A CATEGORICAL STRUCTURE FOR THE VIRTUAL BRAID GROUP

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1. Introduction

This paper gives a new interpretation of the virtual braid group in terms of a strict monoidal category SC that is freely generated by one object $*$ and three morphisms $\mu : * \otimes * \longrightarrow * \otimes *$, $\mu' : * \otimes * \longrightarrow * \otimes *$, and $v : * \otimes * \longrightarrow * \otimes *$. This basic structure, subjected to appropriate relations can be understood via defined morphisms μ_{ij} where this symbol can be interpreted as an abstract string or connection between strands i and j in a diagram that otherwise is an identity on n strands. That is, μ_{ij} is diagrammatically a decorated identity braid where the decoration consists in a connection between the i strand and the j strand. The μ_{ij} satisfy the algebraic Yang-Baxter equation in the sense that for $i < j < k$, $\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}$. The other generators of this category are elements v_i that can be depicted as virtual crossings between strings i and $i+1$. The v_i generate the symmetric group S_n. An n-strand diagram that is a product of these generators is a morphism from $[n]$ to $[n]$ where the symbol $[n]$ is an ordered row of n points that constitute the top or the bottom of a diagram involving n strands. In terms of the definition of the monoidal category $[n] = * \otimes * \cdots * \otimes *$ for a tensor product of n $*$'s.

The virtual braid group on n strands is isomorphic to the group of morphisms in SC from $[n]$ to $[n]$. The point of this categorical formulation of the virtual braid groups is that we see how these groups form a natural extension of the symmetric groups by formal elements that satisfy the algebraic Yang-Baxter equation. The category we desribe is a natural structure for an algebraist interested in exploring formal properties of the algebraic Yang-Baxter equation, and it is directly related to more topological points of view about virtual links and virtual braids.

This paper is an abbreviated version of [20] where we give complete proofs of all the theorems stated here. The present paper is self-contained with a few details of proofs omitted. Our longer paper [20] gives complete proofs of all results and discusses generalizations related to Hopf algebras and quantum link invariants for rotational virtual knots and links.

Without the concept of virtuality, the direct relationship of the algebraic Yang-Baxter equation with the braid groups would not be apparent. We see that from an algebraic

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point of view, the virtual braid group is an entirely natural construction. It is an algebraic structure related to viewing solutions to the algebraic Yang-Baxter equation inside tensor products of algebras where these tensor products are endowed with the natural permutation action of the symmetric group.

We develop this model for the virtual braid group by first recalling its usual definition motivated by virtual knot theory. We then proceed to reformulate the virtual braid group in terms of the above mentioned generators. By the time we reach Theorem 1, we have reformulated the virtual braid group in terms of the new generators. We then use this approach to give a presentation of the pure virtual braid group in Theorem 2.

More precisely, in Section 2 we give a presentation for the virtual braid group in terms of our stringy model. We start by describing the usual presentation of the virtual braid group in terms of classical braid generators and virtual generators that act as permutations between pairs of adjacent strands in the braid. Elementary connection strings (see Figure 6) are defined as elementary pure braids – products of braid generators and virtual generators. We then generalize the notion of connecting string and show that it has the formal diagrammatic property of being stretched and contracted as shown in Figure 8. With these constructions we then rewrite presentations for the virtual braid group and, in Section 3, show how the connection with strings generate the pure virtual braid group with a set of relations that correspond to the algebraic Yang-Baxter equation. See Theorem 2.

In Section 4 we construct the String Category alluded to in the first paragraph of this introduction. In Section 5 we detail the relationship with the algebraic Yang-Baxter equation, show how to use solutions of the algebraic Yang-Baxter equation to obtain representations of the pure virtual braid group and virtual braid group. In our point of view the entire virtual braid group can be seen as a natural extension of the pure virtual braid group by a category of permutation operators. The pure virtual braid groups themselves are seen to be a natural monoidal category associated with solutions of the algebraic Yang-Baxter equation. This gives an essentially categorical point of view for understanding the nature of the virtual braid group. In starting our discussion of the virtual braid group from virtual knot theory we began with the motivation that virtual crossings are artifacts of a planar representation of knots and links that are embedded in thickened surfaces. This is a correct point of view, but it does not speak directly to the algebraic structure of the virtual braid group, where the virtual part of the group is the symmetric group generated by the virtual crossings. In the braiding context the virtual crossings are permutation operators and it is conceptually important to have a point of view in which their role is natural in a categorical and algebraic sense. This is what we have done in reformulating the virtual braid group in terms of the category of string connectors and associated permutation operators.

2. A Stringy Presentation for the Virtual Braid Group

2.1. **The virtual braid group.** Let's begin with a presentation for the virtual braid group. The set of isotopy classes of virtual braids on n strands forms a group, the *virtual braid group* denoted VB_n, that was introduced in [15]. The group operation is the usual braid multiplication (form bb' by attaching the bottom strand ends of b to the top strand

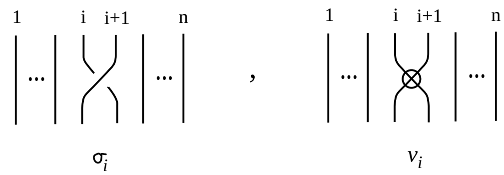

FIGURE 1. The generators of VB_n

ends of b'). VB_n is generated by the usual braid generators $\sigma_1, \ldots, \sigma_{n-1}$ and by the virtual generators $v_1, \ldots, v_{n-1}$, where each virtual crossing v_i has the form of the braid generator σ_i with the crossing replaced by a virtual crossing. See Figure 1 for illustrations. Recall that in virtual crossings we do not distinguish between under and over crossing. Thus, VB_n is an extension of the classical braid group B_n by the symmetric group S_n, whereby v_i corresponds to the elementary transposition $(i, i+1)$.

Among themselves the braid generators satisfy the usual *braiding relations*:

$$\begin{array}{lrcll} \text{(B1)} & \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1}, & \\ \text{(B2)} & \sigma_i\sigma_j & = & \sigma_j\sigma_i, & \text{for } j \neq i \pm 1. \end{array}$$

Among themselves, the virtual generators are a presentation for the symmetric group S_n, so they satisfy the following *virtual relations:*

$$\begin{array}{lrcll} \text{(S1)} & v_iv_{i+1}v_i & = & v_{i+1}v_iv_{i+1}, & \\ \text{(S2)} & v_iv_j & = & v_jv_i, & \text{for } j \neq i \pm 1, \\ \text{(S3)} & {v_i}^2 & = & 1. & \end{array}$$

The *mixed relations* between virtual generators and braiding generators are as follows:

$$\begin{array}{lrcll} \text{(M1)} & v_i\sigma_{i+1}v_i & = & v_{i+1}\sigma_iv_{i+1}, & \\ \text{(M2)} & \sigma_iv_j & = & v_j\sigma_i, & \text{for } j \neq i \pm 1. \end{array}$$

To summarize, the virtual braid group VB_n has the following presentation [15].

$$VB_n = \left\langle \begin{array}{c} \sigma_1, \ldots, \sigma_{n-1}, \\ v_1, \ldots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} (B1), (B2), \\ (S1), (S2), (S3), \\ (M1), (M2) \end{array} \right\rangle \tag{1}$$

It is worth noting at this point that the virtual braid group VB_n does not embed in the classical braid group B_n, since the virtual braid group contains torsion elements (the v_i have order two) and it is well–known that B_n does not. But the classical braid group embeds in the virtual braid group just as classical knots embed in virtual knots. This fact may be most easily deduced from [22], and can also be seen from [24] and [6]. For reference to previous work on virtual knots and virtual braids the reader should consult [4, 5, 9, 10, 11, 15, 16, 17, 13, 14, 21, 22, 24, 25, 27, 28, 29, 18, 19] and references therein. For work on welded braids and welded knots, see [6, 14, 18, 19].

