ΑΠΑΝΤΗΣΕΙΣ ΠΡΟΒΛΗΜΑΤΩΝ

ΚΕΦΑΛΑΙΟ 5

5.2 Προβλήματα

1. $r = 1$

2. $r=1$

3. $r = \frac{2}{3}$

4. $r = 1$

5. $r =1$

6. $r = 1$

7. $r = +\infty$

8. $r = 0$

9. $r = 1$

10. $r = 0$

11. $r = \infty$

12. $r = \frac{1}{4}$

13. αποκλίνει στα ± 1,

14. συγκλίνει στα ± 1

15. $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k+1)!} x^{2k+1} = x - \sin x$

16. $1 + \sum_{k=0}^{\infty} \frac{2}{(2k+1)!} x^{2k+1} = 1 + 2\sin hx$

17. $\sum_{k=1}^{\infty} \frac{x^{2k}}{k!}$

18. $\sum_{n=0}^{\infty} \frac{3^n e^3}{n!} (x-1)^n$

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$

20. $5+(-5)(x+1) + (x+1)^2$

21. $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-2)^n$

22. $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

23. $-\sum_{n=0}^{\infty} \frac{x^{2n}}{2^{n+1}}$

24. $\ln 2 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{24} x^3 - \frac{1}{64} x^4 + \frac{1}{160} x^5$

25. $1 + \frac{\pi}{2} + \left(x - \frac{\pi}{2}\right) - 2\left(x - \frac{\pi}{2}\right)^2 + \frac{2}{3}\left(x - \frac{\pi}{2}\right)^4$

26. $2x^2 - \frac{1}{3} x^4$

27. $1 - x^2 + 2x^4$

28. $\frac{9}{4} + 3\left(x - \frac{1}{2}\right) + \left(x - \frac{1}{2}\right)^2$

29. $\sin(1) + \cos(1)\, x - \frac{1}{2}\sin(1)\, x^2 - \frac{1}{6}\cos(1)\, x^3 + \frac{1}{24}\sin(1)\, x^4 + \frac{1}{120}\cos(1)\, x^5$

30. $1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5$

31. $\ln\frac{4}{3} + \frac{3}{4}\left(x - \frac{1}{3}\right) - \frac{9}{32}\left(x - \frac{1}{3}\right)^2 + \frac{9}{64}\left(x - \frac{1}{3}\right)^3 - \frac{81}{1024}\left(x - \frac{1}{3}\right)^4 + \frac{243}{5720}\left(x - \frac{1}{3}\right)^5$

32. $\sum_{n=4}^{\infty} \frac{(-1)^n x^n}{2n-2}$

33. $\sum_{n=1}^{\infty} \frac{(2n+5)\, x^n}{n+1}$

34. $\sum_{n=2}^{\infty} \frac{2^{n-1}x^n}{(n-1)^2-3}$ **35.** $\sum_{n=3}^{\infty} \frac{2(n-3)}{n-1} x^n$ **36.** $\sum_{n=-1}^{\infty} (n-2)\, nx^n$

37. $F(x) = (2a_2 - a_1 + \lambda a_0) + [6a_3 - 2a_2 + (\lambda - 2)a_1]x$
$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (\lambda - n - n^2)a_n\right]x^n$.

38. $F(x) = 2a_2 + (a_1 + 6a_3)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + n^2 a_n\right]x^n$

40. $1 + 2x + \sum_{n=2}^{\infty} (2^{n-1} + 1 + n)\, x^n$ **41**. $1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \left(\frac{(n+1)!}{(n+1)^2} + 2^n\right) x^n$

42. $\sum_{n=1}^{\infty} \left(\frac{1}{2n} - (n+2)^{n+2}\right) x^n$ **43.** $x + x^2 + x^3 + x^4 + \sum_{n=5}^{\infty} [1 + (n-2)!]\, x^n$

44. $\frac{1}{2} + \frac{1}{3}x + \sum_{n=2}^{\infty} \left(\frac{1}{n+2} + (-2)^{n-1}\right) x^n$

5.3 Προβλήματα

1. όλα τα $x \in IR$ ομαλά

2. όλα τα $x \in IR$ ομαλά

3. –1, 1 ΚΑ

4. 0,1 ΚΑ

5. 0,1 ΜΚΑ

6. 0 ΚΑ

7. 0 ΚΑ, 1 ΜΚΑ

8. –1 ΜΚΑ

9. 0,3 ΚΑ

10. 0, –1 ΚΑ

11. 2, –3 ΚΑ, 0 ΜΚΑ

12. –1, 0, ΟΜ, 1 ΜΚΑ

13. –1, 0, 1 ΚΑ

14. –2, –1, 0, 1 ΚΑ

15. 3i, –3i, –7 ΚΑ

16. 2, ±nπ, n = 1, 2, … ΚΑ

17. –1, ±nπ, n = 1, 2, … ΚΑ, 0 ΜΚΑ

18. –4, –2, 4 ΚΑ, 0 ΜΚΑ

19. –3, –1, 1, 2, –i, i ΚΑ

20. $y(x) = a_0 \sum_{k=0}^{\infty} (k+1)\, x^{2k} + a^1 \sum_{k=0}^{\infty} \frac{2k+3}{3} x^{2k+1}$, $-1 < x < 1$

