Conference Report

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INTRODUCTION

The International Symposium on Operator Theory was held on the old campus of the National Technical University, Athens, Greece, from 26 to 31 August 1985. The organizing committee consisted of Drs. D. Kravvaritis and J. Maroulas of the Department of Mathematics of the National Technical University.

The Symposium had vital financial support from the National Technical University, The Ministry of Culture and Science, and the Ministry of Youth of the Government of Greece, and Olympic Airways.

The international nature of the meeting was demonstrated by the fact that the eleven invited speakers represented ten different countries, while the grand total of fifty-three speakers represented no less than twenty-one different countries.

Lists of invited lectures and contributed papers follow, and this report concludes with synopses of several contributed papers.

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INVITED LECTURES

H. Amann, *University of Zurich, Switzerland*
Semigroups and evolution equations

H. Bart, *Erasmus Universiteit, Rotterdam, The Netherlands*
Transfer functions and operator theory

L. Berg, *Wilhelm Pieck Universität, Rostock, D.R. Germany*
I. General operational calculus
II. Finite operational calculus

P. Hess, *University of Zurich, Switzerland*
I. On the spectrum of elliptic operators with respect to indefinite weights
II. On the stable solutions of periodic-parabolic boundary value problems

H. König, *University of Kiel, F.R. Germany*
Eigenvalues of compact operators with applications to integral operators

T. J. Laffey, *University College of Dublin, Ireland*
Simultaneous reduction of sets of matrices under similarity

P. Lancaster, *University of Calgary, Canada*
I. Common eigenvalues, divisors, and multiples of matrix and operator polynomials: A review
II. Generalized invariant subspaces

V. Pták, *Academy of Sciences, Prague, Czechoslovakia*
I. A maximum problem for operators
II. The infinite companion matrix and Bezoutians

H. Schneider, *University of Wisconsin, Madison, U.S.A.*
Some interrelations of algebraic, geometric, combinatorial, and analytic aspects of Perron-Frobenius theory

C. P. Stegall, *University of Linz, Austria*
Radon-Nikodym property in Banach spaces

J. R. L. Webb, *University of Glasgow, Scotland*
Topological degree and *A*-proper operators

CONTRIBUTED PAPERS

In the case of multiple authors, the one presenting the paper at the Symposium is marked †.

G. A. ANASTASSIOU† and O. SHISHA, *University of Rhode Island, Kingston, U.S.A.*
Monotone approximation with linear differential operators

N. BEBIANO and G. N. DE OLIVEIRA,† *University of Coimbra, Portugal*
On a conjecture concerning the determinant of the sum of two normal matrices

J. BENEDETTO, *University of Maryland, U.S.A.*
Maximum entropy and a related spectrum estimation technique

J. BERKOVITS, *University of Oulu, Finland*
On the degree theory for mappings of monotone type and applications

A. BERMAN, *Technion, Haifa, Israel*
Pole assignment with holdability and M-matrices

R. A. BRUALDI, *University of Wisconsin, Madison, U.S.A.*
The spectral radius of matrices of 0's and 1's

L. BURLANDO, *University of Genova, Italy*
On two subsets of a Banach algebra that are related to continuity of spectrum and spectral radius

P. J. BUSHELL, *University of Sussex, Brighton, England*
The Cayley-Hilbert metric and positive operators

F. CHRYSSOVERGHI, *National Technical University, Athens, Greece*
Nonconvex optimal control of monotone partial differential equations

C.-H. CHU† and N. P. H. JEFFERIES, *Goldsmith's College, University of London, England*
On extreme positive linear maps of operator algebras

G. DASSIOS, *University of Patras, Greece*
Degeneracy of partition modes for dissipative systems

H. R. DOWSON,† *University of Glasgow, Scotland*, and D. KOROS, *Piraeus, Greece*
A model for a scalar-type spectral operator

I. DRIGOJIAS, *Institute of Technological Education, Larissa, Greece*
Approximation property for weighted $H^{(p)}$-spaces

G. FOURNIER, *University of Sherbrooke, Canada*
Periodic solutions of pendulum-like equations

H. GATESOUPE, *University of Nantes, France*
Coherent operators in spaces of periodic evolution

A. GIANNOUSIS, *University of Patras, Greece*
Operator theory in Q-analysis

N. HADJISAVVAS, *Nuclear Research Center "Demokritos," Athens, Greece*
Properties of metrics on the set of statistical operators

D. HUYLEBROUCK and J. V. GEEL,† *University of Gent, Belgium*
Diagonalization of idempotent matrices

E. K. IFANTIS and P. D. SIAFARIKAS,† *University of Patras, Greece*
A functional analytic approach for the study of the zeros of Bessel functions

H. KAMOWITZ, *University of Massachusetts, Boston, U.S.A.*
Compact weighted composition operators

S. KARANASIOS, *National Technical University, Athens, Greece*
Factorization along nest algebra modules

I. KLUVANEK, *Flinders University, Bedford Park, Australia*
Scalar operators

D. KRAVVARITIS and N. STAVRAKAKIS,† *National Technical University, Athens, Greece*
Measurability of inverses of random operators

D. KRAVVARITIS† and N. STAVRAKAKIS, *National Technical University, Athens, Greece*
Nonlinear maximal monotone random operators in Banach spaces

J. MAROULAS, *National Technical University, Athens, Greece*
A theorem on the factorization of matrix polynomials

C. MARTINEZ,† M. SANZ, and L. MARCO, *University of Valencia, Spain*
I. The Taylor nest of fractional powers of operators
II. Fractional powers of operators

K. MATTILA, *University of Stockholm, Sweden*
Hermitian operators and isometric isomorphisms on dual Banach spaces

G. MIMINIS,† *Memorial University, St. John's, Newfoundland, Canada*, and C. C. PAIGE, *McGill University, Montreal, Canada*
Implicitly shifted QR-like algorithms

F. A. OLIVEIRA, *University of Coimbra, Portugal*
Differential equations and interval analysis

G. N. DE OLIVEIRA,† *University of Coimbra, Portugal*, and J. A. DIAS DE SILVA, *Lisbon, Portugal*
Matrices satisfying certain polynomial identities

G. PANTELIDIS† and V. NASSOPOULOS, *National Technical University, Athens, Greece*
Über die Existenz gemeinsame Elemente bester Approximation bezüglich zwei Normen

I. POLYRAKIS, *National Technical University, Athens, Greece*
Strongly exposed points in a base for a cone and characterizations of $l_1(\Gamma)$

R. PUYSTJENS† and D. HUYLEBROUCK, *University of Gent, Belgium*
Generalized inverses of a sum

P. ROZSA, *Technical University of Budapest, Hungary*
Strict band matrices and semiseparable matrices

W. RUESS,† *University of Essen, F.R. Germany* and W. H. SUMMERS, *Fayetteville, Arkansas U.S.A.*
Asymptotic behavior of solutions to the abstract Cauchy problem

B. SCHWARZ† and A. ZAKS, *Technion, Haifa, Israel*
Geometry of matrix differential systems

A. G. SISKAKIS, *University of Illinois, Urbana-Champaign, U.S.A.*
Weighted composition semigroups on Hardy spaces and applications

G. A. STAVRAKAS, *University of Athens, Greece*
Borel maps and K-Souslin sets

H. G. TILLMAN, *University of Munster, F.R. Germany*
Störungstheorie und Iterationsverfahren bei zerlegbaren Operatoren

M. TISMENETSKY, *IBM Israel Scientific Center, Haifa, Israel*
The determinant of block-Toeplitz band matrices

F. WILLIAMSON, *Paris, France*
Some methods for nonlinear problems in the set valued case

A. ZAKS† and B. SCHWARZ, *Technion, Haifa, Israel*
Contractions of the matrix unit disk

INEQUALITIES FOR SPECTRUM ESTIMATION

by JOHN J. BENEDETTO[1]

1. *The Spectrum Estimation Problem*

The spectrum estimation problem is to clarify and quantify the following task: find periodicities in a signal x recorded over a fixed time interval $[-T, T]$ [2].

There are several approaches to this problem, the two leading ones being *windowing methods* and *high-resolution methods*. Uncertainty-principle inequalities play a role with windowing methods, and entropy inequalities play a role with the maximum-entropy high-resolution method. Ultimately, I'd like to compare methods by comparing inequalities, instead of comparing methods by two favorite antipodal means, viz., by means of physical intuition or by means of apparent success of algorithms. (I'm going to deal exclusively with deterministic results at the autocorrelation level, as opposed to dealing first with subtleties involving statistical modeling for data.)

The Fourier transform of $f \in L^1(\mathbb{R})$ is

$$\hat{f}(\gamma) = F(\gamma) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t \gamma} dt, \qquad \gamma \in \hat{\mathbb{R}}$$

($\mathbb{R} = \hat{\mathbb{R}}$ is the real line). The *Wiener-Weyl uncertainty-principle inequality* in terms of variances V is

$$V(f)V(F) \geqslant \frac{1}{16\pi^2} \|f\|_2^4, \tag{1.1}$$

where $V(g) = \|ug(u)\|_2^2$ and $\|g\|_2^2 = \int |g(u)|^2 du$; and *Hirschman's entropy inequality* (with an assist from Beckner [1]) in terms of the differential entropy E (e.g., Kolmogorov [12]) is

$$E(|f|^2) + E(|F|^2) \geqslant 1 - \log 2 > 0, \tag{1.2}$$

where $\|f\|_2 = 1$ and $E(|g|) = -\int |g(u)| \log |g(u)| du$. The inequality (1.2) is not true for the compact-discrete duality; cf. Section 4. Using a result of Shannon, Hirschman observed that (1.2) implies (1.1).

Uncertainty-principle inequalities are used in windowing methods to quantify energy loss of estimators; see, e.g., Section 2. Entropy inequalities are used in high-resolution methods to find an estimator which maximizes the

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entropy rate of the process; this maximization is a mathematical guarantee that the least number of assumptions has been made regarding the information content of the unmeasured data at $|t| > T$; see, e.g., [15; 7, pp. 94–96] (J. Edward and M. Fitelson), and cf. Landau's prediction-theoretic approach (Section 3).

2. *Windowing and Energy Loss*

We begin with a special case of the Bell Labs uncertainty principle [6; 10; 14].

THEOREM 2.1. *Given* $T, \Omega > 0$. *There is* $c = c(T\Omega) \in (0,1)$ *such that for each* T *time-limited function* f (*i.e.*, $\operatorname{supp} f \subseteq [-T, T]$),

$$\|F\chi_\Omega\|_2 \leqslant c(T\Omega)\|f\|_2; \tag{2.1}$$

$c(T\Omega)^2$ *is the largest eigenvalue of* BA, *where* $Af = f\chi_T$ *and* $Bg = (G\chi_\Omega)\check{}$ ($\check{}$ *designates the inverse Fourier transform, and* χ_Ω *is the characteristic function of* $[-\Omega, \Omega]$).

