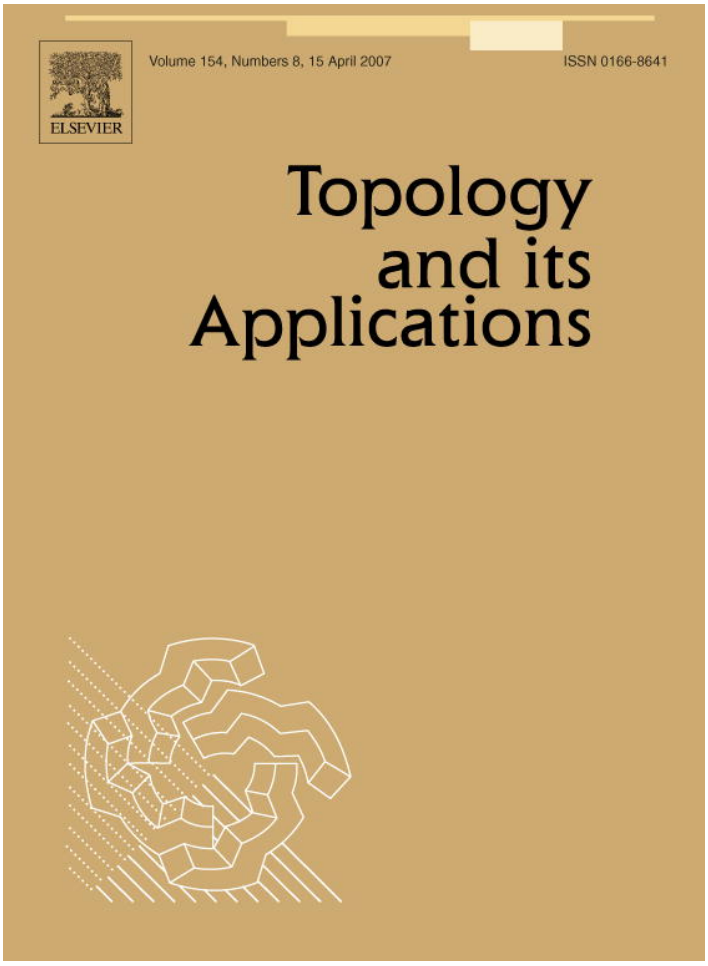
ELSEVIER
Volume 154, Numbers 8, 15 April 2007
ISSN 0166-8641
Topology and its Applications

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Topology and its Applications 154 (2007) 1804–1826
Topology and its Applications
www.elsevier.com/locate/topol

p-Adic framed braids ☆

J. Juyumaya [a,1], S. Lambropoulou [b,*,2]

[a] *Universidad de Valparaíso, Gran Bretaña 1091, Valparaíso, Chile*
[b] *Department of Mathematics, National Technical University of Athens, Zografou campus, GR-157 80 Athens, Greece*

Received 24 August 2006; received in revised form 30 January 2007; accepted 30 January 2007

Abstract

In this paper we define the p-adic framed braid group $\mathcal{F}_{\infty,n}$, arising as the inverse limit of the modular framed braids. An element in $\mathcal{F}_{\infty,n}$ can be interpreted geometrically as an infinite framed cabling. $\mathcal{F}_{\infty,n}$ contains the classical framed braid group as a dense subgroup. This leads to a set of topological generators for $\mathcal{F}_{\infty,n}$ and to approximations for the p-adic framed braids. We further construct a p-adic Yokonuma–Hecke algebra $\mathrm{Y}_{\infty,n}(u)$ as the inverse limit of a family of classical Yokonuma–Hecke algebras. These are quotients of the modular framed braid groups over a quadratic relation. Finally, we give topological generators for $\mathrm{Y}_{\infty,n}(u)$.

MSC: 20C08; 57M27

Keywords: Inverse limits; p-adic integers; p-adic framed braids; p-adic infinite cablings; p-adic Yokonuma–Hecke algebras; p-adic Markov traces

0. Introduction

0.1. Framed knots and links are like classical knots and links but with an integer, the ‘framing’, attached to each component. It is well known that framed links can be used for constructing 3-manifolds using a topological technique called surgery. Then two-manifolds are homeomorphic if and only if any two framed links in S^3 representing them are related through isotopy moves and the Kirby moves or the equivalent Fenn–Rourke moves [2]. In [6] Ko and Smolinsky give a Markov-type equivalence for framed braids corresponding to homeomorphism classes of 3-manifolds. It would be certainly very interesting if one could construct 3-manifold invariants by constructing Markov traces on quotient algebras of the framed braid group and using the framed braid equivalence of [6].

In this paper we introduce the concept of *p-adic framed braids* and we also construct *p-adic quotient algebras*. The p-adic framed braids can be seen as natural infinite cablings of framed braids. Cablings of framed braids have been used for constructing 3-manifold invariants (e.g., by Wenzl [11]). The paper is organized as follows: In Section 2.1 we

☆ Paper presented at the 1017 AMS meeting.

* Corresponding author.
E-mail addresses: juyumaya@uv.cl (J. Juyumaya), sofia@math.ntua.gr (S. Lambropoulou).
URL: http://www.math.ntua.gr/~sofia (S. Lambropoulou).

1 The first author was partially supported by Fondecyt 1050302, Dipuv and the National Technical U. Athens.
2 The second author was partially supported by Fondecyt (International Cooperation), U. Valparaiso and the National Technical U. Athens.

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doi:10.1016/j.topol.2007.01.010

recall the structure of the framed braid group $\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$, where B_n is the classical braid group on n strands. By construction, a framed braid splits into the 'framing part' and the 'braiding part'. Moreover, $\mathcal{F}_n$ is generated by the elementary braids $\sigma_1, \dots, \sigma_{n-1}$ and by the elementary framings $f_1, \dots, f_n$. We further introduce the modular framed braid group $\mathcal{F}_{d,n} = (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$, which has the same presentation as $\mathcal{F}_n$, but with the additional relations:

$$f_i^d = 1.$$

In [13] the Yokonuma–Hecke algebras (abbreviated to Y-H algebras), $\mathrm{Y}_{d,n}(u)$, were introduced by Yokonuma, where u is a fixed non-zero complex number. They appeared originally in the representation theory of finite Chevalley groups and they are natural generalizations of the classical Iwahori–Hecke algebras, see also [10]. In Section 3 we define the Y-H algebra as a finite-dimensional quotient of the group algebra $\mathbb{C}\mathcal{F}_{d,n}$ of the modular framed braid group $\mathcal{F}_{d,n}$ over the quadratic relations:

$$g_i^2 = 1 + (1-u)e_{d,i}(1-g_i),$$

where g_i is the generator associated to the elementary braid σ_i and $e_{d,i}$ are certain idempotents in $\mathbb{C}\mathcal{F}_{d,n}$ (see Sections 3.1 to 3.3). In $\mathrm{Y}_{d,n}(u)$ the relations $f_i^d = 1$ still hold, and they are essential for the existence of the idempotents $e_{d,i}$, because $e_{d,i}$ is by definition a sum involving all powers of f_i and f_{i+1}. In Section 3.4 we give diagrammatic interpretations for the elements $e_{d,i}$ as well as for the quadratic relation (see Figs. 10, 11 and 12).

For relating to framed links and 3-manifolds we would rather not have the restrictions $f_i^d = 1$ on the framings. An obvious idea would be to consider the quotient of the classical framed braid group algebra, $\mathbb{C}\mathcal{F}_n$, over the above quadratic relations. But then, the elements $e_{d,i}$ are not well-defined. Yet, we achieve this aim by employing the construction of inverse limits.

In Sections 1.1 and 1.3 we give some preliminaries on inverse systems and inverse limits and we introduce the concept of topological generators. This is a set, whose span is dense in the inverse limit (see Definition 1). In Sections 1.2, 1.4 and 2.2 we focus on the construction of the p-adic integers $\mathbb{Z}_p$ and their approximations. Let p be a prime number and let C_r be the cyclic group of p^r elements: $C_r \cong \mathbb{Z}/p^r\mathbb{Z}$. Then $\varprojlim C_r = \mathbb{Z}_p$, where the inverse system maps $\theta_s^r : \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^s\mathbb{Z}$ $(r \geqslant s)$ are the natural epimorphisms. $\mathbb{Z}_p$ contains $\mathbb{Z} = \langle \mathbf{t} \rangle$ as a dense subgroup. The element $\mathbf{t}$ is a topological generator for $\mathbb{Z}_p$, and a p-adic integer will be denoted $\mathbf{t}^{\underleftarrow{a}}$, where $\underleftarrow{a} =: (a_1, a_2, \dots)$ with $a_r \equiv a_s \pmod{p^s}$ whenever $r \geqslant s$.

