Some Hecke-type algebras derived from the braid group with two fixed strands

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Abstract We construct some Hecke-type algebras, and most notably the quotient algebra $\mathrm{H}_{2,n}(q)$ of the group-algebra $\mathbb{Z}\left[q^{\pm 1}\right] B_{2,n}$ of the mixed braid group $B_{2,n}$ with two identity strands and n moving ones, over the quadratic relations of the classical Hecke algebra for the braiding generators. The groups $B_{2,n}$ are known to be related to the knot theory of certain families of 3-manifolds, and the algebras $\mathrm{H}_{2,n}(q)$ are aimed for the construction of invariants of oriented knot and links in these manifolds. To this end, one needs a suitable basis of $\mathrm{H}_{2,n}(q)$, and we have singled out a subset Λ_n of this algebra for which we proved it is a spanning set, whereas ongoing research aims at proving it to be a basis.

Key words: mixed braid group on two fixed strands, mixed Hecke algebra, quadratic relation, Hecke-type algebras.

Introduction

It is established that knots and links in arbitrary knot complements, in compact, connected, oriented (c.c.o.) 3-manifolds and in handlebodies may be represented by mixed links and mixed braids in S^3 [4, 9, 13]. The braid structures related to knots and links in the above spaces are the mixed braid groups $B_{m,n}$ and appropriate cosets of theirs [15]. An element in $B_{m,n}$ is a classical braid in S^3 on $m+n$ strands with the first m strands forming the identity braid. The mixed braid groups enable the algebraic formulation of the geometric braid equivalences in the above spaces [9, 16].

In this paper we focus on the mixed braid groups $B_{2,n}$, which are related to knots and links in certain families of 3- manifolds like, for example, the handlebody of genus two, the complement of the 2-unlink in S^3 and the connected sums of two lens spaces, which are of interest also to some biological applications [3]. We define the quotient algebras $\mathrm{H}_{2,n}(q)$, $\mathrm{H}_{2,n}(q, u_1, v_1)$ and $\mathrm{H}_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2})$ of $B_{2,n}$ over the quadratic relations of the classical Iwahori-Hecke algebra for the braiding generators, and polynomial relations for the looping generators. We then focus on $\mathrm{H}_{2,n}(q)$ and present a subset Λ_n of it, indicating the reason it has to be a spanning set for its additive structure. The set Λ_n potentially constitutes a linear basis of $\mathrm{H}_{2,n}(q)$, a fact whose proof is the object of ongoing research.

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The sets Λ_n for $n \in \mathbb{N}$ are destined to provide an appropriate inductive basis for the sequence of algebras $\mathrm{H}_{2,n}(q), n \in \mathbb{N}$, in order to construct Homflypt-type invariants for oriented links in 3-manifolds whose braid structure is encoded by the groups $B_{2,n}$. It is known that the mixed braid groups $B_{1,n}$ have been utilized for constructing Homflypt-type invariants for oriented links the solid torus ([12],[8],[14]) and the lens spaces $L(p,1)$ ([6]), following the original Jones construction of the classical Homflypt polynomial for oriented links in S^3 using the Iwahori Hecke algebra of type A and the Ocneanu trace [10]. For our purposes we first need to construct appropriate algebras related to the mixed braid groups $B_{2,n}$, and then to chose an appropriate inductive bases on them, so that the construction of Oceanu-type Markov traces on these algebras would be possible, which subsequently can be used for the construction of link invariants.

The paper is organized as follows: in §1.1 we recall the definition and a presentation of the mixed braid group $B_{2,n}$ and we define some important elements of it which we call loopings. In §1.2 we define our quotient algebras $\mathrm{H}_{2,n}(q)$, $\mathrm{H}_{2,n}(q, u_1, v_1)$ and $\mathrm{H}_{2,n}(q, u_1, \dots, u_{d_1}, v_1, \dots, v_{d_2})$. In §2 we provide a potential basis Λ_n for the algebra $\mathrm{H}_{2,n}(q)$, and we give the necessary lemmata for proving it to be spanning set of the algebra.