If $x(t, \alpha)$, a real stationary stochastic process, is the statistical model associated with our data on $[-T, T]$ and b is a real even data window for which $\|b\|_2 = 1$ and $\operatorname{supp} b \subseteq [-T, T]$, then

$$S_b(\omega, \alpha) = \left| \int_{-T}^{T} b(t) x(t, \alpha) e^{-2\pi i t \omega} \, dt \right|^2$$

is the periodogram associated with x and b. The expectation of S_b is

$$E\{S_b(\omega)\} = S * B^2(\omega).$$

If $B^2 = B_T^2$ is an approximate identity, then S_{b_T} is an asymptotically unbiased estimator, i.e., $E\{S_{b_T}(\omega)\} \to S$ in a weak* topology.

If $E\{S_b\}$ is our estimator for the power spectrum S, then Theorem 2.1 allows us to compute, for $\operatorname{supp} S \subseteq [-\Omega, \Omega]$, that

$$c(T\Omega)\|S\|_2 \geqslant \|\hat{S}\chi_{2T}\| = \left\| \hat{S}(b*b)\frac{1}{b*b} \right\|_{L^2[-2T,2T]} \geqslant \|\hat{S}(b*b)\|_2$$

$$= \|S * B^2\|_2 = \|E\{S_b\}\|_2,$$

where we've taken $b > 0$ on $(-T, T)$, noting $b*b(t) \leqslant \|b\|_2^2 = 1$. The energy loss occurs for $E\{S_b\}$ or its windowing variants, e.g., S_b itself or means of $S_b s$, since $c(T\Omega) < 1$. Thus, in theory, windowing estimators have inherent resolution problems exhibited by uncertainty-principle inequalities.

Having made this criticism of windowing, we must point out that windowing algorithms have provided reliable, understandable estimation, from Bartlett's consistent estimator (a mean of periodograms), to the estimators of Blackman and Tukey, to the more recent resolution of two close peaks by Thomson [16]; cf. [2, Example 6.2b].

3. *Landau's Maximum-Entropy Point of View (Discrete Data)*

The following result is well known [5, 8].

THEOREM 3.1. *Given $c_0, \dots, c_m \in \mathbf{C}$. Define $\bar{c}_j = c_{-j}$ and the Toeplitz matrix $R_n = (c_{jk})$, $c_{jk} = c_{j-k}$. Assume R_n is positive definite. There is a unique positive absolutely convergent Fourier series $S \sim \sum s_j e^{-2\pi i j\gamma}$, $s_j = \hat{S}(j)$, such that*

(a) *$\forall |j| \leqslant n$, $s_j = c_j$, and*

(b) *for every positively absolutely convergent Fourier series F, for which $c_j = \hat{F}(j)$, $|j| \leqslant n$, we can conclude that*

$$\int_0^1 \log F(\gamma)\, d\gamma \leqslant \int_0^1 \log S(\gamma)\, d\gamma. \tag{3.1}$$

We outline a proof due to Landau [13] which begins, as do several others [5, 8], with Szegö's approach, but which depends ultimately on the Szegö alternative and an intimate knowledge of orthogonality.

Szegö's approach. Let Π_n be the space of nth-degree polynomials $S_\alpha = \sum_0^n a_k z^k$, $\alpha = (a_0, \dots, a_n)$, with inner product $[S_\alpha, S_\beta] = \sum a_j \bar{b}_k c_{j-k}$ and norm $\|S_\alpha\| = [S, S]^{1/2}$. Let $P_0, P_1, \dots$ be an orthogonal basis on $\{\Pi_n\}$ with the orthogonality property $[S_\alpha, P_n] = 0$ for each $S \in \Pi_{n-1}$. It is well known that

$$P_m(\zeta) = 0 \quad \text{implies } |\zeta| < 1. \tag{3.2}$$

Landau's proof of (3.2) is especially simple and beautiful: $P_n(\zeta) = 0$ implies $P_n(z) + \zeta S_{n-1}(z) = zS_{n-1}(z)$ for some $S_{n-1} \in \Pi_{n-1}$; and so, by orthogonality and taking $|z| = 1$, we compute

$$\|P_n + \zeta S_{n-1}\|^2 = \|P_n\|^2 + |\zeta|^2 \|S_{n-1}\|^2$$

and

$$\big\| \|zS_{n-1}(z)\| \big\|^2 = \|S_{n-1}\|^2,$$

which combine to yield $1 - |\zeta|^2 > 0$.

Given autocorrelation data $c_0, \dots, c_n$ as in Theorem 3.1 and using (3.2), it is well known by Szegö's approach [11, p. 43] that

$$S = \frac{p_n}{|P_n|^2}, \qquad P_n(z) = \sum_0^n p_j z^j,$$

is an absolutely convergent Fourier series for which $\hat{S}(j) = c_j$, $|j| \leqslant n$.

Landau's approach. Landau's point of view is that the maximum-entropy power spectrum S of Theorem 3.1 is really that which produces the poorest prediction from the past.

To see explicitly how prediction plays a role, let μ be the power spectrum of a process and define

$$I_m(\mu) = \inf_{\{a_k\}} \int_0^1 \left| 1 - \sum_{1 \leqslant k \leqslant m} a_k e^{-2\pi i k \gamma} \right|^2 d\mu(\gamma).$$

If we ask how well 1 can be approximated by the span of $\{e^{2\pi i k\gamma} : k < 0\}$ in $L^2(\mu)$, i.e., how well the present can be predicted from the past, then we are asking to compute $I_\infty(\mu)$. The *Szegö alternative* theorem gives the answer:

$$I_\infty(\mu) = \exp\left\{ \int_0^1 \log F(e^{2\pi i \gamma})\, d\gamma \right\},$$

where F is the absolutely continuous part of μ and $\log F$ is integrable.

Given the autocorrelation data $c_0, \dots, c_n$ of Theorem 3.1, Landau elegantly showed that $I_\infty(\mu) \leqslant I_\infty(S)$ for any positive measure μ whose Fourier transform $\hat{\mu}$ satisfies $\hat{\mu}(j) = c_j$, $|j| \leqslant n$. The verification of this inequality in terms of evaluating polynomials is not so difficult; and Theorem 3.1 follows by taking logarithms.

4. *An Alternative for Maximum Entropy (Continuous Data)*

If continuous autocorrelation data are given on $[-T, T]$, then the maximum entropy theorem analogous to Theorem 3.1 is more difficult to formulate (cf. [9]); and there does not seem to be a smooth transition via digitization of analog spectrum estimation problems when dealing with maximum entropy. To be more precise, we first ask if the entropy estimator is ever preempted in the continuous case. The answer is in the affirmative if the difference between possible estimators must belong to a given small subspace of power spectra, i.e., such a subspace constraint forces a unique estimator. This point of view gives rise to stability problems. If the difference between estimators must belong to certain large subspaces of power spectra, then the weighted variation (corresponding to entropy integrals) of such differences is necessarily great.

These general remarks are quantified by results such as the following and will be exposited elsewhere in greater detail.

THEOREM 4.1 [4]. *Let $W \geqslant 1$ be a continuous weight on $\hat{\mathbb{R}}$ for which $\log W$ is uniformly continuous. The condition*

$$\int \frac{\log W(\gamma)}{1+\gamma^2}\, d\gamma < \infty \tag{4.1}$$

is valid if and only if for every $\varepsilon > 0$ there is a bounded measure R supported by $[-\varepsilon, \varepsilon]$ for which $\hat{R}W \in L^\infty(\hat{\mathbb{R}})$. This latter multiplier property can be replaced by the condition $\int W d|S| < \infty$, $\hat{R} = S$, if an invariance condition is added to (4.1).

THEOREM 4.2 [3]. *Given $\alpha \in (0,1)$. There is a computable constant $C(\alpha) > 0$ such that if $\{f_\varepsilon\}$ is any family of functions satisfying the conditions,*

(a) $\operatorname{supp} f_\varepsilon \subseteq [-\varepsilon, \varepsilon]$,

(b) $\|f_\varepsilon\|_\infty \geqslant M > 0$,

(c) $\|f_\varepsilon\|_\alpha = \int \| \hat{f}_\varepsilon(\gamma) \| 1 + |\gamma|^\alpha | \, d\gamma < \infty$

for all $\varepsilon > 0$, then

$$\forall \varepsilon > 0, \qquad \|f_\varepsilon\|_\alpha \geqslant \frac{C(\alpha) M}{\varepsilon^{\alpha/4}}.$$

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THE SPECTRAL RADIUS OF MATRICES OF 0'S AND 1'S

by RICHARD A. BRUALDI[2]

Let $\mathfrak{A}_n$ be the set of n-by-n matrices of 0's and 1's, and for a square matrix A let $\rho(A)$ be the spectral radius of A. Let $\mathscr{P}$ be a subset of $\mathfrak{A}_n$, which is assumed to have some structure (i.e. $\mathscr{P}$ is not to be an arbitrary

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subset of $\mathcal{P}$). Let

$$\bar{\rho} = \bar{\rho}(\mathcal{P}) = \max_{A \in \mathcal{P}} \rho(A)$$

and

$$\tilde{\rho} = \tilde{\rho}(\mathcal{P}) = \min_{A \in \mathcal{P}} \rho(A).$$

The following general problem was formulated in [2]: Determine $\bar{\rho}$ and $\tilde{\rho}$ (or at least an upper bound on $\bar{\rho}$ and a lower bound on $\tilde{\rho}$). The reason for this problem is that we will then have upper and lower bounds on the spectral radius of each matrix in $\mathcal{P}$ which may improve on the classical bounds.

Now define a $\mathcal{P}$ as follows. Let τ and τ' be nonnegative integers with $\tau + \tau' = n^2$. Let $\mathcal{P}$ be the set of n-by-n matrices of 0's and 1's with exactly τ 1's—equivalently, exactly τ' 0's. This $\mathcal{P}$ is denoted by both $\mathfrak{A}_n(\tau)$ and $\mathfrak{A}'_n(\tau')$ depending on whether the number of 1's or 0's is emphasized. The rest of this discussion refers to these kinds of $\mathcal{P}$'s, and we report on work of Brualdi and Hoffman [1], Friedland [4], and Brualdi and Solheid [3]. First we note that a theorem of B. Schwarz [5] implies that $\bar{\rho}$ is attained by matrices in $\mathfrak{A}_n(\tau)$ of the form

$$\begin{bmatrix} \text{all 1's} & \\ & \text{all 0's} \end{bmatrix}$$

(i.e., the 1's precede the 0's in both rows and columns), while $\tilde{\rho}$ is attained by matrices of the form

$$\begin{bmatrix} & \text{all 0's} \\ \text{all 1's} & \end{bmatrix}$$

(i.e. the 1's precede the 0's in the rows but follow the 0 's in the columns). In both cases equality may be attained by other matrices (even after permutation similarity).

From Schur's inequality one obtains [1] that for $\tau = k^2$, $\bar{\rho} = k$; moreover for $A \in \mathfrak{A}_n(\tau = k^2)$, $\rho(A) = k$ if and only if A is permutation-similar to a matrix in $\mathfrak{A}_n(\tau)$ with its leading k-by-k principal submatrix equal to a matrix of all 1's.

