

# ΑΠΑΝΤΗΣΕΙΣ ΠΡΟΒΛΗΜΑΤΩΝ

## ΚΕΦΑΛΑΙΟ 10

### 10.2 Προβλήματα

1.  $y(x) \equiv 0$ , μοναδική
2.  $y(x) = c \cos 3x$
3.  $y(x) = c_1 \cos(\pi x) + c_2 \sin(\pi x)$ ,  $c_1, c_2$  αυθαίρετα
4.  $y(x) \equiv 0$ , μοναδική
5.  $y(x) = 2 \sin(kx)/\sin k$ , μοναδική, αν  $k \neq n\pi$ ,  $n = 0, 1, 2, \dots$
6.  $y(x) = \frac{1}{2e}(3 \sin hx - \cos hx)$ , μοναδική
7.  $y(x) = 3x e^{3(x-1)}$ , μοναδική
9.  $k = n\pi$ ,  $n = 1, 2, 3, \dots$ ,  $y(x) = c \sin(\pi x)$ ,  $c$  αυθαίρετο
10. (i)  $b-a = n\pi$ ,  $n = 1, 3, 5, \dots$ , (ii)  $b-a \neq n\pi$ ,  $n = 1, 3, 5, \dots$ , (iii)  $b-a = n\pi$ ,  $n = 2, 4, 6, \dots$
11. όχι λύση
12.  $y(x) = -(3/\pi^2) \sin(\pi x) + (4x/\pi) - 2$  μοναδική
13.  $y(x) = (1/5)e^{-x} - (6/5)\cos 2x$ , μοναδική
14.  $y(x) = (x-1)e^{-x} + x - 2$ , μοναδική
16.  $y(x) = c x^3 \sin(4 \log x)$ ,  $c$  αυθαίρετο
18.  $y(x) = \left(1 - \frac{1}{2} \log x\right)x$
20.  $y(x) = (4/5)x^{1/2}(x-1)$
23.  $y(x) = 2x - \pi \sin(4x)/4$
25.  $y(x) = x e^x$
27.  $y(x) = \frac{c_2 - c_1}{b} x$
28.  $y(x) = x(\pi^2 + 1 - x^2)/6$ ,  $y_{\max} = (\pi^2 + 1)^{3/2}/9\sqrt{3}$
29.  $y'' = -\sin x$ ,  $0 < x < \pi$ ,  $y(0) = 0$ ,  $y(\pi) = 0$  καμία λύση
30.  $y(x) = \sin x$ ,  $y_{\max} = 1$
31.  $y(x) = [1 - (2x-1)^4]/192$ ,  $y_{\max} = 1/192$

### 10.3 Προβλήματα

1.  $\lambda_n = n^2 \pi^2 / 16$ ,  $\varphi_n(x) = \cos(n\pi x / 4)$ ,  $n = 0, 1, 2, \dots$
2.  $\lambda_n = \frac{(2n-1)^2 \pi^2}{4}$ ,  $\varphi_n(x) = \cos\left[\frac{(2n-1)\pi x}{2}\right]$ ,  $n = 1, 2, \dots$
3.  $\lambda_n = \frac{n^2 \pi^2}{81}$ ,  $\varphi_n(x) = \sin\left[\frac{n\pi(x-1)}{9}\right]$ ,  $n = 1, 2, \dots$
4.  $\lambda_n = n^2 \pi^2 + \frac{1}{4}$ ,  $\varphi_n(x) = e_{x/2} \sin(n\pi x)$ ,  $n = 1, 2, \dots$