1 The mixed braid groups $B_{2,n}$ and related Hecke-type algebras

1.1 The mixed braid group $B_{2,n}$ on two mixed strands and other related groups

For each $n \in \mathbb{N}$, the elements of the *mixed braid group $B_{2,n}$ on two fixed strands* are defined to be the braids with $n+2$ strands where the first two of them are straight, and the group operation is by definition the usual braid concatenation. A description of $B_{2,n}$ in terms of generators and relations is the following ([15]):

$$B_{2,n} = \left\langle \begin{array}{l} \tau, \mathcal{T}, \\ \sigma_1, \dots, \sigma_{n-1} \end{array} \middle| \begin{array}{l} \sigma_k \sigma_j = \sigma_j \sigma_k \ (|k-j| > 1), \quad \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1} \ (1 \le k \le n-1) \\ \mathcal{T} \sigma_k = \sigma_k \mathcal{T} \ (k \ge 2), \quad \mathcal{T} \sigma_1 \mathcal{T} \sigma_1 = \sigma_1 \mathcal{T} \sigma_1 \mathcal{T} \\ \tau \sigma_k = \sigma_k \tau \ (k \ge 2), \quad \tau \sigma_1 \tau \sigma_1 = \sigma_1 \tau \sigma_1 \tau, \\ \tau(\sigma_1 \mathcal{T} \sigma_1) = (\sigma_1 \mathcal{T} \sigma_1)\tau \end{array} \right\rangle$$

where $\sigma_i, \tau, \mathcal{T}$ are shown in Figure 1; I, II indicate the two *fixed strands* as they are called, whereas $1, 2, \dots, n$ indicate the *moving strands*. The braids $\tau, \mathcal{T}$ and their inverses are called the *looping generators*, whereas σ_i and its inverse are called the *i-th braiding generators* for $i = 1, 2, \dots, n-1$, whereas i is called the *index* of the i-th braiding generators.

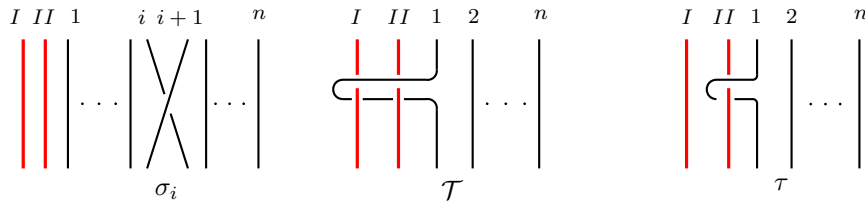**Fig. 1** The generators of $B_{2,n}$.

Below we define the elements $\mathcal{T}_i, \tau_i, \ i = 1, \dots, n$ of $B_{2,n}$ which will be of central importance to us in what follows.

Definition 1 The *looping elements* or just *loopings* $\mathcal{T}_i, \tau_i$ of $B_{2,n}$ are those braids in which all strands are straight except for the i-th moving strand that loops once around the first or the second fixed strand respectively, first going over and then (after the looping) under the rest of the strands to its left (see Figure 2). We use the name *looping* for the inverses of these elements as well, and we say that each one of the loopings $\mathcal{T}_i^{\pm 1}, \tau_i^{\pm 1}$ has *index* i. Formally:
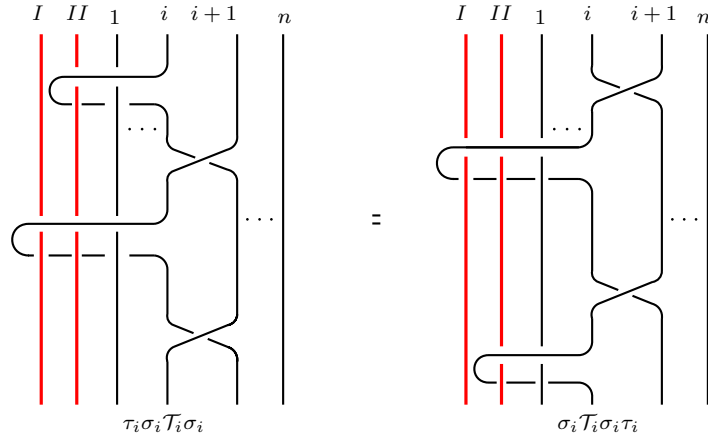
$$\mathcal{T}_1 := \mathcal{T}, \quad \tau_1 := \tau \quad and \; \mathcal{T}_i := \sigma_{i-1}\ldots\sigma_1\mathcal{T}\sigma_1\ldots\sigma_{i-1}, \quad \tau_i := \sigma_{i-1}\ldots\sigma_1\tau\sigma_1\ldots\sigma_{i-1} \; for \; i > 1.$$
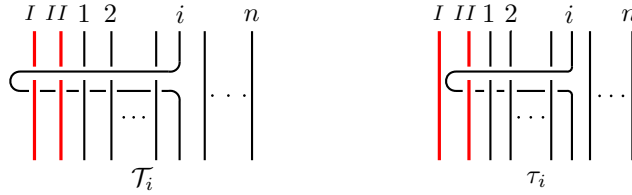
Fig. 2 The looping elements $\mathcal{T}_i, \tau_i$.