21. $y(x) = a_0 \sum_{k=1}^{\infty} \frac{4(-1)^k (x-1)^{3k}}{(3k-1)(3k-4)\, k!} + a_1 \left[(x-1) + \frac{1}{4}(x-1)^4\right]$, $x \in IR$

22. $y(x) = a_0 \left(1 + \frac{x^2}{2} - \sum_{k=2}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \ldots (2k-3)}{2^k k!} x^{2k}\right) + a_1 x$, $x \in IR$

23. $y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$, $x \in IR$

24. $y(x) = a_0\left(1 - 2x_2 + x_4/3\right) + a_1\left(x + \sum_{k=1}^{\infty} \frac{(-3)(-1)\ I\ (2k-5)}{(2k+1)!} x^{2k+1}\right), \quad x \in IR$

25. $y(x) = a^0\left(1 + \sum_{k=1}^{\infty} \frac{1\,.\,4\ \ldots\ (3k-5)(3k-2)}{(3k)!} x^{3k}\right)$

$+ a^1\left(x + \sum_{k=1}^{\infty} \frac{2\,.\,5\ldots(3k-4)(3k-1)}{(3k+1)!} x^{3k+1}\right), \quad x \in IR, \quad a3k+2 = 0, \quad k = 0, 1, \ldots$

26. $y(x) = a_0 \sum_{k=0}^{\infty} x^{2k} + a_1 \sum_{k=0}^{\infty} x^{2k+1} = \frac{a_0 + a_1 x}{1 - x^2}, \quad -1 < x < 1$

27. $y(x) = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!2^k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!!}, \quad x \in IR$

28. $y = a_0 + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)\,3^k}, \quad x \in (-\sqrt{3}, \sqrt{3})$

29. $y(x) = a_0\left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{120}x^5 +\right.$

$\left. + 9 \sum_{k=3}^{\infty} \frac{(-1)^k [(2k-5)!!]^2}{(2k+1)!3^k} x^{2k+1}\right), \quad x \in (-\sqrt{3}, \sqrt{3})$

30. $y(x) = a_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{360}x^5\right.$

$\left. + 3 \sum_{k=3}^{\infty} \frac{(-1)^k (2k-5)!!}{(2k+1)!3^k} x^{2k+1}\right), \quad x \in IR$

31. $y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!3^k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{1\ \ 4\ \ I\ \ (3k+1)}, \quad x \in IR$

32. $y(x) = 1 + x + \frac{x^2}{2} + \frac{x^4}{24} + \ldots, \quad -1 < x < 1$

33. $y(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{15} - \ldots, \quad 0 < x < 2$

34. $y(x) = a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{20} + \ldots\right) + a_1\left(x - \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{20} + \ldots\right), \quad x \in IR$

35. $y = \frac{1}{2}(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{48} + \ldots, \quad 0 < x < 2$

36. $y(x) = 1 - x + x^2 - \frac{x^3}{2} - \frac{x^4}{12} - \frac{x^5}{60} - \frac{x^6}{48} + \ldots, \quad -1 < x < 1$

37. $y = a_0\left(1 - \frac{x^2}{16} - \frac{x^3}{96} + \frac{5x^4}{1536} + \ldots\right) + a_1\left(x - \frac{x^3}{24} - \frac{x^4}{192} + \ldots\right)$

38. $y(x) = 2(x-1) + (x-1)^2 + 2(x-1)^3 + \frac{(x-1)^4}{6} + \ldots\ ,\quad 0 < x < 2$

39. $y(x) = 1 - \frac{(x-\pi)^3}{6} + \frac{(x-\pi)^5}{120} + \frac{(x-\pi)^6}{180} + \ldots\ ,\quad x \in IR$

40. $y''(0) = 0,\quad y'''(0) = -2,\quad y^{(4)}(0) = 0$

41. $y''(0) = 0,\quad y'''(0) = -a_0,\quad y^{(4)}(0) = -4a_1$

42. $y''(1) = 0,\quad y'''(1) = -6,\quad y^{(4)}(1) = 42$

43. ανώμαλο σημείο (*Υπόδειξη:* να γίνει χρήση του Θεωρήματος Picard).