5.  $\lambda_n = -n^2 \pi^2$ ,  $\varphi_n(x) = e^{-x} \sin(n \pi x)$ ,  $n = 1, 2, \dots$
6.  $\lambda_n = \frac{9}{8} + \frac{n^2 \pi^2}{2}$ ,  $\varphi_n(x) = e^{3x/2} \sin(n \pi x)$ ,  $n = 1, 2, \dots$
8.  $\lambda_n = n^2 \pi^2$ ,  $\varphi_n(x) = \sin(n \pi \log x)$ ,  $n = 1, 2, \dots$
9.  $\lambda_n = \frac{(2n-1)^2 \pi^2}{4 (\log 2)^2}$ ,  $\varphi_n(x) = \cos \frac{(2n-1) \pi \log x}{2 \log 2}$ ,  $n = 1, 2,$
10. Δεν έχει πραγματικές ιδιοτιμές      12.  $\lambda_0 = 0$ ,  $\varphi_0(x) = x - 1$
13.  $\lambda_n = n^4$ ,  $\varphi_n(x) = \sin(n x)$ ,  $n = 1, 2, \dots$
14.  $\lambda_n = k_n^4$ , όπου  $\cos h k_n \cos k_n = 1$ ,  $n = 1, 2, \dots$   
 $\varphi_n(x) = \sin h [k_n (1-x)] + \sin [k_n (1-x)] + \cos h k_n \sin (k_n x)$   
 $- \sin k_n \cos h (k_n x) + \cos k_n \sin h (k_n x) - \sin h k_n \cos (k_n x)$
15. (i)  $\lambda = -k^2 < 0$ ,  $\varphi(x) = e^{-2x} \sin h [3k(2-x)] / \sin h (6k)$   
(ii)  $\lambda = 0$ ,  $\varphi(x) = (2-x) e^{-2x} / 2$ ,  
(iii)  $\lambda = k^2 > 0$ ,  $k \neq n \pi / 6$ ,  $\varphi(x) = e^{-2x} \sin [3k(2-x)] / \sin (6k)$ ,  $n \in \mathbb{N}$
16.  $\omega_1 = \sqrt{(1+45\pi^2)/4} \rho$ ,  $\varphi_1(x) = c x^{-1/2} \sin(\pi \log x)$
18. (ii)  $\varphi_n(x) = 1 - \cos(2n \pi x)$ ,  $n = 1, 2, 3, \dots$   
 $\varphi_n(x) = (\sin k_n - k_n)(1 - \cos k_n x) - (1 - \cos k_n)(\sin k_n x - k_n)$   
όπου  $\tan(k_n/2) = k_n/2$

#### 10.4 Προβλήματα

1.  $(x e^{-x} y)' + k e^{-x} y = 0$ ,  $\mu(x) = e^{-x}$
2.  $(e^{-x^2} y)' + 2k e^{-x^2} y = 0$ ,  $\mu(x) = e^{-x^2}$
3.  $(\sqrt{1-x^2} y)' + k^2 (1-x^2)^{-1/2} y = 0$ ,  $\mu(x) = (1-x^2)^{-1/2}$
4.  $y' + \lambda x^{-1} y = 0$ ,  $\mu(x) = x^{-1}$       5.  $(x e^{-x} y)' + \lambda e^{-x} y = 0$ ,  $\mu(x) = e^{-x}$
6.  $((x^2+1)y)' + \lambda x^{-2} y = 0$ ,  $\mu(x) = x^{-2}$       7.  $[(\sin x)y]' + \lambda (\sin x)y = 0$ ,  $\mu(x) = 1$
8.  $(e^{-x^2/2} y)' + \lambda e^{-x^2/2} y = 0$ ,  $\mu(x) = e^{-x^2/2}$
9.  $(e^{-x} y)' + \lambda e^{-x} y = 0$ ,  $\mu(x) = e^{-x}$       10.  $(x^2 y)' + \lambda x^2 y = 0$ ,  $\mu(x) = x^2$