As is shown in Figure 3, the defining relation $\tau\sigma_1\mathcal{T}\sigma_1 = \sigma_1\mathcal{T}\sigma_1\tau$ of $B_{2,n}$ which is now written as $\tau_1\sigma_1\mathcal{T}_1\sigma_1 = \sigma_1\mathcal{T}_1\sigma_1\tau_1$, holds in general for all $i = 1, 2 \ldots, n-1$ (just slide the τ_i looping to pass through the $\mathcal{T}_i$ looping):

$$\tau_i\sigma_i\mathcal{T}_i\sigma_i = \sigma_i\mathcal{T}_i\sigma_i\tau_i.$$

Fig. 3 The relation $\tau_i\sigma_i\mathcal{T}_i\sigma_i = \sigma_i\mathcal{T}_i\sigma_i\tau_i$ in $B_{2,n}$.

Clearly, $B_{2,n}$ is generated by the set $\{\mathcal{T}_i, \tau_i, \sigma_i | i = 1, \ldots, n\}$ as well. Also, clearly, the group $B_{2,n}$ is a subgroup of the usual braid group B_{2+n} in $2 + n$ strands [14]. In turn, $B_{2,n}$ contains some important subgroups. One of them is the *pure mixed braid group on two fixed strands* $P_{2,n}$ that consists of all pure braids in $B_{2,n}$. For a further study of the structure of the groups $B_{2,n}, P_{2,n}$ and their generalizations $B_{m,n}, P_{m,n}$ as well as the underling Coxeter-type groups see [2].

Others important subgroups of $B_{2,n}$ are those generated by $\mathcal{T}, \sigma_1, \ldots, \sigma_{n-1}$ and by $\tau, \sigma_1, \ldots, \sigma_{n-1}$, which are isomorphic to the *mixed braid group on one fixed strand* $B_{1,n}$ defined in terms of generators and relations in an analogous manner as $B_{2,n}$ and which, in fact, is the Artin braid group of type B. Indeed, the defining relations $\mathcal{T}\sigma_1\mathcal{T}\sigma_1 = \sigma_1\mathcal{T}\sigma_1\mathcal{T}$ and $\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$ of $B_{2,n}$ are of the same type as the four-term defining relations of the Artin braid group of type B.

1.2 The algebra $H_{2,n}(q)$ and other related algebras

We define the algebra $H_{2,n}(q)$ for each $n \in \mathbb{N}$ as a quotient of an appropriate group-algebra of B_{2n} over appropriate quadratic relations. Namely:

Definition 2 The *mixed Hecke algebra on two fixed strands* $H_{2,n}(q)$ is defined as the unital associative algebra:

$$H_{2,n}(q) := \frac{\mathbb{Z}\left[q^{\pm 1}\right] B_{2,n}}{\langle \sigma_i{}^2 - (q-1)\,\sigma_i - q \cdot 1, \ i = 1, 2, \ldots, n-1 \rangle},$$

where q is a variable.

In general we use the same notation for the elements of $B_{2,n}$ when considered as elements of $H_{2,n}(q)$, except for σ_i which we denote g_i, $i = 1, \ldots, n$. $H_{2,n}(q)$ has equivalently a presentation with generators $\tau, \mathcal{T}, g_1, \ldots, g_{n-1}$ and relations:

$$
\begin{array}{llll}
(b_1) & g_k g_{k+1} g_k = g_{k+1} g_k g_{k+1} & \text{for } 1 \leq k \leq n-1 \\
(b_2) & g_k g_j = g_j g_k & \text{for } |k - j| > 1 \\
(\mathcal{T}_1) & \mathcal{T} g_k = g_k \mathcal{T} & \text{for } k \geq 2 \\
(\mathcal{T}_2) & \mathcal{T} g_1 \mathcal{T} g_1 = g_1 \mathcal{T} g_1 \mathcal{T} \\
(\tau_1) & \tau g_k = g_k \tau & \text{for } k \geq 2 \\
(\tau_2) & \tau g_1 \tau g_1 = g_1 \tau g_1 \tau \\
(m) & \tau(g_1 \mathcal{T} g_1) = (g_1 \mathcal{T} g_1)\tau \\
(q) & g_i^2 = (q-1)\,g_i + q \cdot 1 \quad \text{for } \text{i=1,2,\ldots,n-1}
\end{array}
\tag{1}
$$