5.4 Προβλήματα

1. $y(x) = a_0\left(1 - \frac{(1+1)}{2!}x^2 + \frac{(1+1)(1-2)(1+3)}{4!}x^4 + \ldots\right) + a_1 x$

2. $y(x) = a_0\left(1 - 3x^2\right) + a_1\left(x - \frac{(2-1)(2+2)}{3!}x^3 + \frac{(2-1)(2+2)(2-3)(2+4)}{5!}x^5 + \ldots\right)$

3. $y(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

6. (i) $y_1(x, b) = 1 + \sum_{k=1}^{\infty} \frac{-b(4-b)(16-b)\ldots\left((2k-2)^2-b\right)}{(2k)!} x^{2k}$,

$y_2(x, b) = x + \sum_{k=1}^{\infty} \frac{(1-b)(9-b)\ldots\left((2k-1)^2-b\right)}{(2k+1)!} x^{2k+1}$

(iv) $T_0 = 1,\ T_1 = x,\ T_2 = 2x^2-1,\ T_3 = 4x^3-3x,\ T_4 = 8x^4-8x^2+1$..

7. $b = n(n+2),\quad U_0 = 1,\ U_1 = 2x,\quad U_2 = 4x^2-1,\quad U_3 = 8x^3-4x$.

5.5Α Προβλήματα

1. $y = c_1 + c_2 \ln x$

2. $y = c_1 x^3 + c_2 x^{-4}$

3. $y = c_1 x^{1/2} + c_2 x^{-3/2}$

4. $y = x[c_1\cos(\ln x) + c_2\sin(\ln x)]$

5. $y = c_1\cos\left(\sqrt{\frac{3}{2}}\ln x\right) + c_2\sin\left(\sqrt{\frac{3}{2}}\ln x\right)$

6. $y = c_1 + c_2 x^2 + c_3 x^2 \ln x$

7. $y = c_1 + c_2 \ln x + c_3 (\ln x)^2$

8. $y = c_1 x^{-1} + x^{1/2}\left\{c_2\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right\}$

9. $y = c_1 x^3 + c_2 x^{-2} - \frac{1}{6}$

10. $y = \left(c_1 + c_2 \ln x\right)x^{-1} + \frac{4}{9}\sqrt{x}$

11. $y = c_1 x + c_2 x^{-5} + x^2/7$

12. $y = c_1 x^{-5/2} + c_2 x^{-3}$

13. $y = c_1x^3\cos(2\ln x)+c_2x^3\sin(2\ln x)$ **14.** $y = c_1x+c_2x^{-1}\cos(3\ln x)+c_3x^{-1}\sin(3\ln x)$

15. $y = c_1x^{-2}+c_2x^{-2}\ln x+c_3x^{-2}(\ln x)^2$ **16.** $y = c_1x + c_2x^2 + \frac{4}{15}x^{-1/2}$

17. $y = 2x^4+x^{-3}$ **18.** $y = \frac{31}{17}x + \frac{3}{17}x^{-2}\cos(5\ln x) - \frac{76}{85}x^{-2}\sin(5\ln x)$

19. $y = \frac{5}{3}x^2 - \frac{2}{3}x^5$ **20.** $y = \frac{13}{10}x^3 + \frac{13}{15}x^{-2} - \frac{x}{6}$

21. (i) $y = (1-\ln(x+1))(x+1)$ (ii) $y = [1-4\ln(1-x)](1-x)^3$

(iii) $y = |x-2|^{1/2}\left\{3\cos\left(\frac{\sqrt{3}}{2}\ln|x-2|\right) + \frac{5}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\ln|x-2|\right)\right\}$

5.5B Προβλήματα

1. $y(x) = c_1\sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{(2n+1)!}x^n + c_2|x|^{-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{(2n)!}x^n$

2. $y(x) = c_1 x^{-1/2}\sin x + c_2 x^{-1/2}\cos x$

3. $y(x) = c_1 x^{1/4}\left\{1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,[5.9\ldots(4n+1)]}x^n\right\} + c_2\left\{1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!\,[3.7\ldots(4n-1)]}\right\}$

4. $y(x) = c_1 x^{\sqrt{2}}\left\{1 + \sum_{n=1}^{\infty}\frac{1}{n!\,(1+2\sqrt{2})(2+2\sqrt{2})\ldots(n+2\sqrt{2})}x^n\right\}$

$+ c_2 x^{-\sqrt{2}}\left\{1 + \sum_{n=1}^{\infty}\frac{1}{n!\,(1-2\sqrt{2})(2-2\sqrt{2})\ldots(n-2\sqrt{2})}x^n\right\}$

