.

First we consider the case where $\int_{\mathcal{R}^n} g(x)dx < 0$.

Theorem 2.1 *Suppose that $\int_{\mathcal{R}^n} g(x)dx < 0$. Then*
(i) $\lambda_\star > 0$, and
(ii) $\lambda^\star = 0$, when $n = 1, 2$.

Proof (i) Choose $R_1 > R_0$ such that $\int_{B_{R_1}} g(x)dx < 0$. By [1] (Lemma 2.7) there exists a constant $K > 0$ such that

$$\int_{B_{R_1}} | \bigtriangledown u(x)|^2 dx \geq K \int_{B_{R_1}} u^2(x)dx$$

for all $u \in H^1(B_{R_1})$ which satisfy $\int_{B_{R_1}} g(x)u^2(x)dx > 0$. Suppose that $R > R_1$ and $u \in H^1_0(B_R)$ such that $\int_{B_R} g(x)u^2(x)dx > 0$. Then $u \in H^1(B_{R_1})$, and since $g(x) < 0$ for $R_1 \leq |x| \leq R$, $\int_{B_{R_1}} g(x)u^2(x)dx > 0$. Hence

$$\begin{aligned}\int_{B_R} | \bigtriangledown u(x)|^2 dx &\geq \int_{B_{R_1}} | \bigtriangledown u(x)|^2 dx \\ &\geq K \int_{B_{R_1}} u^2(x)dx \geq KK_1 \int_{B_{R_1}} g(x)u^2(x)dx, \\ &\geq KK_1 \int_{B_R} g(x)u^2(x)dx\end{aligned}$$

for a suitable constant K_1.Thus

$$\frac{\int_{B_R} | \bigtriangledown u(x)|^2 dx}{\int_{B_R} g(x)u^2(x)dx} \geq KK_1 \text{ for all } u \in H^1_0(B_R) \text{ such that } \int_{B_R} g(x)u^2(x)dx > 0,$$

and so $\lambda_1(R) \geq KK_1$. Since this holds for all $R > R_1$, we conclude that $\lambda_\star \geq KK_1 > 0$.

(ii) We consider the case $n = 2$. Suppose that $\varepsilon > 0$ is arbitrary. Since $\int_{\mathcal{R}^n} g(x)dx < 0$, there exists $R > R_0$ and a constant $a > 0$, such that $\int_{B_R} h(x)dx > a$.

We define a radially symmetric function u as follows

$$\begin{array}{lll} u(r) = & 1, & \text{if} \quad r \leq R \\ u'(r) = & -(\varepsilon/r), & \text{if} \quad R \leq r \leq Z \\ u(r) = & 0, & \text{if} \quad r \geq Z, \end{array}$$

where Z is chosen so that u is continuous. Then u is a decreasing function for $R \leq r \leq Z$. Since $u(r) = -\varepsilon \ \ln r + \beta$, for some positive constant β, we must have that

$$1 = -\varepsilon \ln R + \beta, \qquad 0 = -\varepsilon \ln Z + \beta,$$

and so

$$\varepsilon \ (\ln Z - \ln R) = 1.$$

Then, if $r > Z$,

$$\begin{aligned} \textstyle\int_{B_r} | \bigtriangledown u(x)|^2 dx &= \textstyle\int_0^r r u_r^2(r)dr = \int_R^Z (\varepsilon^2/r)dr \\ &= \varepsilon^2(\ln Z - \ln R) = \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \textstyle\int_{B_r} h(x)\hat{f}'(0)u^2(x)dx &\geq \hat{f}'(0) \textstyle\int_{B_R} h(x)u^2(x)dx \\ &= \hat{f}'(0) \textstyle\int_{B_R} h(x)dx \geq \hat{f}'(0)a. \end{aligned}$$

Hence, i f $r > Z$, then $\lambda_1(r) \leq \varepsilon/\hat{f}'(0)a$. Since $\varepsilon > 0$ is arbitrary, it follows that $\lambda^\star = 0$.