(here (b) stands for "braid", (m) for "mixed" and (q) for "quadratic"). The elements $\tau, \mathcal{T}$ and their inverses will be called the *looping generators* of the algebra. The elements $g_1, \ldots, g_{n-1}$ and their inverses will be called the *braiding generators* of the algebra, whereas i will be the *index* of g_i, g_i^{-1}.

Since the classical Artin braid group B_n embeds naturally in $B_{2,n}$, we have that the classical Iwahori-Hecke Algebra $H_n(q)$ is a subalgebra of $H_{2,n}(q)$ in a natural way as well. Furthermore, note that the relations $(\mathcal{T}_2)$ and (τ_2) are of the same type as the well-known four-term defining relation of the Artin braid group of type B which is realized here by the mixed braid group $B_{1,n}$ with one fixed strand, hence it embeds naturally in $B_{2,n}$. So, the algebra $H_{2,n}(q)$ extends naturally the mixed Hecke algebra $H_{1,n}(q)$ introduced in [14] as "generalized Hecke algebra" of type B. The algebra $H_{2,n}(q)$ clearly contains two subalgebras isomorphic to $H_{1,n}(q)$.

We can define a lot of other interesting related algebras, a few of which as follows:

Definition 3 The algebra $H_{2,n}(q, u_1, v_1)$ is defined as:

$$H_{2,n}(q, u_1, v_1) := \frac{\mathbb{Z}\left[q^{\pm 1}, u_1^{\pm 1}, v_1^{\pm 1}\right] B_{2,n}}{\langle (q), \ \mathcal{T}^2 = (u_1 - 1)\mathcal{T} + u_1 \cdot 1, \ \tau^2 = (v_1 - 1)\tau + v_1 \cdot 1 \rangle},$$

where distinct variables u_1, v_1 are associated to $\mathcal{T}, \tau$.

Note that the relations $(\mathcal{T}_2)$ and (τ_2) are of the same type as the defining relations of the Hecke algebra of type B [7]. Furthermore, it is clear from the quadratic relations for the looping generators in Definition 3 that the algebra $H_{2,n}(q, u_1, v_1)$ extends the classical Hecke algebra of type B. In fact $H_{2,n}(q, u_1, v_1)$ contains two subalgebras isomorphic to the Hecke algebra of type B.

Definition 4 The *cyclotomic algebra* $H_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2})$ is defined as:

$$H_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2}) := \frac{\mathbb{Z}[q^{\pm 1}, u_1^{\pm 1}, \ldots, u_{d_1}^{\pm 1}, v_1^{\pm 1}, \ldots, v_{d_2}^{\pm 1}] B_{2,n}}{\langle (q), \ (\mathcal{T} - u_1) \ldots (\mathcal{T} - u_{d_1}) = 0, \ (\tau - v_1) \ldots (\tau - v_{d_1}) = 0 \rangle},$$

where $q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2}$ are variables and the last two relations are called *cyclotomic relations* for $\mathcal{T}$ and τ respectively (see Figure 4).
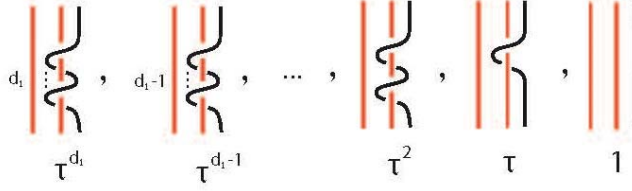
Fig. 4 The mixed braids involved in the cyclotomic relation of the loop generator τ

Analogously to the algebras defined above, by relations for $(\mathcal{T}_2)$ and (τ_2) and the defining relation for $\mathcal{T}, \tau$ in Definition 4 it follows that the algebra $H_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2})$ extends naturally the Ariki-Koike algebra of type B [1], and in fact it contains two subalgebras isomorphic to the Ariki-Koike algebra.