Further, for Markov–type theorems for virtual braids, giving sets of moves on virtual braids that generate the same equivalence classes as the oriented virtual link types of their closures, see [14] and [19]. Such theorems are important for understanding the structure and classification of virtual knots and links.

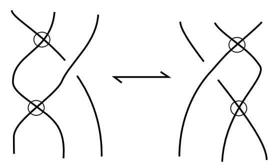

FIGURE 2. The local detour

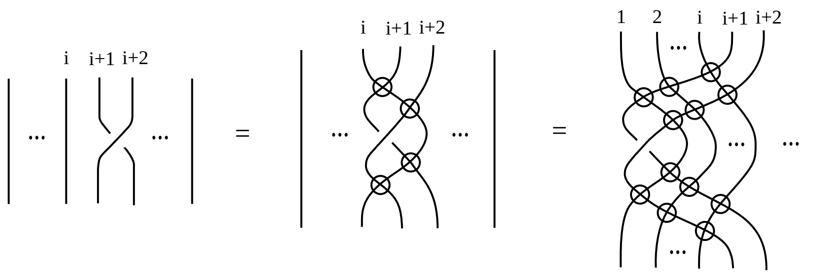

FIGURE 3. Detouring the crossing σ_{i+1}

The second mixed relation in the presentation of the virtual braid group will be called the *local detour relation* and it is illustrated in Figure 2. Note that the following relations are also local detour moves for virtual braids and they are easy consequences of the above.

$$
\begin{array}{rcl}
v_i v_{i+1} \sigma_i{}^{\pm 1} & = & \sigma_{i+1}{}^{\pm 1} v_i v_{i+1}, \\
\sigma_i{}^{\pm 1} v_{i+1} v_i & = & v_{i+1} v_i \sigma_{i+1}{}^{\pm 1}.
\end{array} \tag{2}
$$

This set of relations taken together define the basic local isotopies for virtual braids. Note that each relation is a braided version of a local virtual link isotopy. The local detour move is written equivalently:

$$\sigma_{i+1} = v_i v_{i+1} \sigma_i v_{i+1} v_i. \tag{3}$$

Notice that Eq. 3 is the braid detour move of the ith strand around the crossing between the $(i+1)$-st and the $(i+2)$-nd strand (see first two illustrations in Figure 3) and it provides an inductive way of expressing all braiding generators in terms of the first braiding generator σ_1 and the virtual generators $v_1, \ldots, v_{n-1}$ (see first and last illustrations in Figure 3), that is:

$$\sigma_j = (v_{j-1} \ldots v_2 v_1)\,(v_j \ldots v_3 v_2)\,\sigma_1\,(v_2 v_3 \ldots v_j)\,(v_1 v_2 \ldots v_{j-1}). \tag{4}$$

In [18] we derive the following reduced presentation for VB_n:

$$
VB_n = \left\langle \begin{array}{c} \sigma_1, \\ v_1, \ldots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} (S1), (S2), (S3) \\ \sigma_1 v_j = v_j \sigma_1, \quad \text{for } j > 2 \\ v_1\sigma_1 v_1\, v_2\sigma_1 v_2\, v_1\sigma_1 v_1 = v_2\sigma_1 v_2\, v_1\sigma_1 v_1\, v_2\sigma_1 v_2 \\ \sigma_1\, v_2 v_3 v_1 v_2 \sigma_1 v_2 v_1 v_3 v_2 = v_2 v_3 v_1 v_2 \sigma_1 v_2 v_1 v_3 v_2\, \sigma_1 \end{array} \right\rangle \tag{5}
$$

The local detour move gives rise to a generalized *detour move*, by which any box in the braid can be detoured to any position in the braid, see Figure 4.

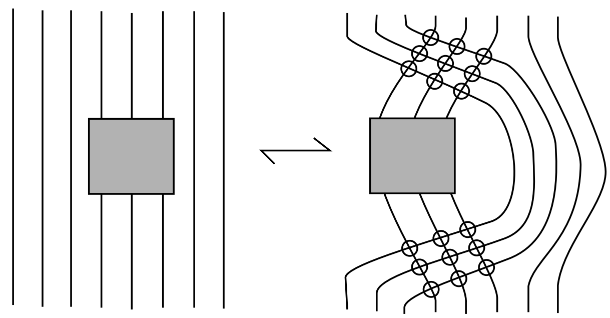

FIGURE 4. Detouring a box

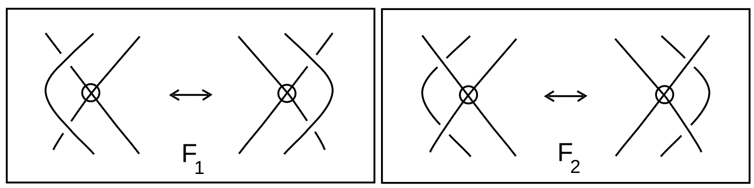

FIGURE 5. The forbidden moves

Finally, it is worth recalling that in virtual knot theory there are "forbidden moves" involving two real crossings and one virtual. More precisely, there are two types of forbidden moves: One with an over arc, denoted F_1 and another with an under arc, denoted F_2. See [15] for explanations and interpretations. Variants of the forbidden moves are illustrated in Figure 5. So, relations of the types:

$$\sigma_i v_{i+1} \sigma_i^{-1} = \sigma_{i+1}^{-1} v_i \sigma_{i+1} \quad (F1) \qquad \text{and} \qquad \sigma_i^{-1} v_{i+1} \sigma_i = \sigma_{i+1} v_i \sigma_{i+1}^{-1} \quad (F2) \tag{6}$$

are not valid in virtual knot theory.

2.2. We now wish to describe a new set of generators and relations for the virtual braid group that makes it particularly easy to describe the pure virtual braid group, VP_n. In order to accomplish this aim, we introduce the following elements of VP_n, for $i = 1, \ldots, n-1$.

$$\mu_{i,i+1} := \sigma_i v_i \tag{7}$$

We indicate $\mu_{i,i+1}$ by a connecting string between the i-th and $(i+1)$-st strands with a dark vertex on the i-th strand, a dark vertex on the $(i+1)$-st strand, and an arrow from left to right. View Figure 6. The inverses $\mu_{i,i+1}^{-1} = v_i \sigma_i^{-1}$ have same directional arrows but are indicated by using white vertices. Note that, by detouring it to the leftmost position of the braid, we can write $\mu_{i,i+1}$ in terms of μ_{12} conjugated by a virtual word:

$$\mu_{i,i+1} = (v_{i-1} \ldots v_2 v_1)(v_i \ldots v_3 v_2)\mu_{12}(v_2 v_3 \ldots v_i)(v_1 v_2 \ldots v_{i-1}). \tag{8}$$

We also introduce the elements

$$\mu_{i+1,i} := v_i \sigma_i = v_i \mu_{i,i+1} v_i \tag{9}$$

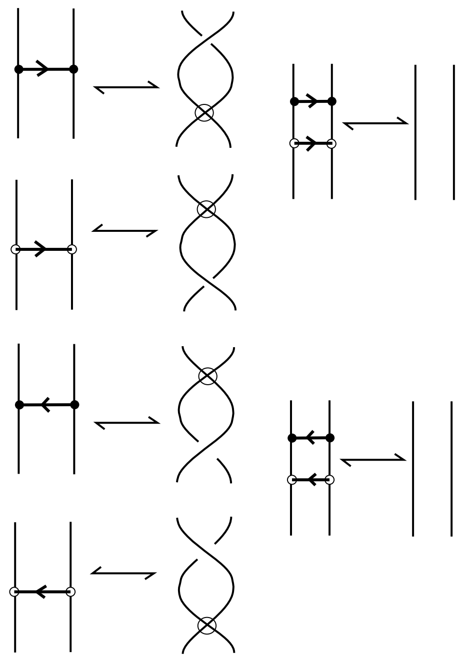

FIGURE 6. The elementary connecting strings $\mu_{i,i+1}$, $\mu_{i+1,i}$ and their inverses

