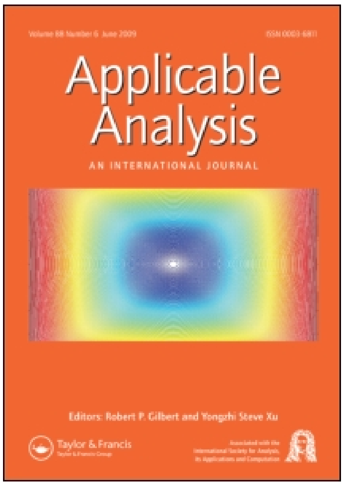

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Compact attractors for weak dynamical systems

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Compact Attractors for Weak Dynamical Systems

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Abstract Some abstract ideas on dissipative systems for dynamical systems are extended to weak dynamical systems. Applications are given for a linearly damped nonlinear wave equation and some distributed control problems.

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INTRODUCTION

At the present time, there is a rather extensive theory of dissipative dynamical systems on an infinite dimensional Banach space. In particular, there are applicable criteria for the existence of a compact attractor and the determination of the asymptotic behavior of the solutions. Many of these ideas originated in the study of delay equations and have recently found applications in partial differential equations (for some references, see, for example, Hale, Magalhães and Oliva[15] or Hale[14]).

In the late 1960's, efforts were being made to apply Lyapunov theory and the invariance principle to dynamical systems on a Banach space (Hale[13]). However, due to certain difficulties which appeared in some applications, Slemrod[19-20] introduced the concept of a weak dynamical system and used this concept for a discussion of the wave equation and a nonlinear partial differential equation similar to the ordinary differential equation of van der Pol.

Ball[4] showed that the equation for a linearly damped beam defines a weak dynamical system and he used this fact to show that the solutions strongly approach equilibrium points. Artstein and Slemrod[1], using ideas from Ball[5], prove the existence of orbits weakly connecting certain equilibrium points in the beam equation. Lopes and Ceron[18] has used the weak topology to show that every solution of the beam equation with nonlinear damping approaches an equilibrium point. For a linearly damped wave equation, Ball[5] has similar results.

Other situations where convergence in the weak topology plays an important role is in linear thermoelasticity (Dafermos[11], Slemrod and Infante[22], Lopes[17]), viscoelasticity of the Boltzmann type (Dafermos[9-11]), the linearized theory of simple fluids (Slemrod[21]) and conservation laws (Lax[16], Dafermos[12]).

Ball and Slemrod[6-7] made extensive use of weak dynamical systems in studying feedback stabilization of distributed semilinear control systems. This application also will be considered below.

Much of the difficulty in the above problems stems from the fact that the Lyapunov function is not continuous but only lower semicontinuous. Ball[5] (see, also Dafermos[12]) overcame the difficulties involved and showed that a version of the invariance principle was valid.

The purpose of the present paper is to extend some of the abstract ideas on dissipative dynamical systems to weak dynamical systems and to point out a few applications. More specifically, in Section 2, we prove the existence of a weak compact attractor with the basic hypotheses being weak point dissipative and orbits of bounded sets bounded. In Section 3, we discuss the implications for a linearly damped nonlinear wave equation considered by Ball[5] for which the nonlinearity is not compact. In Section 4, we discuss the distributed control problems of Ball and Slemrod[6-7].

WEAK DYNAMICAL SYSTEMS AND ATTRACTORS

Let X be a reflexive Banach space. We denote by $\check{X}$ the space X endowed with the weak topology.

A *weak dynamical system* on $\check{X}$ is a function T: $\mathbb{R} \times \check{X} \to \check{X}$ with the following properties:

(i) T(t): $x \to T(t)x$ is weakly sequentially continuous for fixed $t \in \mathbb{R}^+$ (i.e., if $x_n \rightharpoonup x$, then $T(t)x_n \rightharpoonup T(t)x$).

(ii) $T(\cdot)x$: $t \to T(t)x$ is continuous from $\mathbb{R}^+$ into $\check{X}$ for fixed $x \in X$.

(iii) $T(0)x = x$ for all $x \in X$, and

(iv) $T(t + \tau)x = T(t)T(\tau)x$ for all $t,\tau \in \mathbb{R}^+$, $x \in X$.