For $\tau = k^2 + 1$, we have [1]: $\bar{\rho} = k$, and for $A \in \mathfrak{A}_n(\tau = k^2 + 1)$, $\rho(A) = k$ if and only if A satisfies the property above, or $k = 1$ and A is permutation-similar to the matrix in $\mathfrak{A}_n(\tau = 2)$ whose leading 2-by-2 principal submatrix equals

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

or $k = 2$ and A is permutation-similar to the matrix in $\mathfrak{A}_n(\tau = 5)$ whose leading 3-by-3 principal submatrix equals

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Friedland [4] obtained the following asymptotic result. Consider $\mathfrak{A}_n(\tau = k^2 + l)$ where $l \geqslant 2$. Then there exists an integer $M(l)$ such that for $k \geqslant M(l)$, $\bar{\rho}$ equals the spectral radius of the matrix A in $\mathfrak{A}_n(\tau = k^2 + l)$ whose leading $(k+1)$-by-$(k+1)$ submatrix equals

$$\left[\begin{array}{c|c} \text{all 1's} & \left.\begin{matrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} \left\lfloor \frac{l}{2} \right\rfloor \\ \hline \underbrace{1 \ \cdots \ 1 \ \ 0 \ \cdots \ 0}_{\left\lfloor \frac{l}{2} \right\rfloor} & 0 \end{array}\right]$$

When $l = 2k$, the above matrix A always satisfies $\rho(A) = \bar{\rho}$. When $l = 2k - 3$, $\bar{\rho}$ is the spectral radius of the matrix in $\mathfrak{A}_n(\tau = k^2 + 2k - 3)$ whose leading $(k+1)$-by-$(k+1)$ principal submatrix equals

$$\begin{bmatrix} \text{all 1's} & & \\ & 0 & 0 \\ & 0 & 0 \end{bmatrix}$$

We now turn to some recent results [3] on $\tilde{\rho}$. Now we need to know the number of 1's or of 0's as a function of n. For if the number of 1's is small compared to n, then $\tilde{\rho} = 0$. We now use the notation $\mathfrak{A}'_n(\tau')$, where τ' denotes the number of 0's. There are two easy cases. When

$$\tau' \geqslant \binom{n+1}{2},$$

$\tilde{\rho}=0$, since there is then a matrix in $\mathfrak{A}_n(\tau')$ with 0's on and above the main diagonal. When

$$\binom{n}{2}\leqslant\tau'<\binom{n+1}{2},$$

then $\tilde{\rho}>0$, hence $\tilde{\rho}\geqslant 1$, and thus $\tilde{\rho}=1$, since there is a matrix in $\mathfrak{A}'_n(\tau')$ with 0's above the main diagonal. This leaves

$$\tau'<\binom{n}{2}:$$

roughly, less than half the entries are 0's. When less than one-quarter of the entries are 0's, we have the following [3]: Let

$$0\leqslant\tau'\leqslant\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil.$$

Then

$$\tilde{\rho}=\tfrac{1}{2}\left(n+\sqrt{n^2-4\tau'}\right).$$

Moreover, for $A\in\mathfrak{A}'_n(\tau')$, $\rho(A)=\tilde{\rho}$ if and only if there are nonnegative integers r and s with $n=r+s$ such that A is permutation-similar to

$$\begin{array}{cc} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} r \\ s \end{array} & \left[\begin{array}{c|c} \text{all 1's} & * \\ \hline \text{all 1's} & \text{all 1's} \end{array}\right] \end{array}.$$

Now suppose $1\leqslant k\leqslant n$, and write $n=qk+l$ where $q\geqslant 1$ and $0\leqslant l<k$. Let $\tau'=\tau'_{n,k}$, where

$$\tau'_{n,k}=\frac{q(q-1)}{2}k^2+qkl.$$

Thus $\tau'_{n,k}$ is the number of 0's in the matrix

$$A_{n,k}=\begin{bmatrix} J_k & & & \text{all 0's} \\ & \ddots & & \\ & & J_k & \\ \text{all 1's} & & & J_l \end{bmatrix},$$

where J_k and J_l are k-by-k and l-by-l matrices, respectively, of all 1's. Clearly

$\rho(A_{n,k}) = k$, and indeed we have [3] $\tilde{\rho} = k$. Necessary and sufficient conditions for a matrix $A \in \mathfrak{A}'_n(\tau'_{n,k})$ to satisfy $\rho(A) = \tilde{\rho} = k$ are described in [3]. As a corollary, for any τ' the minimum spectral radius can be bounded between two consecutive integers. More precisely, let

$$\tau' \leqslant \binom{n}{2}$$

and determine k between 1 and $n-1$ such that

$$\tau'_{n,k+1} \leqslant \tau' < \tau'_{n,k}.$$

Then [3], the minimal spectral radius for $\mathfrak{A}'_n(\tau')$ satisfies $k < \tilde{\rho} \leqslant k+1$.

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DEGENERACY OF PARTITION MODES FOR DISSIPATIVE SYSTEMS by GEORGE DASSIOS[3]

The problem of propagation of thermoelastic waves in an isotropic and homogeneous infinite medium is mathematically formulated as a Cauchy problem for a coupled system of three hyperbolic and one parabolic linear partial differential equations of the second order. The parabolic nature of the heat equation imposes on the system a dissipative character which describes the mechanism of gradual transferring of energy from the mechanical to the irreversable thermal form, during each wave cycle.

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The mathematical problem consists in determining the functions

$$\mathbf{u}:\mathbb{R}^3\times[0,+\infty)\to\mathbb{R}^3,\qquad \mathbf{u}\in C^{(2)}$$

and

$$\Theta:\mathbb{R}^3\times[0,+\infty)\to\mathbb{R}^3,\qquad \Theta\in C^{(2)}$$

which satisfy the differential equations

$$\frac{\mu}{\rho}\Delta\mathbf{u}+\frac{\lambda+\mu}{\rho}\nabla(\nabla\cdot\mathbf{u})-\mathbf{u}_{tt}=\frac{\gamma}{\rho}\nabla\Theta, \tag{1}$$

$$\Delta\Theta-\frac{1}{k}\Theta_t=n\nabla\mathbf{u}_t \tag{2}$$

and the Cauchy data

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \tag{3}$$

$$\mathbf{u}_t(\mathbf{x},0)=\mathbf{u}_1(\mathbf{x}), \tag{4}$$

$$\Theta(\mathbf{x},0)=\Theta_0(\mathbf{x}), \tag{5}$$

where $\mu,\lambda,\rho,\gamma,k,n$ are physical constants and $\mathbf{u}_0,\mathbf{u}_1,\Theta_0$ are sufficiently smooth functions with compact support in $\mathbb{R}^3$. Let $R>0$ be such that

$$[\operatorname{supp}\mathbf{u}_0]\cup[\operatorname{supp}\mathbf{u}_1]\cup[\operatorname{supp}\Theta_0]\subset\bar{B}(0;R). \tag{6}$$

Three kinds of energy norms are connected with this problem: the kinetic energy

$$K(t)=\frac{1}{2}\int_{\mathbb{R}^3}\rho|\mathbf{u}_t(\mathbf{x},t)|^2\,d^3x, \tag{7}$$

the strain energy

$$W(t)=\frac{1}{2}\int_{\mathbb{R}^3}\left[\mu\|\nabla\mathbf{u}(x,t)\|^2+(\lambda+\mu)|\nabla\cdot\mathbf{u}(\mathbf{x},t)|^2\right]d^3\mathbf{x}, \tag{8}$$

and the thermal energy

$$P(t)=\frac{1}{2}\int_{\mathbb{R}^3}\frac{\varepsilon(\lambda+2\mu)}{n^2k^2}|\Theta(\mathbf{x},t)|^2\,d^3x \tag{9}$$

where ε is a dimensionless coupling constant taking values in the interval $[0,6\sqrt{3}-10)$. The above energy-norms satisfy the fundamental energy equation

$$\frac{d}{dt}[K(t)+W(t)+P(t)]+X_T(t)=0, \tag{10}$$

where

$$X_T(t)=\frac{\lambda_0}{T_0}\int_{\mathbb{R}^3}|\nabla\Theta(\mathbf{x},t)|^2\,d^3x \tag{11}$$

is the dissipation function and λ_0, T_0 are also constants of physical interest.
Define the *nth directional moment of* $\mathbf{f}$ *in the direction* $\mathbf{a}$ by

$$M_{\mathbf{f}}^{(n)}(\mathbf{a})=2\pi^{-3/2}\int_{\mathbb{R}^3}[\mathbf{a}\cdot\mathbf{f}(\mathbf{x})](\mathbf{a}\cdot\mathbf{x})^n\,d^3x, \tag{12}$$

where $u\in\mathbb{N}_0=\{0,1,2,\dots\}$. Also define the *order of symmetry of* $\mathbf{f}$ by

$$v=\min\big\{m\in\mathbb{N}_0|\exists\mathbf{a}_0\in\mathbb{R}^3\colon M_{\mathbf{f}}^{(n)}(\mathbf{a}_0)\neq 0,\quad M_{\mathbf{f}}^{(n)}(\mathbf{a})=0,$$
$$\forall\mathbf{a}\in\mathbb{R}^3,\quad n=0,1,2,\dots,m-1\big\}. \tag{13}$$

In [4] we proved that, after we eliminate the solenoidal part of the solution which gives rise to the transverse wave, then as $t\to+\infty$,

$$K(t)=\left(1+\frac{1}{\varepsilon}\right)\Gamma_2 t^{-(m+\frac{3}{2})}+o\big(t^{-(m+\frac{3}{2})}\big), \tag{14}$$

$$W(t)=\left(\Gamma_1+\frac{1}{\varepsilon}\Gamma_2\right)t^{-(m+\frac{3}{2})}+o\big(t^{-(m+\frac{3}{2})}\big), \tag{15}$$

$$P(t)=\left(\frac{1}{\varepsilon}\Gamma_1+\Gamma_2\right)t^{-(m+\frac{3}{2})}+o\big(t^{-(m+\frac{3}{2})}\big), \tag{16}$$

where Γ_1 and Γ_2 are some known constants that depend on the initial data and

$$m = \min\{ v_0 + 1, v_1, v_\Theta \}, \tag{17}$$

v_0, v_1, v_Θ being the orders of symmetry for $\mathbf{u}_0$, $\mathbf{u}_1$, and Θ_0, respectively.

We introduce the vector

$$(K_\infty, W_\infty, P_\infty) = \left(\left(1 + \frac{1}{\varepsilon} \right) \Gamma_2, \Gamma_1 + \frac{1}{\varepsilon}\Gamma_2, \frac{1}{\varepsilon}\Gamma_1 + \Gamma_2 \right), \tag{18}$$

which is a constant vector for each given set of Cauchy data. The vector $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$ is called a *partition vector* whenever it belongs to the orthogonal complement of the vector subspace spanned by $(K_\infty, W_\infty, P_\infty)$.