We shall now explain briefly our constructions. Section 2 deals with the construction of the p-adic framed braids. More precisely, in Section 2.3 we consider the inverse system (C_r^n, π_s^r) indexed by $\mathbb{N}$, where the map $\pi_r^s : C_r^n \to C_s^n$ $(r \geqslant s)$ acts componentwise as the natural epimorphism θ_s^r. Then $\varprojlim C_r^n \cong \mathbb{Z}_p^n$ (see Proposition 3) and $\mathbb{Z}_p^n$ contains $\mathbb{Z}^n = \langle \mathbf{t}_1, \dots, \mathbf{t}_n \rangle$ as a dense subgroup (see Lemma 2). We then consider the inverse system $(\mathcal{F}_{p^r,n}, \pi_s^r \cdot \mathrm{id})$ indexed by $\mathbb{N}$, where the map $\pi_s^r \cdot \mathrm{id}$ acts on the framing part of a modular framed braid as described above, and trivially on the braiding part (Section 2.4). So, we define the *p-adic framed braid group $\mathcal{F}_{\infty,n}$* (Definition 3) as

$$\mathcal{F}_{\infty,n} = \varprojlim \mathcal{F}_{p^r,n}.$$

Geometrically, a p-adic framed braid is an infinite sequence of modular framed braids with the same braiding part and such that the framings of the ith strands in each element of the sequence give rise to a p-adic integer. See Section 2.5 and left-hand side of Fig. 1 for an illustration, where $(a_1, a_2, \dots), (b_1, b_2, \dots) \in \mathbb{Z}_p$. In Theorem 1 the natural identification

$$\mathcal{F}_{\infty,n} \cong \mathbb{Z}_p^n \rtimes B_n$$

is established. This is used in Section 2.5, where we give geometric interpretations of the p-adic framed braids as classical braids with framings p-adic integers. See Fig. 4. We can then say that a p-adic framed braid splits into the 'p-adic framing part' and the 'braiding part'. So, a p-adic framed braid can be also interpreted as an infinite framed cabling of a braid in B_n, such that the framings of each infinite cable form a p-adic integer. See right-hand side of Fig. 1. Of course, the closure of a p-adic framed braid defines an oriented p-adic framed link. Fig. 2 illustrates an example.

The identification in Theorem 1 implies also that there are no modular relations for the framing in $\mathcal{F}_{\infty,n}$. Moreover, that the classical framed braid group $\mathcal{F}_n$ sits in $\mathcal{F}_{\infty,n}$ as a dense subset. Hence, the set $A = \{\mathbf{t}_1, \sigma_1, \dots, \sigma_{n-1}\} \subset \mathcal{F}_n$ is a set of topological generators for $\mathcal{F}_{\infty,n}$. So, by Theorem 1, a p-adic framed braid is a word of the form:

$$\mathbf{t}_1^{\underleftarrow{a_1}} \mathbf{t}_2^{\underleftarrow{a_2}} \dots \mathbf{t}_n^{\underleftarrow{a_n}} \cdot \sigma,$$

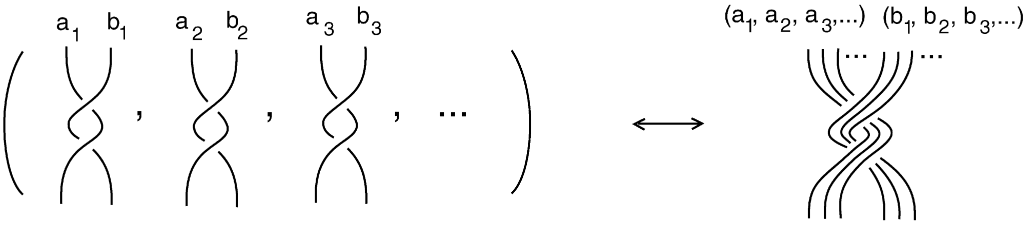

Fig. 1. A p-adic framed braid as an infinite framed cabling.

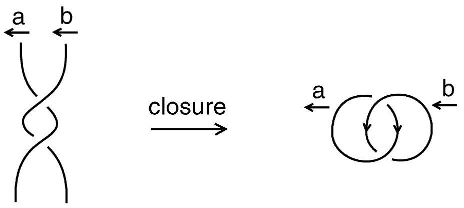

Fig. 2. A p-adic framed braid and a p-adic framed link.

where $\underleftarrow{a_1}, \ldots, \underleftarrow{a_n}$ are the p-adic framings and $\sigma \in B_n$. In Section 2.6 we give approximations of p-adic framed braids by sequences of classical framed braids. See Figs. 8 and 9 for examples.

Section 3 deals with the construction of the p-adic Yokonuma–Hecke algebras. More precisely, in Section 3.5 we define the *p-adic Yokonuma–Hecke algebra* $\mathrm{Y}_{\infty,n}(u)$ as the inverse limit of the inverse system $(\mathrm{Y}_{p^r,n}(u), \varphi_s^r)$ of classical Y-H algebras, indexed by $\mathbb{N}$ (Definition 5):

$$\mathrm{Y}_{\infty,n}(u) = \varprojlim \mathrm{Y}_{p^r,n}(u).$$

The above inverse system is induced by the inverse system $(\mathbb{C}\mathcal{F}_{p^r,n}, \phi_s^r)$, where ϕ_s^r is the 'linear span' of $\pi_s^r \cdot \mathrm{id}$ at the level of the group algebra, using also our definition of the Y-H algebras as finite-dimensional quotients of the group algebras $\mathbb{C}\mathcal{F}_{d,n}$. $\mathrm{Y}_{\infty,n}(u)$ is an infinite-dimensional algebra, in which the framing restrictions $f_i^d = 1$ do not hold. Finally, in Section 3.6, Theorem 3, we give the set of topological generators $\{\mathbf{t_1}, g_1, \ldots, g_{n-1}\}$ for $\mathrm{Y}_{\infty,n}(u)$, satisfying the quadratic relations:

$$g_i^2 = 1 + (1-u)e_i(1-g_i),$$

where the element e_i is also an idempotent and its approximation involves the 'framing' generators $\mathbf{t}_i, \mathbf{t}_{i+1}$.

It is, perhaps, worth stressing that the quadratic relations satisfied in the classical as well as in the p-adic Y-H algebras involve the framing, by definition of the elements e_i. One could also define 'framed' Iwahori–Hecke algebras (see Section 3.7) by taking quotients of the group algebras $\mathbb{C}\mathcal{F}_{d,n}$ or $\mathbb{C}\mathcal{F}_n$ over the well-known Hecke algebra quadratic relations:

$$g_i^2 = (q-1)g_i + q.$$

The structure of these algebras is clearly not as rich as that of the Y-H algebras.

In [4] linear Markov traces have been constructed by the first author for the classical Y-H algebras of any index. In a sequel paper we use these traces to extend the construction to a p-adic linear Markov trace on the p-adic Y-H algebras. We then normalize all these traces according to the Markov equivalence for classical framed and p-adic framed braids to construct isotopy invariants of classical and p-adic framed links. We also adapt the Markov traces constructed in [7] by the second author for obtaining a simpler family of framed link invariants.

We hope that this new concept of p-adic framed braids and p-adic framed links that we propose, as well as our framed link invariants will be useful for constructing new 3-manifold invariants.

0.2. As usual we denote by $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N} = \{1, 2, \ldots\}$ the set of complex numbers, the integers and the natural numbers, respectively. We also denote $\mathbb{Z}/d\mathbb{Z}$ the additive group of integers modulo d. Throughout the paper we fix a prime number p and a $u \in \mathbb{C}\backslash\{0, 1\}$. Finally, whenever two objects a, b are identified we shall write $a \doteq b$.

0.3. Let H be a group and let $H^n = H \times \cdots \times H$ (n times). The symmetric group S_n of the permutations of the set $\{1, 2, \ldots, n\}$ acts on H^n by permutation, that is:

$$\sigma \cdot (h_1, \ldots, h_n) = (h_{\sigma(1)}, \ldots, h_{\sigma(n)}) \quad \forall \sigma \in S_n.$$

We define on the set $H^n \times S_n$ the operation:

$$(h, \sigma) \cdot (h', \tau) = \big(h\,\sigma(h'), \sigma\tau\big).$$

Then, the set $H^n \times S_n$ with the above operation is a group, the *semidirect product* $H^n \rtimes S_n$.