A *dynamical system* is a function which satisfies (iii), (iv) and (i), (ii) in the strong topology.

For any set $B \subset X$, we define the *weak* ω*-limit set* $\check{\omega}(B)$ *of* B by

$$\check{\omega}(B) = \bigcap_{\sigma \geq 0} \left[w - c\ell \bigcup_{t \geq \sigma} T(t)B \right].$$

A set $J \subset \check{X}$ is said to be *invariant* under T, if $T(t)J = J$, $t \geq 0$.

A set is *maximal* w-*compact invariant* if it is w-compact, invariant and maximal with respect to these properties. A set $J \subset \check{X}$ is a *weak compact attractor* for T(t) in $\check{X}$, if J is maximal, w-compact, invariant and *weakly attracts* the bounded sets of $\check{X}$; that is, for any bounded set $B \subset X$ and any $\epsilon > 0$, there is a $t_0 = t_0(\epsilon,B,J)$ such that $T(t)B \subset \check{N}_\epsilon(J)$, for $t \geq t_0$ where $\check{N}_\epsilon(J)$ is a weak ϵ-neighborhood of J.

The w-dynamical system T(t) is said to be *weak point* (*bounded*) *dissipative* if there is a bounded set $K \subset \check{X}$, which weakly attracts the points (bounded sets) of X.

A set J is said to be *weakly stable* if, for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that

$$T(t)\check{N}_\delta(J) \subset \check{N}_\epsilon(J), \text{ for all } t \geq 0.$$

The next lemma consists of a generalization of Ball's[4] Theorem 4 and is used in the proof of the theorems below.

Lemma 2.1. *Let* X *be a separable, reflexive Banach space*, T: $\mathbb{R}^+ \times X \to X$ *a* w-*dynamical system and* B *a bounded subset of* X. *If* $\gamma^+(B)$ *is a bounded subset of* X, *then* $\breve{\omega}(B)$ *is nonempty*, w-*compact and invariant and* $\breve{\omega}(B)$ w-*attracts* B. *If* B *is* w-*connected, then* $\breve{\omega}(B)$ *is also* w-*connected*.

Proof: Since X is reflexive and $\gamma^+(B)$ is bounded, the set $A = w - c\ell\gamma^+(B)$ is w-compact. Also, for any $\sigma > 0$, we have

$$\bigcup_{t\geq\sigma} T(t)B \subset \bigcup_{t\geq 0} T(t)B = \gamma^+(B) \subset A \ .$$

Hence

$$w\text{-}c\ell \bigcup_{t\geq\sigma} T(t)B \subset A$$

and

$$\breve{\omega}(B) = \bigcap_{\sigma\geq 0} \left[w - c\ell \bigcup_{t\geq\sigma} T(t)B \right] \subset A.$$

Thus $\breve{\omega}(B)$ *is nonempty*. Since X is separable, by Dafermos's[10] Prop. 2.2 we have that $\breve{\omega}(B)$ is w-compact and invariant. The proof that $\breve{\omega}(B)$ w-attracts B *is the same as for dynamical systems* in the strong topology.

If B is connected and $\breve{\omega}(B)$ is not connected *in* $\breve{X}$, there exist M_1, M_2, A_1, A_2 with A_1, A_2 w-open such that $M_1 \cap M_2 = \emptyset$, $A_1 \cap A_2 = \emptyset$, $M_1 \subset A_1$, $M_2 \subset A_2$ and

$$\breve{\omega}(B) = M_1 \cup M_2.$$

From the continuity property of T, we conclude that there exist sequences $\{t_j\}$, $\{\varphi_j\} \subset B$, with $t_j \to \infty$, as $j \to \infty$ such that

$$\{T(t_j)\varphi_j\} \subset A\backslash(A_1 \cup A_2).$$

But $A\backslash(A_1 \cup A_2)$ is w-closed, hence

$$\breve{\omega}(B) \cap (A\backslash(A_1 \cup A_2)) \neq \emptyset$$

which is a contradiction.

The following theorems give sufficient conditions for the existence of a weak-compact attractor and its stability.