We indicate $\mu_{i+1,i}$ by a connecting string between the i-th and $(i+1)$-st strands, with a dark vertex on the i-th strand, a dark vertex on the $(i+1)$-st strand, and an arrow from right to left (reversing the direction from $\mu_{i,i+1}$), view Figure 6. An illustration of Eq. 9 (see top of Figure 7) explains the reversing of the direction of the arrow in the graphical interpretation of $\mu_{i+1,i}$. The inverses $\mu_{i+1,i}^{-1} = \sigma_i^{-1} v_i$ have same directional arrows but are indicated by using white vertices. Note that an analogous equation to Eq. 8 holds:

$$\mu_{i+1,i} = (v_{i-1} \dots v_2 v_1)(v_i \dots v_3 v_2)\mu_{21}(v_2 v_3 \dots v_i)(v_1 v_2 \dots v_{i-1}) \tag{10}$$

Definition 1. The pure virtual braids $\mu_{i,i+1}, \mu_{i+1,i}$ and their inverses shall be called *elementary connecting strings.*

From Eqs. 7 and 9 follow directly the relations:

$$v_i \mu_{i+1,i} = \mu_{i,i+1} v_i \quad \text{and} \quad \mu_{i+1,i}^{-1} v_i = v_i \mu_{i,i+1}^{-1}, \tag{11}$$

also illustrated in Figure 7.

Further, we generalize the notion of a connecting string and define, for $i < j$, the element μ_{ij} (a connecting string from strand i to strand j) by the formula

$$\mu_{ij} := v_{j-1} v_{j-2} \dots v_{i+1}\, \mu_{i,i+1}\, v_{i+1} \dots v_{j-2} v_{j-1}. \tag{12}$$

In a diagram μ_{ij} is denoted by a connecting string from strand i to strand j, with dark vertices on these two strands and an arrow pointing from left to right, view Figure 8.

We also generalize, for $i < j$, the elements $\mu_{i+1,i}$ to the elements:

$$\mu_{ji} := t_{ij}\, \mu_{ij}\, t_{ij} \tag{13}$$

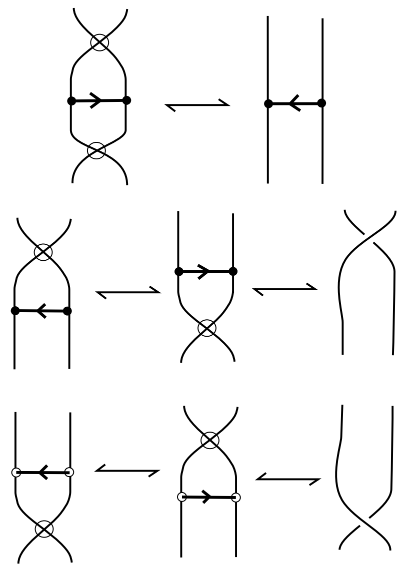

FIGURE 7. Relations between the elementary connecting strings

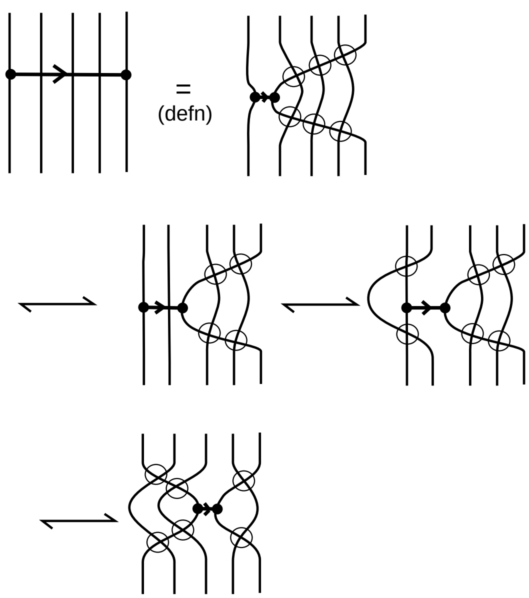

FIGURE 8. Connecting strings

where $t_{ij} = v_i v_{i+1} \dots v_j \dots v_{i+1} v_i$ is the element of S_n (generated by the v_i's) that interchanges strands i and j, leaving all other strands fixed. We denote μ_{ji} by a connecting string from strand i to strand j, with dark vertices, and an arrow pointing from right to left. Figure 9 illustrates the example $\mu_{31} = v_2 v_1 v_2 \mu_{13} v_2 v_1 v_2$. It is easily verified that

$$\mu_{ji} = v_{j-1} v_{j-2} \dots v_{i+1}\, \mu_{i+1,i}\, v_{i+1} \dots v_{j-2} v_{j-1} \tag{14}$$

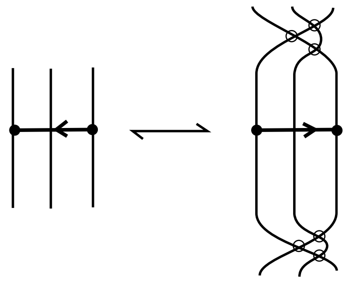

FIGURE 9. The exchange of labels between μ_{ij} and μ_{ji}

The inverses of the elements μ_{ij} and μ_{ji} have same directional arrows respectively, but white dotted vertices.

Definition 2. The elements μ_{ij}, μ_{ji} and their inverses shall be called *connecting strings.*

With the above conventions we can speak of connecting strings μ_{rs} for any r, s. It is important to have the elements μ_{ji} when $j > i$, but in the algebra they are all defined in terms of the μ_{ij}. The importance of having the elements μ_{ji} will become clear when we restrict to the pure virtual braid group.

Remark 1. In the definition of μ_{ij} if we consider $\mu_{i,i+1}$ as a virtual box inside the virtual braid we can use the (generalized) detour moves to bring it to any position, as Figure 8 illustrates. This means that the contraction of μ_{ij} to $\mu_{i,i+1}$ may be pulled anywhere between the i-th and the j-th strands. By the same reasoning the contraction of μ_{ji} to $\mu_{i+1,i}$ may be also pulled anywhere between the i-th and the j-th strands.

2.3. We shall next give some relations satisfied by the connecting strings. Before that we need the following remark.

Remark 2. The symmetric group S_n clearly acts on VB_n by conjugation. By their definition (Eqs. 7, 9, 12, 14), this action on connecting strings translates into permuting their indices, that is, a permutation $\tau \in S_n$ acting on μ_{rs} will change it to $\mu_{\tau(r),\tau(s)}$. This means that S_n acts by conjugation also on the subgroup of VB_n generated by the μ_{ij}'s. Moreover, by Eqs. 8, 9, all connecting strings may be obtained by the action of S_n on μ_{12}. Note that for $\sigma \in S_n$ we regard σ both as a product of the elements v_i and as a permutation of the set $\{1, 2, 3, \ldots, n\}$.

Further, any relation in VB_n transforms into a valid relation after acting on it an element of S_n. In particular, a commuting relation between connecting strings will be transformed to a new commuting relation between connecting strings.