11.  $y'' + \lambda x^{-2} y = 0$  ,  $\mu(x) = x^{-2}$       12.  $c_n = \sqrt{2/\pi}$   
 13.  $c_0 = \sqrt{1/\pi}$  ,  $c_n = \sqrt{2/\pi}$  ,  $n \geq 1$   
 24.  $\lambda_n = [(2n-1)\pi / (2L)]^2$  ,  $\varphi_n = c_n \sin [(2n-1)\pi x / (2L)]$  ,  $n \in \mathbb{N}$ .  
 25.  $\sqrt{\lambda_n} = -\tan(L\sqrt{\lambda_n})$  ,  $\varphi_n(x) = A_n \sin x \sqrt{\lambda_n}$  ,  $n = 1, 2, 3, \dots$   
 26.  $\lambda_n = 1 + n^2 \pi^2$  ,  $\varphi_n(x) = \sin(n\pi \ln x) / x^{-1}$  ,  $n = 1, 2, 3, \dots$   
 27.  $\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\ln 3}\right)^2$  ,  $\varphi_n(x) = (x+2) \sin[(n\pi / \ln 3) \ln(x+2)]$  ,  $n = 1, 2$   
 28.  $\lambda_n = \left[1 + (2n\pi / \ln 2)\right]^2 / 12$  ,  $\varphi_n(x) = \sin \frac{[(n\pi / \ln 2) \ln(1+x)]}{\sqrt{1+x}}$  ,  $n \geq 1$   
 29.  $\lambda_0 = 0$  ,  $\varphi_0(x) = 1$  ,  $\lambda_n = n^2 + \frac{9}{4}$  ,  $\varphi_n(x) = e^{3x/2} (\sin(nx) - \frac{2}{3}n \cos nx)$  ,  $n \in \mathbb{N}$   
 30.  $\lambda_n = 16n^2$  ,  $\varphi_n(x) = \sin[4n(\tan^{-1} x)]$  ,  $n = 1, 2, 3, \dots$   
 31.  $\lambda_n = n^2 \pi^2$  ,  $\varphi_n(x) = \cos n\pi x$  ,  $\psi_n(x) = \sin n\pi x$  ,  $n = 1, 2, 3, \dots$   
 32.  $\lambda_n = 4n^2 \pi^2$  ,  $\varphi_n(x) = \cos 2n\pi x$  ,  $\psi_n(x) = \sin 2n\pi x$  ,  $n = 1, 2, 3, \dots$   
 33.  $\lambda_n = n^2$  ,  $\varphi_n(x) = \sin(nx)$  ,  $\psi_n(x) = \cos(nx)$  ,  $n = 0, 1, 2, 3, \dots$   
 34.  $\lambda_n = 4n^2$  ,  $\varphi_n(x) = \sin(2nx)$  ,  $\psi_n(x) = \cos(2nx)$  ,  $n = 0, 1, 2, 3, \dots$   
 35.  $\lambda > 0$  ,  $\varphi_\lambda(x) = \sin(\sqrt{\lambda} \ln x)$       36.  $\lambda > 0$  ,  $\varphi_\lambda(x) = \cos(\sqrt{\lambda} x)$   
 37.  $\lambda_n = 2n(2n-1)$  ,  $\varphi_n(x) = P_{2n}(x)$  ,  $P_n$  πολυώνυμο Legendre τάξης  $n$ .  
 38.  $\lambda_n = n^2$  ,  $\varphi_n(x) = T_n(x)$  ,  $T_n$  πολυώνυμο Chebysh εν τάξης  $n$ .  
 39.  $\lambda_0 = 0$  ,  $\varphi_0(x) \equiv 1$  ,  $\lambda_n$  : η θετική ρίζα τάξης  $n$  της εξίσωσης  $J'_0(\sqrt{\lambda}) = 0$   
 ( $J_0$  συνάρτηση Bessel μηδενικής τάξης και πρώτου είδους),  
 44.  $a = \pi \sqrt{\lambda} / \sqrt{3(k-1)A}$  .

## 10.5 Προβλήματα

7.  $\varphi_0(x) = 1/\sqrt{\pi}$  ,  $\varphi_n(x) = \sqrt{2/\pi} \cos(nx)$  ,  $n = 1, 2, 3, \dots$   
 8.  $\lambda_n = (1 + n^2 \pi^2 / L^2)^{1/2}$  ,  $\varphi_n(x) = \sqrt{2L e^{-x} / (n^2 \pi^2 + L^2)}$   
 9.  $\lambda_n = \frac{(2n-1)^2 \pi^2}{4(\log b)^2}$  ,  $\varphi_n(x) = \sqrt{\frac{2}{\log b}} \cos\left[\frac{(2n-1)\pi \log x}{2 \log b}\right]$