The mixed Hecke algebra $H_{1,n}(q)$, the Iwahori-Hecke algebra of type B, and the Ariki-Koike algebra of type B are all related to the knot theory of the solid torus and the lens spaces ([12],[8],[14],[5],[6]). Note that each one of the three types of algebras that we define here surjects naturally onto its corresponging B-type algebra, for example via the following mappings respectively:

- $\mathcal{T} \mapsto 1, \tau \mapsto \tau, g_i \mapsto g_i$ surjects $H_{2,n}(q)$ onto $H_{1,n}(q)$.
- $\mathcal{T} \mapsto 1, \tau \mapsto \tau, g_i \mapsto g_i$ and specializing u_1 to 1, surjects $H_{2,n}(q, u_1, v_1)$ onto the Hecke algebra of type B.
- $\mathcal{T} \mapsto 1, \tau \mapsto \tau, g_i \mapsto g_i$ and specializing u_i to 1 for $i = 1, \ldots, d_1$, surjects $H_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2})$ onto the Ariki-Koike algebra of type B.

Finally, let us note that the algebras $H_{2,n}(q, u_1, v_1)$ and $H_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2})$ can be viewed as quotient algebras of $H_{2,n}(q)$, if in Definitions 2–4 we use $\mathbb{Z}[q^{\pm 1}, u_1^{\pm 1}, \ldots, u_{d_1}^{\pm 1}, v_1^{\pm 1}, \ldots, v_{d_2}^{\pm 1}]$ as a common ring of coefficients for all three algebras.

2 A spanning set and potential basis for the algebra $H_{2,n}(q)$.

We still call $\mathcal{T}_i, \tau_i$ and their inverses as *looping elements* or *loopings* when we cosider them as elements of $H_{2,n}(q)$, and similarly we call i as their *index*. For $\mathcal{T}_i, \tau_i$ as elements of the algebra we have

$$\mathcal{T}_i = g_{i-1} \ldots g_1 \mathcal{T} g_1 \ldots g_{i-1} \quad and \quad \tau_i = g_{i-1} \ldots g_1 \tau g_1 \ldots g_{i-1}.$$

Our aim is to provide a "nice" form for any element w of $H_{2,n}(q)$ using these looping elements and the g_i's, so that a possible spanning set of the algebra reveals itself. Since w is a $\mathbb{Z}[q^{\pm 1}]$-linear combination of images in the algebra $H_{2,n}(q)$ of braids in $B_{2,n}$, one has to think about only the case of putting an image of a braid w in a "nice" form.

Now let us recall that the set $\{\mathcal{T}_i, \tau_i, \sigma_i | i = 1, \ldots, n\}$ generates $B_{2,n}$, thus an arbitrary braid w of $B_{2,n}$ is written as a finite product of elements of this set and their inverses; as a matter of fact, it can be written so in many ways. Previous work done on specific subsets of $B_{2,n}$, shows that considering their elements as belonging to appropriate related algebras similar to $H_{2,n}(q)$, we can put them in canonical forms useful for constructing Markov traces over these algebras. For example whenever w is a product of only the g_i's (considered as an element of the algebra) then w actually belongs to $H_n(q)$ and as such it is subjected to the canonical form of the classical Hecke algebra $H_n(q)$ of type A, given by V.F.R. Jones [10]. Also, whenever w is a product of only τ_i, g_i's (thus containing no $\mathcal{T}_i$'s), it actually belongs to $H_{1,n}(q)$ (as mentioned in the previous section, this is the generalized Hecke algebra of type B), and therefore it is subjected to the canonical form given in [14]. Such a w is written as a finite $\mathbb{Z}[q^{\pm 1}]$-linear combination of products of τ_i's and g_i's with the τ_i's appearing first, and moreover with the indices of the τ_i's in increasing order from left to right.