The following theorems are simple consequences of the above relations:

THEOREM 1. *In the generic case, where the orders of symmetry of the initial data are arbitrary, there exists exactly one partition vector given by* $(1 - \varepsilon, -1, \varepsilon)$.

THEOREM 2. *In the nongeneric cases where*

(i) $v_0 + 1 = m,\ m < v_1,\ m < v_\Theta$,
(ii) $v_1 = m,\ m < v_0 + 1,\ m < v_\Theta$,
(iii) $v_\Theta = m,\ m < v_0 + 1,\ m < v_1$,

besides the partition vector $(1 - \varepsilon, -1, \varepsilon)$ *of Theorem* 1 *there exists a second linearly independent partition vector which is given by*

(i′) $\left(1 + \varepsilon^2 \left(\frac{\varepsilon}{2} \right)^{m + \frac{1}{2}}, -1, -1 \right)$
(ii′) $(1, -1, -1)$
(iii′) $\left(1 + \left(\frac{\varepsilon}{2} \right)^{m + \frac{1}{2}}, -1, -1 \right)$,

respectively.

THEOREM 3. *In the nongeneric cases where*

(i) $v_\Theta = v_0 + 1 = m,\ m < v_1$,
(ii) $v_\Theta = v_1 = m,\ m < v_0 + 1$,
(iii) $v_1 = v_0 + 1 = m,\ m < v_\Theta$,

the only partition vector that exists is the one given in Theorem 1.

As a result of the above theorems we observe that in the cases of Theorem 1 and Theorem 3 there is only one mode of partition of energy, which is described by

$$W_\infty = (1-\varepsilon)K_\infty + \varepsilon P_\infty, \tag{19}$$

and it states that as $t \to +\infty$ the strain energy is partitioned into a convex combination of the kinetic and the thermal energy.

In the particular nongeneric cases given by Theorem 2, there are infinitely many modes of partition of energy, exhibiting the structure of a two-dimensional vector space. In these cases any linear combination of the vector $(1-\varepsilon, -1, \varepsilon)$ and the vector given by (i′), (ii′), or (iii′) is also a partition vector. In other words, there exists a partition degeneracy which is characteristic of the dissipative system of thermoelasticity.

Other physically interesting dissipative systems such as the generalized thermoelasticity, the magnetoelasticity in a conducting medium, and the coupled magnetothermoelasticity are possible areas for further investigation of the asymptotic behavior of the energy norms.

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A FUNCTIONAL-ANALYTIC APPROACH FOR THE STUDY OF THE ZEROS OF BESSEL FUNCTIONS

by E. K. IFANTIS and P. D. SIAFARIKAS[4]

1. *Introduction*

The singular differential equation

$$z^2\frac{dy}{dz} + (\alpha_0 + \alpha_1 z)y = h(z), \tag{1.1}$$

where $h(z) = \sum_{n=1}^{\infty} h_n z^{n-1}$ is analytic in some neighborhood of zero, has been the subject of several investigations and generalizations [1, 2]. If $h(z)$

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belong to the Hardy-Lebesgue space $H_2(\Delta)$, i.e., the Hilbert space of functions $y(z)=\sum_{n=1}^{\infty}y_n z_{\infty}^{n-1}$ which are analytic in $\Delta=\{z:|z|<1\}$ and satisfy the condition $\sum_{n=1}^{\infty}|y_n|^2<+\infty$, then a necessary and sufficient condition for Equation (1) to have solutions in $H_2(\Delta)$ is the following [3]:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\alpha_0^{n-1}\frac{h_n}{\Gamma(\alpha_1+n-1)}=0, \qquad \alpha_0\neq 0. \tag{1.2}$$

It is easy to see that for $\alpha_0=-\rho/2$, $\alpha_1=\mu+1$, and

$$h(z)=-\frac{\rho}{2}\exp\left(-\frac{\rho}{2}z\right) \qquad \left[h_n=\frac{(-1)^n}{(n-1)!}\left(\frac{\rho}{2}\right)^n\right],$$

the left-hand side of Equation (1.2) is the ordinary Bessel function $J_\mu(\rho)$. Thus it follows from the above result that $\rho\neq 0$ is a zero of the Bessel function $J_\mu(z)$ if and only if the equation

$$z^2y'(z)+\left(-\frac{\rho}{2}+(\mu+1)z\right)y(z)=-\frac{\rho}{2}\exp\left(-\frac{\rho}{2}z\right), \qquad y(0)=1.$$

has a solution in $H_2(\Delta)$.

On the other hand it is well known [4] that the study of Equation (1.1) in $H_2(\Delta)$ is equivalent to the study of an operator equation in an abstract Hilbert space H with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. This equation has the form $(V+K)f=h$, $h\in H$, where V is the shift operator ($Ve_n=e_{n+1}$, $n=1,2,\dots$) and K is compact. In the case of Bessel functions the above equation can be transformed to an eigenvalue equation of the form $L_\mu T_0 g=(2/\rho)g$, $g\in H$, where L_μ is the diagonal operator

$$L_\mu e_n=\frac{1}{n+\mu}e_n, \qquad n=1,2,\dots, \; n\neq-\mu,$$

and $T_0=V+V^*$ (V^* is the adjoint of V). In case $\mu>-1$ the operator $L_\mu T_0$ is similar to the compact self-adjoint operator $S_\mu=L_\mu^{1/2}T_0L_\mu^{1/2}$, where $L_\mu^{1/2}$ is the diagonal operator

$$L_\mu^{1/2}e_n=\frac{1}{\sqrt{n+\mu}}e_n, \qquad n=1,2,\dots.$$

More precisely we have the following:

THEOREM 1.1 [3]. *$\rho \neq 0$ is a zero of the Bessel function $J_\mu(z)$ if and only if $2/\rho$ is an eigenvalue of the operator $L_\mu T_0$ or the operator $T_0 L$.*

COROLLARY 1.1 [3]. *For $\mu > -1$, $\rho \neq 0$ is a zero of $J_\mu(z)$ if and only if $2/\rho$ is an eigenvalue of the compact and self-adjoint operator S_μ.*

Theorem 1.1 and Corollary 1.1 lead us to an operator approach in an abstract separable Hilbert space for the study of the zeros of the Bessel function $J_\mu(z)$. In section 2 we give some results concerning the real and complex zeros of the Bessel function $J_\mu(z)$.

2. *Main Results*

I. Using some properties of the operators $L_\mu T_0$ and S_μ, we are led easily to some alternative proofs of well-known properties of zeros of Bessel functions (Lommel-Hurwitz theorem, Rayleigh's formula, Burget's hypothesis [5]) and some results which are not presented as new although we were unable to find them in the literature. These results are the following [3].

(1) For every real ρ there exists a sequence of real numbers μ_n, $|\mu_n| \to +\infty$, such that $J_{\mu_n}(\rho) = 0$, $n = 1,2,\ldots$.
(2) If μ is nonreal then the function $J_\mu(z)$ has no real zeros,....
(3) If $\mu = \mu_1 + i\mu_2$, $\mu_2 \neq 0$, $\rho = \rho_1 + i\rho_2$, $\rho_2 \neq 0$, and $J_\mu(\rho) = 0$, then

$$|\rho_1| > |1 + \mu_1|, \qquad \mu_1 > -1,$$
$$|\rho_2| \geqslant |\mu_2|$$

II. Let $J_\mu(z)$ and $J_\nu(z)$ be ordinary Bessel functions of order μ and ν respectively. Based on the results of Theorem 1.1 and Corollary 1.1 and using different functional-analytic techniques, we have proved an inequality which relates the first positive zero of the ordinary Bessel function $J_\nu(z)$ and the absolute value of the real part of any zero of $J_\mu(z)$. More precisely, we have proved the following:

THEOREM 2.1 [6]. *For $\nu > \operatorname{Re}\mu > -1$, $\mu \in \mathbb{C}$, any zero $\rho = \operatorname{Re}\rho + i\operatorname{Im}\rho \neq 0$ of $J_\mu(z)$ satisfies the inequality*

$$|\operatorname{Re}\rho| > \rho_{\nu,1}\frac{1+\operatorname{Re}\mu}{1+\nu}, \tag{2.1}$$

where $\rho_{\nu,1}$ is the first positive zero of $J_\nu(z)$.

Some lower bounds for the absolute value of the complex zeros of $J_\mu(z)$ follow easily from the proof of Theorem 2.1. These lower bounds are the following:

$$|\rho| > \rho_{\nu,1} \qquad \text{for} \quad \operatorname{Re}\mu > \nu, \tag{2.2}$$

$$|\rho| > \frac{\rho_{\nu,1}}{1+\nu}\left[(1+\operatorname{Re}\mu)^2 + (\operatorname{Im}\mu)^2\right]^{1/2} \qquad \text{for} \quad -1 < \operatorname{Re}\mu < \nu, \tag{2.3}$$

$$|\rho| > \rho_{\nu,1}\min\left\{\frac{\varepsilon}{k+\nu}, \frac{1-\varepsilon}{k+\nu+1}\right\}, \quad \nu > -1,$$

$$\text{for} \quad \mu = -k-\varepsilon, \quad 0<\varepsilon<1, \quad k = 1,2,\dots. \tag{2.4}$$

From (2.1) or (2.3) it follows that if μ is real and $\nu > \mu > -1$, then the first positive zeros of $J_\mu(z)$ and $J_\nu(z)$ satisfy the inequality

$$\frac{\rho_{\mu,1}}{\rho_{\nu,1}} > \frac{1+\mu}{1+\nu}. \tag{2.5}$$

A number of simple lower and upper bounds for the first positive zero of $J_\nu(z)$ follow immediately from the inequality (2.5). These bounds are better than many well-known lower and upper bounds found recently by several authors (see the references of [6]).

III. In Ref. [7] was proved the following.

THEOREM 3.1 [7]. *For $\nu > -1$ every zero $\rho(\nu) \neq 0$ of $J_\nu(z)$ satisfies the differential equation*

$$\frac{d\rho(\nu)}{d\nu} = \rho(\nu)(L_\nu x(\nu), x(\nu)), \qquad \nu > -1, \tag{3.1}$$

where $x(\nu)$ is a normalized element of H.