When $n = 1$ a similar but simpler argument, in which $u(r) = -\varepsilon r + \beta$ on its nonconstant part, can be used. $\diamond$

It does not seem possible to show, by using an argument similar to that above, that $\lambda^\star = 0$ when $n \geq 3$; it is however easy to obtain supersolutions for all $\lambda > 0$ when $n \geq 3$ as for this case, since $u(x) = 1/|x|^{n-2}$ satisfies $\Delta u(x) = 0$, then

$$\underline{u}(x) = \begin{cases} 1, & \text{if} \quad |x| \leq R_0 \\ (R_0/|x|)^{n-2}, & \text{if} \quad |x| > R_0 \end{cases},$$

is always a supersolution.

As the following theorem shows, it is also possible to have $\lambda^\star = 0$ for $n = 1, 2$ and so obtain supersolutions for arbitrarily small λ, without assuming that $\int_{\mathcal{R}^n} g(x)dx < 0$.

Theorem 2.2 *(i) If* $n = 1$ *and* $\lim_{|x|\to\infty} |x|^2 g(x) = -\infty$*, then* $\lambda^\star = 0$*.*
(ii) If $n = 2$ *and* $\lim_{|x|\to\infty} |x|^2[\ln(|x|)]^2 g(x) = -\infty$*, then* $\lambda^\star = 0$*.*

Proof (i) First we consider the case $n = 1$. Let $\varepsilon > 0$ be arbitrary. Then there exists $R > R_0$ such that $h(x) > 1/\varepsilon x^2$ if $|x| \geq R$. We define a test function as follows

$$\phi(x) = \begin{cases} 0, & \text{if} \quad -\infty \leq x \leq R_0 \\ \frac{x-R_0}{R-R_0}, & \text{if} \quad R_0 \leq x \leq R \\ 1, & \text{if} \quad R \leq x \leq 2R \\ \frac{3R-R_0-x}{R-R_0}, & \text{if} \quad 2R \leq x \leq 3R - R_0 \\ 0, & \text{if} \quad x \geq 3R - R_0 \end{cases},$$

Then

$$\int_{\mathcal{R}} [\phi_x(x)]^2 dx = \frac{2}{R - R_0}$$

and

$$\int_{\mathcal{R}} h(x)[\phi(x)]^2 dx \geq \int_R^{2R} h(x)dx \geq \frac{1}{\varepsilon} \int_R^{2R} \frac{1}{x^2} dx = \frac{1}{2\varepsilon R}$$

Hence

$$\frac{\int_{\mathcal{R}} [\phi_x(x)]^2 dx}{\int_{\mathcal{R}} h(x)\hat{f}'(0)\phi^2(x)dx} \leq \frac{4\varepsilon R}{\hat{f}'(0)(R - R_0)} \leq \frac{4\varepsilon}{\hat{f}'(0)},$$

and so $\lambda^\star \leq 4\varepsilon/\hat{f}'(0)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\lambda^\star = 0$.

(ii) A similar argument shows that $\lambda^\star = 0$ when $n = 2$ by replacing the functions

$$\frac{x - R_0}{R - R_0} \quad \text{and} \quad \frac{3R - R_0 - x}{R - R_0},$$

by the functions

$$\frac{\ln|x| - \ln R_0}{\ln R - \ln R_0} \quad \text{and} \quad \frac{3\ln R - \ln R_0 - \ln|x|}{\ln R - \ln R_0},$$

respectively, and using the ranges $R_0 \leq |x| \leq R$, $R \leq |x| \leq R^2$, and $R^2 \leq |x| \leq R^3/R_0$. $\diamond$

Suppose now that $\int_{\mathcal{R}^n} g(x)dx \geq 0$. It is proved in [1] (Lemma 4.4) that when $n = 1, 2$ and $\int_{\mathcal{R}^n} g(x)dx > 0$, then $\lambda_\star = 0$. We now discuss the more delicate case where $\int_{\mathcal{R}^n} g(x)dx = 0$.