Lemma 1. *The following relations hold in VB_n for all i.*

(1) $v_i\mu_{i,i+1} = \mu_{i+1,i}v_i \quad , \quad v_i\mu_{i+1,i} = \mu_{i,i+1}v_i$
(2) $v_{i+1}\mu_{i,i+1} = \mu_{i,i+2}v_{i+1} \quad , \quad v_{i+1}\mu_{i+1,i} = \mu_{i+2,i}v_{i+1}$
(3) $v_{i-1}\mu_{i,i+1} = \mu_{i-1,i+1}v_{i-1} \quad , \quad v_{i-1}\mu_{i+1,i} = \mu_{i+1,i-1}v_{i-1}$
(4) $v_j\mu_{i,i+1} = \mu_{i,i+1}v_j \quad , \quad v_j\mu_{i+1,i} = \mu_{i+1,i}v_j, \quad j \neq i-1,\, i,\, i+1.$

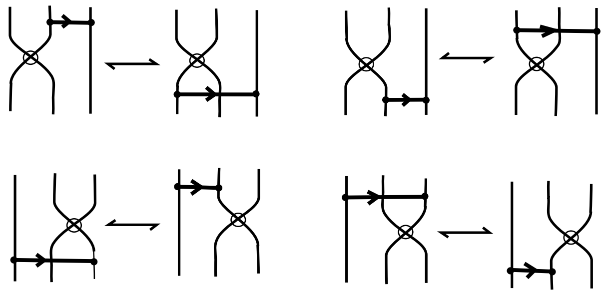

FIGURE 10. Slide moves

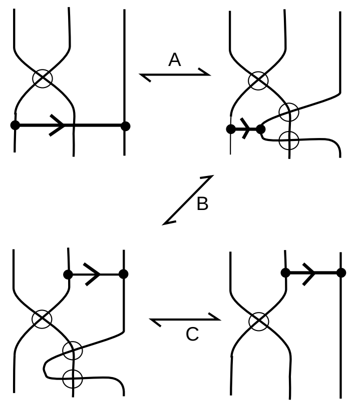

FIGURE 11. Proving a local slide move

The above local relations generalize to similar ones involving different indices. Relations 1 are generalized by Eq. 13, reflecting the mutual reversing of μ_{ij} *and* μ_{ji}*, recall Figures 7 and 9. Relations 2 and 3 are the* local slide moves*, as illustrated in Figure 10, and they generalize to the* slide moves *coming from the defining equations:* $\mu_{i+1,k} = v_i \mu_{ik} v_i$ *for any* $k < i$ *or* $k > i+1$*. Relations 4 and their generalizations:* $v_j \mu_{ik} = \mu_{ik} v_j$ *for any* $k \neq i$ *and* $j \neq i-1$, i, $k-1$, k*, are all commuting relations. All these relations result from the action of any* $\tau \in S_n$ *on* μ_{12}*:*

$$\boldsymbol{\tau^{-1}\,\mu_{12}\,\tau = \mu_{\tau(1),\tau(2)}}\ . \tag{15}$$

Proof. All relations 1,2 and 3 follow directly from the definitions of the elements μ_{ij} and μ_{ji}. For example, $v_{i+1}\mu_{i,i+1} = \mu_{i,i+2}v_{i+1}$ is equivalent to the defining relation $\mu_{i,i+2} = v_{i+1}\mu_{i,i+1}v_{i+1}$. Figure 11 illustrates the proof of a local slide move. Relations 4 follow immediately from the commuting relations (S2) and (M2) of VB_n. The generalizations of all types of moves follow from the local ones after using detour moves. Finally, the derivation of all relations from the action of S_n on μ_{12} is explained in Remark 2 and, more precisely, by the Eqs. 8, 12, 10, 14. □

Lemma 2. *The following commuting relations among connecting strings hold in* VB_n*.*

(1) $\boldsymbol{\mu_{12}\mu_{34} = \mu_{34}\mu_{12}}$

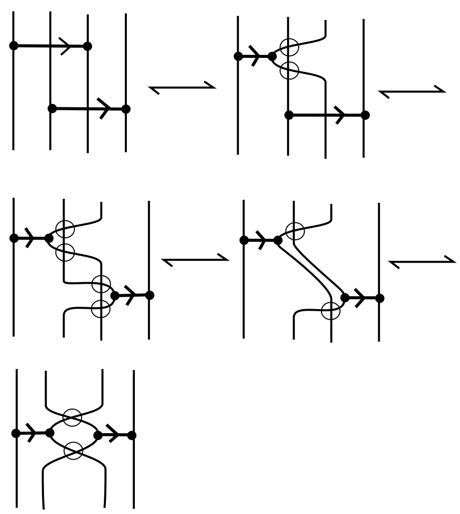

FIGURE 12. A local commuting relation

(2) $\mu_{14}\mu_{23} = \mu_{23}\mu_{14}$ *(action by* (324))
(3) $\mu_{13}\mu_{24} = \mu_{24}\mu_{13}$ *(action by* (23))

The above local relations generalize to commuting relations of the form:

$$\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i,j\} \cap \{k,l\} = \emptyset. \tag{16}$$

All the above commuting relations result from relation 1 by actions of permutations (indicated for relations 2, 3 to the right of each relation). Moreover, for any choice of four strands there are exactly 24 *such commuting relations that preserve the four strands.*

Proof. Relation 1 clearly rests on the virtual braid commuting relations (B2) and (M2). We shall show how relation 2 reduces to relation 1. In the proof we underline in each step the generators of VB_n on which virtual braid relations are applied.

$$\begin{array}{rcl}
\mu_{i,i+3}\mu_{i+1,i+2} & = & v_{i+2}v_{i+1}\mu_{i,i+1}\underline{v_{i+1}v_{i+2}\mu_{i+1,i+2}} \\
& \overset{detour}{=} & v_{i+2}v_{i+1}\underline{\mu_{i,i+1}\mu_{i+2,i+3}}v_{i+1}v_{i+2} \\
& \overset{(1)}{=} & \underline{v_{i+2}v_{i+1}\mu_{i+2,i+3}}\mu_{i,i+1}v_{i+1}v_{i+2} \\
& \overset{detour}{=} & \mu_{i+1,i+2}v_{i+2}v_{i+1}\mu_{i,i+1}v_{i+1}v_{i+2} \\
& = & \mu_{i+1,i+2}\mu_{i,i+3}.
\end{array}$$

Figure 12 illustrates how relation 3 also reduces to relation 1. Notice now that relations 2 and 3 can be derived from relation 1 by conjugation by the permutations (324) and (23) respectively. Let us see how this works specifically for relation 2: the indices of relation 1 against the indices of relation 2 induce the permutation $(324) = v_2v_3$. This means that conjugating relation 1 by the word v_2v_3 will yield relation 2.

Notice also that there are 24 commuting relations in total involving the strands $1, 2, 3, 4$ and indices in any order. Likewise for any choice of four strands. The derivation of all relations from the action of S_n on relation 1 is clear from Remark 2. $\square$

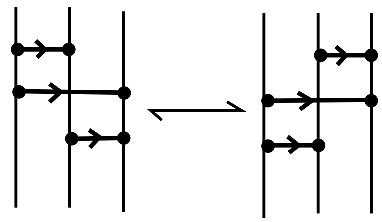

FIGURE 13. The stringy braid relation

Lemma 3. *The following* stringy braid relations *hold in* VB_n.

(1) $\boldsymbol{\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}}$
(2) $\mu_{21}\mu_{23}\mu_{13} = \mu_{13}\mu_{23}\mu_{21}$ (*action by* (12))
(3) $\mu_{13}\mu_{12}\mu_{32} = \mu_{32}\mu_{12}\mu_{13}$ (*action by* (23))
(4) $\mu_{32}\mu_{31}\mu_{21} = \mu_{21}\mu_{31}\mu_{32}$ (*action by* (13))
(5) $\mu_{23}\mu_{21}\mu_{31} = \mu_{31}\mu_{21}\mu_{23}$ (*action by* (123))
(6) $\mu_{31}\mu_{32}\mu_{12} = \mu_{12}\mu_{32}\mu_{31}$ (*action by* (132))

The above relations generalize to three-term relations of the form:

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \textit{for any distinct } i, j, k. \tag{17}$$

All six relations stated above result from the action on relation 1 by permutations of S_n*, which only permute the indices* $\{1, 2, 3\}$*. These permutations are indicated to the right of each relation. Moreover, for any choice of three strands there are exactly six relations analogous to the above, which all result from relation 1 by actions of appropriate permutations that preserve the three strands each time.*

Proof. Figure 13 illustrates relation 1. Relation 1 rests on the braid relations (B1) of VB_n. See also Figure 14 for a pictorial proof. We omit the remaining details of this proof. □

Another remark is now due.