Theorem 2.2. *Let* X *be a reflexive separable Banach space. Let* T: $\mathbb{R}^+ \tilde{X} \to \tilde{X}$ *be a* w-*dynamical system, which is* w-*bounded dissipative and let* K *be a bounded* w-*closed set which* w-*attracts all bounded sets of* X. *If* $J = \cap_{t\geq 0} T(t)K$, *then the following statements are true*:

(i) $J = \breve{\omega}(K)$ *and* J *is independent of* K.

(ii) J *is* w-*connected.*

(iii) J *is a* w-*compact atttractor.*

(iv) J *is* w-*stable.*

Proof (i) Suppose H is a bounded subset of X. Since K w-attracts H, $\gamma^+(H)$ is bounded, w-cℓ $\gamma^+(H)$ is w-compact and $\breve{\omega}(H)$ exists. For any integer n, there is a σ_n such that $T(t)H \subset \breve{N}_{1/n}(K)$, $t \geq \sigma_n$. Since

$$\breve{\omega}(H) = \bigcap_{n \geq 0} \text{w-c}\ell \bigcup_{t\geq\sigma_n} T(t)H$$

it follows that $\breve{\omega}(H) \subset K$. Since this is true for an arbitrary bounded set of X, we have $\breve{\omega}(K) \subset K$. From here, it follows that $\breve{\omega}(K) \subset T(t)K$ for each $t \geq 0$. Hence, $\breve{\omega}(K) \subset \cap_{t\geq 0} T(t)K$. But obviously $\cap_{t\geq 0} T(t)K \subset \breve{\omega}(K)$. Therefore, $\breve{\omega}(K) = \cap_{t\geq 0} T(t)K$. If K_1 is another set in X with the same properties as K, then

$$\breve{\omega}(K_1) \subset \breve{\omega}(K) \subset \breve{\omega}(K_1)$$

i.e., A is independent of K.

To prove that J is w-connected, let $J_1 = \overline{co}\ J$. Then J_1 is connected, bounded w-compact and J attracts J_1. Since J is bounded, $\gamma^+(J_1)$ is bounded and thus $\breve{\omega}(J_1)$ exists, is w-compact, invariant, connected (by Lemma 2.1) and $\breve{\omega}(J_1) \subset J$. But, obviously $\breve{\omega}(J_1) \supset \breve{\omega}(J)$ and since J is invariant $\breve{\omega}(J) = J$. Thus $\breve{\omega}(J_1) = J$ and J is w-connected.

We prove next that J w-attracts bounded sets of X. Since $\breve{\omega}(K) \subset K$, $\breve{\omega}(K)$ is w-compact. If $\breve{\omega}(K)$ does not w-attract K, then there is an $\epsilon > 0$ such that, for each $\delta \in \mathbb{R}^+$, we can find z_δ, such that

$$z_\delta \in T(\delta)K \text{ and } z_\delta \notin \breve{N}_\epsilon(\breve{\omega}(K))$$

i.e., $z_\delta = T(\delta)x_\delta$, $x_\delta \in K$ and $d(z_\delta, \breve{\omega}(K)) > \epsilon$. But $\{z_\delta\} \subset \gamma^+(K) \subset (\text{w-c}\ell\ \gamma^+(K))$, which is w-compact. Hence, we can extract a sequence $\{z_n\}$ with $z_n \rightharpoonup z$, and $z \notin \breve{\omega}(K)$, which is a contradiction.

Thus $J = \breve{\omega}(K)$ w-attracts K. Since K w-attracts bounded sets of X, it follows that J w-attracts the bounded sets of X.

The set J is invariant. In fact, since $J = \cap_{t \geq 0} T(t)K = \breve{\omega}(K)$ it is obvious that $T(t)J = J$, $\forall t \geq 0$.

To show that J is maximal, suppose B is a bounded invariant. Since J w-attracts B and $T(t)B = B$, $t \geq 0$, it follows that $B \subset \breve{N}_\epsilon(J)$ for any $\epsilon > 0$. Hence, $B \subset J$. This completes the proof of part (iii) of the theorem.