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Theorem 1 below tells us how to bring any w of $B_{2,n}$ to a similar "nice" form when considered as an element of the algebra $\mathrm{H}_{2,n}(q)$. At the same time we get a spanning set Λ_n for the additive structure of $\mathrm{H}_{2,n}(q)$ as a $\mathbb{Z}[q^{\pm 1}]$-module. What the Theorem actually says is that every element of the algebra $\mathrm{H}_{2,n}(q)$ is written as a finite $\mathbb{Z}[q^{\pm 1}]$-linear combination of products of $\mathcal{T}_i, \tau_i$'s and g_i's with the $\mathcal{T}_i, \tau_i$'s appearing first, and moreover with the indices of the τ_i's in increasing order from left to right. To achieve this for the image of a braid given in a product form we can try first to push all g_i's (i.e. the images of the σ_i's) at the end using braid isotopies (at the braid level) together with the quadratic relations in the algebra $\mathrm{H}_{2,n}(q)$ (see Lemma 1). And then we can similarly try to push all loopings with big indices after those with smaller ones (see Lemma 2). Working out specific examples one soon realizes that pushing the g_i's is always possible, and that pushing the loopings with big indices after those with small ones can be almost always achieved, except that in the process some new g_i's might be created, and pushing them anew to the end might increase the indices of the loopings from which it passes, leaving quite open the question of whether the indices of the loopings can indeed be ordered. We deal with this issue in Lemma 3.

Theorem 1 *Any element in $\mathrm{H}_{2,n}(q)$ can be written as a finite $\mathbb{Z}[q^{\pm 1}]$-linear combination of the form (suppressing the coefficient in $\mathbb{Z}[q^{\pm 1}]$ of each term):*

$$\sum (\Pi_1 \Pi_2 \cdots \Pi_n) G$$

where G is a finite product of braiding generators, and Π_i is a finite product of only the loopings $\mathcal{T}_i, \tau_i, \mathcal{T}_i^{-1}, \tau_i^{-1}$ for all i. Thus the following is a spanning set of the algebra $\mathrm{H}_{2,n}(q)$:
$\Lambda_n := \{ \Pi_1 \Pi_2 \cdots \Pi_n G \mid G = $ finite product of braiding generators, and $\Pi_i = $ finite product of only the loopings $\mathcal{T}_i, \tau_i, \mathcal{T}_i^{-1}, \tau_i^{-1}, \forall i \}$.

The definition of the looping elements and braiding generators can be repeated for the other algebras which we defined in §1.2, and the proof of Theorem 1 can be repeated unaltered step by step to get:

Theorem 2 *Let $A = \mathrm{H}_{2,n}(q, u_1, v_1), R = \mathbb{Z}[q^{\pm 1}, u_1^{\pm 1}, v_1^{\pm 1}]$ or $A = \mathrm{H}_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2}), R = \mathbb{Z}[q^{\pm 1}, u_1^{\pm 1}, \ldots, u_{d_1}^{\pm 1}, v_1^{\pm 1}, \ldots, v_{d_2}^{\pm 1}]$. Then any element in A can be written as a finite R-linear combination of the form (suppressing the coefficient in R of each term):*

$$\sum (\Pi_1 \Pi_2 \cdots \Pi_n) G$$

where G is a finite product of braiding generators, and Π_i is a finite product of only the loopings $\mathcal{T}_i, \tau_i, \mathcal{T}_i^{-1}, \tau_i^{-1}$ for all i. Thus the following is a spanning set of the algebra R:
$\Lambda_n := \{ \Pi_1 \Pi_2 \cdots \Pi_n G \mid G = $ finite product of braiding generators, and $\Pi_i = $ finite product of only the loopings $\mathcal{T}_i, \tau_i, \mathcal{T}_i^{-1}, \tau_i^{-1}, \forall i \}$.

Below we list the necessary lemmata for the proof of Theorem 1 which is quite technical since it has to deal carefully with the indices appearing in a given product of loopings as well as with the possible recursion phenomena that might arise during the process. We provide the actual proof in [11]. The lemmata equip us with specific formulas for pushing braiding generators to the right of a product of loopings, and also for pushing loopings with high indices to the right of loopings with lower indices in a product of loopings. The lemmata also explain how we can deal with recursion phenomena.

Lemma 1. *Let us call $A = q^{-1} - 1, B = q - 1$. And let us denote the identity element of $\mathrm{H}_{2,n}(q)$ by 1. Then the following hold in $\mathrm{H}_{2,n}(q)$:*

(1) $g_i^{-1} = q^{-1} g_i + A \cdot 1$

(2) $g_i \mathcal{T}_j^{\pm 1} = \mathcal{T}_j^{\pm 1} g_i$ $g_i \tau_j^{\pm 1} = \tau_j^{\pm 1} g_i$ *whenever* $j \neq i, i+1$

(3) $g_i \mathcal{T}_i = q^{-1} \mathcal{T}_{i+1} g_i + A \mathcal{T}_{i+1}$ $g_i \tau_i = q^{-1} \tau_{i+1} g_i + A \tau_{i+1}$