Theorem 2.3 *Suppose that* $\int_{\mathcal{R}^n} g(x)dx = 0$. *Then* $\lambda_\star = 0$, *if*
(i) $n = 1$ *and* $\int_{\mathcal{R}}(\ln|x|)^2 g(x)dx$ *converges, or*
(ii) $n = 2$ *and* $\int_{\mathcal{R}^2}[\ln(|\ln|x||)]^2 g(x)dx$ *converges.*

Proof (i) We consider a test function of the form

$$u(x) = \begin{cases} 1, & \text{for} \quad |x| \leq R_0 \\ \zeta - \delta\ln|x|, & \text{for} \quad R_0 \leq |x| \leq R \\ 0, & \text{for} \quad |x| \geq R \end{cases},$$

where δ, ζ are constants chosen so that u is a continuous function,
i.e., $\delta = 1/(\ln R - \ln R_0)$ and $\zeta = \ln R/(\ln R - \ln R_0)$ and show that

$$\lim_{R\to\infty} \frac{\int_{\mathcal{R}}[u_x(x)]^2 dx}{\int_{\mathcal{R}} g(x)f'(0)u^2(x)dx} = 0.$$

Now

$$\int_{\mathcal{R}}[u_x(x)]^2 dx = 2\int_{R_0}^{R} \left(\frac{\delta}{x}\right)^2 dx = 2\delta^2\left[\frac{1}{R_0} - \frac{1}{R}\right] = \frac{2\delta^2(R - R_0)}{RR_0},$$

and

$$\begin{aligned} &\int_{\mathcal{R}} g(x)f'(0)u^2(x)dx \\ &= f'(0)\left\{\int_{|x|\leq R_0} g(x)dx + \int_{R_0\leq|x|\leq R} g(x)(\zeta - \delta\ln|x|)^2 dx\right\} \\ &= f'(0)\left\{\int_{|x|\leq R_0} g(x)dx + \int_{R_0\leq|x|\leq R} g(x)[\zeta^2 - 2\delta\zeta\ln|x| + \delta^2(\ln|x|)^2]dx\right\}. \end{aligned}$$

Hence

$$\begin{aligned}
\frac{\int_{\mathcal{R}} g(x) f'(0) u^2(x) dx}{\int_{\mathcal{R}} [u_x(x)]^2 dx} &= \tfrac{f'(0)RR_0}{2(R-R_0)} \Big\{ (\ln R - \ln R_0)^2 \textstyle\int_{|x|\le R_0} g(x)dx \\
&\quad + \textstyle\int_{R_0 \le |x| \le R} g(x)[(\ln R)^2 - 2\ln R \ln|x| + (\ln|x|)^2] dx \Big\} \\
&= \tfrac{f'(0)RR_0}{2(R-R_0)} \Big\{ (\ln R)^2 \textstyle\int_{|x|\le R} g(x)dx + (\ln R_0)^2 \int_{|x|\le R_0} g(x)dx \\
&\quad + \textstyle\int_{R_0 \le |x| \le R} (\ln|x|)^2 g(x) dx - 2\ln R_0 \ln R \int_{|x|\le R_0} g(x)dx \\
&\quad - 2\ln R \textstyle\int_{R_0 \le |x| \le R} \ln|x| g(x)dx \Big\} \\
&\equiv \tfrac{f'(0)RR_0}{2(R-R_0)} \{A_1(R) + A_2 + A_3(R) + A_4(R) + A_5(R)\}.
\end{aligned}$$

Since $\int_{\mathcal{R}} g(x)dx = 0$, $A_1(R) > 0$, for all $R > R_0$. Since we are assuming that $\int_{\mathcal{R}} (\ln|x|)^2 g(x)dx$ converges, $A_3(R)$ is uniformly bounded. Finally

$$A_4(R) + A_5(R) = -2\ln R_0 \ln R \int_{|x|\le R} \psi(x) g(x) dx,$$

where

$$\psi(x) = \begin{cases} 1, & \text{for} \quad 0 \le |x| \le R_0 \\ \ln|x| / \ln R_0, & \text{for} \quad R_0 \le |x| \le R \end{cases},$$

Since $\int_{\mathcal{R}} g(x)dx = 0$, it is clear that $\int_{\mathcal{R}} \psi(x) g(x) dx < 0$ and so $\int_{|x|\le R} \psi(x) g(x) dx$ is negative and uniformly bounded away from zero when R is sufficiently large. Hence $\lim_{R\to\infty} (A_4(R) + A_5(R)) = \infty$, and so

$$\lim_{R\to\infty} \frac{\int_{\mathcal{R}} [u_x(x)]^2 dx}{\int_{\mathcal{R}} g(x) f'(0) u^2(x) dx} = 0.$$

Thus $\lambda_\star = 0$.