To prove part (iv), suppose J is *not* w-stable. Then there exists $\epsilon > 0$ such that, for each $\delta > 0$, there exists a $t_0 = t_0(\delta,\epsilon)$ for which $T(t_0)\breve{N}_\delta(J) \not\subset \breve{N}_\epsilon(J)$. This implies that there exists a sequence $\{y_j\}$ with $y_j \rightharpoonup y \in J$ and $\{n_j\} \subset N$, with $n_j \to \infty$ as $j \to \infty$, such that $T(n_j)y_j \in \breve{N}_\epsilon(J)$ and $T(n_j+1)\ y_j \notin \breve{N}_\epsilon(J)$. The set $\{y_j,y\}$ is w-compact, and hence, bounded in X. Since J w-attracts bounded sets of X, it follows that $\breve{\omega}(\{y_j,y\}) \subset J$. Thus, we may assume without loss of generality that $T(n_j)y_j \rightharpoonup z \in J$ as $j \to \infty$. Since $z \in \breve{\omega}(\{y_j,y\}) \subset J$ and T(t) is w-continuous, we have $T(n_j)y_j — z \in J$ and $T(1)z \notin \breve{N}_\epsilon(J)$, which is a contradiction. This completes the proof of the theorem.

Theorem 2.3. *Let* X *be a separable reflexive Banach space,* $T: \mathbb{R}^+ \times \tilde{X} \to \tilde{X}$ *be a* w-*dynamical system which is* w-*point dissipative. Also, assume that* $\gamma^+(B)$ *is a bounded set of* X *for each bounded set* $B \subset X$. *Then* T *is* w-*bounded dissipative.*

Proof: If B is a bounded set in X which w-attracts points of X, then the set H = w-cℓ B, is w-compact and w-attracts points of X. By Lemma 2.1, the set $J = \breve{\omega}(H)$ is nonempty, w-compact, invariant and J w-attracts H. Thus, J w-attracts B. In particular, J w-attracts points of X.

There exists a w-neighborhood V of J in $\breve{X}$, for which $\gamma^+(V)$ is bounded. Indeed, if this is not true, there exist sequences $\{x_j\} \subset V$ and integers k_j with $k_j \to \infty$ as $j \to \infty$ such that

$$\lim_{j\to\infty} \inf \{d_w(x_j,y),\ \ y \in J\} = 0$$

and

$$|x^*(T(k_j)x_j)| \to \infty, \quad \text{as } j \to \infty. \tag{2.1}$$

Since J is w-compact, we can assume $x_j \rightharpoonup z \in J$. Take $K = \{x_j, z;\ j \geq 1\}$. Since K is w-compact, it is bounded and $\gamma^+(K)$ is bounded. This contradicts (2.1). Hence there is a w-neighborhood V of J such that $\gamma^+(V)$ is bounded. Since J w-attracts points of X, for each $x \in X$, there is a weak neighborhood 0_x and an integer n_0 such that

$$T(n)0_x \subset \gamma^+(V), \text{ for } n \geq n_0.$$

For each bounded set $H \subset X$, the set w-cℓ H is w-compact; that is, there exists a finite covering of H by w-neighborhoods of finite many points of w-cℓ H and a w-open neighborhood H_1 of w-cℓ H such that $\gamma^+(V)$ w-attracts H_1. Hence $\gamma^+(V)$ w-attracts H. This completes the proof of the lemma.

Corollary 2.4. *Let* X *be a separable reflexive Banach space*, $T: \mathbb{R}^+ \times X \to X$ *be a* w-*dynamical system with* $T(t,\cdot)$ w-*point dissipative. Also assume that* $\gamma^+(B)$ *is bounded, when* B *is a bounded subset of* X. *Then*

a. *There exists a* w-*compact attractor* J.

b. J *is* w-*stable* and w-*connected.*

The following theorems are useful in some particular applications.

Theorem 2.5. *Let* X *be a reflexive, separable Banach space. Let* $T:\mathbb{R}^+ \times \tilde{X} \to \tilde{X}$ *be a* w-*dynamical system. Suppose that orbits of bounded sets are bounded in* X *and that there exists a* w-*compact, invariant and* w-*stable subset* J *of* X, *which* w-*attracts points of* X. *Then* J *is the* w-*attractor in* X *under* T.