1. Inverse limits and p-adic integers

1.1. An *inverse system* (X_i, ϕ^i_j) of topological spaces (groups, rings, algebras, et cetera) indexed by a directed set I, consists of a family $(X_i;\ i \in I)$ of topological spaces (groups, rings, algebras, et cetera) and a family $(\phi^i_j : X_i \to X_j;\ i, j \in I,\ i \geqslant j)$ of continuous homomorphisms, such that

$$\phi^i_i = \mathrm{id}_{X_i} \quad \text{and} \quad \phi^j_k \circ \phi^i_j = \phi^i_k \quad \text{whenever } i \geqslant j \geqslant k.$$

If no other topology is specified on the sets X_i, they are regarded as topological spaces with the discrete topology. In particular, finite sets are compact Hausdorff spaces.

The *inverse limit* $\varprojlim X_i$ of the inverse system (X_i, ϕ^i_j) is defined as:

$$\varprojlim X_i := \Big\{ z \in \prod X_i;\ \big(\phi^i_j \circ \varpi_i\big)(z) = \varpi_j(z) \text{ whenever } i \geqslant j \Big\},$$

where the map ϖ_i denotes the natural projection of the Cartesian product $\prod X_i$ onto X_i. It turns out that $\varprojlim X_i$ is uniquely defined, and it is non-empty if each X_i is a non-empty compact Hausdorff space. $\varprojlim X_i$ is a topological group (ring, algebra, etc.) with operation induced in $\prod X_i$ componentwise by the group (ring, algebra, etc.) operations. Moreover, in this case, $\varprojlim X_i$ is always non-empty.

As a topological space, $\prod X_i$ is endowed with the product topology, so $\varprojlim X_i$ inherits the induced topology. A *basis* of open sets in $\varprojlim X_i$ contains elements of the form

$$\varpi_i^{-1}(U_i) \cap \varprojlim X_i,$$

where U_i open in X_i. Then, any open set in $\varprojlim X_i$ is a union of sets of the form

$$\varpi_{i_1}^{-1}(U_1) \cap \cdots \cap \varpi_{i_n}^{-1}(U_n) \cap \varprojlim X_i, \tag{1.1}$$

where $i_1, \ldots, i_n \in I$ and U_r open in X_{i_r} for each r.

A *morphism* between two inverse systems (X_i, ϕ^i_j) and (Y_i, ψ^i_j), both indexed by the same directed set I, is a collection of continuous homomorphisms

$$(\rho_i : X_i \to Y_i;\ i \in I)$$

such that $\psi^i_j \circ \rho_i = \rho_j \circ \phi^i_j$, for all $i \in I$. A morphism $(\rho_i;\ i \in I)$ from the inverse system (X_i, ϕ^i_j) to the inverse system (Y_i, ψ^i_j) induces a morphism between the inverse limits:

$$\varprojlim \rho_i : \varprojlim X_i \to \varprojlim Y_i$$

by setting

$$\varprojlim \rho_i\big((x_i)\big) := \big(\rho_i(x_i)\big).$$

If we have embeddings ι_i from X_i into Y_i, these induce a natural embedding $\varprojlim \iota_r : \varprojlim X_i \to \varprojlim Y_i$. Moreover, if the following sequence

$$0 \to X_i \xrightarrow{\iota_i} Y_i \xrightarrow{\varphi_i} Z_i \to 0$$

is exact for any i, then the sequence

$$0 \to \varprojlim X_i \xrightarrow{\varprojlim \iota_r} \varprojlim Y_i \xrightarrow{\varprojlim \varphi_r} \varprojlim Z_i \tag{1.2}$$

is also exact.

Let now J be a subset of the index set I, such that for every $i \in I$ there is a $j \in J$ with $j \geqslant i$. Then J gives rise to the same inverse limit. This is used in the following: Let X and Y be the inverse limits of the inverse systems $(X_i, \phi_k^i;\ i \in I)$ and $(Y_j, \psi_m^j;\ j \in I)$, respectively. Then we have

$$X \times Y \cong \varprojlim{}_{(i,i)}(X_i \times Y_i) \cong \varprojlim{}_{(i,j)\in I\times I}(X_i \times Y_j). \tag{1.3}$$

The isomorphism between $X \times Y$ and $\varprojlim_{(i,i)}(X_i \times Y_i)$ identifies pairs of sequences $((x_i),(y_i)) \in X \times Y$ with the sequence $(x_i, y_i) \in \varprojlim_{(i,i)}(X_i \times Y_i)$. Clearly, the above generalize to any finite Cartesian product of inverse limits.

Finally, let $X_i = X$ for all i and ϕ_j^i the identity for all i, j. Then $\varprojlim X$ can be identified naturally with X (identifying a constant sequence $(x, x, \ldots)$ with $x \in X$).

1.2. Our working example for the notion of inverse limit will be the construction of the p-adic integers. Let p be a prime number, which will be fixed throughout the paper, and let $\mathbb{Z}/p^r\mathbb{Z}$ be the additive group of integers modulo p^r. An element $a_r \in \mathbb{Z}/p^r\mathbb{Z}$ can be written uniquely in the form

$$a_r = k_0 + k_1 p + k_2 p^2 + \cdots + k_{r-1} p^{r-1} + p^r\mathbb{Z},$$

where $k_0, \ldots, k_{r-1} \in \{0, 1, \ldots, p-1\}$. For any $r, s \in \mathbb{N}$ with $r \geqslant s$ we consider the following natural epimorphisms:

$$\begin{aligned} &\theta_s^r : \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^s\mathbb{Z}, \\ &\theta_s^r\big(k_0 + k_1 p + k_2 p^2 + \cdots + k_{r-1}p^{r-1} + p^r\mathbb{Z}\big) = k_0 + k_1 p + k_2 p^2 + \cdots + k_{s-1}p^{s-1} + p^s\mathbb{Z} \end{aligned} \tag{1.4}$$

("cutting out" $r - s$ terms). We obtain, thus, the inverse system $(\mathbb{Z}/p^r\mathbb{Z}, \theta_s^r)$ of topological groups, indexed by $\mathbb{N}$. Its inverse limit, $\varprojlim \mathbb{Z}/p^r\mathbb{Z}$, is the group of *p-adic integers*, denoted $\mathbb{Z}_p$. $\mathbb{Z}_p$ is a non-cyclic subgroup of $\prod(\mathbb{Z}/p^r\mathbb{Z})$ and it contains no elements of finite order. $\mathbb{Z}_p$ can be identified with the set of sequences:

$$\mathbb{Z}_p = \big\{(a_r);\ a_r \in \mathbb{Z},\ a_r \equiv a_s \pmod{p^s} \text{ whenever } r \geqslant s\big\}. \tag{1.5}$$

Clearly, for the $(n+1)$st entry of an element $(a_r) \in \mathbb{Z}_p$ there are p choices, namely:

$$a_{r+1} \in \{a_r + \lambda p^r;\ \lambda = 0, 1, \ldots, p-1\}. \tag{1.6}$$

On the contrary, there is no choice for the entries before, as $a_s \equiv a_r \pmod{p^s}$ for all $s = 1, \ldots, r-1$. Elements in $\mathbb{Z}_p$ shall be usually denoted as

$$\underleftarrow{a} := (a_1, a_2, a_3, \ldots) \in \mathbb{Z}_p. \tag{1.7}$$

1.3. Contrary to embeddings between inverse systems, if each component $\rho_i : X_i \to Y_i$ of a morphism between two inverse systems is onto, the induced map $\varprojlim \rho_i$ between the inverse limits is not necessarily onto.

For example, consider the inverse systems $(\mathbb{Z}, \mathrm{id})$ and $(\mathbb{Z}/p^r\mathbb{Z}, \theta_s^r)$, both indexed by $\mathbb{N}$, and for each $s \in \mathbb{N}$ define the canonical epimorphism

$$\rho_s : \mathbb{Z} \to \mathbb{Z}/p^s\mathbb{Z}. \tag{1.8}$$

Then $(\rho_s;\ s \in \mathbb{N})$ is a morphism between the two inverse systems. The first inverse limit is isomorphic to $\mathbb{Z}$, while the second is the set of p-adic integers $\mathbb{Z}_p$. Note that the image of $\varprojlim \mathbb{Z}$ in $\mathbb{Z}_p$ under $\varprojlim \rho_s$ consists in all constant tuples of integers. On the other hand, the tuple (b_r), where $b_r = 1 + p + \cdots + p^{r-1}$ is in $\mathbb{Z}_p$ but is not constant.