(ii) When $n = 2$, a similar argument shows that $\lambda_\star = 0$ by replacing the test function $\zeta - \delta \ln|x|$ by $\zeta - \delta \ln(|\ln|x||)$. $\diamond$

These results, that $\lambda_\star = 0$ when $\int_{\mathcal{R}^n} g(x)dx \ge 0$, cannot be extended to the case where $n \ge 3$ because the nonexistence results in the next section imply that when $n \ge 3$ and g is sufficiently small at infinity then $\lambda_\star > 0$, no matter what the sign of $\int_{\mathcal{R}^n} g(x)dx$.

3 Existence and nonexistence of solutions.

We shall first consider the case of $n = 1, 2$ and show how the existence theory for $(1.1)_\lambda$ is heavily dependent on the sign of $\int_{\mathcal{R}^n} g(x)dx$. The following results are simple consequences of the sufficient conditions for $\lambda_\star = 0$, $\lambda^\star = 0$ obtained in Section 2.

Theorem 3.1 *Suppose that $n = 1$. Then $(1.1)_\lambda$ has a solution for all $\lambda > 0$, provided that*
(i) $\int_{\mathcal{R}} g(x)dx > 0$
or
$\int_{\mathcal{R}} g(x)dx = 0$ *and* $\int_{\mathcal{R}} (\ln|x|)^2 g(x)dx$ *converges,*
and
(ii) $\lim_{|x|\to\infty} |x|^2 g(x) = -\infty$.

A similar result holds when $n = 2$ and the hypotheses that $\int_{\mathcal{R}} (\ln|x|)^2 g(x)dx$ converges and $\lim_{|x|\to\infty} |x|^2 g(x) = -\infty$ are replaced by $\int_{\mathcal{R}^2} [\ln(|\ln|x||)]^2 g(x)dx$ converges and $\lim_{|x|\to\infty} |x|^2[\ln(|x|)]^2 g(x) = -\infty$.

The $\int_{\mathcal{R}} g(x)dx > 0$ case of Theorem 3.1 was proved in [1] by constructing a supersolution, using phase plane analysis.

The hypotheses of Theorem 3.1 can be satisfied provided g is small, but not too small at infinity, e.g. $g(x) \sim 1/x^\alpha$, when $n = 1$, and $g(x) \sim 1/x^2(\ln x)^\alpha$, when $n = 2$, where $1 < \alpha < 2$.

It was shown in Section 2 that, when $\int_{\mathcal{R}^n} g(x)dx < 0$, then $\lambda^\star = 0$ but $\lambda_\star > 0$. It is still possible a priori that some other form of subsolution can be constructed or that solutions of $(1.1)_\lambda$ exist for $\lambda < \lambda_\star$. The following result shows, however, that solutions cannot exist for arbitrarily small $\lambda > 0$.

Theorem 3.2 *Suppose that $n = 1, 2$, $\int_{\mathcal{R}^n} g(x)dx < 0$ and f satisfies $2f(u) - uf'(u) > 0$ for $0 < u < 1$. Then there exists $\lambda_0 > 0$ such that $(1.1)_\lambda$ has no solution u whenever $0 < \lambda < \lambda_0$.*