A principal result which follows easily from (3.1) is the following:

COROLLARY 3.1 [7]. *Every positive zero $\rho(\nu)$ of $J_\nu(z)$ satisfies the inequality*

$$\frac{d\rho(\nu)}{d\nu} > 1, \qquad -1 < \nu < +\infty. \tag{3.2}$$

The differential inequality (3.2) has attracted recently the attention of many authors. McCann and Love [8], in order to prove the inequality

$$\rho_{\nu,1} > \rho_{0,1} + \nu, \qquad 0 < \nu < +\infty, \tag{3.3}$$

have proved (3.2) in the interval $0 < \nu < 0.05$. More recently Elbert, Gatteshi, and Laforgia [9] have defined a function $j_{\nu,k}$ for every $k > 0$, which for $k = 1,2,\ldots$ is the function $\rho_{\nu,k}$, and proved that the inequality

$$\frac{dj_{\nu,k}}{d\nu} > 1 \tag{3.4}$$

holds in the intervals $(-1, -\frac{1}{2}), (-\frac{1}{2}, 0), (0, \infty)$ under several assumptions which require the knowledge of lower bounds on $j_{\nu,k}$. In all the above, $\rho_{\nu,k}$ means the kth positive zero of $J_\nu(z)$. From (3.2) it follows immediately that

$$\rho_{\nu,k} - \nu > \rho_{\mu,k} - \mu, \qquad \nu > \mu > -1. \tag{3.5}$$

This relation unifies and improves a number of lower and upper bounds given for the positive zeros of $J_\nu(z)$. In fact, for $\mu = 0$ in (3.5) we have

$$\rho_{\nu,k} > \rho_{0,k} + \nu, \qquad 0 < \nu < +\infty. \tag{3.6}$$

This is a well-known lower bound. For $k = 1$ it was proved by McCann and Love [8], and for $k \geqslant 1$ by Laforgia and Muldoon [10]. Also, for $\mu = \frac{1}{2}$ in (3.5) we have

$$\rho_{\nu,k} > k\pi - \tfrac{1}{2} + \nu, \qquad \tfrac{1}{2} < \nu < +\infty. \tag{3.7}$$

In general the lower bound (3.7) is better than (3.6). In particular, for $k = 1$ we have

$$\rho_{\nu,1} > 2.64159 + \nu > \rho_{0,1} + \nu, \qquad \nu > \tfrac{1}{2}. \tag{3.8}$$

Note that the lower bound $\rho_{\nu,1} > \rho_{0,1} + \nu$ is better than a number of lower bounds found recently by many authors (see p. 261 of Ref. [6]).

Finally, from Theorem 3.1 one obtains easily the following

COROLLARY 3.2 [7]. *For every positive zero $\rho_{\nu,k}$ of $J_\nu(z)$ the function*

$$\frac{\rho_{\nu,k}}{1+\nu} \tag{3.9}$$

is a strictly decreasing function in $-1<\nu<+\infty$.

This means that

$$\frac{\rho_{\mu,k}}{1+\mu} > \frac{\rho_{\nu,k}}{1+\nu}, \qquad \nu>\mu>-1. \tag{3.10}$$

For $\nu=0$, (3.10) gives the inequality

$$\rho_{\mu,k} > \rho_{0,k}(1+\mu), \qquad -1<\mu<0, \tag{3.11}$$

which for $k=1$ is the well-known lower bound $\rho_{\mu,1} > \rho_{0,1}(1+\mu)$, $-1<\mu<0$, given by Laforgia and Muldoon [10]. Also, for $\nu=\frac{1}{2}$ in (3.10) we have

$$\rho_{\mu,k} > \tfrac{2}{3}k\pi(1+\mu), \qquad -1<\mu<\tfrac{1}{2}. \tag{3.12}$$

For $k=1$ in (3.12) we obtain the lower bound $\rho_{\mu,1} > \frac{2}{3}\pi(1+\mu)$, $-1<\mu<\frac{1}{2}$, which in the interval $(0,\frac{1}{2})$ is better than the well-known lower bound $\rho_{\mu,1} > \rho_{0,1}+\nu$.

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WEIGHTED COMPOSITION OPERATORS
by HERBERT KAMOWITZ[5]

We present a brief summary of some properties of a class of bounded linear operators on Banach function spaces. Let X denote a locally compact space or a measure space, B a Banach space or Banach algebra of complex-valued functions on X, φ a function $X \to X$, and u an element of B. If $C_\varphi : f(x) \to f(\varphi(x))$ maps B into B, then C_φ is called a *composition operator* on B, and if $uC_\varphi : f(x) \to u(x)f(\varphi(x))$ maps B into B, then uC_φ is called a *weighted composition operator* on B.

Varying B or φ leads to widely studied classes of operators such as multipliers (if $\varphi =$ identity), weighted shift operators on sequence spaces, or composition operators on $H^p(\Omega)$, where Ω is a region in $\mathbb{C}$ or $\mathbb{C}^n$.

The aim here is twofold: to state some background results of ours and to give references to related results.

Our interest in this type of problem arose in a Banach-algebra setting. In [4] we showed that if T is a nonperiodic automorphism of a commutative semisimple Banach algebra, then the spectrum of T, $\sigma(T)$, is a connected set containing the unit circle. This is a problem of the type under consideration, since every nonzero endomorphism T of a commutative semisimple Banach algebra B induces a continuous function φ on the maximal ideal space $\mathcal{M}_B$ of B such that $\widehat{Tf}(x) = \hat{f}(\varphi(x))$ for all $x \in \mathcal{M}_B$. The result in [4] led to considering the spectra of other endomorphisms such as endomorphisms of algebras of analytic functions on disks and annuli, and we obtained complete results when the fixed point of the inducing map φ is in the interior of the region [5, 7]. From here we turned to looking at the spectra of composition operators on H^p of the unit disk and again obtained complete results when

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the fixed point of φ is in the interior of the disk [6]. Much more extensive results in general may be found in a very comprehensive paper by C. Cowen [2].

Regarding weighted composition operators (on Banach algebras, weighted endomorphisms), there are results on the spectra of weighted endomorphisms of the disk algebra A when the map φ is a Möbius function [8]. In this case the spectrum of uC_φ depends on the behavior of u at the fixed points of φ. The type of transformation—parabolic, hyperbolic, or elliptic—enters into the proofs. The result is that if φ is nonperiodic, then $\sigma(uC_\varphi)$ is either an annulus, a circle, or a disk. Further, in [9] we characterized compact weighted endomorphisms on A.

If X is a compact set, $\varphi: X \to X$, and $u \in C(X)$, then uC_φ is a compact weighted endomorphism on $C(X)$ if, and only if, for each connected component C of $\{x \mid u(x) \neq 0\}$ there is an open set $V \supset C$ on which φ is constant [10]. If $X = I = [0,1]$, say, and φ is differentiable, then the last condition can be written $u\varphi' = 0$. Recently we have extended this to weighted composition operators on $W_{m,p}[I]$, a Sobolev space, and to $C^{(m)}(I^n)$, the Banach algebra of functions on I^n which have continuous partial derivatives through order m. For $W_{m,p}$, again, uC_φ is compact if, and only if, $u\varphi' = 0$ [11], and for $C^{(m)}(I^n)$, a weighted endomorphism uC_φ is compact if and only if $u\nabla\varphi_i = 0$ for each $i = 1,2,\dots,n$ [12].

We close by giving several references.

(1) Antonevic [1] discusses the spectra of uC_φ on $W_{m,p}[I]$ when φ is a homeomorphism of I.

(2) Kitover [13] has results on weighted endomorphisms of uniform algebras.

(3) Gorin [3] also has results on the spectra of weighted endomorphisms of uniform algebras, with particular attention to the disk algebra and the case where φ is a linear fractional transformation.

Finally, for the most comprehensive results on H^p we again refer to [2].

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SCALAR OPERATORS
by IGOR KLUVÁNEK[6]

Let E be a Banach space, $L(E)$ the space of all bounded linear operators on E, and I the identity operator on E.

An operator $T \in L(E)$ is of scalar type in the sense of Dunford if there exists a space Ω, a σ-algebra $\mathscr{S}$ of subsets of Ω, a multiplicative set function $P: \mathscr{S} \to L(E)$ which is σ-additive in the strong operator topology such that $P(\Omega) = I$, and a P-integrable function f such that

$$T = \int_\Omega f \, \mathrm{d}P \tag{1}$$

—equivalently, if there exists a σ-complete Boolean algebra $\mathscr{A}$ of bounded projections on E such that T belongs to the closed linear span of $\mathscr{A}$ in $L(E)$.

For short, we shall call such operators σ-scalar, departing thus to some extent from the established terminology.

The theory of σ-scalar operators is beautiful, rich, and fruitful. The monograph [1] is of course an indispensable reference.

However, the requirement that $\mathscr{S}$ be a σ-algebra and P be strongly σ-additive or, alternatively, that the Boolean algebra $\mathscr{A}$ be σ-complete,

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excludes many operators of interest from the class of σ-scalar operators. Moreover, this requirement may be considered extraneous in some sense; it seems that many results of the theory could be obtained without it. For, if the integral (1) exists, then there exist $\mathscr{S}$-simple functions f_j, $j = 1, 2, \ldots$, such that

$$\sum_{j=1}^{\infty} \left\| \int_\Omega f_j \, \mathrm{d}P \right\| < \infty, \tag{2}$$

the equality

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega) \tag{3}$$

holds for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty, \tag{4}$$

and

$$\int_\Omega f \mathrm{d}P = \sum_{j=1}^{\infty} \int_\Omega f_j \, \mathrm{d}P \tag{5}$$

So the integral (1) can be defined purely in terms of the operator-norm convergence. This well-known fact leads us to the following definitions.

An operator $T \in L(E)$ is called a *scalar operator* if there exists a Boolean algebra $\mathscr{A}$ of bounded projections on E such that T belongs to the operator-norm closed linear span of $\mathscr{A}$ in $L(E)$.

Let $\mathscr{R}$ be an algebra of subsets of a space Ω. An additive set function $P: \mathscr{R} \to L(E)$ is called *very additive* if

$$\sum_{j=1}^{\infty} \int_\Omega f_j \, \mathrm{d}P = 0$$

for any $\mathscr{R}$-simple functions f_j, $j = 1, 2, \ldots$, satisfying the condition (2), such that

$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which the inequality (4) holds.

Let $P: \mathscr{R} \to L(E)$ be a very additive set function. A scalar-valued function f on Ω is said to be *P-integrable* if there exist $\mathscr{R}$-simple functions f_j, $j = 1, 2, \ldots$, satisfying the condition (2) such that the equality (3) holds for every $\omega \in \Omega$ for which the inequality (4) does. The integral of the function f is then defined by (5). The definition of integral is unambiguous because P is very additive.

Then the Beppo Levi theorem for the so-defined integral holds. That is to say, if f_j, $j = 1, 2, \ldots$, are P-integrable functions, satisfying the condition (2), and f is a function on Ω such that the equality (3) holds for every $\omega \in \Omega$ for which the inequality (4) does, then the function f is P-integrable and the equality (5) holds.

Now it turns out that an operator $T \in L(E)$ is scalar if and only if there exist a space Ω, and algebra $\mathscr{R}$ of subsets of Ω, a very additive and multiplicative set function $P: \mathscr{R} \to L(E)$ such that $P(\Omega) = I$, and a P-integrable function f on Ω such that the equality (1) holds.

An interesting class of scalar operators which are not σ-scalar are the translations in L^p spaces and some other L^p-multiplier operators, $1 < p < \infty$, $p \neq 2$. The failure of translations to be σ-scalar is illuminated in [2].