Remark 3. The forbidden moves are naturally forbidden also in the stringy category. For example, the forbidden relations $F1, F2$ of Eq. 6 translate into the following corresponding *forbidden stringy relations* $SF1, SF2$:

$$\mu_{i,i+2}\mu_{i+1,i+2} = \mu_{i+1,i+2}\mu_{i,i+2} \quad (SF1) \qquad \text{and} \qquad \mu_{i,i+2}\mu_{i,i+1} = \mu_{i,i+1}\mu_{i,i+2} \quad (SF2) \tag{18}$$

which, together with all similar relations arising from conjugating the above by permutations, are not valid in the stringy category. See Figure 15 for illustrations.

2.4. The stringy presentation. We will now define an abstract stringy presentation for VB_n that starts from the concept of connecting string and recaptures the virtual braid group. By Eq. 7 we have

$$\sigma_i = \mu_{i,i+1}v_i \tag{19}$$

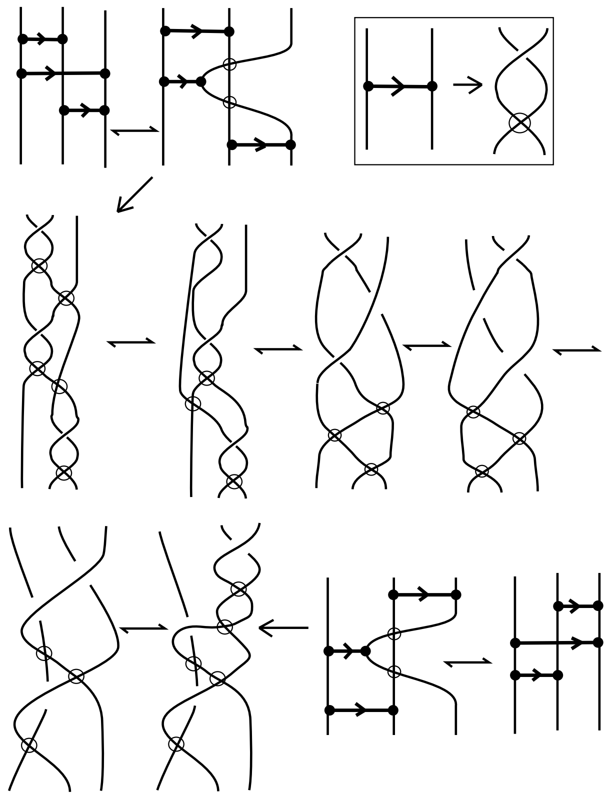

FIGURE 14. Proof of the stringy braid relation

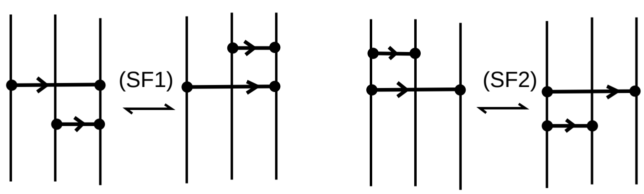

FIGURE 15. Stringy forbidden moves

so, the connecting strings μ_{ij} can be taken as an alternate set of generators of the virtual braid group, along with the virtual generators v_i. The relations in this new presentation consist in the results we proved above in Lemmas 1, 2, 3 describing the interaction of connecting strings with virtual crossings, the commutation properties of connecting strings, the stringy braiding relations, and the usual relations $(S1), (S2), (S3)$ in the symmetric group S_n. For the work below, recall that we have defined the element $t_{ij} = v_i v_{i+1} \ldots v_j \ldots v_{i+1} v_i$ that corresponds to the transposition (ij) in S_n.

In any presentation of a group G containing the elements $\{v_1, \ldots, v_{n-1}\}$ and the relations $(S1), (S2), (S3)$ among them, we have an action of the symmetric group S_n on the group G defined by conjugation by an element τ in S_n, expressed in terms of the v_i:

$$g^{\tau} = \tau g \tau^{-1}$$

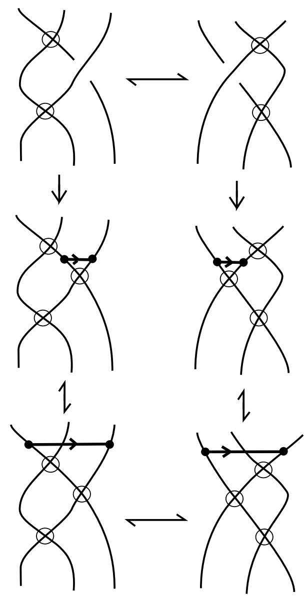

FIGURE 16. The detour moves correspond to the slide moves in the stringy category

for g in G. In particular, we can consider $t_{ij}\, g\, t_{ij}$ as the action by the transposition t_{ij} on an element g of G. We will use this action to define a stringy model of the virtual braid group.

Definition 3. Let VS_n denote the following stringy group presentation.

$$VS_n = \left\langle \begin{array}{l} \mu_{ij}, \quad 1 \le i \ne j \le n, \\ v_1, \dots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} \tau \mu_{ij} \tau^{-1} = \mu_{\tau(i),\tau(j)}, \quad \tau \in S_n \\ \mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12} \\ \mu_{12}\mu_{34} = \mu_{34}\mu_{12} \\ (S1), (S2), (S3) \end{array} \right\rangle \tag{20}$$

We can now state the following theorem.

Theorem 1. *The stringy group VS_n is isomorphic to the virtual braid group VB_n.*

Proof. First we define a homomorphism $F : VB_n \longrightarrow VS_n$ by $F(v_i) = v_i$ and $F(\sigma_i) = \mu_{i,i+1} v_i$, and extend the map to be a homomorphism on words in the generators of the virtual braid group. In order to show that this map is well-defined, we must show that it preserves the relations in the virtual braid group. We omit the details of this verification.

We now define an inverse mapping $G : VS_n \longrightarrow VB_n$ by $G(v_i) = v_i$ and $G(\mu_{i,i+1}) = \sigma_i v_i$. At this stage we have two pieces of work to accomplish: We must extend G to all of VB_n and we must show that G is well-defined and that it preserves the relations in the group presentation. We omit the details of this verification. $\square$

Finally, we also give below a reduced presentation for VB_n, which derives immediately from (5).

Proposition 1. *The following is a reduced stringy presentation for VB_n:*

$$(21) \quad VB_n = \left\langle \begin{array}{c} \mu_{12}, \\ v_1, \ldots, v_{n-1} \end{array} \;\middle|\; \begin{array}{l} \mu_{12} v_j = v_j \mu_{12}, \quad \textit{for } j > 2 \\ \mu_{12}\, v_2 \mu_{12} v_2\, v_1 v_2 \mu_{12} v_2 v_1 = v_1 v_2 \mu_{12} v_2 v_1\, v_2 \mu_{12} v_2\, \mu_{12} \\ \mu_{12}\, v_2 v_3 v_1 v_2 \mu_{12} v_2 v_1 v_3 v_2 = v_2 v_3 v_1 v_2 \mu_{12} v_2 v_1 v_3 v_2\, \mu_{12} \\ (S1), (S2), (S3) \end{array} \right\rangle$$

Note that the second relation is the stringy braid relation 1 of Lemma 3 and the third relation is the commuting relation 1 of Lemma 2.

3. The Pure Virtual Braid Group

3.1. A presentation for the pure virtual braid group. From presentation Eq. 1 of VB_n we have a surjective homomorphism

$$\pi : VB_n \longrightarrow S_n$$

defined by

$$\pi(\sigma_i) = \pi(v_i) = v_i.$$

For a virtual braid b, we refer to $\pi(b)$ as the *permutation associated with the virtual braid* b, and we define the *pure virtual braid group* VP_n to be the kernel of the homomorphism π. Hence, VP_n is a normal subgroup of VB_n of index $n!$. So, $VP_n \cdot S_n = VB_n$. Moreover, $VP_n \cap S_n = \{id\}$. Hence, $VB_n = VP_n \rtimes S_n$. Equivalently, we have the exact sequence

$$1 \longrightarrow VP_n \longrightarrow VB_n \longrightarrow S_n \longrightarrow 1.$$

A presentation for VP_n can be now derived immediately from the stringy presentation of VB_n as an application of the Reidemeister-Schreier process [7, 23, 26]. To see this, we first need the following.