5. $y(x) = c_1x^3\left\{1 + \sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{10.13\ldots(3n+7)}x^n\right\} + c_2x^{2/3}\left\{1 + \sum_{n=1}^{\infty}\frac{(3n-4)(3n-1)}{n!3^n}x^n\right\}$

6. $y(x) = c_1 x^{\sqrt{5}}\left\{1 + \sum_{n=1}^{\infty}\frac{(1+\sqrt{5})(2+\sqrt{5})\ldots(n+\sqrt{5})}{n!\,(1+2\sqrt{5})(2+2\sqrt{5})\ldots(n+2\sqrt{5})}x^n\right\}$

$+ c_2 x^{-\sqrt{5}}\left\{1 + \sum_{n=1}^{\infty}\frac{(1-\sqrt{5})(2-\sqrt{5})\ldots(n-\sqrt{5})}{n!\,(1-2\sqrt{5})(2-2\sqrt{5})\ldots(n-2\sqrt{5})}x^n\right\}$

7. $y(x) = c_1\left(x^{1/3} + \frac{1}{5}x^{4/3}\right) + c_2x^{-1/3}\sum_{n=0}^{\infty}\frac{(-1)^n 10}{3^n n!\,(3n-2)(3n-5)}x^n$

8. $y(x) = c_1 x^{-1/4}\sum_{n=0}^{\infty}a_n\left(-\frac{1}{4}\right)x^n + c_2x^{-1}\sum_{n=0}^{\infty}a_n(-1)x^n,$

όπου $a_n(\lambda) = -\frac{2(n+\lambda-1)}{(n+\lambda)(4n+4\lambda+5)+1}a_{n-1}$

9. $y(x) = c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(1+2\sqrt{2})\ldots(n+2\sqrt{2})}x^{n+\sqrt{2}}$

$$+ c_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(1-2\sqrt{2})\ldots(n-2\sqrt{2})} x^{n-\sqrt{2}},$$

10. $y(x) = C_1 y_1 + C_2 \overline{y_1}$, όπου

$$y_1(x) = (\cos(\ln x) + i \sin(\ln x)) \left\{ 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,(1+2i)(2+2i)\ldots(n+2i)} \right\}$$

11. $$y(x) = c_1 \left\{ 1 + \sum_{n=1}^{\infty} \frac{2^n (x+2)^n}{n!\,5.7\ldots(2n+3)} \right\}$$

$$+ c_2 (x+2)^{-3/2} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2^n (x+2)^n}{n!\,(-1)\,1.3\ldots(2n-3)} \right\}, \quad -\infty < x < -2, \; -2 < x < +\infty.$$

12. $$y(x) = c_1 \sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)!!} x^n + c_2 x^{-1/2} \sum_{n=0}^{\infty} \frac{1}{n!\,(2n-1)!!} x^n$$

13. $$y(x) = c_1 x^{3/2} \left\{ 1 + 3 \sum_{n=1}^{\infty} \frac{x^n}{n!\,(2n+3)!!} \right\} + c_2 \left\{ 1 - x - \sum_{n=2}^{\infty} \frac{x^n}{n!\,(2n-3)!!} \right\}$$

14. $$y(x) = c_1 x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!\,4.7\ldots(3n+1)} + c_2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!\,2.5\ldots(3n-1)}$$

15. $$y(x) = c_1 x \left\{ 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,7.11\ldots(4n+3)} \right\} + c_2 x^{-1/2} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,1.5\ldots(4n+1)}$$

16. $$y(x) = c_1 x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!\,2^n} \left(\equiv x^{1/2} e^{-x/2}\right) + c_2 \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n-1)!!} \right\}$$

17. $$y(x) = c_1 x^{1/2} \sum_{n=0}^{\infty} \frac{1}{n!\,2^n} x^{2n} + c_2 \frac{\sin(3x)}{x}$$

18. $y(x) = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x}, \; x > 0$

19. $y(x) = c_1 \dfrac{\cos(3x)}{x} + c_2 \dfrac{\sin(3x)}{x}, \quad x \neq 0$ **20.** $y(x) = c_1 \dfrac{\cos h\,(2x)}{x} + c_2 \dfrac{\sin h\,(2x)}{x}$

21. $y(x) = c_1 \dfrac{\cos(x/2)}{x} + c_2 \dfrac{\sin(x/2)}{x}$ **22.** $y(x) = c_1 \cos x^2 + c_2 \sin x^2$

23. $y(x) = c_1 x^{1/2} \cos h\, x + c_2 x^{1/2} \sin h\, x$

24. $y(x) = (c_1 - c_2)\, e^{-x} + c_2 x^{-1} e^x, \; 0 < |x| < \infty$

25. $y(x) = c_1 \sqrt{x}\, e^x + c_2 e^{2x}$ **26.** $y(x) = c_1 (1-x)^{-1} - c_2 x^{-1/2} (1-x)^{-1}$