Yet, we have the following very important result.

Lemma 1. *(See [8], Lemma 1.1.7.) Let (X_i, ϕ_j^i) be an inverse system of topological spaces indexed by a directed set I and let $\rho_i : X \to X_i$ be compatible surjections from a topological space X onto the spaces X_i $(i \in I)$. Then, either $\varprojlim X_i = \emptyset$ or the induced mapping $\rho = \varprojlim \rho_i : \varprojlim X \to \varprojlim X_i$ maps $\varprojlim X$ onto a dense subset of $\varprojlim X_i$.*

Proof. For the proof of Lemma 1 consider a non-empty open set V in $\varprojlim X_i$ of the form (1.1). We have to show that $\rho(X) \cap V \neq \emptyset$. Indeed, let $i_0 \geqslant i_1, \ldots, i_n$ and let $y = (y_i) \in V$. Choose $x \in X$ so that $\rho_{i_0}(x) = y_{i_0}$. Then $\rho(x) \in V$. □

For example, let ρ_i denote the restriction on a subset $A \subset \varprojlim X_i$ of the canonical projection of $\varprojlim X_i$ onto X_i. Recall that $\varprojlim A$ can be identified with A. Then we have the following.

Corollary 1. *If for a subset $A \subset \varprojlim X_i$ we have $\rho_i(A) = X_i$ for all $i \in I$, then $\rho(\varprojlim A)$ is dense in $\varprojlim X_i$, where $\rho = \varprojlim \rho_i$.*

Since $\mathbb{Z}$ projects onto each factor $\mathbb{Z}/p^r\mathbb{Z}$ via the canonical epimorphism (1.8), we obtain the following, as an application of Corollary 1.

Corollary 2. *$\mathbb{Z}$ is dense in $\mathbb{Z}_p$.*

This means that every p-adic integer can be approximated by a sequence of constant sequences. In Section 1.4 we study further this approximation.

Definition 1. (cf. [8] §2.4.) Let G_i be a group (ring, algebra, et cetera) for all $i \in I$. A subset $S \subset \varprojlim G_i$ is a set of *topological generators* of $\varprojlim G_i$ if the span $\langle S \rangle$ is dense in $\varprojlim G_i$. If, moreover, S is finite, $\varprojlim G_i$ is said to be *finitely generated*.

For example, the element $(1, 1, \ldots)$ is a topological generator of $\mathbb{Z}_p$, since, by Corollary 2, the cyclic subgroup $\langle (1, 1, \ldots) \rangle = \mathbb{Z}$ is dense in $\mathbb{Z}_p$.

1.4. As a topological space, $\mathbb{Z}_p$ is endowed with the induced topology of $\prod(\mathbb{Z}/p^r\mathbb{Z})$, which builds up from the discrete topology of each factor $\mathbb{Z}/p^r\mathbb{Z}$. Thus, a basic open set in $\mathbb{Z}_p$ is of the form $\{\varpi_i^{-1}(U_i);\ U_i \subseteq \mathbb{Z}/p^i\mathbb{Z}\}$, where ϖ_i is the restriction of the natural projection of $\mathbb{Z}_p$ onto $\mathbb{Z}/p^i\mathbb{Z}$. So, for any given element $a = (a_1, a_2, \ldots) \in \mathbb{Z}_p$ we have that $a \in \varpi_i^{-1}(\{a_i\})$ for all i. Moreover,

$$\begin{aligned}
\varpi_1^{-1}\big(\{a_1\}\big) &= \big\{\big(a_1, x_2, x_3, x_4, \ldots\big);\ x_2 \equiv a_1 \ (\mathrm{mod}\ p),\ x_n \equiv x_m \ (\mathrm{mod}\ p^m),\ n \geqslant m\big\}, \\
\varpi_2^{-1}\big(\{a_2\}\big) &= \big\{(a_1, a_2, y_3, y_4, \ldots);\ y_3 \equiv a_2 \ (\mathrm{mod}\ p^2),\ y_n \equiv y_m \ (\mathrm{mod}\ p^m),\ n \geqslant m\big\} \\
&\ \vdots
\end{aligned}$$

As we can see, $\varpi_1^{-1}(\{a_1\}) \supsetneq \varpi_2^{-1}(\{a_2\}) \supsetneq \cdots$.

This implies, in particular, that for $a_i \in \mathbb{Z}$ the constant sequence $(a_i, a_i, \ldots) \in \mathbb{Z}_p$ is contained in infinitely many open sets, each set being a refinement of the previous one. Recall now, from Corollary 2, that the set of constant sequences is dense in $\mathbb{Z}_p$. Thus, every element $\underleftarrow{a} = (a_1, a_2, a_3, \ldots) \in \mathbb{Z}_p$ can be approximated by a sequence of constant sequences, the following:

$$\begin{aligned}
&(a_1, a_1, a_1, \ldots) \in \varpi_1^{-1}\big(\{a_1\}\big), \\
&(a_1, a_2, a_2, a_2, \ldots) = (a_2, a_2, a_2, \ldots) \in \varpi_2^{-1}\big(\{a_2\}\big), \\
&(a_1, a_2, a_3, a_3, \ldots) = (a_3, a_3, a_3, \ldots) \in \varpi_3^{-1}\big(\{a_3\}\big), \\
&\quad \vdots
\end{aligned} \tag{1.9}$$

since $a_r \equiv a_1 \ (\mathrm{mod}\ p)$, for $r \geqslant 1$, $a_r \equiv a_2 \ (\mathrm{mod}\ p^2)$, for $r \geqslant 2$, and so on. Indeed, $(a_1, a_2, a_3, \ldots)$ and $(a_i, a_i, a_i, \ldots) = (a_1, a_2, \ldots, a_{i-1}, a_i, a_i, \ldots)$ are both in $\varpi_i^{-1}(\{a_i\})$ for all i. Finally, $\varpi_1^{-1}(\{a_1\}) \supsetneq \varpi_2^{-1}(\{a_2\}) \supsetneq \cdots$, justifying the approximation claim. We shall write:

$$\underleftarrow{a} = \lim_k (a_k). \tag{1.10}$$

For more details and further reading on inverse limits and the p-adic integers see, for example, [1,8,9,12].

2. p-Adic framed braids

The aim of this section is to introduce the notion of p-adic framed braids. These are similar to the classical framed braids but, instead of integral framing, each strand may be colored with a p-adic integer.

2.1. Before starting with our construction we need to digress briefly and recall the definition and the structure of the classical framed braid group (see, for example, [6]) and the modular framed braid group.

We consider the group $\mathbb{Z}^n$ with the usual operation:

$$(a_1,\dots,a_n)(b_1,\dots,b_n) := (a_1+b_1,\dots,a_n+b_n). \tag{2.1}$$

$\mathbb{Z}^n$ is generated by the 'elementary framings':

$$f_i := (0,\dots,0,1,0,\dots,0)$$

with 1 in the ith position. Then, an element $a=(a_1,\dots,a_n)\in\mathbb{Z}^n$ can be expressed as:

$$a = f_1^{a_1} f_2^{a_2}\dots f_n^{a_n}.$$

Let also B_n be the classical braid group on n strands. B_n is generated by the elementary braids $\sigma_1,\dots,\sigma_{n-1}$, where σ_i is the positive crossing between the ith and the $(i+1)$st strand. The σ_i's satisfy the well-known braid relations: $\sigma_i\sigma_j=\sigma_j\sigma_i$, if $|i-j|>1$ and $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$. Recall the symmetric group S_n, generated by the $n-1$ elementary transpositions $s_i:=(i,i+1)$, and let further π be the natural projection of B_n on S_n. We let $\sigma(j)$ denote $\pi(\sigma)(j)$ for any $j=1,2,\dots,n$. In particular, $\sigma_i(j)=s_i(j)$. Using π we define the *framed braid group $\mathcal{F}_n$* as:

$$\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n,$$

where the action of B_n on $a=(a_1,\dots,a_n)\in\mathbb{Z}^n$ is given by permutation of the indices:

$$\sigma(a) = (a_{\sigma(1)},\dots,a_{\sigma(n)}) \quad (\sigma\in B_n). \tag{2.2}$$

In the above notation, the action of B_n on $\mathbb{Z}^n$ is given by the multiplicative formula:

$$\sigma\big(f_1^{a_1} f_2^{a_2}\dots f_n^{a_n}\big) = f_1^{a_{\sigma(1)}} f_2^{a_{\sigma(2)}}\dots f_n^{a_{\sigma(n)}} \quad (\sigma\in B_n).$$