Lemma 4. *The subgroup VP_n of VB_n is generated by the elements μ_{ij} for all $i \neq j$.*

Proof. Indeed, by Eqs. 7 and 9, $\sigma_i = \mu_{i,i+1} v_i = v_i \mu_{i+1,i}$. So, any element $b \in VB_n$ can be written as a product in the μ_{ij}'s and the v_k's. Furthermore, by the slide relations of Lemma 1, all μ_{ij}'s can pass to the top of the braid, leaving at the bottom a word τ in the v_k's, such that $\tau = \pi(b)$. Thus, if $b \in VP_n$ then τ must be the identity permutation. This completes the proof of the Lemma. □

We can now give a stringy presentation of VP_n.

Theorem 2. *The following is a presentation for the pure virtual braid group.*

$$(22) \qquad VP_n = \left\langle \; \mu_{rs}, \quad r \neq s \;\middle|\; \begin{array}{l} \mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \textit{for all distinct } i, j, k \\ \mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i, j\} \cap \{k, l\} = \emptyset \end{array} \right\rangle$$

Proof. Having reformulated the presentation of the virtual braid group, the proof is now a direct application of the Reidemeister-Schreier technique [7, 23, 26]. The relations in VP_n arise as conjugations of the relations in VB_n by coset representatives of VP_n in VB_n, which are the elements of S_n. The relations $(S1), (S2), (S3)$ describe S_n and are used for choosing the coset representatives. We now describe the process from the point of view of covering spaces. We have $VP_n \subset VB_n$ as a normal subgroup with the subgroup S_n acting on it by conjugation. VP_n is the fundamental group of the covering space E of a cell complex B with fundamental group VB_n, where E has group of deck transformations S_n. Since the elements of the symmetric group lift to paths in the covering space, the relations $\tau\mu_{ij}\tau^{-1} = \mu_{\tau(i),\tau(j)}$ serve to describe the action of the symmetric group on the loops in the covering space (these loops are the lifts of the elements μ_{ij}). We choose basic relations in VP_n to be the lifts at a specific basepoint of the braiding relation $\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}$ and the commuting relation $\mu_{12}\mu_{34} = \mu_{34}\mu_{12}$. All other relations are obtained from these by the action of S_n, and all relations constitute the two orbits of the basic relations under this action. For example the relations

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}$$

constitute the orbit under the action of S_n on the single basic braiding relation

$$\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12}.$$

The same pattern applies to the commuting relations. This gives the statement of the Theorem and completes the proof. □

3.2. Semi-Direct Product Structure. The virtual braid group and the pure virtual braid group can be described in terms of semi-direct products of groups, just as is begun in the paper by Bardakov [1] and continued in [8]. In this section we remark that these decompositions are based on the following algebra: The Yang-Baxter relation has the generic form

$$\mu_{i,i+1}\mu_{i,i+2}\mu_{i+1,i+2} = \mu_{i+1,i+2}\mu_{i,i+2}\mu_{i,+1}$$

which is abstractly in the form

$$ABC = CBA$$

and can be rewritten in the form $B^{-1}ABC = B^{-1}CBA$ or

$$A^B = C^B AC^{-1}.$$

This allows one to rewrite some of the Yang-Baxter relations in terms of the conjugation action of the group on itself, and this is the key to the structural work pioneered by Bardakov.

4. A String Category for the Virtual Braid Group

In this section we summarize our results by pointing out that the string connectors and the virtual crossings can be regarded as generators of a category whose algebraic structure yields the virtual braid group and the pure virtual braid group.

For this purpose we define a strict monoidal category with generating morphisms μ_{ij} where this symbol is interpreted as an abstract string or connection between strands i and j in a diagram that otherwise is an identity braid on n strands just as defined in the previous sections. The other generators of this category are morphisms v_i that are interpreted as virtual crossings between strings i and $i+1$. The generators v_i have all the relations for transpositions generating the symmetric group. Compositions of these elements generate the morphisms of the category. The relations among these morphisms are exactly the relations described for the v_k and the μ_{ij} in the previous sections.

Consider the strict monoidal category freely generated by one object $*$ and three morphisms $\mu : * \otimes * \longrightarrow * \otimes *$, $\mu' : * \otimes * \longrightarrow * \otimes *$, and $v : * \otimes * \longrightarrow * \otimes *$. Let $\mu_{12} = \mu \otimes id_*$, $\mu_{21} = \mu' \otimes id_*$, $v_1 = v \otimes id_*, v_2 = id_* \otimes v$ and let

$$v_i = id_* \otimes \cdots \otimes id_* \otimes v \otimes id_* \otimes \cdots \otimes id_*$$

where v occurs in the i-th place in this tensor product. More generally, it is understood that μ_{12} can stand for $\mu \otimes id_* \otimes \cdots \otimes id_*$ and that μ_{21} can stand for $\mu' \otimes id_* \otimes \cdots \otimes id_*$ for an arbitrary number of tensor factors.

Quotient this category by the following relations.

(1) $\mu\mu' = id_{*\otimes*} = \mu'\mu$,
(2) $vv = id_*$,
(3) $\mu_{12}v_j = v_j\mu_{12}, \quad$ for $j > 2$,
(4) $\mu_{12}\, v_2\mu_{12}v_2\, v_1v_2\mu_{12}v_2v_1 = v_1v_2\mu_{12}v_2v_1\, v_2\mu_{12}v_2\, \mu_{12}$,
(5) $\mu_{12}\, v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2 = v_2v_3v_1v_2\mu_{12}v_2v_1v_3v_2\, \mu_{12}$,
(6) $v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1}$,
(7) $v_iv_j = v_jv_i$, for $j \neq i \pm 1$.

This quotient is called the *String Category* and denoted SC. The category SC is still strict monoidal.

To recapture the connecting string morphisms, we follow the formalism of the previous sections. Define

$$\mu_{i,i+1} = id_* \otimes \cdots \otimes id_* \otimes \mu \otimes id_* \otimes \cdots \otimes id_*$$

where μ occurs in the i and $i+1$ places in the tensor product and define

$$\mu_{i+1,i} = id_* \otimes \cdots \otimes id_* \otimes \mu' \otimes id_* \otimes \cdots \otimes id_*$$

where μ' occurs in the i and $i+1$ places in the tensor product. Define, for $i < j$, the element μ_{ij} by the formula

$$\mu_{ij} = v_{j-1}v_{j-2}\cdots v_{i+1}\,\mu_{i,i+1}\,v_{i+1}\cdots v_{j-2}v_{j-1}. \tag{23}$$

and define

$$\mu_{ji} = v_{j-1}v_{j-2}\cdots v_{i+1}\,\mu_{i+1,i}\,v_{i+1}\cdots v_{j-2}v_{j-1}. \tag{24}$$

Remark 4. Note that, in this notation, relation 4 becomes the algebraic Yang-Baxter equation

$$\mu_{12}\mu_{13}\mu_{23} = \mu_{23}\mu_{13}\mu_{12},$$

and relation 5 becomes the commuting relation

$$\mu_{12}\mu_{34} = \mu_{34}\mu_{12}.$$

Then one has, as consequences, the general algebraic Yang-Baxter equation and commuting relations, as we have described them in earlier sections of the paper.

$$\mu_{ij}\mu_{ik}\mu_{jk} = \mu_{jk}\mu_{ik}\mu_{ij}, \quad \text{for all distinct } i, j, k$$

and

$$\mu_{ij}\mu_{kl} = \mu_{kl}\mu_{ij}, \quad \{i, j\} \cap \{k, l\} = \emptyset.$$

The morphisms v_i effect the action of the symmetric group and the category models the pure virtual braid group in the following precise sense:

By Proposition 1, for any positive integer n, the group of endomorphisms of the object $*^{\otimes n}$ is isomorphic to VB_n. In particular, any monoidal functor

$$F : SC \longrightarrow Mod_k$$

gives rise to a representation of VB_n :

$$f \in End_{SC}(*^{\otimes n}) \simeq VB_n \longmapsto F(f) \in End_k(A^{\otimes n})$$

where $A = F(*)$.