(4) $g_i \mathcal{T}_i^{-1} = \mathcal{T}_{i+1}^{-1} - A \mathcal{T}_{i+1}^{-1} + A \mathcal{T}_i^{-1}$ $g_i \tau_i^{-1} = \tau_{i+1}^{-1} - A \tau_{i+1}^{-1} + A \tau_i^{-1}$

(5) $g_i \mathcal{T}_{i+1} = q \mathcal{T}_i g_i + B \mathcal{T}_{i+1}$ $g_i \tau_{i+1} = q \tau_i g_i + B \tau_{i+1}$

(6) $g_i \mathcal{T}_{i+1}^{-1} = q^{-1} \mathcal{T}_i^{-1} g_i + A \mathcal{T}_i^{-1}$ $g_i \tau_{i+1}^{-1} = q^{-1} \tau_i^{-1} g_i + A \tau_i^{-1}$.

(7) (The passage property) Any product $g_k^\epsilon t_l^\zeta$ *($\epsilon, \zeta \in \{1, -1\}$, t_l a looping) can be written as a finite linear combination of the form (suppressing the coefficient in $\mathbb{Z}[q^{\pm 1}]$ of each term on the right-hand side):*

$$g_i^\epsilon t_l^\zeta = \sum t_l^\zeta g_i^\epsilon + \sum t_i^\zeta + \sum t_i^\zeta g_i^\epsilon + \sum t_{i+1}^\zeta.$$

(where possibly some of the terms are missing).

(8) (The big passage property) Let Π be a finite product of k in number loopings with indices in the interval $[m, M]$, and let $i \in [m, M-1]$. Then $g_i^{\pm 1} \Pi$ can be written as a finite linear combination of the form (suppressing the coefficient in $\mathbb{Z}[q^{\pm 1}]$ of each term on the right-hand side):

$$g_i^{\pm 1} \Pi = \sum \Pi_1 g_i^{\pm 1} + \sum \Pi_2$$

(where possibly some terms are missing) with each Π_1, Π_2 a product of k in number loopings with indices in $[m, M]$.

Part (2) of the lemma can be seen in the braid level via trivial braid isotopies, and parts (2)-(6) can also be seen pictorially after at most two applications of the quadratic relation to the braids of the left-hand side. These are summarized in part (7). Since on both sides of parts (2)-(6) each term contains a single looping and in part (7) the index of the looping either does not change at all or if it does, it decreases by 1 but then never below the index i of the braiding generator, or else it increases by 1 but then never by 1 above the index i of the braiding generator, we get part (8) at once.

The following lemma describes how we can push a looping with high index to the right of a looping with smaller index.

Lemma 2. *For $j < i$ and $\epsilon, \zeta \in \{1, -1\}$ each one of the words $\mathcal{T}_i^\epsilon \mathcal{T}_j^\zeta, \mathcal{T}_i^\epsilon \tau_j^\zeta, \tau_i^\epsilon \mathcal{T}_j^\zeta, \tau_i^\epsilon \tau_j^\zeta$ can be written as a linear combination of the form (suppressing the coefficient in $\mathbb{Z}[q^{\pm 1}]$ of each term on the right-hand side):*

1. $\mathcal{T}_i^\epsilon \mathcal{T}_j^\zeta = \mathcal{T}_j^\zeta \mathcal{T}_i^\epsilon, \ \tau_i^\epsilon \tau_j^\zeta = \tau_j^\zeta \tau_i^\epsilon, \ \mathcal{T}_i^\epsilon \tau_j^\zeta = \tau_j^\zeta \mathcal{T}_i^\epsilon$

2. $\tau_i^\epsilon \mathcal{T}_j^\epsilon = \mathcal{T}_j^\epsilon \tau_i^\epsilon + \mathcal{T}_j^\epsilon \tau_i^\epsilon G^\epsilon + \tau_j^\epsilon \mathcal{T}_i^\epsilon G^\epsilon,$ *where* $G = g_j g_{j+1} \ldots g_{i-2} g_{i-1}^{-1} g_{i-2}^{-1} \ldots g_{j+1}^{-1} g_j^{-1}.$

3. $\tau_i^\epsilon \mathcal{T}_j^{-\epsilon} = \mathcal{T}_j^{-\epsilon} \tau_i^\epsilon + \mathcal{T}_j^{-\epsilon} \tau_j^\epsilon G^\epsilon + \tau_i^\epsilon \mathcal{T}_i^{-\epsilon} G^\epsilon,$ *where* $G = g_j g_{j+1} \ldots g_{i-2} g_{i-1} g_{i-2} \ldots g_{j+1} g_j.$

The proof of this lemma is easy, as part (1) can be seen at the braid level via braid isotopies, and the last two parts can also be seen at the braid level as a double application of the quadratic relation at the obvious crossings so that the i-looping can be moved above the j-looping.