Any word in $\mathcal{F}_n$ splits, by construction, into the 'framing' part and the 'braiding' part. That is, it can be written in the form

$$f_1^{k_1} f_2^{k_2}\dots f_n^{k_n}\cdot\sigma, \quad \text{where } k_i\in\mathbb{Z},\ \sigma\in B_n. \tag{2.3}$$

The multiplication in $\mathcal{F}_n$ is defined using the action of B_n on $\mathbb{Z}^n$ as follows:

$$\big(f_1^{a_1} f_2^{a_2}\dots f_n^{a_n}\cdot\sigma\big)\big(f_1^{b_1} f_2^{b_2}\dots f_n^{b_n}\cdot\tau\big) := f_1^{a_1+b_{\sigma(1)}} f_2^{a_2+b_{\sigma(2)}}\dots f_n^{a_n+b_{\sigma(n)}}\cdot\sigma\tau. \tag{2.4}$$

Geometrically, an element of $\mathcal{F}_n$ is a classical braid on n strands, with each strand decorated on the top by an integer, its framing. An element of $\mathbb{Z}^n$, when this is seen as a subgroup of $\mathcal{F}_n$, is identified with the identity braid on n strands, each strand being decorated by the corresponding integer of the element. For example, the element f_i is the identity braid with framing 1 on the ith strand and 0 elsewhere, while $f_1^{a_1} f_2^{a_2}\dots f_n^{a_n}$ is the identity braid with framings $a_1,a_2,\dots,a_n$. On the other hand, a braid in B_n, when this is seen as a subgroup of $\mathcal{F}_n$, is meant as a framed braid with all framings 0. Geometrically, the multiplication in the group $\mathcal{F}_n$ is the usual concatenation in B_n together with collecting the total framing of each strand to the top of the final braid. See Fig. 3 for an illustration.

Definition 2. The *d-modular* (*or simply modular*) *framed braid group* on n strands is defined as $\mathcal{F}_{d,n} := (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$.

The group $\mathcal{F}_{d,n}$ can be considered as the quotient of $\mathcal{F}_n$ by imposing the relations

$$f_i{}^d = 1 \quad (i=1,\dots,n).$$

Clearly, $\mathcal{F}_{d,n}$ has the same geometric interpretation as $\mathcal{F}_n$, only that the framings of the n strands are taken from the cyclic group $\mathbb{Z}/d\mathbb{Z}$. Note now that in $\mathcal{F}_n$ or in $\mathcal{F}_{d,n}$ the f_i's can be deduced from f_1, setting, for example:

$$f_i := \sigma_{i-1}\dots\sigma_1 f_1 \sigma_1^{-1}\dots\sigma_{i-1}^{-1}.$$

Then we have the following.

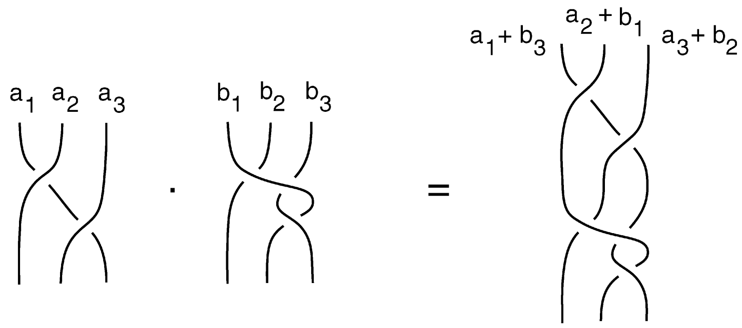

Fig. 3. Multiplication of framed braids.

Proposition 1. *$\mathcal{F}_n$ has a presentation with generators $f_1, \sigma_1, \ldots, \sigma_{n-1}$ and relations*:

$$f_1\sigma_j = \sigma_j f_1 \quad \textit{for } j > 1,$$
$$f_1\sigma_1 f_1\sigma_1^{-1} = \sigma_1 f_1\sigma_1^{-1} f_1,$$
$$\sigma_i(\sigma_{i-1}\ldots\sigma_1 f_1\sigma_1^{-1}\ldots\sigma_{i-1}^{-1})\sigma_i^{-1} = \sigma_i^{-1}(\sigma_{i-1}\ldots\sigma_1 f_1\sigma_1^{-1}\ldots\sigma_{i-1}^{-1})\sigma_i \quad \textit{for all } i$$

together with the usual braid relations among the σ_i's.

Proposition 2. *$\mathcal{F}_{d,n}$ has the same presentation as $\mathcal{F}_n$, but with the extra relation $f_1^d = 1$.*

2.2. In order to define the p-adic framed braids we would rather pass to multiplicative notation for $\mathbb{Z}/p^r\mathbb{Z}$. Let C_r denote the multiplicative cyclic group of order p^r, generated by the element t_r. That is,

$$C_r := \langle t_r;\ t_r^{p^r} = 1\rangle.$$

Then $\mathbb{Z}/p^r\mathbb{Z} \cong C_r$. The maps (1.4) of the inverse system (C_r, θ_s^r) are now defined by:

$$\begin{aligned} \theta_s^r : C_r &\to C_s, \\ t_r &\mapsto t_s \end{aligned} \tag{2.5}$$

whenever $r \geqslant s$. In this notation: $\theta_s^r(t_r^{k_0+k_1p+\cdots+k_{r-1}p^{r-1}}) = t_s^{k_0+k_1p+\cdots+k_{s-1}p^{s-1}}$. We have:

$$\mathbb{Z}_p = \varprojlim C_r$$

and we can write:

$$\mathbb{Z}_p = \Big\{(t_1^{a_1}, t_2^{a_2}, \ldots) \in \prod C_i;\ a_r \in \mathbb{Z},\ a_r \equiv a_s \pmod{p^s} \text{ whenever } r \geqslant s\Big\}.$$

The element

$$\mathbf{t} := (t_1, t_2, \ldots) \in \varprojlim C_r \tag{2.6}$$

corresponds to $(1, 1, \ldots)$ in the additive notation, so, following the notation of (1.7), we shall write: $\mathbf{t}^{\underleftarrow{a}} := (t_1^{a_1}, t_2^{a_2}, \ldots)$ for elements in $\varprojlim C_r = \mathbb{Z}_p$. The element $\mathbf{t}$ generates in $\varprojlim C_r$ the constant sequences. So, we shall write $\mathbb{Z} = \langle \mathbf{t}\rangle$. By Corollary 2, $\mathbb{Z}$ is dense in $\varprojlim C_r$ and $\mathbf{t}$ is a topological generator of $\varprojlim C_r$, so an element $(t_1^{a_1}, t_2^{a_2}, \ldots) \in \mathbb{Z}_p$ can be approximated by the sequence $(\mathbf{t}^{a_k})$ of elements in $\mathbb{Z}$. So, we shall write:

$$\mathbf{t}^{\underleftarrow{a}} = \big(t_1^{a_1}, t_2^{a_2}, \ldots\big) = \lim_k(\mathbf{t}^{a_k}). \tag{2.7}$$

For example, for $\underleftarrow{b} = (1, 1+p, 1+p+p^2, \ldots)$ in the additive notation we write $\mathbf{t}^{\underleftarrow{b}}$ in the multiplicative notation, and we have that it can be approximated by the sequence $(\mathbf{t}, \mathbf{t}^{1+p}, \mathbf{t}^{1+p+p^2}, \ldots)$. That is:

$$\mathbf{t}^{\underleftarrow{b}} = \big(t_1, t_2^{1+p}, t_3^{1+p+p^2}, \ldots\big) = \lim_k(\mathbf{t}^{1+p+\cdots+p^k}). \tag{2.8}$$

With the above notation and according to Section 1.2, if $\mathbf{t}^{\underleftarrow{b}} = (t_1^{b_1}, t_2^{b_2}, \ldots)$ is another element in $\mathbb{Z}_p$, the multiplication in $\mathbb{Z}_p$ is defined as follows:

$$\mathbf{t}^{\underleftarrow{a}}\mathbf{t}^{\underleftarrow{b}} := \mathbf{t}^{\underleftarrow{a}+\underleftarrow{b}} = \left(t_1^{a_1+b_1}, t_2^{a_2+b_2}, \ldots\right) \tag{2.9}$$

and we have the approximation:

$$\mathbf{t}^{\underleftarrow{a}}\mathbf{t}^{\underleftarrow{b}} = \lim_k \left(\mathbf{t}^{a_k+b_k}\right). \tag{2.10}$$