Remark 5. For each natural number n, the symbols

$$[n] = * \otimes * \otimes \cdots \otimes *$$

with n $*$'s are the objects in the category. One can regard $[n]$ as an ordered row of n points that constitute the top or the bottom of a diagram involving n strands. Diagrammatically, μ_{ij} consists in n parallel strands with a string connector between the i-th and j-th strands directed from i to j. Similarly, v_i corresponds to a diagram of n strands where there is a virtual crossing between the i-th and $(i+1)$-st strands. An n-strand diagram that is a product of these generators is regarded as a morphism from $[n]$ to $[n]$ for n any natural number. Note that we interpret μ_{ij} and v_i diagrammatically according to the conventions previously established in this paper.

The virtual braid group on n strands is isomorphic to the group of morphisms in the String Category from $[n]$ to $[n]$. The point of this categorical formulation of the virtual braid groups is that we see how these groups form a natural extension of the symmetric groups by formal elements that satisfy the algebraic Yang-Baxter equation. The category we desribe is a natural structure for an algebraist interested in exploring formal properties of the algebraic Yang-Baxter equation. It should be remarked that the relationship between the relations in the pure virtual braid group and the algebraic Yang-Baxter equation was also pointed out in [3]. See also [2] Remark 10, page 7. We have taken this observation further to point out that the virtual braid group is a direct result of forming a convenient category associated with the algebraic Yang-Baxter equation.

For the reader who would like to take the String Category as a starting point for the theory of virtual braids, here is a description of how to read our figures for that purpose. Figure 1 illustrates the permutation generators v_i for the String Category. The braiding elements σ_i will be defined in terms of the string generators. Elementary connecting strings are given in Figure 6. Note that, it is implicit in Figure 6 how to define the braiding elements σ_i by composing string generators with permutations (virtual crossings). See also Figure 7, which illustrates basic relationships among string generators, permutations and braiding operators. Figure 8 illustrates the general connecting strings and their relations with the permutation operators. In particular, Figure 8 shows how any string connection can be written in terms of a basic string generator and a product of permutations. Figure 9 illustrates how μ_{ij} and μ_{ji} are related diagrammatically. Figures 10, 11 and 12 show the basic slide relations between string connections and permutations. Figure 13 illustrates the algebraic Yang-Baxter relation as it occurs for the string connectors.

5. Representations of the Virtual and Pure Virtual Braid Groups

5.1. Let A be an algebra over a commutative ground ring k. Let $\rho \in A \otimes A$ be an element of the tensor product of A with itself. Then ρ has the form given by the following equation

$$\rho = \sum_{i=1}^{N} e_i \otimes e^i \tag{25}$$

where e_i and e^j are elements of the algebra A. We will write this sum symbolically as

$$\rho = \sum e \otimes e' \tag{26}$$

where it is understood that this is short-hand for the above specific summation.

We then define, for $i < j$, $\rho_{ij} \in A^{\otimes n}$ by the equation

$$\rho_{ij} = \sum 1_A \otimes \cdots \otimes 1_A \otimes e \otimes 1_A \otimes \cdots \otimes 1_A \otimes e' \otimes 1_A \otimes \cdots \otimes 1_A \tag{27}$$

where the e occurs in the i-th tensor factor and the e' occurs in the j-th tensor factor.

If $i > j$ we define ρ_{ij} by reversing the roles of e and e' as shown in the next equation

$$\rho_{ij} = \sum 1_A \otimes \cdots \otimes 1_A \otimes e' \otimes 1_A \otimes \cdots \otimes 1_A \otimes e \otimes 1_A \otimes \cdots \otimes 1_A \tag{28}$$

where e' occurs in the j-th tensor factor and e occurs in the i-th tensor factor.

We say that ρ is a *solution to the algebraic Yang-Baxter equation* if it satisfies the equation

$$\rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12} \tag{29}$$

in $A^{\otimes n}$. It is immediately obvious that if ρ satisfies the algebraic Yang-Baxter equation, then, for any pairwise distinct i, j, k we have

$$\rho_{ij}\rho_{ik}\rho_{jk} = \rho_{jk}\rho_{ik}\rho_{ij}, \tag{30}$$

and that the equations obtained from this particular equation by permuting the indices i, j, k remain true. All such equations derive from permutations of any given instance of the algebraic Yang-Baxter equation.

The following proposition is an immediate consequence of our presentation for the pure virtual braid group.

Proposition 2. *Let VP_n denote the pure virtual braid group with generators μ_{ij} and relations as given in Theorem 2 of Section 3. Let A be an algebra over a commutative ground ring k, with an invertible algebraic solution to the Yang-Baxter equation denoted by $\rho \in A \otimes A$ as described above. Define $rep : VP_n \longrightarrow A^{\otimes n}$ by the equation*

$$rep(\mu_{ij}) = \rho_{ij}.$$

Then rep is a representation of the the pure virtual braid group to the tensor algebra $A^{\otimes n}$.

Proof. Note that it follows at once from the definitions of the ρ_{ij} that $\rho_{ij}\rho_{kl} = \rho_{kl}\rho_{ij}$ whenever the sets $\{i, j\}$ and $\{k, l\}$ are disjoint. Thus, we have shown that the ρ_{ij} satisfy all the relations in the pure virtual braid group. This completes the proof of the Proposition. □

Next, we show how to obtain representations of the full virtual braid group. To this purpose, consider the algebra $Aut(A^{\otimes n})$ of linear automorphisms of $A^{\otimes n}$ as a module over the ground ring k. Assume that we are given an invertible solution to the algebraic Yang-Baxter equation, $\rho \in A \otimes A$, and define $\tilde{\rho}_{ij} : A^{\otimes n} \longrightarrow A^{\otimes n}$ by the equation $\tilde{\rho}_{ij}(\alpha) = \rho_{ij}\alpha$ where $\alpha \in A^{\otimes n}$. Since ρ is invertible, $\tilde{\rho}_{ij} \in Aut(A^{\otimes n})$. Let $P_{ij} : A^{\otimes n} \longrightarrow A^{\otimes n}$ be the mapping that interchanges the i-th and j-th tensor factors. Then $P_{ij} \in Aut(A^{\otimes n})$. We let P_i denote $P_{i,i+1}$. We now define $Rep : VB_n \longrightarrow Aut(A^{\otimes n})$ by the equations

$$Rep(\mu_{ij}) = \tilde{\rho}_{ij}$$

and

$$Rep(v_i) = P_i.$$

Here we use our presentation (20) for the virtual braid group.