Assume that $1 < p < \infty$. Let $\mathscr{R}$ be the family of all sets in $\Omega = \mathbb{R}$ whose characteristic functions are L^p-multipliers. For each set $X \in \mathscr{R}$, let $P(X)$ be the multiplier operator generated by the characteristic function of the set X. That is to say, if $1 < p \leqslant 2$ and $\varphi \in L^p(\mathbb{R})$, then $P(X)\varphi = \psi$, where ψ is the element of $L^p(\mathbb{R})$ whose Fourier transform is the Fourier transform of φ multiplied by the characteristic function of X. If $2 \leqslant p < \infty$, then $P(X)$ is the adjoint of the similarly defined operator in $L^q(\mathbb{R})$, where $p^{-1} + q^{-1} = 1$.

Because the multiplier operator generated by a bounded measurable function f is the zero operator if and only if f vanishes almost everywhere (with respect to the Lebesgue measure), it is easy to see that the set function $P: \mathscr{R} \to L(X)$ is very additive. Obviously, it is multiplicative and $P(\mathbb{R}) = I$.

The domain, $\mathscr{R}$, of P is rather rich. It is known to contain all intervals and all sets of the form $X_{t,J} = \{\omega : \exp(it\omega) \in J\}$, where $t \in \mathbb{R}$ and J is a connected subset of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. What is more, there exists a constant c_p such that $\|P(X)\| \leqslant c_p$ for each interval $X \subset \mathbb{R}$, and $\|P(X_{t,J})\| \leqslant c_p$ for each t and J. Of course, $\mathscr{R}$ contains all sets belonging to the algebra generated by intervals and the sets $X_{t,J}$. And still more sets belong to $\mathscr{R}$.

These facts then imply, via the Beppo Levi theorem, that every function of finite variation is P-integrable. Similarly, every periodic function with finite variation in a period is P-integrable. In particular, if $s \in \mathbb{R}$ and $f(\omega) = \exp(is\omega)$, for $\omega \in \mathbb{R}$, then the function f is P-integrable. The multiplier operator generated by f is of course the translation by s. There are also many P-integrable functions which neither have finite β-variation, $\beta \geqslant 1$, nor are periodic and have finite β-variation in a period, nor are a Lipschitzian of any order $\alpha \leqslant 1$.

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THE INFINITE COMPANION MATRIX
by VLASTIMIL PTÁK[7]

The infinite companion matrix of a polynomial was introduced by the author [1] in the course of his investigation of the maximum problem, which was discussed in the lecture [5]. The notion of the infinite companion turned out to be useful also on a number of other occasions—see e.g. the notes [3, 4].

Here we limit ourselves to listing its most important properties. For the proofs we refer the reader to the papers [1, 2, 3, 4]. A systematic account is being prepared.

Let p be a polynomial of degree n,

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

with $a_n = 1$. We denote by $C(p)$ the companion matrix of the polynomial p,

$$C(p) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}.$$

The infinite companion matrix $C^{\infty}(p)$ corresponding to the polynomial p has n columns numbered $0, 1, \ldots, n-1$ and an infinite number of rows $0, 1, 2, \ldots$.

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(1) The entries in the rth row

$$t_{r,0} \quad t_{r,1} \quad \cdots \quad t_{r,n-1}$$

are defined as follows. If $m_r(\lambda)$ is the remainder obtained upon dividing λ^r by $p(\lambda)$, then

$$m_r(\lambda) = \sum_{j=0}^{n-1} t_{r,j}\lambda^j.$$

The notation $C(p)$ and $C^\infty(p)$ will be abbreviated to C and C^∞ unless there is a danger of a misunderstanding.

(2) The jth column c_j of $C^\infty(p)$ is the solution of the recursive relation

$$a_0 x_r + a_1 x_{r+1} + \cdots + a_n x_{r+n} = 0$$

with the initial conditions

$$x_j = 1$$

and $x_k = 0$ for all $0 \leqslant k \leqslant n-1$ different from j.

(3) It follows from (1) that, given any linear operator A for which $p(A) = 0$ and any nonnegative r, then

$$A^r = \sum_{k=0}^{n-1} t_{rk} A^k.$$

(4) Given any $r = 0, 1, 2, \ldots$, the matrix consisting of the n consecutive rows of $C^\infty(p)$ starting with the row of index r equals $C(p)^r$.

(5) In terms of the roots of p, the entries may be described as follows: if $\alpha_1, \ldots, \alpha_n$ are the roots of p then

$$t_{rj} = (-1)^{n-j-1} \sum \binom{q(e_1, \ldots, e_n)^{-1}}{n-j-1} \alpha_1^{e_1} \cdots \alpha_n^{e_n},$$

the summation ranging over all n-tuples of nonnegative integers $e_1, \ldots, e_n$ such that $\Sigma e = r - j$, while $q(e_1, \ldots, e_n)$ stands for the number of those e which are positive.

The next property of the infinite companion is formulated in terms of the generating function. Accordingly we shall assume that all roots of p are less than one in modulus.

(6) If $F(z,y)$ is the function

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n-1} t_{rj}y^j z^r,$$

then

$$\begin{aligned}F(z,y) &= \frac{1}{1-zy}\left(1-\frac{p(y)}{p(1/z)}\right)\\ &= \frac{1}{1-zy}\left(1-\frac{z^n p(y)}{p_1(z)}\right)\\ &= \frac{1}{p_1(z)}z^{n-1}\frac{p(1/z)-p(y)}{1/z-y},\end{aligned}$$

where p_1 is the reciprocal polynomial

$$p_1(z)=z^n p\left(\frac{1}{z}\right).$$

The generating function $F(z,y)$ of the matrix C^{∞} is closely related to the kernel of a projection (or interpolation) operator in the Hardy space H^2, the Hilbert space of all functions holomorphic in the open unit disk such that their Taylor coefficients a_n satisfy $\Sigma|a_n|^2<\infty$.

The space H^2 may be decomposed into the direct sum of the set P_n of all polynomials of degree $\leqslant n-1$ and the subspace $M(p)$ of all multiples of the polynomial p.

(7) Denote by R the projection operator onto P_n corresponding to this decomposition. Then, for any $m\in H^2$,

$$\begin{aligned}(Rm)(s) &= \frac{1}{2\pi i}\int F\left(\frac{1}{t},s\right)m(t)\frac{dt}{t}\\ &= \frac{1}{2\pi i}\int\left(1-\frac{p(s)}{p(y)}\right)\frac{m(y)}{y-s}\,dy.\end{aligned}$$

In this manner F appears as an interpolation operator: indeed, Rm is a polynomial of degree $\leqslant n-1$ which coincides with m at each root α of p; this, of course, including multiplicities, so that Rm and m coincide at α together with their derivatives up to order $k-1$ if p is divisible by $(z-\alpha)^k$.

(8) If q is an arbitrary positive integer, we denote by C_q^∞ the matrix consisting of the first q rows of C^∞, that is, the rows with indices $0, 1, \ldots, q-1$. Also, we denote by $\mathcal{S}_p$ the q-by-infinity matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \end{pmatrix}$$

then

$$\mathcal{S}_q C^\infty = C_q^\infty C.$$

In a similar manner

$$\mathcal{S} C^\infty = C^\infty C$$

if $\mathcal{S}$ is the infinite analog of $\mathcal{S}_q$.

(9) Explicit formulas for the solution of Lyapunov-type equations may be obtained using the intertwining relation of item (8). Given two monic polynomials p_1 and p_2 of degree n_1 and n_2 respectively, we are looking for an n_2-by-n_1 matrix X such that

$$X - C(p_2)^* X C(p_1) = E(n_2, n_1),$$

where $E(n_2, n_1)$ is the n_2-by-n_1 matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If E stands for the infinite analog of these matrices, $E(\cdot,\cdot)$, we note that $1 - \mathcal{S}^*\mathcal{S} = E$. Setting $N_1 = C^\infty(p_1)$, $N_2 = C^\infty(p_2)$, we have

$$\begin{aligned} N_2^* N_1 - C_2^* N_2^* N_1 C_1 &= N_2^* N_1 - N_2^* \mathcal{S}^* \mathcal{S} N_1 = N_2^*(1 - \mathcal{S}^*\mathcal{S}) N_1 \\ &= N_2^* E N_1 = E(n_2, n_1). \end{aligned}$$

It follows that $N_2^* N_1$ is the explicit solution.

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NONLINEAR RANDOM EQUATIONS IN BANACH SPACES

by DIMITRIOS KRAVVARITIS and NICOS STAVRAKAKIS[8]

Let X, Y be Banach spaces and Ω a measurable space. Let $T : \Omega \times X \to Y$ be a random operator (i.e., for each $x \in X$, $T(\cdot)x$ is measurable) and $\eta : \Omega \to Y$ a measurable mapping. The random equation corresponding to the double $[T, \eta]$ asks for a measurable mapping $\xi : \Omega \to X$ such that

$$T(\omega)\xi(\omega) = \eta(\omega) \qquad \text{for all} \quad \omega \in \Omega.$$

The systematic study of random equations as initiated by Špaček [19] and Hanš [7,8]. They proved random fixed-point theorems for contraction mappings. Later various results on random fixed-point theorems for single-valued or multivalued mappings were given by Bharucha-Reid [1], Itoh [9, 11, 12], and Engl [4, 5, 6]. Recently, Kannan and Salehi [13] studied random Hammerstein equations with monotone operators. Their method of proving the measurability of solutions was based on the uniqueness of solutions of these equations. In [10] Itoh proved the existence of solutions of random equations with monotone operators. He used the theory of measurable selections of measurable multifunctions to obtain random solutions (not necessarily unique).

We state now some definitions from nonlinear functional analysis and random-operator theory.

Let X be a reflexive Banach space, X^* its dual space, and (x^*, x) the pairing between $x^* \in X^*$ and $x \in X$. Let D be a subset of X, and L an

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operator from D into X^*. Then L is said to be: (1) *monotone* if

$$(Lx - Ly, x - y) \geqslant 0 \qquad \text{for all} \quad x, y \in D, \tag{$*$}$$

(2) *maximal monotone* if it is monotone and there is no proper extension of L that is also a monotone operator, (3) *strictly monotone* if the equality in $(*)$ implies that $x = y$, (4) *of type* M if for any sequence $\{x_n\}$ in D for which $x_n \to x$ in X, $Lx_n \to x^*$ in X^*, and $\limsup(Lx_n, x_n - x) \leqslant 0$, we have $x^* = Lx$, (5) *demicontinuous* if for any sequence $\{x_n\}$ in D with $x_n \to x \in D$, it follows that $Lx_n \to Lx$.