2.3. Consider now the direct product $C_r^n := C_r \times \cdots \times C_r$ (n times). This is an Abelian group with the usual product operation defined componentwise, generated by the n elements

$$t_{r,i} := (1, \ldots, 1, t_r, 1, \ldots, 1), \tag{2.11}$$

where t_r is in the ith position and where 1 is the unit element in C_r. In this notation:

$$\left(t_r^{m_1}, t_r^{m_2}, \ldots, t_r^{m_n}\right) = t_{r,1}^{m_1} \cdot t_{r,2}^{m_2} \ldots t_{r,n}^{m_n} \quad \text{in } C_r^n. \tag{2.12}$$

Moreover, C_r^n has the presentation:

$$C_r^n = \left\langle t_{r,1}, \ldots, t_{r,n};\ t_{r,i}\, t_{r,j} = t_{r,j}\, t_{r,i} \text{ and } t_{r,i}^{p^r} = (1, \ldots, 1)\right\rangle. \tag{2.13}$$

Using the maps (2.5) of the inverse system (C_r, θ_s^r) and (2.12) we define componentwise the maps:

$$\begin{array}{rcl} \pi_s^r : C_r^n & \to & C_s^n \\ t_{r,i} & \mapsto & t_{s,i} \end{array}$$

whenever $r \geqslant s$. Then:

$$\pi_s^r\left(t_{r,1}^{m_1} \cdot t_{r,2}^{m_2} \ldots t_{r,n}^{m_n}\right) = t_{s,1}^{m_1 \,(\text{mod } p^s)} \cdot t_{s,2}^{m_2 \,(\text{mod } p^s)} \cdots t_{s,n}^{m_n \,(\text{mod } p^s)}. \tag{2.14}$$

The maps π_s^r are obviously group epimorphisms, so (C_r^n, π_s^r) is an inverse system of topological groups, indexed by $\mathbb{N}$, and so the inverse limit $\varprojlim C_r^n$ exists.

Proposition 3. $\varprojlim C_r^n \cong (\varprojlim C_r)^n = \mathbb{Z}_p^n$.

Proof. It follows immediately from (1.3). $\square$

Notice now that an element $w \in \varprojlim C_r^n$ can be written as:

$$\begin{aligned} w &= \left(\left(t_1^{a_{11}}, t_1^{a_{12}}, \ldots, t_1^{a_{1n}}\right),\ \left(t_2^{a_{21}}, t_2^{a_{22}}, \ldots, t_2^{a_{2n}}\right), \ldots\right), \\ &= \left(t_{1,1}^{a_{11}}\, t_{1,2}^{a_{12}} \ldots t_{1,n}^{a_{1n}}, t_{2,1}^{a_{21}}\, t_{2,2}^{a_{22}} \ldots t_{2,n}^{a_{2n}}, \ldots\right) \quad \text{(by (2.12))} \\ &= \left(t_{1,1}^{a_{11}}, t_{2,1}^{a_{21}}, \ldots\right) \cdot \left(t_{1,2}^{a_{12}}, t_{2,2}^{a_{22}}, \ldots\right) \ldots \left(t_{1,n}^{a_{1n}}, t_{2,n}^{a_{2n}}, \ldots\right) \quad \text{(by product operation)} \\ &= \left(t_{r,1}^{a_{r1}}\, t_{r,2}^{a_{r2}} \ldots t_{r,n}^{a_{rn}}\right)_r. \end{aligned}$$

An explicit isomorphism between $\varprojlim C_r^n$ and $\mathbb{Z}_p$ is then given by the map:

$$w \mapsto \left(\left(t_1^{a_{11}}, t_2^{a_{21}}, \ldots\right), \left(t_1^{a_{12}}, t_2^{a_{22}}, \ldots\right), \ldots, \left(t_1^{a_{1n}}, t_2^{a_{2n}}, \ldots\right)\right)$$

Thus, we have the identification:

$$\left(t_{r,1}^{a_{r1}}\, t_{r,2}^{a_{r2}} \ldots t_{r,n}^{a_{rn}}\right)_r \doteq \left(\left(t_1^{a_{11}}, t_2^{a_{21}}, \ldots\right), \left(t_1^{a_{12}}, t_2^{a_{22}}, \ldots\right), \ldots, \left(t_1^{a_{1n}}, t_2^{a_{2n}}, \ldots\right)\right). \tag{2.15}$$

In particular, the following elements get identified, for $i = 1, \ldots, n$:

$$\varprojlim C_r^n \ni (t_{r,i})_r \doteq \left((1, 1, \ldots), \ldots, (t_1, t_2, \ldots), \ldots, (1, 1, \ldots)\right) \in (\varprojlim C_r)^n,$$

where the sequence $(t_1, t_2, \ldots)$ is in the ith position. Set now $\mathbf{1} := (1, 1, \ldots)$ and $\mathbf{t} = (t_1, t_2, \ldots)$ (recall (2.6)) in $\varprojlim C_r$ and denote:

$$\mathbf{t}_i := (\mathbf{1}, \ldots, \mathbf{1}, \mathbf{t}, \mathbf{1}, \ldots, \mathbf{1}) \in (\varprojlim C_r)^n, \tag{2.16}$$

where $\mathbf{t}$ is in the ith position. Then we have the identifications:

$$\varprojlim C_r^n \ni (t_{r,i})_r \doteq \mathbf{t}_i \in \mathbb{Z}_p^n. \tag{2.17}$$

Thus, with the above notation and with the notation of (2.7) we can rewrite the identification (2.15) as follows, for $\underleftarrow{a_i} = (a_{ri})_r$:

$$\varprojlim C_r^n \ni w = \left(t_{r,1}^{a_{r1}}\, t_{r,2}^{a_{r2}}\, \dots\, t_{r,n}^{a_{rn}}\right)_r \doteq \left(\mathbf{t}^{\underleftarrow{a_1}}, \mathbf{t}^{\underleftarrow{a_2}}, \dots, \mathbf{t}^{\underleftarrow{a_n}}\right) = \mathbf{t}_1^{\underleftarrow{a_1}} \mathbf{t}_2^{\underleftarrow{a_2}} \dots \mathbf{t}_n^{\underleftarrow{a_n}} \in \mathbb{Z}_p^n. \tag{2.18}$$

Lemma 2. *The identification in $\varprojlim C_r^n$ of the set $X = \{\mathbf{t}_1, \dots, \mathbf{t}_n\} \subset \mathbb{Z}_p^n$ is a set of topological generators of $\varprojlim C_r^n$. Equivalently, the identification in $\varprojlim C_r^n$ of the subgroup $\mathbb{Z}^n = \langle X \rangle$ of $\mathbb{Z}_p^n$ is dense in $\varprojlim C_r^n$.*

Proof. By Corollary 2 and by Definition 1, $\langle \mathbf{t}_i \rangle$ is clearly dense in the ith factor $(\{\mathbf{1}\} \times \cdots \times \{\mathbf{1}\} \times \mathbb{Z}_p \times \{\mathbf{1}\} \times \cdots \times \{\mathbf{1}\})$ of $\mathbb{Z}_p^n$. The result now follows from Corollary 2 and the identification (2.18). □

For example, by (2.16) and (2.17), and by the approximation (2.7), we have the approximation of $(t_{r,i}^{a_r})_r \in \varprojlim C_r^n$:

$$\left(t_{r,i}^{a_r}\right)_r \doteq \mathbf{t}_i{}^{\underleftarrow{a}} = \lim_k \left(\mathbf{t}_i{}^{a_k}\right) \doteq \lim_k \left[\left(t_{r,i}^{a_k}\right)_r\right]. \tag{2.19}$$

In general, for an element in $\mathbb{Z}_p^n$ we have, by (2.18), (2.7) and (2.19), the following approximation, where $\underleftarrow{a_i} = (a_{ri})_r$:

$$\mathbb{Z}_p^n \ni \mathbf{t}_1^{\underleftarrow{a_1}} \mathbf{t}_2^{\underleftarrow{a_2}} \dots \mathbf{t}_n^{\underleftarrow{a_n}} = \lim_k \left(\mathbf{t}_1^{a_{k1}} \mathbf{t}_2^{a_{k2}} \dots \mathbf{t}_n^{a_{kn}}\right) = \lim_k \left(\mathbf{t}^{a_{k1}}, \mathbf{t}^{a_{k2}}, \dots, \mathbf{t}^{a_{kn}}\right). \tag{2.20}$$