Proposition 3. *With the above conventions the mapping $Rep : VB_n \longrightarrow Aut(A^{\otimes n})$ is a representation of the virtual braid group to a subgroup of $Aut(A^{\otimes n})$.*

Proof. It is clear that the elements P_i obey all the relations in the symmetric group S_n. Thus it remains to show that letting $\lambda = Rep(\tau)$ where τ is an element of S_n, the relations

$$\lambda\rho_{ij}\lambda^{-1} = \tilde{\rho}_{\tau(i),\tau(j)}, \quad \tau \in S_n$$

are satisfied in $Aut(A^{\otimes n})$. Since ρ_{ij} is defined via the placement of the e and e' factors in the summation for ρ on the i-th and j-th strands, these relations are immediate. This completes the proof of the proposition. □

Remark 6. The method we have described for constructing a representation of the virtual braid group from an algebraic solution to the Yang-Baxter equation generalizes the well-known construction of a representation of the classical Artin braid group from a solution to the Yang-Baxter equation in braided form. In the usual method for constructing the classical representation, one composes the algebraic solution with a permutation, obtaining a solution to the braiding equation $(B1)$. This is the same as our relation

$$\sigma_i = \mu_{i,i+1}v_i$$

between the braiding element σ_i and the stringy generator $\mu_{i,i+1}$ for the pure virtual braid group. Without the concept of virtuality, the direct relationship of the algebraic Yang-Baxter equation with the braid groups would not be apparent. We see that from an algebraic point of view, the virtual braid group is an entirely natural construction. It is the universal algebraic structure related to viewing solutions to the algebraic Yang-Baxter equation inside tensor products of algebras and endowing these tensor products with the natural permutation action of the symmetric group.

Solutions to the algebraic version of the Yang-Baxter equation are usually thought of as deformations of the identity mapping on a two-fold tensor product $A \otimes A$. We think of a braiding operator as a deformation of a transposition, and so one goes between the algebraic and braided versions of such operators by composition with a transposition.

The Artin braid group B_n is motivated by a combination of topological considerations and the desire for a group structure that is very close to the structure of the symmetric group S_n. We have seen that the virtual braid group VB_n is motivated at first by a natural extension of the Artin braid group in the context of virtual knot theory, but now we see a different motivation for the virtual braid group. Given that one studies the algebraic Yang-Baxter equation in the context of tensor powers of an algebra A, it is thoroughly natural to study the compositions of algebraic braiding operators placed in two out of the n tensor lines (the stringy generators) and to let the permutation group of the tensor lines act on this algebra. As we have seen in (20), this is precisely the virtual braid group. Viewed in this way, the virtual braid group has nothing to do with the plane and nothing to do with virtual crossings. It is a natural group associated with the structure of algebraic braiding.

5.2. A Representation Category for the Virtual Braid Group. We now give a categorical interpretation of virtual knot theory and the virtual braid group in terms of these representation modules. For A as above, let $End(A^{\otimes n})$ denote the linear endomorphisms of $A^{\otimes n}$ as a module over k. View $End(A^{\otimes n})$ as a category with generating morphisms:

(1) $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n \in A^{\otimes n}$ acting on $A^{\otimes n}$ by left multiplication,
(2) the elements of the symmetric group S_n, generated by transpositions of adjacent tensor factors.

This category has one object. In making the representation of VB_n we have used the stringy generators μ_{ij} and mapped them to sums of morphisms of the first type above. The virtual braid group VB_n described via (20), can be viewed as a category with one object and generators μ_{ij} and v_k. Of course any associative algebra can be seen as a single object category with morphisms the elements of the algebra. But here we have a pictorial representation of the morphisms as stringy braid diagrams. These diagrams can be generalized to include the algebraic category $End(A^{\otimes n})$ by letting algebra elements decorate the lines and taking the transpositions of the form $P_{i,i+1}$ as represented by v_i via a diagram of lines i and $i+1$ virtually crossing over one another. In this view the virtual crossing is interpreted as a generator of the symmetric group. The virtual crossings have not disappeared. They have become part of the embedded symmetry of the structure of the virtual braid group. This is in sharp contrast to the role of the virtual crossings in the original form of the virtual braid group. There the virtual crossings appear as artifacts of the presentation of virtual knots in the plane where those knots acquire extra crossings that are not really part of the essential structure of the virtual knot. Nevertheless, these same crossings appear crucially in the virtual braid group, and turn into the generators of the symmetric group embedded in the virtual braid group. With the use of the full set of μ_{ij} in (20) the detour moves and other remnants of the virtual crossings as artifacts have completely disappeared into the permutation action. We will continue the categorical discussion for the virtual braid group after first discussing certain aspects of knot theory and the tangle categories.

The representations of VB_n that we have here derived can be interpreted as follows.

Theorem 3. *Let $\rho \in A \otimes A$ be a solution of the algebraic Yang-Baxter equation, where A is an algebra over a commutative ring k. Define a monoidal functor*

$$F_A : SC \longrightarrow Mod_k$$

by setting $F_A(*) = A$, $F_A(\mu) = \tilde{\rho}$, and $F_A(v_i) = P$, where the endomorphisms $\tilde{\rho}$ and P of $A \otimes_k A$ are given by

$$\tilde{\rho}(x \otimes y) = \rho(x \otimes y)$$

and

$$P(x \otimes y) = y \otimes x$$

for all $x, y \in A$.

Proof. The proof follows from the previous discussion. □

Remark 7. In the case A is a bialgebra (so that the category Mod_A of modules over A is monoidal), it would be interesting to address the following question: When does the above functor $F_A : SC \longrightarrow Mod_k$ lift to a monoidal functor $\tilde{F}_A : SC \longrightarrow Mod_A$ (that is such that $U \circ \tilde{F}_A = F_A$ as monoidal functors, where $U : Mod_A \longrightarrow Mod_k$ is the forgetful functor)?

5.3. Virtual Hecke Algebra. From the point of view of the theory of braids the Hecke algebra $H_n(q)$ is a quotient of the group ring $\mathbb{Z}[q, q^{-1}][B_n]$ of the Artin braid group by the ideal generated by $\sigma_i^2 - z\sigma_i - 1$ where $z = q - q^{-1}$. This corresponds to the identity $\sigma_i - \sigma_i^{-1} = z1$, which is sometimes regarded diagrammatically as a skein identity for calculating knot polynomials. By the same token, we define the *virtual Hecke algebra* $VH_n(q)$ to be the quotient of the group ring $\mathbb{Z}[q, q^{-1}][VB_n]$ by the ideal generated by $\sigma_i^2 - z\sigma_i - 1$ for $i = 1, 2, \ldots n-1$. There are difficulties in extending structure theorems for the Hecke algebra to corresponding structure theorems for the virtual Hecke algebra, but some matters of representations do generalize directly. In particular, if R is a solution to the Yang-Baxter equation with $R : W \otimes W \longrightarrow W \otimes W$, where W is a module over $\mathbb{Z}[q, q^{-1}]$, then one has a corresponding representation $Rep : VH_n(q) \longrightarrow Aut(W^{\otimes n})$. This representation is specified as follows.

$$Rep(\sigma_i) = \sum 1 \otimes \cdots \otimes 1 \otimes R \otimes 1 \otimes \cdots \otimes 1 \tag{31}$$

where R operates in the i-th and $i+1$-st tensor factors, and

$$Rep(v_i) = \sum 1 \otimes \cdots \otimes 1 \otimes P \otimes 1 \otimes \cdots \otimes 1 \tag{32}$$

where P acts to permute the i-th and $(i+1)$-st tensor factors. It is easy to see that this gives a representation of the virtual Hecke algebra.

One can hope that the presence of such representations would shed light on the existence of a generalization of the Ocneanu trace [12] on the Hecke algebra to a corresponding trace and link invariant using the virtual Hecke algebra. At this point there is an issue about the nature of the generalization. One can aim for a trace on the virtual Hecke algebra that is compatible with the Markov Theorem for virtual knots and links as formulated in [14, 19]. This is the trace that is most difficult to achieve. A simpler trace is possible by working in rotational virtual knot theory [15]. See [20] for a discussion of unoriented quantum invariants for rotational virtuals. We will report on the relation of this approach with the Markov Theorem for virtuals in a separate paper.

Another line of investigation is suggested by translating the basic Hecke algebra relation into the language of stringy connections. We have $\sigma = \mu v$ for the abstract relation between a braiding generator, a connector and a virtual element. Thus, the Hecke relation $\sigma^2 = z\sigma + 1$ becomes

$$\mu v \mu = z\mu + v,$$

and it is possible to work in the presentation (20) of the virtual braid group to find a structure theory for the virtual Hecke algebra.

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