Proof: Since J is w-stable, for every $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$T(t)(\breve{N}_{\delta_1}(J)) \subset \breve{N}_{\epsilon_1}(J) \quad \text{for all} \quad t \geq 0,$$

Thus

$$\gamma^+(\breve{N}_{\delta_1}(J)) \subset \breve{N}_{\epsilon_1}(J). \tag{2.2}$$

Since T(t) is w-continuous and J w-attracts points of X, for any $\epsilon_2 > 0$ and any $x \in X$, there exists $\delta_2 = \delta_2(\epsilon_2,x) > 0$ and $t_0 = t_0(x,J,\epsilon_2)$ such that

$$T(t)\breve{N}_{\delta_2}(x) \subset \breve{N}_{\epsilon_2}(J) \subset \gamma^+(\breve{N}_{\epsilon_2}(J)) \quad \text{for all} \quad t \geq t_0\,. \tag{2.3}$$

Now, if B is a bounded subset of X, then $\text{w-}c\ell\, B = \breve{B}$ is a w-compact set in X. Therefore there exists an integer k, such that

$$B \subset \breve{B} \subset \bigcup_{i=1}^{k} \breve{N}_{\delta_2}(x_i)$$

where $x_i \in \breve{B}$ for $i = 1, \ldots, k$. If we take $\epsilon_2 = \delta_1$, relations (2.2), (2.3) imply that

$$T(t)B \subset T(t)\breve{B} \subset T(t)\left[\bigcup_{i=1}^{k} \breve{N}_{\delta_2}(x_i)\right] \subset \gamma^+(\breve{N}_{\epsilon_2}(J))$$
$$= \gamma^+(\breve{N}_{\delta_1}(J)) \subset \breve{N}_{\epsilon_1}(J), \text{ for all } \quad t \geq t_1,$$

where

$$t_1 = \max_{i=1,\ldots,N} t(x_i,J,\delta_1) = t_1(B,J,\delta_1).$$

Hence, for any $\epsilon_1 > 0$, there exists a δ_1 ($= \epsilon_2$) such that, for all bounded subsets B of X, there exists $t_1 = t_1(B,J,\delta_1)$ such that

$$T(t)B \subset \tilde{N}_{\epsilon_1}(J) \quad \text{for all} \quad t \geq t_1$$

i.e., J w-attracts bounded subsets of X. Therefore Theorem 2.2 implies that the set

$$J = \bigcap_{t \geq 0} T(t)J$$

is the w-attractor in X under T.

Let us make some remarks about the meaning of the above results for linear C^0-semigroups. Suppose $T(t):X \longrightarrow X$ is a linear w-dynamical system on a reflexive separable Banach space X. If {0} is stable in X and, for every $x \in X$, $T(t)x \rightharpoonup 0$ as $t \longrightarrow \infty$, we claim that {0} is the w-compact attractor. In fact, the stability property implies that $\gamma^+(B)$ is bounded in X if B is bounded in X. Thus, Corollary 2.4 implies that there is a w-compact attractor A, and A is w-stable. If $A \neq \{0\}$, then there is an $x \in A$, $x \neq 0$. Since T(t) is linear, it follows that $C\ell\{\alpha T(t)x,\ t \geq 0\}$ is w-compact and invariant. Thus, $\alpha x \in A$ for any $\alpha \in R$. Thus, A is not compact, which is a contraction.

It is not known if a corresponding result holds in the nonlinear case. More precisely, suppose $T(t):X \longrightarrow X$ is a w-dynamical system on a reflexive separable Banach space X. Also suppose that $\gamma^+(B)$ is bounded in X if B is bounded in X. Suppose J is a w-compact invariant set which w-attracts points of X and J is stable in X. Is J weakly stable? If so, then Theorem 2.5 implies that J is the x-compact attractor.