Consequently, for the product of two elements in $\mathbb{Z}_p^n$ we have by (2.10) the following approximation, where $\underleftarrow{b_i} = (b_{ri})_r$:

$$\left(\mathbf{t}_1^{\underleftarrow{a_1}} \dots \mathbf{t}_n^{\underleftarrow{a_n}}\right)\left(\mathbf{t}_1^{\underleftarrow{b_1}} \dots \mathbf{t}_n^{\underleftarrow{b_n}}\right) = \lim_k \left(\mathbf{t}_1^{a_{k1}+b_{k1}} \dots \mathbf{t}_n^{a_{kn}+b_{kn}}\right). \tag{2.21}$$

Hence, for an element $w \in \varprojlim C_r^n$, $w = (t_{r,1}^{a_{r1}}\, t_{r,2}^{a_{r2}}\, \dots\, t_{r,n}^{a_{rn}})_r \doteq \mathbf{t}_1^{\underleftarrow{a_1}} \mathbf{t}_2^{\underleftarrow{a_2}} \dots \mathbf{t}_n^{\underleftarrow{a_n}}$ we obtain, by (2.18), (2.19) and (2.20), the approximation:

$$\varprojlim C_r^n \ni \left(t_{r,1}^{a_{r1}}\, t_{r,2}^{a_{r2}} \dots t_{r,n}^{a_{rn}}\right)_r = \lim_k \left[\left(t_{r,1}^{a_{k1}}\, t_{r,2}^{a_{k2}} \dots t_{r,n}^{a_{kn}}\right)_r\right] \tag{2.22}$$

and for the product of two elements in $\varprojlim C_r^n$ we have the approximation:

$$\left(t_{r,1}^{a_{r1}} \dots t_{r,n}^{a_{rn}}\right)_r \left(t_{r,1}^{b_{r1}} \dots t_{r,n}^{a_{bn}}\right)_r = \lim_k \left[\left(t_{r,1}^{a_{k1}+b_{k1}} \dots t_{r,n}^{a_{kn} b_{kn}}\right)_r\right]. \tag{2.23}$$

2.4. p-Adic framed braids. In order to introduce the inverse limits in the construction of framed braids we start the construction from the beginning. Consider the Cartesian product $C_r^n \times B_n$. Using the maps (2.14), define for any $r, s \in \mathbb{N}$ with $r \geqslant s$ the surjective maps:

$$\begin{aligned} \pi_s^r \times \mathrm{id} : C_r^n \times B_n &\to C_s^n \times B_n, \\ \left(t_{r,1}^{a_{r1}}\, t_{r,2}^{a_{r2}}\, \dots\, t_{r,n}^{a_{rn}}, \sigma\right) &\mapsto \left(t_{s,1}^{a_{s1}}\, t_{s,2}^{a_{s2}}\, \dots\, t_{s,n}^{a_{sn}}, \sigma\right) \end{aligned} \tag{2.24}$$

for any $\sigma \in B_n$ and for any exponents satisfying $a_{ri} \equiv a_{si} \pmod{p^s}$. Then we have the following.

Proposition 4. $(C_r^n \times B_n, \pi_s^r \times \mathrm{id})$ *is an inverse system of topological spaces, indexed by $\mathbb{N}$ and we have*:

$$\varprojlim \left(C_r^n \times B_n\right) \cong \varprojlim C_r^n \times B_n \cong \mathbb{Z}_p^n \times B_n.$$

Moreover, the identification in $\varprojlim (C_r^n \times B_n)$ of $\mathbb{Z}^n \times B_n$ is dense in $\varprojlim (C_r^n \times B_n)$ and $\mathbb{Z}^n \times B_n$ is dense in $\mathbb{Z}_p^n \times B_n$.

Proof. Since the maps π_s^r are maps of the inverse system (C_r^n, π_s^r), it follows immediately that $(C_r^n \times B_n, \pi_s^r \times \mathrm{id})$ is an inverse system of topological spaces. An element in $\varprojlim(C_r^n \times B_n)$ is a sequence of the form $((w_1, \sigma), (w_2, \sigma), \ldots)$, where $\sigma \in B_n$ and where $w_1 \in C_1^n, w_2 \in C_2^n, \ldots$, such that $\pi_s^r(w_r) = w_s$ whenever $r \geqslant s$. Identifying it with the pair of sequences $((w_1, w_2, \ldots), (\sigma, \sigma, \ldots)) \in \varprojlim C_r^n \times \varprojlim B_n$, where $\varprojlim B_n$ arises as the inverse limit of the trivial inverse system (B_n, id), induces the bijection between $\varprojlim(C_r^n \times B_n)$ and $\varprojlim C_r^n \times B_n$:

$$\varprojlim\bigl(C_r^n \times B_n\bigr) \ni \bigl((w_1, \sigma), (w_2, \sigma), \ldots\bigr) \doteq \bigl((w_1, w_2, \ldots), \sigma\bigr) \in \varprojlim C_r^n \times B_n, \tag{2.25}$$

where the natural identification between $\varprojlim B_n$ and B_n is induced by the identification $(\sigma, \sigma, \ldots) = \sigma$. So the assertion $\varprojlim(C_r^n \times B_n) \cong \varprojlim C_r^n \times B_n$ is proved. Moreover, by (2.15), $\varprojlim C_r^n \times B_n \cong \mathbb{Z}_p^n \times B_n$.

By Lemma 2, and by Corollary 2, the identification of $\mathbb{Z}^n = \langle \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n\rangle$ in $\varprojlim C_r^n$ projects surjectively on each factor C_r^n of the inverse system (C_r^n, π_s^r). Extending the projection by the identity map on B_n implies that the identification of $\mathbb{Z}^n \times B_n$ projects surjectively on each factor $C_r^n \times B_n$ of the inverse system $(C_r^n \times B_n, \pi_s^r \times \mathrm{id})$. Hence, by Corollary 1, the identification of $\mathbb{Z}^n \times B_n$ is dense in $\varprojlim(C_r^n \times B_n)$. □

Consider now the action of the group B_n on the group C_r^n by permutation, as defined in (2.2). For the case $d = p^r$ and with the above notation, we have that $C_r^n \rtimes B_n = \mathcal{F}_{p^r,n}$, the modular framed braid group with the operation (2.4) (in additive notation).

Remark 1. The generator f_i of $\mathcal{F}_{p^r,n}$ (Proposition 2) in the additive notation corresponds to the generator $t_{r,i}$ of C_r^n. The generators of $C_r^n \rtimes B_n = \mathcal{F}_{p^r,n}$ are the n elementary framings $t_{r,1}, \ldots, t_{r,n}$ and the $n-1$ elementary braids $\sigma_1, \ldots, \sigma_{n-1}$.

Further, use the maps (2.24) of the inverse system $(C_r^n \times B_n, \pi_s^r \times \mathrm{id})$ to define:

$$\begin{aligned} \pi_s^r \cdot \mathrm{id} : \mathcal{F}_{p^r,n} &\to \mathcal{F}_{p^s,n}, \\ (t_{r,i}, \mathrm{id}) &\mapsto (t_{s,i}, \mathrm{id}), \\ \bigl((1, \ldots, 1), \sigma_i\bigr) &\mapsto \bigl((1, \ldots, 1), \sigma_i\bigr), \end{aligned} \tag{2.26}$$

whenever $r \geqslant s$.

Lemma 3. $(\mathcal{F}_{p^r,n}, \pi_s^r \cdot \mathrm{id})$ *is an inverse system of topological groups, indexed by* $\mathbb{N}$.