27. $y(x) = c_1 x^{-1} \cos(2x) + c_2 x^{-1} \sin(2x)$

28. $y(x) = c_1 \left\{\left(1 + \frac{x}{5} - \frac{3}{100} x^2 + \dots\right) \cos(\ln x) + \frac{x}{25} (10 + x + \dots) \sin(\ln x)\right\}$

$+ c_2 \left\{\left(1 + \frac{x}{5} - \frac{3x^2}{100} + \dots\right) \sin(\ln x) - \frac{x}{25} (10 + x + \dots) \cos(\ln x)\right\}$

29. $y_1(x) = \sqrt{x}\,, \quad y_2(x) = \sqrt{x} \left\{\ln x + \sum_{n=1}^{\infty} \frac{1}{n\, n!} x^n\right\}$

30. $N = 0 \Rightarrow y(x) = 1\,, \quad N = 1 \Rightarrow y(x) = x^{-1} (1 - x)\,,$

$N = 2 \Rightarrow y(x) = x^{-2} \left(1 - 2x + \frac{1}{2} x^2\right), \; N = 3 \Rightarrow y(x) = x^{-1} \left(1 - 3x + \frac{3}{2} x^2 - \frac{1}{6} x^3\right)$

31. $y(x) = c_1 F\left(2, -1, \frac{3}{2}, x\right) + c_2 x^{-1/2} F\left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, x\right)$

$= c_1 \left(1 - \frac{4}{3} x\right) + c_2 x^{-1/2} F\left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, x\right)$

32. $y = c_1 F\left(\frac{1}{2}, 1, \frac{1}{2}, -x\right) + c_2 (-x)^{\frac{1}{2}} F\left(1, \frac{3}{2}, \frac{3}{2}, -x\right) = c_1 \frac{1}{1+x} + c_2 \frac{(-x)^{1/2}}{1+x}$

33. $y(x) = c_1 F\left(2, 2, \frac{1}{2}, \frac{x+1}{2}\right) + c_2 \left(\frac{x+1}{2}\right)^{1/2} F\left(\frac{5}{2}, \frac{5}{2}, \frac{3}{2}, \frac{x+1}{2}\right)$

34. $y(x) = c_1 F\left(1, 1, \frac{14}{5}, \frac{3-x}{5}\right) + c_2 \left(\frac{3-x}{5}\right)^{-\frac{9}{5}} F\left(-\frac{4}{5}, -\frac{4}{5}, -\frac{4}{5}, \frac{3-x}{5}\right)$

35. $y(x) = c_1 F\left(1, -1, -1/2, 1 - e^x\right) + c_2 \left(1 - e^x\right)^{\frac{3}{2}} F\left(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}, 1 - e^x\right)$

5.5Γ Προβλήματα

1. $(\lambda_1 = 0, \lambda_2 = -1)\; y(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x)$ με $\varphi_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!\,(n+1)!}\,,$

$\varphi_2(x) = -\varphi_1(x) \ln|x| + |x|^{-1} \left\{1 + \sum_{n=1}^{\infty} \frac{\varphi(n) + \varphi(n-1)}{n!\,(n-1)!} (-1)^{n+1} x^n\right\}$

όπου $\varphi(n) = A J_1(n) + B Y_1(n)$, A, B αυθαίρετα, $|x|>0$.

2. $(\lambda_1 = \lambda_2 = 1)$

$y(x) = c_1 |x| (1 + x) + c_2 \left\{|x| (1 + x) \ln|x| - 2|x|^2 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n\,(n+1)} x^n\right)\right\}$

3. $(\lambda_1 = 0, \lambda_2 = 1)$, $y(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x)$

όπου $\varphi_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1}\,, \quad \varphi_2(x) = -\varphi_1(x) \ln|x| + \sum_{n=0}^{\infty} b_n x^n\,,$

με $b_0 = 1, b_1 = 0$ και $n\, b_{n+1} + b_n = \frac{(-1)^n}{n!}\,, \quad n = 1, 2, \dots, \quad |x| > 0\,.$

4. $(\lambda_1 = \lambda_2 = 0)$, $y(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x)$,

όπου $\varphi_1(x) = \sum_{n=0}^{\infty} \frac{4^n x^n}{(n!)^2}$, $\varphi_2(x) = \varphi_1(x)\ln x - \sum_{n=1}^{\infty} \frac{2^{2n+1}}{(n!)^2} H_n x^n$

5. $(\lambda_1 = \lambda_2 = 1)$, $y(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x)$,

όπου $\varphi_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$, $\varphi_2(x) = \varphi_1(x)\ln x - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} H_n x^{n+1}$, $x > 0$