APPLICATION TO A LINEARLY DAMPED WAVE EQUATION

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, Δ be the Laplacian, $g \in L^2(\Omega)$, $f \in C^1(\mathbb{R},\mathbb{R})$ and suppose there are positive constants β, c such that:

$$|f'(u)| \leq c(|u|^2 + 1) \tag{3.1}$$

$$\overline{\lim_{|u|\to\infty}} f(u)/u \leq 0. \tag{3.2}$$

Consider the wave equation

$$\begin{cases} u_{tt} + 2\beta u_t - \Delta u = f(u) - g & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{3.3}$$

Theorem 3.1. *If* $X = H_0^1(\Omega) \times L^2(\Omega)$, *then problem* (3.3) *defines a weak dynamical system on* X. *Furthermore, there is a* w-*compact attractor* J *in* X, J *is* w-*connected and* w-*stable.*

Proof: If $w_0 = (u_0,v_0) \in X$, let $(u(t), u_t(t)) \in X$ be the solution of (3.3) through w_0 and let $W(t,w_0) = (u(t), u_t(t))$. The function $W: \mathbb{R}^+ \times X \to X$ is defined for $t \geq 0$ and is a dynamical system of X (see Babin and Vishik[2-3], Ball[5], Lopes and Ceron[18]). Furthermore, the map $W(t,\cdot)$: $X \to X$ is bounded dissipative (see Babin and Vishik[3], Lopes and Ceron[18]).

Since W is a dynamical system on X, the conditions (ii), (iii), (iv) of the definition of weak dynamical systems is satisfied. To show (i) is satisfied, observe that f: $H_0^1(\Omega) \to L^2(\Omega)$ is weakly sequentially continuous; that is,

$$u_n \xrightarrow{H_0^1} u \Rightarrow f(u_n) \xrightarrow{L^2} f(u).$$

Since the map $h(v) = 2\beta v$, $h: L^2 \to L^2$ is also weakly sequentially continuous, it follows from Ball and Slemrod's[6] Theorem 2.3 that (i) is satisfied and W is a weak dynamical system on X.

Since $W(t,\cdot)$ is bounded dissipative, there is a bounded set B in X such that B attracts (in the topology of X) bounded sets of X. Since $\gamma^+(B)$ is relatively compact in the weak topology, $\breve{\omega}(B)$ exists and w-attracts B. Therefore, $\breve{\omega}(B)$ w-attracts any bounded set in X and in particular $W(t,\cdot)$ is w-point dissipative. The proof of the theorem is completed by applying Corollary 2.4.

Remark 3.2. If in addition to the hypotheses of Theorem 3.1, it is assumed that the equilibrium points of (3.3) are isolated, then Ball's[5] Theorem 5.16 (p. 261) has shown that the ω-limit set of any point in X (in the topology of X) is an equilibrium point. If E is the set of equilibrium points and all equilibrium points are hyperbolic and E_0 (E_1) is the set of stable (unstable) equilibrium points, $E = E_0 \cup E_1$, then each $(\varphi,0) \in E_0$ is w-connected by an orbit to a point in E_1. This follows from the w-connectedness of the attractor J.

APPLICATION TO DISTRIBUTED PARAMETER CONTROL PROBLEMS

Ball and Slemrod[6-7] have given some nice results on the w-stabilization of distributed parameter problems. The purpose of this section is to show that their proofs together with the general results of Section 2 on w-compact attractors yield stronger results.

Consider the abstract evolutionary equation

$$u_t = Au + f(u) \tag{4.1}$$

where A is the infinitesimal generator of a C^0-semigroup e^{At} on a real Hilbert space H with inner product $<\ ,\ >$ and $f: H \to H$ is a given function. Let $T(t)u_0$ be the solution of (4.1) through u_0 if it exists.

Theorem 4.1. *Suppose*

(i) e^{At} *is dissipative; that is,* $\|e^{At}\| \leq 1$, $t \geq 0$,

(ii) $f: H \to H$ *is locally Lipschitz,*

(iii) $u_n \rightharpoonup u \Rightarrow f(u_n) \to f(u)$,

(iv) $\langle f(\varphi),\varphi\rangle \leq 0$ *for all* $\varphi \in H$.

Then $T(t): H \to H$ *is a* C^0*-semigroup on H and* $T(t)x$ *is a weak dynamical system on H. If the set* M *defined by*

$$M = \{\varphi \in H: \langle T(t)\varphi, f(T(t)\varphi)\rangle = 0, \quad t \in \mathbb{R}^+\}$$

is bounded, then there is a w*-compact attractor* J *for* (4.1), J *is* w*-connected and* w*-stable. For each* $\varphi \in H$, $\tilde{\omega}(\varphi) \subset M$.