In the following Ω will denote a measurable space with a σ-algebra $\mathscr{A}$. Ω is called *complete* if there exists a complete σ-finite measure defined on $\mathscr{A}$. We denote by $B(\Omega, X)$ the set of all measurable mappings $\xi: \Omega \to X$ such that $\sup\{\|\xi(\omega)\|: \omega \in \Omega\} < \infty$. A random operator $T: \Omega \times D \to X^*$ is said to be *coercive* if there exists a function $c: \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{r \to \infty} c(r) = +\infty$, such that $(T(\omega)x, x) \geqslant c(\|x\|)\|x\|$ for all $\omega \in \Omega$ and $x \in D$. A random operator T is said to be monotone (demicontinuous, etc.) if for each $\omega \in \Omega$, $T(\omega)$ is monotone (demicontinuous, etc.).

We present now some existence theorems for nonlinear random equations with operators of monotone type. The measurability of solutions depends mainly on the selection theorem of Kuratowski and Ryll-Nardzewski [16]. For corresponding deterministic results we refer to Browder [3], Brezis [2], and Pascali and Sburlan [18].

The next existence theorem for random equations with maximal monotone operators was proved in [15].

THEOREM 1. *Let Ω be complete, X be a separable reflexive Banach space, and D be a subset of X with $0 \in D$. Let $T: \Omega \times D \to X^*$ be a coercive, demicontinuous maximal monotone random operator such that $\sup\{\|T(\omega)0\|: \omega \in \Omega\} < \infty$. Then for each $\eta \in B(\Omega, X^*)$ there exists $\xi \in B(\Omega, X)$ such that*

$$T(\omega)\xi(\omega) = \eta(\omega) \qquad \textit{for all} \quad \omega \in \Omega.$$

The proof of this result is based on the Debrunner-Flor lemma for monotone random operators, given also in [15]. Itoh [10] has proved an existence theorem for maximal monotone random operators $T: \Omega \times D \to X^*$, where Ω is a measurable space, not necessarily complete, and D a dense linear subspace of X.

The following existence theorem for nonlinear random equations with densely defined operators of type (M) was obtained in [14].

THEOREM 2. *Let X be a separable reflexive Banach space and D be a subset of X. Let $T: \Omega \times D \to X^*$ be a bounded, coercive random operator of type* (M). *Suppose that there exists a dense linear subspace X_0 of X which is contained in D such that for each finite-dimensional subspace F of X_0, the random operator $T: \Omega \times F \to X^*$ is demicontinuous. Then for each $\eta \in B(\Omega, X^*)$ there exists $\xi \in B(\Omega, X)$ such that*

$$T(\omega)\xi(\omega) = \eta(\omega) \qquad \text{for all} \quad \omega \in \Omega.$$

In [14] there is also studied a random Hammerstein equation involving a linear monotone operator and an operator of type (M), as well as nonlinear random inequalities.

Now we present two new existence theorems for nonlinear random equations. The key to our results is the next

LEMMA. *Let X be a separable Banach space and Y be a Banach space. Let $T: \Omega \times X \to Y$ be a continuous random operator such that for each $\omega \in \Omega$, $T(\omega)$ is invertible and its inverse $T(\omega)^{-1}$ is demicontinuous. Then the operator $S: \Omega \times Y \to X$ defined by $S(\omega)y = T(\omega)^{-1}y$ $(\omega \in \Omega,\ y \in Y)$ is random.*

This lemma extends a result in [17], where both operators T and S were assumed to be continuous.

We give an existence theorem for nonlinear random Hammerstein equations involving monotone and noncoercive operators.

THEOREM 3. *Let X be a separable reflexive Banach space. Let $K: \Omega \times X^* \to X$ be a linear monotone random operator and $A: \Omega \times X \to X^*$ a continuous, strictly monotone random operator. Suppose that the operator $[I + K(\omega)A(\omega)]^{-1}$ is bounded. Then for each measurable mapping $\eta: \Omega \to X$ there exists a unique measurable mapping $\xi: \Omega \to X$ such that*

$$\xi(\omega) + K(\omega)A(\omega)\xi(\omega) = \eta(\omega) \qquad \text{for all} \quad \omega \in \Omega.$$

We consider a random equation which contains operators of the form $L + T$, where L is an unbounded linear monotone random operator and T a random operator of type (M).

THEOREM 4. *Let X be a separable reflexive Banach space and D be a dense linear subspace of X. Let $L: \Omega \times D \to X^*$ be a linear maximal mono-*

tone random operator, such that for each $\omega \in \Omega$, $L(\omega)$ *is one-to-one and onto. Let* $T: \Omega \times X \to X^*$ *be a random operator which is bounded, coercive, and of type* (M). *Then for each* $\eta \in B(\Omega, X^*)$ *there exists* $\xi \in B(\Omega, X)$ *such that*

$$L(\omega)\xi(\omega) + T(\omega)\xi(\omega) = \eta(\omega) \qquad \text{for all} \quad \omega \in \Omega.$$

In proving this theorem we apply an approximation procedure by considering, for each $\omega \in \Omega$, the Yosida approximants $L_\varepsilon(\omega)$ $(\varepsilon > 0)$ of $L(\omega)$ (cf. Pascali and Sburlan [18]).

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WEAK* CONTINUITY OF HERMITIAN OPERATORS AND ISOMETRIC ISOMORPHISMS ON DUAL BANACH SPACES

by KIRSTI MATTILA[9]

Introduction

In the first part of this paper we will consider weak* continuity of the Hermitian operators in the decomposition $T^* = H + iK$ on the dual space of a Banach space (See Problem 1 below). In the second part we will consider Banach spaces X such that the canonical projection of X either is Hermitian or satisfies some weaker condition. We will then obtain results on weak* continuity of the Hermitian operators on X^* (Theorem 5) and on weak* continuity of certain isometric isomorphisms (Theorems 4 and 6).

Let X and Y be complex Banach spaces. The dual space of X will be denoted by X^*. Further, $B(X,Y)$ will be the space of all bounded linear operators from X to Y, and $B(X) = B(X,X)$. It is well known that an operator $S \in B(Y^*, X^*)$ is continuous in the weak* topology if and only if S is the adjoint of an operator $T \in B(X,Y)$. The adjoint of T will be denoted by T^*.

An operator $T \in B(X)$ is called *Hermitian* if $\|e^{itT}\| = 1$ for all real numbers t. It is easy to see that T is Hermitian if and only if T^* is Hermitian. Moreover, T is Hermitian if and only if the numerical range $V(T) = \{ f(Tx) : x \in X,\ f \in X^*,\ \|x\| = \|f\| = f(x) = 1\}$ is a subset of the real numbers. An account of numerical ranges and Hermitian opreators is given in [3] and [4].

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1. *The Decomposition* $H + iK$

PROBLEM 1. If H and K are Hermitian operators on X^* and if the operator $H + iK$ is weak* continuous, are H and K weak* continuous?

This question was asked by E. Behrends in [2], where he also proved that this is true if $T^* = H + iK$ is normal (i.e. if $HK = KH$). For the following Banach spaces the answer is affirmative.

THEOREM 1. *Assume that either*

(i) *X is the range of a projection of norm one on X^{**} (in particular, X may be a dual space), or*

(ii) *X is a C^*-algebra with unit.*

Then, if $T \in B(X)$ and $T^* = H + iK$ for some Hermitian operators H and K, it follows that H and K are weak continuous.*

On the other hand, Theorem 2 below will give an affirmative answer to Problem 1 for a class of operators including the normal operators.

DEFINITION 1. An operator T $\in$ B(X) is called **-hyponormal* if $T = H + iK$ for some Hermitian operators H and K and the inequality

$$\|e^{z\bar{T}}e^{-\bar{z}T}\| \leqslant 1, \qquad (*)$$

where $\bar{T} = H - iK$, holds for all complex numbers z.

It was proved in [1] that subnormal operators on a Hilbert space have the property $(*)$. Normal operators on a Banach space are obviously *-hyponormal. It can be shown that *-hyponormal operators are hyponormal [i.e., they satisfy the condition $V(i(HK - KH)) \subset \{t \in \mathbb{R} : t \geqslant 0\}$].

THEOREM 2. *Let T be an operator on X such that*

(i) *T^* is *-hyponormal (let $T^* = H + iK$) and*

(ii) *the operator $HK - KH$ is weakly compact.*

*Then H and K are weak * continuous.*

In proving Theorem 2 we also prove the following generalization of Fuglede's theorem:

THEOREM 3. *Let T be a *-hyponormal operator on Y, and U a *-hyponormal operator on X, where X and Y are Banach spaces. If $TS = S\bar{U}$ for some $S \in B(X,Y)$, then $\bar{T}S = SU$. (For the notation $\bar{T}$ see Definition 1.)*

The proofs of Theorems 1, 2, and 3 are given in Reference [10].

2. *The Canonical Projection, Hermitian Operators, and Isometric Isomorphisms*

Let i_X denote the canonical embedding of X into X^{**}, and let $\hat{X}$ be the range of i_X. The *canonical projection* of X is the operator $P_X = i_{X^*} \circ i_X^*$. It is well known that P_X is a projection on X^{***} whose range is $\widehat{(X^*)}$ and whose kernel is the annihilator of $\hat{X}$ in X^{***}. Clearly $\|P_X\| = 1$.

DEFINITION 2. A Banach space X is said to have the property

(H) if every Hermitian operator on X^* is weak* continuous;
(I) if every isometric isomorphism on X^* is weak* continuous;
(UP) (we also say that X is the *unique predual* of X^*) if whenever Y is a Banach space such that Y^* is isometrically isomorphic to X^*, then every isometric isomorphism of Y^* onto X^* is weak* continuous.

We have the implications (UP) $\Rightarrow$ (I) $\Rightarrow$ (H).

The results of Harmand and Lima in [8] showed that the canonical projection and the properties in Definition 2 are related. By [8], the Banach space X is an M-ideal in X^{**} if and only if P_X is an L-projection. If X is an M-ideal in X^{**}, then by [8, Proposition 4.2] X and X^* have the property (I). It follows from [9, Theorem 2.6] and [6, Theorems 15 and 18] or from Theorem 6 that X^* is even the unique predual of its dual. An L-projection is a Hermitian operator. There are Banach spaces where the canonical projection is Hermitian, but not an L-projection. Such are for example the spaces

$$c_0 \oplus_{l_1} c_0$$

and the l_p-sum

$$\left(\sum c_0\right)_{l_p} \quad \text{for} \quad p > 1.$$

PROBLEM 2. Which of the properties (H) and (I) does X have if

(i) P_X is Hermitian, or
(ii) $I - 2P_X$ is an isometric isomorphism, or
(iii) $\|I - P_X\| = 1$?

It is easy to see that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). For the cases (i) and (ii) we can prove the following theorems:

THEOREM 4. *Assume that P_X is Hermitian.*

(i) *If Y is a Banach space such that $I - 2P_Y$ is an isometric isomorphism and if T is an isometric isomorphism of Y^* onto X^*, then T is weak* continuous.*
(ii) *X has the property* (I).

THEOREM 5. *If $I - 2P_X$ is an isometric isomorphism, then X has the property* (H).

Finally, the following results for X and the dual space of X are obtained from [5, Proposition 2 and Theorem] and [6, Theorems 15 and 18].