6. $(\lambda_1 = \lambda_2 = -1)$, $y(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x)$,

όπου $\varphi_1(x) = x^{-1} + 3$, $\varphi_2(x) = \varphi_1(x)\ln x - \left\{9 + \sum_{n=2}^{\infty} \frac{(-3)^n x^{n-1}}{n(n-1)\,n!}\right\}$, $x > 0$

7. $(\lambda_1 = \lambda_2 = 0)$ $y(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x)$,

όπου $\varphi_1(x) = \sum_{n=0}^{\infty} \frac{3^n x^n}{(n!)^2}$, $\varphi_2(x) = \varphi_1(x)\ln x - \sum_{n=1}^{\infty} \frac{2\;3^n}{(n!)^2} H_n x^n$, $x > 0$

8. $(\lambda_1 = 0, \lambda_2 = -1)$ $y(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x)$,

όπου $\varphi_1(x) = 2 + x$, $\varphi_2(x) = \varphi_1(x)\ln x + x^{-1} - 3 - 4x + \sum_{n=3}^{\infty} \frac{2(-1)^n x^{n-1}}{n!\,(n-1)(n-2)}$

9. $(\lambda_1 = 0, \lambda_2 = 2)$ $y(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x)$

όπου $\varphi_1(x) = x^2$ και $\varphi_2(x) = \varphi_1(x)\ln x - 1 + 2x + 2\sum_{n=3}^{\infty} \frac{(-x)^n}{n!\,(n-2)}$, $x > 0$.

10. $(\lambda_1 = 0, \lambda_2 = -1)$ $y(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x)$,

$\varphi_1(x) = x^{-1} \sum_{n=0} \frac{1}{(2n)!} x^{2n} = \frac{1}{x}\cos h\,x$, $\varphi_2(x) = x^{-1} \sum_{n=0} \frac{1}{(2n+1)!} x^{2n+1} = \frac{1}{x}\sin h\,x$

11. $(\lambda_1 = 4, \lambda_2 = 0)$ $y(x) = c_1\left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right) + c_2 \sum_{n=0}^{\infty} (n+1)\,x^{n+4}$

12. $(\lambda_1 = \lambda_2 = 0)$ $y(x) = c_1\,\varphi_1(x) + c_2\,\{\varphi_1(x)\ln x$
$+ \varphi_1(x)\left(-x + \frac{1}{4}x^2 - \frac{1}{3\;3!}x^3 + \frac{1}{4\;4!}x^4 \ldots\right)\}$, όπου $\varphi_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

13. $(\lambda_1 = \lambda_2 = 0)$ $y(x) = c_1\,\varphi_1(x) + c_2\,\{\varphi_1(x)\ln x$
$+ \varphi_1(x)\left(2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \ldots\right)\}$, όπου $\varphi_1(x) = \sum_{n=0} \frac{(-1)^n}{(n!)^{2n}} x^n$

14. $(\lambda_1 = \lambda_2 = 1)$ $y(x) = c_1 x\,e^{-x} + c_2\,x\,e^{-x}\left\{\ln x + x + \frac{1}{4}x^2 + \frac{1}{3\;3!}x^3 + \ldots\right\}$

15. $(\lambda_1 = 2, \lambda_2 = 0)$ $\quad y(x) = c_1 x^2 + c_2 \left\{ \frac{1}{2} x^2 \ln x - \frac{1}{2} + x - \frac{1}{3!} x^3 + \ldots \right\}$

16. $(\lambda_1 = 7, \lambda_2 = 0)$

$$y(x) = c_1 \left\{ 1 - \frac{x}{2} + \frac{x^2}{10} - \frac{x^3}{120} \right\} + c_2 \left\{ x^7 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 4\, 5\, 6 \ldots (n+3)}{n!\, 8\, 9\, 10 \ldots (n+7)} x^{n+7} \right\}$$

17. $(\lambda_1 = 2, \lambda_2 = -2)$

$$y(x) = c_1 \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!\,(n+4)!} + c_2 \left\{ \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!\,(n+4)!} - 6\, x^{-2} \left(1 - \frac{x}{3} + \frac{x^2}{12} - \frac{x^3}{36} + \ldots \right) \right\}$$

18. $(\lambda_1 = \lambda_2 = -1)$

$$y(x) = c_1 x^{-1} (1 - x) + c_2 \left\{ x^{-1} (1 - x) \ln x + x^{-1} \left[3x - \frac{x^2}{4} - \sum_{n=3}^{\infty} \frac{(n-2)!}{(n!)^2} x^n \right] \right\}$$