Finally, if $M = \{0\}$, *then* $J = \{0\}$.

Proof: Ball and Slemrod's[6] Theorem 2.4 and the proof of that theorem shows that T(t)x is a dynamical system on H, orbits of bounded sets are bounded, T(t)x is a weak dynamical system on H and $\tilde{\omega}(\varphi) \subset M$ for all $\varphi \in H$. The existence and properties of the w-attractor J follows from Corollary 2.4. If M is w-stable, then $J = \{0\}$ from Theorem 2.5.

We remark that, if $M = \{0\}$, then the proof of Theorem 2.4 in Ball and Slemrod[6] imply that {0} is stable. We do not know if this implies {0} is w-stable so that the w-compact attractor would be {0}.

The stabilization problem is defined in Ball and Slemrod[6] as follows: Suppose e^{At} is a contraction semigroup on H such that $\langle A\psi,\psi\rangle \leq 0$ for all $\psi \in D(A)$ and B is a (possibly nonlinear) operator from H to H, suppose v(t) is a real valued function for $t \geq 0$ and consider the system

$$u_t = Au + v(t)Bu. \tag{4.2}$$

The system (4.2) is said to be *stabilizable* (*weakly stabilizable*) if there exists a continuous feedback control v: $H \to \mathbb{R}$ with $v(t) = v(u(t))$, such that the equation

$$u_t = Au + v(u)B(u(t)) \qquad (4.3)$$

satisfies the following properties:

(i) for each $u_0 \in H$, there is a unique solution $u(t,u_0)$ of (4.3) defined for all $t \in \mathbb{R}^+$,

(ii) $\{0\}$ is a stable equilibrium of (4.3),

(iii) $u(t,u_0) \to 0$ ($\rightharpoonup 0$) as $t \to \infty$ for all $u_0 \in H$.

Theorem 4.2. *If* B: $\tilde{H} \to H$ *is sequentially continuous and*

$$M = \{\varphi \in H: \langle e^{At}\varphi, B(e^{At}\varphi)\rangle = 0 \text{ for all } t \in \mathbb{R}^+\}$$

is bounded, then there is a w-*compact attractor* J *for* (4.3) *with* $v(u) = -\langle u,B(u)\rangle$. *If, in addition,* $M = \{0\}$, *then* (4.2) *is weakly stabilizable with this feedback control. If* $M=\{0\}$ *and is* w-*stable, then* $\{0\}$ *is the* w-*compact attractor.*

The proof of this theorem follows from the proof in Ball and Slemrod's[6] Theorem 3.1 making use of Theorem 4.1.

As an application of the previous results, we consider with Ball and Slemrod[6] the following hyperbolic equation. Let V be a real Hilbert space with inner product $\langle\ ,\ \rangle_V$. Let P be a densely defined positive definite self-adjoint linear operator on V such that P^{-1} is everywhere defined and compact. If $V^{1/2} = D(P^{1/2})$, then $V^{1/2}$ is a Hilbert space under the inner product

$$\langle w_1,w_2\rangle_{V^{1/2}} = \langle P^{1/2}w_1,P^{1/2}w_2\rangle_V .$$

Suppose v(t) is a real valued control and consider the wave equation

$$y_{tt} + Py + v(t)y = 0. \qquad (4.5)$$

This equation can be written as a special case of (4.3) with

$$H = V^{1/2} \times V$$

$$u = \begin{bmatrix} y \\ z \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -P & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix}. \tag{4.6}$$

The operator A is skew adjoint on $D(A) = D(P) \times V^{1/2}$ and $B: H \to H$ is compact.

Choose the feedback control

$$v(u) = -\langle u,Bu\rangle_H = -\langle z,y\rangle_V = \langle y(t), y_t(t)\rangle_V. \tag{4.7}$$

From *Theorem* 4.2, we have that (4.5) is weakly stabilizable with feedback control (4.7). If $M = \{0\}$ and is w-stabilizable, then $\{0\}$ is the w-compact attractor.