The lemma that follows is the last one that we need for the proof of Theorem 1, and it says that a certain class of words actually satisfies the theorem. These words have the odd property that whenever we apply all the previous formulas in order to write them as sums of monomials in the way demanded by the theorem, they are written so except from the fact that one of the monomials is the word itself. Fortunately, the coefficients appearing in these equalities are well behaved and we can solve for the given word so that it is indeed expressed in way described in the theorem. This recursion phenomenon is possible only because one of the monomials on the right-hand side in case (3) of Lemma 2 still starts with an i-looping instead of a j-looping.

In the statement of the lemma it is convenient to write $[i, j]$ in the bottom of a product of looping or braiding generators to indicate that their indices lie in the interval $[i, j]$, and to write $< i, j >$ to indicate that these indices are also in increasing order (from left to right).

Lemma 3. *Let us denote elements in $\{\mathcal{T}_i^{\pm 1}, \tau_i^{\pm 1}\}$ indiscreetly by t_i. Then each one of the words $\tau_M^\epsilon \mathcal{T}_M^{-\epsilon} t_m^\zeta$ with $m < M$ and $\epsilon, \zeta \in \{-1, 1\}$ can be written as a finite linear combination of the form (suppressing the coefficient in $\mathbb{Z}[q^{\pm 1}]$ of each term on the right-hand side):*

$$\tau_M^\epsilon \mathcal{T}_M^{-\epsilon} t_m^\zeta = \sum \left(t_m t_{m_1} t_{m_2} \right)_{<m,M>} G_{[m, M-1]}$$

where each G is a finite product of $g_i^{\pm 1}$'s (notice the crucial fact that every term of the last sum starts with an m-looping).

The proof of this lemma is not as immediate as the proofs in the previous lemmata. We have to examine all possible cases separately, applying the quadratic relations appropriately and using isotopies at the braid level. The reader is referred to [11] for full details of the proof, as well as for the proof of Theorem 1 which is a consequences of Lemmata 1–3.

Remark: In [11], we also conjecture that the set Λ_n is a linear basis for the algebra $\mathrm{H}_{2,n}(q)$. This is not straightforward to prove, as the algebra $\mathrm{H}_{2,n}(q)$ is infinite dimensional. Nevertheless we can get some insight of how Λ_n behaves, by examining its counterparts in the other algebras $\mathrm{H}_{2,n}(q, u_1, v_1)$ and $\mathrm{H}_{2,n}(q, u_1, \ldots, u_{d_1}, v_1, \ldots, v_{d_2})$, defined in this paper, and for which these counterparts also constitute spanning sets (Theorem 2). Although these algebras are infinite dimensional too, the exponents of the loopings in the elements of the above spanning sets are bounded, a fact that makes these algebras easier to study.

3 Conclusion and further research

In this paper we have defined some Hecke-type algebras related to the mixed braid group $B_{2,n}$ on two fixed strands, and we have focused on one of them, namely on the mixed Hecke algebra $\mathrm{H}_{2,n}(q)$ which is defined as the quotient of the group-algebra $\mathbb{Z}[q^{\pm 1}]B_{2,n}$ over the quadratic relations of the usual Hecke algebra. These algebras are related to the knot theory of various 3-manifolds whose knot structure is encoded by the mixed braid groups $B_{2,n}$, such as handelebodies of genus two, and connected sums of lens spaces. We have given here a subset Λ_n of $\mathrm{H}_{2,n}(q)$ and provided the necessary lemmata along with hints for their truth, for proving that Λ_n is a spanning set for the additive structure of the algebra [11]. We conjecture that Λ_n is actually a basis for $\mathrm{H}_{2,n}(q)$ and this the subject of current research. Then, based on previous work done on similar Hecke-type algebras, we expect that we can use Λ_n for the construction of knot invariants in the above 3-manifolds.

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