10.  $\lambda_n = \left[ \frac{(2n-1)\pi}{2} \right]^2, \quad \varphi_n(x) = \sqrt{2} \sin \left[ \frac{(2n-1)\pi x}{2} \right]$
11.  $\lambda_n = n^2 \pi^2, \quad \varphi_n(x) = \sqrt{2} e^x \sin(n\pi x)$
12.  $f(x) = x \sim \sum_{n=1}^{\infty} \left\{ 2 [(-1)^n - 1] \cos(n\pi x) (n^2 \pi)^{-1} \right\} + \frac{\pi}{2}$
13.  $f(x) \sim \sum_{n=1}^{\infty} 2n\pi(1 - b \cos n\pi) \sin \left( \frac{n\pi \log x}{\log b} \right) (n^2 \pi^2 + \log^2 b)^{-1}$
14.  $f(x) \sim \sum_{n=1}^{\infty} 8n\pi(1 - e^{a/2} \cos n\pi) e^{-x/2} \sin \left( \frac{n\pi x}{a} \right) (4n^2 \pi^2 + a^2)^{-1}$
15.  $f(x) \sim 1 + 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sin(n\pi x) / n$
16.  $f(x) \sim 2\pi e^{3x/2} \sum_{n=1}^{\infty} n [1 + (-1)^{n-1} e^{-3/2}] \sin(n\pi x) [n^2 \pi^2 + 9/4]^{-1}$
17.  $f(x) \sim (2/\pi) \sum_{n=1}^{\infty} (-1)^{n-1} \sin(n\pi \log x) / n$
18.  $f(x) \sim (16L^2/\pi^3) \sum_{n=1}^{\infty} [\pi(-1)^{n+1} (2n-1) - 2] \sin[(2n-1)\pi x/2L] (2n-1)^{-3}$

### 10.6 Προβλήματα

1.  $y(x) = 2 \sum_{n=1}^{\infty} \left\{ (-1)^{n+1} \sin(n\pi x) / (n^2 \pi^2 - 2) n \pi \right\}$
2.  $y(x) = 2 \sum_{n=1}^{\infty} \left\{ (2 \cos \sqrt{\lambda_n} - 1) \cos \sqrt{\lambda_n} x / \lambda_n (\lambda_n - 2) / (1 + \sin^2 \sqrt{\lambda_n}) \right\}$   
(βλέπε Παράδειγμα 4 της Ενότητας 10.3)
3.  $y(x) = 8 \sum_{n=1}^{\infty} \left\{ \frac{\sin(n\pi/2) \sin(n\pi x)}{(n^2 \pi^2 - 2) n^2 \pi^2} \right\}$     4.  $y(x) = \frac{\cos(\pi x)}{2\pi^2} + \frac{(2x-1)}{2\pi^2} + c \sin(\pi x)$
5.  $y(x) = c \sin \pi x - \frac{x \sin \pi x}{2\pi}$     6.  $y(x) = \left( \frac{2}{\pi^3} \right) \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n-1} \sin(n\pi x)}{n^3} \right\}$
7.  $y(x) = 1 + \sin x + \frac{(\cos 1 - 1) \cos x}{\sin 1}$
8.  $y(x) = \frac{2 \cosh(x-1)}{\cosh 1} - \sum_{n=1}^{\infty} \frac{[(-1)^{n-1} e^{-x} + (2n-1)\pi]}{[4 + (2n-1)^2 \pi^2]} \sin \left[ \frac{(2n-1)\pi x}{2} \right]$
10.  $y(x) = 3(1-x)e^{-x} - \left( \frac{2e^{-x}}{\pi} \right) \sum_{n=1}^{\infty} \left\{ \frac{[1 - (-1)^n e^{-2}] \sin(n\pi x)}{n(n^2 \pi^2 + 4)} \right\}$

$$11. y(x) = \frac{1-x}{2} (3-x^2) - \left(\frac{8x^2}{\pi}\right) \sum_{n=1}^{\infty} \left\{ \frac{\sin[(2n-1)\pi \log x / \log 2]}{(2n-1)[1+(2n-1)^2 \pi^2 / (\log 2)^2]} \right\}$$

$$13. y(x) = \frac{e^2 \cos(\ln x)}{\sin 2} + \left(\frac{8}{\pi^2}\right) \sum_{n=1}^{\infty} \left\{ \frac{4 \cos[(2n-1)\pi \log x / 2]}{(2n-1)^2 [(2n-1)^2 \pi^2 - 4]} \right\}$$

$$15. y(x) = c - \cos(\pi x) / \pi^2, \quad c \in \mathbb{R} \qquad 16. y(x) = c \sin(4 \log x) + \frac{\sin(3 \log x)}{7}$$

17. όχι λύση