19. $\left(\lambda_1 = \frac{3}{2}, \lambda_2 = -\frac{3}{2}\right)$

$$y(x) = c_1 x^{3/2} \sum_{n=0}^{\infty} \frac{1}{(n+3)!} x^n + c_2 x^{-3/2} \left(1 + x + \frac{1}{2} x^2 \right)$$

$$= c_1 x^{-3/2} \left(e^x - 1 - x - \frac{x^2}{2} \right) + c_2 x^{-3/2} \left(1 + x + \frac{x^2}{2} \right)$$

20. $(\lambda_1 = \lambda_2 = 1)$ $\quad y(x) = c_1 x + c_2 \left\{ x \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\ n!} x^{n+1} \right\}$

21. (ii) $k = n$, ακέραιος (iii) $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = 1 - 2x + \frac{x^2}{2}$,

(iv) $y_1 = 1$, $\quad y_2 = \ln x + \left[x + \frac{x^2}{2\ 2!} + \frac{x^3}{3\ 3!} + \frac{x^4}{4\ 4!} + \ldots \right]$,

(v) $y_1 = 1 - x$, $\quad y_2 = (1 - x) \ln x + \left[3x - \frac{x^2}{4} - \frac{x^3}{36} - \frac{x^4}{288} - \ldots \right]$.

22. (ii) $k \neq -\frac{1}{4} + \frac{m^2}{4}$, $m \in \mathbb{Z}$, $\varphi_1(x) = x^{\lambda_1} \sum_{n=0}^{\infty} a_n x^n$, $\varphi_2(x) = x^{\lambda_2} \sum_{n=0}^{\infty} a_n x^n$,

(iii) $k = -\frac{1}{4}$, $\varphi_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n$, $\varphi_2(x) = \varphi_1(x) \ln x + x^{1/2} \sum_{n=1}^{\infty} b_n x^n$

(iv) $k = -\frac{1}{4} + \frac{m^2}{4}$, $\quad m \neq 0$,

29. $y(x) = c_1 J_3(3x) + c_2 Y_3(3x)$

30. $$y(x) = \begin{cases} c_1 J_n(x\sqrt{7}) + c_2 Y_n(x\sqrt{7}), & \text{για } \ v = n \in \mathbb{Z} \\ c_1 J_v(x\sqrt{7}) + c_2 Y_{-v}(x\sqrt{7}), & \text{για } \ v \notin \mathbb{Z}. \end{cases}$$

31. $y(x) = c_1 \int J_3(x\sqrt{6})\,dx + c_2 \int Y_3(x\sqrt{6})\,dx + c_3$.

32. $y(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$

33. $y(x) = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)$

34. $y(x) = c_1 J_2(3x) + c_2 Y_2(3x)$

35. $y(x) = c_1 x^4 J_1(x) + c_2 x^4 Y_1(x)$

36. $y(x) = c_1 x^{-1} J_2\left(\frac{\sqrt{x}}{2}\right) + c_2 x^{-1} Y_2\left(\frac{\sqrt{x}}{2}\right)$

37. $y(x) = c_1 x^2 J_4(x^2) + c_2 x^2 Y_4(x^2)$

41. $y_1(x) = x^{1/2} J_{1/2}(x)$, $y_2(x) = x^{1/2} J_{-1/2}(x)$

42. $y_1(x) = x^{-1} J_{-1}(x)$, $y_2(x) = x^{-1} J_1(x) = -x^{-1} J_{-1}(x)$

43. $y_1(x) = \sqrt{x}\, J_{3/2}(x)$, $y_2(x) = \sqrt{x}\, J_{-3/2}(x)$

5.6 Προβλήματα

1. Κ.Α.

2. ΜΚΑ

3. ΚΑ

4. ΜΚΑ

5. ΚΑ

6. ΚΑ

7. ΚΑ

8. ΚΑ

9. ΜΚΑ

10. ΜΚΑ

11. ΜΚΑ

12. $$y(x) = c_1\left(1 + \frac{2}{x} + \frac{7}{3x^2} + \frac{112}{45x^3} + \dots\right) + c_2 x^{-1/2}\left(1 + \frac{4}{3x} + \frac{22}{15x^2} + \frac{484}{315x^3} + \dots\right), \quad |x| > 1$$

13. $$y(x) = c_1\left(1 - \frac{1}{3x} + \frac{1}{30x^2} - \frac{1}{630x^2} + \dots\right) + c_2 x^{1/2}\left(1 - \frac{1}{x} + \frac{1}{6x^2} - \frac{1}{90x^2} + \dots\right), \quad |x| > 0$$