Ball and Slemrod's[6] Theorem 4.1 shows that $M = \{0\}$ with A and B in (4.6) if and only if the eigenvalues of P are *simple.* These results are directly applicable to the wave equation ($P = -\Delta$ on a bounded domain) with Dirichlet boundary conditions if the eigenvalues of $-\Delta$ are simple. They also apply to the beam equation ($P = +y_{xxxx}$, on $0 < x < 1$) with clamped ends ($y = y_x = 0$, at $x = 0,1$) or simply supported ends ($y = y_{xx} = 0$, at $x = 0,1$). *Condition (iii) of Theorem* 4.1 prevents the application of the previous results to certain types of control problems.

Further results have been given by Ball and Slemrod[7] for the hyperbolic problem

$$y_{tt} + Py + \langle C(y),y_t\rangle C(y) = 0 \tag{4.8}$$

where $P: D(P) \subset V \to V$ is the same as in (4.5) and $C: V^{1/2} \to V$ is locally Lipschitz.

Equation (4.8) is equivalent to a system

$$u_t = Au + f(u)$$

$$u = \begin{bmatrix} y \\ z \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -P & 0 \end{bmatrix}, \quad f(u) = \begin{bmatrix} 0 \\ -\langle C(y), z\rangle_V C(y) \end{bmatrix}$$

which defines a C^0-semigroup $T(t)$ on $H = V^{1/2} \times V$ for which orbits of bounded sets are bounded. Let us now make the following hypotheses:

$U: \mathbb{R}^+ \times H \to H$, $(t,u) \longrightarrow T(t)u$ is a weak dynamical system.

For any $\varphi \in H$, the weak ω-limit set $\tilde{\omega}(\varphi)$ exists. Furthermore, if $E(\varphi) = \|\varphi\|_H^2$ and $u(t) = T(t)\varphi$, then

$$E(u(t)) - E(\varphi) = -\int_0^t \langle C(u(s)), u_t(s)\rangle_V^2 ds.$$

If $\psi \in \tilde{\omega}(\varphi)$ and $u(t_n) \rightharpoonup \psi$, where $t_n \to \infty$, as $n \to \infty$, then Ball and Slemrod's[7] Lemma 3.1 prove that $u(t + t_n) \rightharpoonup e^{At}\psi$. Since $E(e^{At}\psi) = E(\psi)$, $E(u(t_n)) \to E(\psi)$, $E(u(t+t_n)) \to E(e^{At}\psi)$, it follows that

$$\lim_{n\to\infty} \int_0^t \langle C(u(s + t_n)), u(s + t_n)\rangle_V^2 = 0, \quad t \geq 0.$$

Let N be the set in H such that

$$N = \{\psi \in H: \exists\varphi \in H \text{ and a sequence } t_n \to \infty \text{ as } n \to \infty \tag{4.9}$$

such that $T(t_n)\varphi \rightharpoonup \psi$ as $n \to \infty$ and

$$\lim_{n\to\infty} \int_0^t \langle C(T(s + t_n)\varphi), T(s + t_n)\varphi\rangle_V^2 = 0, \ t \geq 0\}.$$

Using Corollary 2.4, we can then state the following generalization of Theorem 4.1.

Theorem 4.3. *If the system* (4.8) *defines a weak dynamical system on* $H = V^{1/2} \times V$ *and the set* N *in* (4.9) *is bounded, then there is a* w-*compact attractor* J *in* H *and* J *is* w-*stable. Furthermore, if* N = {0} *and is* w-*stable, then* J = {0}.

To show that N = {0} is very difficult and is the main essence of the paper of Ball and Slemrod[7]. They have given conditions on the operator C to ensure that this is the case and these conditions are sufficiently general to apply to the beam equation

$$u_{tt} + \Delta^2 u + \langle \Delta u, u_t \rangle_V \Delta u = 0, \text{ in } \Omega$$

$$u = \Delta u = 0 \text{ , in } \partial\Omega$$

where Ω is a bounded domain, $V = H^2(\Omega) \cap H^1_0(\Omega)$ and the eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$ are considered to be simple. Condition (iii) in Theorem 4.1 is not satisfied in this case.

Many other interesting examples also are contained in Ball and Slemrod[7].

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