Proof We shall prove the result for the case $n = 2$; a similar but simpler proof holds when $n = 1$. Suppose that u is a solution of $(1.1)_\lambda$. It is shown in [7] (Lemma 3.2, Lemma 3.3) that $\bigtriangledown u \in L^2(\mathcal{R}^2)$, $gf(u) \in L^1(\mathcal{R}^2)$ and $\lim_{R\to\infty} \int_{\partial B_R} u(x)|\bigtriangledown u(x)|\, dS = 0$. Firstly we shall show that $\int_{\mathcal{R}^2} g(x)u^2(x)dx > 0$. Let $w = (u^2/f(u))$. Since u is subharmonic whenever $|x| > R_0$, it follows from the Hadamard Three Circles Theorem ([5], p.130) that $\sup_{|x|=R} u(x)$ is a decreasing function of R and so there exists a constant $k^\star$ such that $u(x) \leq k^\star < 1$ for all $x \in \mathcal{R}^2$. Hence w is a smooth bounded function on $\mathcal{R}^2$. Thus

$$\begin{aligned}
\lim_{R\to\infty} \quad & \lambda \textstyle\int_{B_R} g(x)u^2(x)dx \\
& = \lim_{R\to\infty} \lambda \textstyle\int_{B_R} g(x)f(u)wdx = \lim_{R\to\infty} \int_{B_R}(-\Delta u)wdx \\
& = \lim_{R\to\infty} \left\{- \textstyle\int_{\partial B_R}(\partial u/\partial n)wdS + \int_{B_R} \bigtriangledown u \bigtriangledown wdx\right\} \\
& = \lim_{R\to\infty} \Big\{- \textstyle\int_{\partial B_R}(\partial u/\partial n)u(f(u)/u)dS \\
& \quad + \textstyle\int_{B_R} | \bigtriangledown u|^2 u[2f(u) - uf'(u)][f^{-2}(u)]dx\Big\}.
\end{aligned}$$

Since

$$|\int_{\partial B_R} \frac{\partial u}{\partial n} u \frac{f(u)}{u} dS| \leq |\int_{\partial B_R} | \bigtriangledown u| u \frac{f(u)}{u} dS| \to 0 \text{ as } R \to \infty,$$

and $2f(u) - uf'(u) > 0$ for $0 < u < 1$, it follows that $\int_{\mathcal{R}^2} g(x)u^2(x)dx > 0$.

Now choose a ball B such that $\int_B g(x)dx < 0$ and $g(x) < 0$ whenever $x \notin B$. By [1](Lemma 2.7) there exists $K > 0$ such that $\int_B g(x)v^2(x)dx > K \int_B v^2(x)dx$ for all $v \in H^1(B)$ such that $\int_B g(x)v^2(x)dx > 0$. Clearly we have $\int_B g(x)u^2(x)dx > 0$ and so

$$\begin{aligned}
\textstyle\int_{\mathcal{R}^2} | \bigtriangledown u(x)|^2 dx \quad & \geq \textstyle\int_B | \bigtriangledown u(x)|^2 dx \geq K \int_B u^2(x)dx \\
& \geq KK^\star \textstyle\int_B g(x)f(u)u(x)dx \geq \gamma \int_{\mathcal{R}^2} g(x)f(u)u(x)dx,
\end{aligned}$$

where $K^\star$ is chosen so that $g(x)f(u) \leq u/K^\star$ for all $x \in B$ and $0 < u < 1$ and $\gamma = KK^\star$ depends only on f and g, but is independent of u. On the other hand multiplying $(1.1)_\lambda$ by u and integrating gives that

$$\int_{\mathcal{R}^2} | \bigtriangledown u(x)|^2 dx - \lim_{R\to\infty} \int_{\partial B_R} \frac{\partial u}{\partial n} udS = \lambda \int_{\mathcal{R}^2} g(x)f(u)u(x)dx.$$

Thus we must have $\lambda \geq \gamma$ and so the proof is complete.$\diamond$

The hypothesis $2f(u) - uf'(u) > 0$ is a weak concavity condition and is satisfied by the cubic function f arising in the population genetics model, viz., $f(u) = u(1-u)[h(1-u)+(1-h)u]$. A more precise nonexistence result is obtained in [7](Theorem 4.3), but under more restrictive assumptions on f than those used here.