THEOREM 6. *If $\|I - P_X\| = 1$, then $X \not\supset l_1$ and X^* has the property* (UP).

The proofs of Theorems and 4 and 5 will be given elsewhere.

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BAND MATRICES AND SEMI-SEPARABLE MATRICES[10]

by P. RÓZSA[11]

Sparse matrices, in particular band matrices, play an important role both in theoretical and in practical problems. Band matrices are characterized by the property that their nonzero elements are to be found in a certain neighborhood of the main diagonal only. It is known that the inverse of a tridiagonal matrix with nonzero super- and subdiagonal elements is a one-pair matrix [3]. It is not obvious, however, what happens when the matrix is slowly "filled up," i.e. how the elements of the inverse behave when the width of the band grows. By making use of the theory of linear difference equations, T. Oohashi has dealt with this problem in [4]. W. W. Barrett and Ph. J. Feinsilver in [2] related the vanishing of a certain set of minors in a matrix to the vanishing of a corresponding set of minors in the inverse. As a corollary they obtained the first theorem of E. Asplund [1].

Let us recall some basic definitions and theorems from [1].

Definition 1. A band matrix of grade p is a square matrix $[a_{ij}]$ whose elements satisfy $a_{ij} = 0$ for $j > i + p$.

Definition 2. A Green's matrix of grade p is a square matrix $[a_{ij}]$ whose submatrices have rank $\leqslant p$ if their elements belong to the part of a_{ij} for which $j + p > i$.

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THEOREM 1 (Asplund). *A nonsingular square matrix is a band matrix of grade p if and only if its inverse is a Green's matrix of grade p.*

THEOREM 2 (Asplund). *A nonsingular square matrix is a band matrix of grade p with nonvanishing elements in the pth diagonal above the main diagonal if and only if its inverse is the sum of a matrix of rank p and a band matrix of grade $-p$ (and hence also a Green's matrix of grade p).*

Definitions 1 and 2 can be extended:

DEFINITION 3. A square matrix is called a strict band matrix of grade p if it is a band matrix of grade p with nonvanishing elements in the pth diagonal above the main diagonal.

DEFINITION 4. A square matrix is called a *strict band matrix of grades* $\begin{Bmatrix} p \\ q \end{Bmatrix}$ if it is a strict band matrix of grade p and its transpose is a strict band matrix of grade q.

DEFINITION 5. A square matrix is called a *Green's matrix of grades* $\begin{Bmatrix} p \\ q \end{Bmatrix}$ if it is a Green's matrix of grade p and its *transpose* is a Green's matrix of grade q.

As an example, let us consider the tridiagonal matrices with nonzero super- and subdiagonal elements, i.e. the strict band matrices of grades $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$. The inverse of a symmetrical strict tridiagonal matrix is a *one-pair matrix* (see [3, p. 90]) or *separable* matrix, with elements r_{ij} defined as

$$r_{ij} = \begin{cases} u_j v_j & \text{for} \quad i \leqslant j, \\ v_i u_j & \text{for} \quad i \geqslant j. \end{cases}$$

Obviously this is a Green's matrix of grades $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, since each submatrix above the subdiagonal and below the superdiagonal has rank 1. In other words, since $[u_i v_j]$ is a *one-rank matrix*, we may say that a one-pair matrix can be obtained from a one-rank matrix by cutting it in two halves and reflecting it across the main diagonal.

More generally, the elements of a Green's matrix of grades $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ whose inverse has nonzero super- and subdiagonal elements can be written as

$$r_{ij} = \begin{Bmatrix} u_i v_j & \text{for} \quad i \leqslant j \\ z_i w_j & \text{for} \quad i \geqslant j \end{Bmatrix}, \qquad u_i v_i = z_i w_i,$$

i.e., $[r_{ij}]$ can be considered as the half of a one-rank matrix $[u_i v_j]$ *above* the main diagonal and the half of $[z_i w_j]$ *below* the main diagonal. Thus we may say that such a Green's matrix of grades $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ can be regarded as two semi-pair matrices: one above and one below the main diagonal.

Inspired by the relation between the semipair matrices and the strict band matrices of grades $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, the question can be raised whether there exists a similar relation between an arbitrary strict band matrix of grades $\begin{Bmatrix} p \\ q \end{Bmatrix}$ and a certain number of semipair matrices. In order to answer this question let us introduce the following definition.

DEFINITION 6. The matrix $R = [r_{ij}]$ with

$$r_{ij} = \begin{cases} \sum\limits_{\nu=1}^{p} u_i^{(\nu)} v_j^{(\nu)} & \text{for} \quad i < j + p, \\ \sum\limits_{\mu=1}^{q} z_i^{(\mu)} w_j^{(\mu)} & \text{for} \quad i > j - q \end{cases} \tag{1a}$$

is called a $\begin{Bmatrix} p \\ q \end{Bmatrix}$ *semiseparable matrix* if

$$\sum_{\nu=1}^{p} u_i^{(\nu)} v_j^{(\nu)} - \sum_{\mu=1}^{q} z_i^{(\mu)} w_j^{(\mu)}$$

$$\begin{cases} = 0 & \text{for} \quad -p < j - i < q, \\ \neq 0 & \text{for} \quad j - i = -p \text{ and } j - i = q. \end{cases} \tag{1b}$$

Making use of this definition, the main result can be formulated in the following theorem.

THEOREM 3. *A nonsingular square matrix is a strict band matrix of grade* $\begin{Bmatrix} p \\ q \end{Bmatrix}$ *if and only if its inverse is a* $\begin{Bmatrix} p \\ q \end{Bmatrix}$ *semiseparable matrix.*

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BOREL MAPS AND $\mathscr{K}$-SOUSLIN SETS

by G. A. STAVRAKAS[12]

1. *Introduction*

In this note we are concerned with topological spectra, in relation with $\mathscr{K}$-analytic sets and in particular with Lusin and Souslin sets (see below for definitions). Also, we examine properties of spaces which are images of $\mathscr{K}$-Souslin topological vector spaces under a continuous injective linear map.

Let E and F be topological algebras. By a *topological algebra* (respectively, *locally convex*) we mean an algebra E equipped with a Hausdorff topology such that the (underlying) vector space is a topological (respectively, locally convex) vector space and the multiplication in E is separately continuous (considered as a bilinear map of $E \times E$ into E).

Let $\mathscr{M}(E, F)$ be the *generalized spectrum* of E (for given F) (i.e. the set of nonzero continuous algebra homomorphisms of E into F) topologized as a subset of $\mathscr{L}_s(E, F)$, the space of continuous linear maps between the topological vector spaces E and F, equipped with the topology of simple convergence in E. If E and F are unital algebras, then the elements of $\mathscr{M}(E, F)$ are assumed to be identity-preserving [18].

$\mathscr{K}$-analytic sets or $\mathscr{K}$-Souslin sets (every $\mathscr{K}$-analytic set is $\mathscr{K}$-Souslin) constitute a class of sets larger than Borel sets, with important properties in their applications to integration and Radon measure theory. We know that

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every Lusin or Souslin space (they belong to the class of $\mathscr{K}$-Souslin spaces) is a Radon space, i.e., on these spaces every finite Borel measure is a Radon measure. Also, we note that the measurability on these spaces is obtained on spaces which are not locally compact. This is very useful, because some of the function spaces in probability theory are not locally compact. For basic definitions and theorems on $\mathscr{K}$-Souslin sets see [1, Chapter 3], [3, Chapter IX], [4, I], [15, II], [17, Appendix], [5, Chapter 8], [8], [6], [9].

A topological space is said to be *polish* if it is separable, if there is a distance on the space compatible with the topology, and if the space is complete. Let $X_{[\tau]}$ be a Hausdorff topological space. $X_{[\tau]}$ is said to be *Souslin* (*Lusin*) if it is the continuous (injective) image of a polish space.

2. *On Spaces with Separable Model*

DEFINITION 2.1 [13]. Let V be a metrizable topological vector space. A (topological vector) Fréchet space E is said to be a *model* of V if $V = \varphi(E)$, where φ is a continuous injective linear map. Equivalently, we say that V has a model if there is a Fréchet space E and a continuous isomorphism of E onto V. If the model E of V is separable, we have the definition of a separable model.

PROPOSITION 2.2 [2, p. 193; 14, p. 84; 15, p. 110]. *Let V be a metrizable topological vector space with separable model E. Then the closed unit balls of the weak* duals of E and V are polish sets.*

Now, we describe necessary conditions such that sets of Borel linear maps coincide with sets of continuous linear functions. We introduce the following symbolisms. If V, W are topological vector spaces $\mathscr{L}_0(V, W)$ is the set of surjective linear continuous maps. $\mathscr{B}_0(V, W)$ is the set of surjective linear Borel maps and $E_p(V, W)$ is the set of surjective linear maps with Borel graph. We state the following:

LEMMA. *Let E, F be locally convex spaces. Furthermore, let E be a Souslin and Baire space. If the linear map $f\colon E \to F$ is Borel, then f is a linear continuous map.*

Proof. See [10, p. 80], [17, p. 555]. ∎

PROPOSITION 2.3. *Let V, W be locally convex spaces with separable models. Furthermore, suppose that V is a Baire space. Then*

$$\mathscr{L}_0(V, W) = E_p(V, W) = \mathscr{B}_0(V, W)$$

Proof. By the Lemma and [15, Lemmas 12, 13]. ■

3. *On Topological Algebras*
The *Gel'fand transform* of xE is the map

$$\hat{x}: \mathcal{M}(E,V) \to V: h \to \hat{x}(h) = h(x),$$

and the *generalized Gel'fand map* is the (algebraic) morphism G which is defined by the relation

$$G: E \to C_c(\mathcal{M}(E,V),V): x \to G(x) = \hat{x}$$

The image of E under the Gel'fand map is said to be the Gel'fand transform of E, and we symbolize it with

$$E^{\wedge} = G(E) \subseteq C(\mathcal{M}(E))$$

if $V = \mathbb{C}$.

Corollary 3.1. *Let V be a metrisable topological algebra of a separable model E. Then, the Gel'fand transform algebra $\hat{E}$ is a Souslin space and there is a Borel map between the spaces V and $C_c(\mathcal{M}(E))$.*

Proof. See [3, Chapter IX, §6, Lemma 7] and [12, p. 153]. ■

Proposition 3.2 [6, p. 66; 7]. *If $E(\mathbb{C})$ is a unitary separable commutative Banach algebra with separable spectrum, the space $C_c(\mathcal{M}(E),\mathbb{R})$ is Souslin.*

Theorem 3.3. *Let X be a locally compact and $C_c(X)$ be a separable metrizable topological algebra. Then X is a polish space, and if E is a unitary closed subalgebra of $C_c(X)$, its spectrum $\mathcal{M}(E)$ is a Lusin space.*

Proof. [11, pp. 312, 185, 167], [15, II], [18]. ■

Proposition 3.4. *If E is an arbitrary subalgebra of $C(X)$, there are Lusin subsets in its spectrum.*

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