14. $$y(x) = c_1 x^a\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^n a(a-1)\dots(a-2n+1)}{2^n n!(2a-1)\dots(2a-2n+1)} x^{-2n}\right] + c_2 x^{-(a+1)}\left[1 + \sum_{n=1}^{\infty} \frac{(a+1)\dots(a+2n)}{2^n n!(2a+3)\dots(2a+2n+1)} x^{-2n}\right]$$

15. $y(x) = c_1 \sum_{n=0}^{\infty} \dfrac{x^{-2n}}{2^n\, n!\, (2n-1)(2n-3)(2n-5)} + c_2 \left[x^{-1} - \dfrac{2}{3}x^{-3} + \dfrac{1}{15}x^{-5}\right]$

16. $y(x) = c_1 x^{-3/2} \left\{1 + \sum_{n=1}^{\infty} \dfrac{(-2)^n (n+1)\, x^{-n}}{5\; 7\; 9 \cdots (2n+3)}\right\} + c_2 \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1} (2n-1)\, x^{-n}}{n!}$

17. $y(x) = c_1 \left(x^2 + 2x + 3\right) + c_2\, x^{-1} \sum_{n=0}^{\infty} (n+4)\, x^{-n}$

18. $y(x) = c_1\, x^{-1} + c_2\, x^{-1/2} \sum_{n=0}^{\infty} \dfrac{x^{-n}}{2n-1}$

19. $y(x) = c_1\, x^{-1} \sum_{n=0}^{\infty} (n+1)(2n+3)(2n+5)\, x^{-n}$

$+ c_2\, x^{-1/2} \sum_{n=0}^{\infty} (n+1)(n+2)(2n+1)\, x^{-n}$

20. $y(x) = c_1\, x^{-1} \sum_{n=2}^{\infty} (-1)^{n+1}\, n(n-1)\, x^{-n} +$

$+ c_2 \left\{x^{-1} \ln\left(x^{-1}\right) \sum_{n=2}^{\infty} (-1)^{n+1}\, n(n-1)\, x^{-n} + x^{-1} + x^{-2}\right.$

$\left. + x^{-1} \sum_{n=2}^{\infty} (-1)^{n+2} \left(n^2 + n - 1\right) x^{-n}\right\}$

21. $y(x) = c_1\, x^{-1} \sum_{n=1}^{\infty} n\, x^{-n} + c_2\, x^{-1} \left\{\ln\left(x^{-1}\right) \sum_{n=1}^{\infty} n\, x^{-n} + \sum_{n=0}^{\infty} x^{-n}\right\}$

22. $y(x) = c_1 \cos\left(2\, x^{-1}\right) + c_2 \sin\left(2\, x^{-1}\right)$

5.7 ΠΡΟΒΛΗΜΑΤΑ ΕΠΙΣΚΟΠΗΣΗΣ

4. (i) όχι (ii) $\varphi_2(x) = x\, e^{x^2}$

(iii) $\varphi_1(x) = \left(\dfrac{8}{3} - \dfrac{5x}{3}\right) e^{x^2-1}$, $\varphi_2(x) = (8 - 5x)\, e^{x^2-1}$

9. $(\lambda_1 = \lambda_2 = 0)$

$y_1(x) = 1 + \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}\, a(a+1)\,[1 \cdot 2 - a(a+1)] \ldots [n(n-1) - a(a+1)]}{2^n\, (n!)^2} (x-1)^n$

$= F\left(-a,\, a+1,\, 1,\, \dfrac{1-x}{2}\right)$

10. $\left(\lambda_1 = \dfrac{1}{2},\, \lambda_2 = 0\right)$ $y(x) = c_1\, |x-1|^{1/2} \{1 +$

$\sum_{n=1}^{\infty} \dfrac{(-1)^n (1+2\alpha) \ldots (2n-1+2\alpha)(1-2\alpha) \ldots (2n-1-2\alpha)}{2^n\, (n+1)!} (x-1)^n \} +$

$$+ c_2 \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \alpha (1+\alpha) \ldots (n-1+\alpha)(-\alpha)(1-\alpha) \ldots (n-1-\alpha)}{n!\ 1\ 3\ 5 \ldots (2n-1)} (x-1)^n \right\}$$

11. $y(x) = c_1 F\left(\rho, \sigma, \tau, \frac{x-\alpha_1}{\alpha_2-\alpha_1}\right)$

$$+ c_2 \left(\frac{x-\alpha_1}{\alpha_2-\alpha_1}\right)^{1-\tau} F\left(\rho-\tau+1, \sigma-\tau+1, 2-\tau, \frac{x-\alpha_1}{\alpha_2-\alpha_1}\right),$$

όπου $\rho + \sigma + 1 = \frac{b}{a}$, $\tau = \frac{b(\alpha_1 - \alpha_3)}{a(\alpha_1 - \alpha_2)}$, $\rho\,\sigma = \frac{c}{a}$.