The above theorem implies that, when $n = 1, 2$, there exist examples of functions g decaying arbitrarily fast at infinity such that no solution of $(1.1)_\lambda$ exists if λ is sufficiently small. Moreover, at least in the radial case, the solutions obtained under the hypotheses of Theorem 3.1 will approach 0 as $|x|$ tends to infinity (see [2]). We now consider the case where $n \geq 3$ and g is small at infinity and show that completely different results hold.

We shall assume that the function g is small in the following sense

$$(\mathbf{G_1}) \quad |g(x)| \leq k(1+|x|^2)^{-\alpha} \quad \text{for some constants} \quad k > 0, \quad \text{and} \quad \alpha > 1.$$

Our proofs shall make use of various properties of Newtonian potentials, which are proved in [4].

Theorem 3.3 *Suppose that* $n \geq 3$ *and* g *satisfies* (G_1). *Then there exists* $\lambda_0 > 0$ *such that , whenever* $0 < \lambda < \lambda_0$, *there does not exist a solution* u *of* $(1.1)_\lambda$ *with* $\lim_{|x|\to\infty} u(x) = 0$.

Proof Suppose that u is a solution of $(1.1)_\lambda$ such that $\lim_{|x|\to\infty} u(x) = 0$. It follows from [4] (Lemma 2.3) that u satisfies

$$u(x) = c_n \int_{\mathcal{R}^n} \frac{g(y)f(u(y))}{|x-y|^{n-2}} dy, \tag{3.1}$$

where $c_n = [n(n-2)\omega_n]^{-1}$ and ω_n is the volume of the unit ball in $\mathcal{R}^n$. Since $|g(y)f(u(y)| \leq k_1|y|^{-2\alpha}$ at ∞, we have $|u(x)| \leq k_1|x|^{-\beta}$, where $\beta = n-2$ if $2\alpha > n$ and $\beta = 2 - 2\alpha$ if $2\alpha < n$. Since $f(u(y)) \leq c^\star u(y)$, for some constant $c^\star$, it can be proved by using a bootstrapping argument that $u(x) \leq k_2|x|^{2-n}$, for some constant k_2. Moreover it can be shown, by differentiating (3.1) and using arguments similar to those used above, that $| \bigtriangledown u(x)| \leq k_3|x|^{1-n}$.

Since g satisfies (G_1), g lies $L^{n/2}(\mathcal{R}^n)$. By the estimates on u obtained above we have $\bigtriangledown u \in L^2(\mathcal{R}^n)$, $u \in L^q(\mathcal{R}^n)$, where $q = 2n/(n-2)$. Hence there exists a sequence $\{u_k\} \subset C_0^\infty(\mathcal{R}^n)$, such that $\bigtriangledown u_k \to \bigtriangledown u$

in $L^2(\mathcal{R}^n)$ and $u_k \to u$ in $L^q(\mathcal{R}^n)$. Thus $u_k^2 \to u^2$ in $L^{n/(n-2)}(\mathcal{R}^n)$ and so $lim_{k\to\infty} \int_{\mathcal{R}^n} |g|u_k^2\, dx = \int_{\mathcal{R}^n} |g|u^2\, dx$. By Hardy's inequality there exists $d > 0$ such that

$$\int_{\mathcal{R}^n} | \bigtriangledown v(x)|^2 dx \geq d \int_{\mathcal{R}^n} \frac{v^2(x)}{1 + |x|^2} dx$$

for all $v \in C_0^\infty(\mathcal{R}^n)$. Hence

$$\begin{aligned}\textstyle\int_{\mathcal{R}^n} \quad & | \bigtriangledown u(x)|^2 dx \\ & = \lim_{k\to\infty} \textstyle\int_{\mathcal{R}^n} | \bigtriangledown u_k(x)|^2 dx \geq d \textstyle\int_{\mathcal{R}^n} u_k^2(x)/(1 + |x|^2) dx \\ & \geq (d/k) \lim_{k\to\infty} \textstyle\int_{\mathcal{R}^n} |g(x)|u_k^2(x) dx = (d/k) \textstyle\int_{\mathcal{R}^n} |g(x)|u^2(x) dx.\end{aligned}$$

But multiplying $(1.1)_\lambda$ by u and integrating gives

$$\int_{\mathcal{R}^n} | \bigtriangledown u(x)|^2 dx - \lim_{R\to\infty} \int_{\partial B_R} \frac{\partial u}{\partial n} u dS = \lambda \int_{\mathcal{R}^n} g(x) f(u) u(x) dx.$$

Since the integral over the boundary tends to 0, it follows that

$\int_{\mathcal{R}^n} | \bigtriangledown u(x)|^2 dx \leq \lambda c^\star \int_{\mathcal{R}^n} |g(x)|u^2(x) dx,$

for all $0 < u < 1$. This is impossible if $\lambda c^\star < d/k$, i.e., if $\lambda < h(d/c^\star k)$ and so the proof is complete. $\diamond$

Finally we prove, for arbitrarily small λ, the existence of solutions of $(1.1)_\lambda$ which approach nonzero constants as $|x| \to \infty$.

Theorem 3.4 *Suppose that* $n \geq 3$ *and* g *satisfies* (G_1). *Let* c *be given constant such that* $0 < c < 1$. *If* λ *is sufficiently small, there exists a solution* u *of* $(1.1)_\lambda$ *satisfying* $\lim_{|x|\to\infty} u(x) = c$.

Proof It is easy to check that u is a solution of $(1.1)_\lambda$, with $\lim_{|x|\to\infty} u(x) = c$ if and only if $v = u - c$ is a solution of

$$
\begin{aligned}
&-\Delta v = \lambda g(x)\tilde{f}(v), \qquad x \in \mathcal{R}^n \\
&\lim_{|x|\to+\infty} v(x) = 0.
\end{aligned} \tag{3.2}
$$

where $\tilde{f}(v) = f(v+c)$. Thus $\tilde{f}(0) > 0$, $\tilde{f}(1-c) = 0$ and $\tilde{f}(v) > 0$ for $0 < v < 1-c$. We prove the existence of a solution of (3.2) by constructing appropriate sub and supersolutions. Clearly the function

$$
\bar{v}(x) = \begin{cases} 1-c, & \text{for} \quad 0 \le |x| \le R_0 \\ (1-c)\left\{R_0/|x|\right\}^{n-2}, & \text{otherwise} \end{cases}
$$

defines a positive supersolution.

Suppose $g = g_+ + g_-$ where g_+ and g_- denote the positive and negative parts of g. Then the equation $-\Delta\phi = g_-(x)$ has a negative solution

$$
\phi(x) = c_n \int_{\mathcal{R}^n} \frac{g_-(y)}{|x-y|^{n-2}} dy,
$$

such that $\lim_{|x|\to+\infty} \phi(x) = 0$. Since $\phi(x)$ is bounded we can find $\epsilon > 0$ such that $\epsilon\phi(x) + c > 0$ and so $\tilde{f}(\epsilon\phi(x)) > 0$, for all $x \in \mathcal{R}^n$. Let $\underline{v}(x)) = \epsilon\phi(x)$; we shall show that $\underline{v}(x))$ is a subsolution of (3.2) provided λ is sufficiently small.

If $x \in \mathcal{R}^n$ and $g(x) \ge 0$, then

$$
-\Delta\underline{v}(x) = \epsilon g_-(x) = 0 \le \lambda g(x)\tilde{f}(\underline{v}(x)),
$$

whereas if $g(x) < 0$

$$
-\Delta\underline{v}(x) = \epsilon g(x), \quad \text{and} \quad \lambda g(x)\tilde{f}(\underline{v}(x)) > \lambda\mu g(x),
$$

where μ is such that $f(u) \le \mu$ for all $0 \le u \le 1$. Thus $\underline{v}$ is a subsolution provided that $\epsilon g(x) < \lambda\mu g(x)$ whenever $g(x) < 0$, i.e., provided that $\lambda < \epsilon/\mu$ and so the proof is complete.$\diamond$

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