Proof. On the level of the sets $C_r^n \times B_n$, the map $\pi_s^r \cdot \mathrm{id}$ is $\pi_s^r \times \mathrm{id}$. We shall show that $\pi_s^r \cdot \mathrm{id}$ is a group homomorphism. Indeed, let $(x, \sigma), (y, \tau) \in C_r^n \rtimes B_n$. Then we have:

$$\begin{aligned} \bigl(\pi_s^r \cdot \mathrm{id}\bigr)\bigl[(x, \sigma), (y, \tau)\bigr] &= \bigl(\pi_s^r \cdot \mathrm{id}\bigr)\bigl(x\,\sigma(y), \sigma\tau\bigr) = \bigl(\pi_s^r\bigl(x\,\sigma(y)\bigr), \sigma\tau\bigr) \\ &\overset{(\text{in } C_r^n)}{=} \bigl(\pi_s^r(x)\,\pi_s^r\bigl(\sigma(y)\bigr), \sigma\tau\bigr) \overset{(\pi_s^r\circ\sigma=\sigma\circ\pi_s^r)}{=} \bigl(\pi_s^r(x)\,\sigma\bigl(\pi_s^r(y)\bigr), \sigma\tau\bigr) \\ &= \bigl(\pi_s^r(x), \sigma\bigr) \cdot \bigl(\pi_s^r(y)\tau\bigr) = \bigl(\pi_s^r \cdot \mathrm{id}\bigr)(x, \sigma) \cdot \bigl(\pi_s^r \cdot \mathrm{id}\bigr)(y, \tau). \end{aligned}$$

Hence, $(\mathcal{F}_{p^r,n}, \pi_s^r \cdot \mathrm{id})$ is an inverse system of topological groups. □

Definition 3. The *p-adic framed braid group on n strands $\mathcal{F}_{\infty,n}$* is defined to be the inverse limit of the inverse system $(\mathcal{F}_{p^r,n}, \pi_s^r \cdot \mathrm{id})$, that is:

$$\mathcal{F}_{\infty,n} := \varprojlim \mathcal{F}_{p^r,n} = \varprojlim\bigl(C_r^n \rtimes B_n\bigr).$$

Elements of $\mathcal{F}_{\infty,n}$ shall be denoted $\underset{\leftarrow}{\beta}$.

Remark 2. $\mathcal{F}_{\infty,n}$ could have alternatively been defined as the semidirect product $\mathbb{Z}_p^n \rtimes B_n$. In fact, the two groups are isomorphic, as the following theorem states. Our definition, though, leads naturally to the construction of the p-adic Yokonuma–Hecke algebras, since the classical Yokonuma–Hecke algebras are quotients of the modular framed braid groups (see Section 3).

Theorem 1. *There are group isomorphisms*:

$$\mathcal{F}_{\infty,n} \cong \mathbb{Z}_p^n \rtimes B_n \cong \varprojlim C_r^n \rtimes B_n.$$

Moreover, $\mathcal{F}_n$ is dense in $\mathbb{Z}_p^n \times B_n$ and the identification in $\mathcal{F}_{\infty,n}$ of $\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$ is dense in $\mathcal{F}_{\infty,n}$. Finally, the identification in $\mathcal{F}_{\infty,n}$ of the set $A = \{\mathbf{t}_1, \sigma_1, \ldots, \sigma_{n-1}\} \subset \mathcal{F}_n$ is a set of topological generators of $\mathcal{F}_{\infty,n}$.

Proof. The second isomorphism is clear from Proposition 3. We will prove the first one. On the right-hand side B_n acts on $\mathbb{Z}_p^n$ by permutation, that is, a $\sigma \in B_n$ permutes accordingly the positions of an n-tuple of p-adic integers. We consider the bijection:

$$\alpha : \mathcal{F}_{\infty,n} \to \mathbb{Z}_p^n \rtimes B_n$$

defined by combining (2.25) and (2.15). More precisely:

$$\big((w_1,\sigma),\ (w_2,\sigma),\ \ldots\big) \overset{\alpha}{\mapsto} \big(\big[(w_{11}, w_{21}, \ldots), (w_{12}, w_{22}, \ldots), \ldots, (w_{1n}, w_{2n}, \ldots)\big], \sigma\big)$$

where $w_r = (w_{r1}, w_{r2}, \ldots, w_{rn}) \in C_r^n$.

Claim. *α is a group homomorphism.*

Indeed, let $x = ((w_1,\sigma),\ (w_2,\sigma),\ \ldots)$ and $y = ((\mu_1,\tau),\ (\mu_2,\tau),\ \ldots) \in \mathcal{F}_{\infty,n}$, where $\mu_r = (\mu_{r1}, \mu_{r2}, \ldots, \mu_{rn}) \in C_r^n$. Then:

$$\begin{aligned} xy &= \big((w_1,\sigma),\ (w_2,\sigma),\ \ldots\big) \cdot \big((\mu_1,\tau),\ (\mu_2,\tau),\ \ldots\big) \\ &= \big((w_1,\sigma)(\mu_1,\tau), (w_2,\sigma)(\mu_2,\tau), \ldots\big) \\ &= \big(\big(w_1\,\sigma(\mu_1), \sigma\tau\big), \big(w_2\,\sigma(\mu_2), \sigma\tau\big), \ldots\big) \\ &= \big(\big[(w_{11}\mu_{1\sigma(1)}, \ldots, w_{1n}\mu_{1\sigma(n)}), \sigma\tau\big],\ \big[(w_{21}\mu_{2\sigma(1)}, \ldots, w_{2n}\mu_{2\sigma(n)}), \sigma\tau\big], \ldots\big). \end{aligned}$$

Hence,

$$\alpha(xy) = \big(\big[(w_{11}\mu_{1\sigma(1)}, w_{21}\mu_{2\sigma(1)}, \ldots), \ldots, (w_{1n}\mu_{1\sigma(n)}, w_{2n}\mu_{2\sigma(n)}, \ldots)\big], \sigma\tau\big).$$

On the other hand:

$$\begin{aligned} \alpha(x)\,\alpha(y) &= \big(\big[(w_{11}, \ldots), \ldots, (w_{1n}, \ldots)\big], \sigma\big) \cdot \big(\big[(\mu_{11}, \ldots), \ldots, (\mu_{1n}, \ldots)\big], \tau\big) \\ &= \big(\big[(w_{11}, \ldots), \ldots, (w_{1n}, \ldots)\big]\,\sigma\big[(\mu_{11}, \ldots), \ldots, (\mu_{1n}, \ldots)\big], \sigma\tau\big) \\ &= \big(\big[(w_{11}, \ldots), \ldots, (w_{1n}, \ldots)\big]\big[(\mu_{1\sigma(1)}, \ldots), \ldots, (\mu_{1\sigma(n)}, \ldots)\big], \sigma\tau\big) \\ &= \big(\big[(w_{11}, \ldots)\,(\mu_{1\sigma(1)}, \ldots), \ldots, (w_{1n}, \ldots)\,(\mu_{1\sigma(n)}, \ldots)\big], \sigma\tau\big) \\ &= \big(\big[(w_{11}\mu_{1\sigma(1)}, \ldots), \ldots, (w_{1n}\mu_{1\sigma(n)}, \ldots)\big], \sigma\tau\big) = \alpha(xy). \end{aligned}$$

Further, $\mathbb{Z}^n \rtimes B_n$ is identical as set to $\mathbb{Z}^n \times B_n$. By Proposition 4, $\mathbb{Z}^n \times B_n$ is dense in $\mathbb{Z}_p^n \times B_n$, which in turn is identical as set to $\mathbb{Z}_p^n \rtimes B_n$. With similar reasoning the identification in $\mathcal{F}_{\infty,n}$ of $\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$ is dense in $\mathcal{F}_{\infty,n}$.

For the last statement of the theorem, we only need to observe that the generators (2.16) of $\mathbb{Z}^n$ are the multiplicative versions of the generators f_i of $\mathcal{F}_n$ given in Section 2.1. Therefore, the span $\langle A \rangle$ is isomorphic to the classical framed braid group $\mathcal{F}_n$. So, the identification of A in $\mathcal{F}_{\infty,n}$ is a set of topological generators for $\mathcal{F}_{\infty,n}$. □

In the sequel we will not distinguish between $\mathbb{Z}_p^n \rtimes B_n$ and $\mathcal{F}_{\infty,n}$.

Remark 3. The fact that $\mathbb{Z}_p$ and B_n contain no elements of finite order imply that $\mathcal{F}_{\infty,n} \cong \mathbb{Z}_p^n \rtimes B_n$ contains no elements of finite order either. In particular, the modular relations for the framing are not valid in $\mathcal{F}_{\infty,n}$.

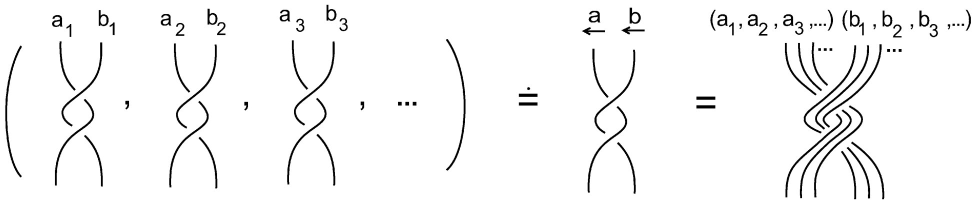

Fig. 4. A p-adic framed braid.

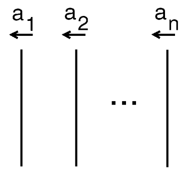

Fig. 5. A p-adic identity framed braid.

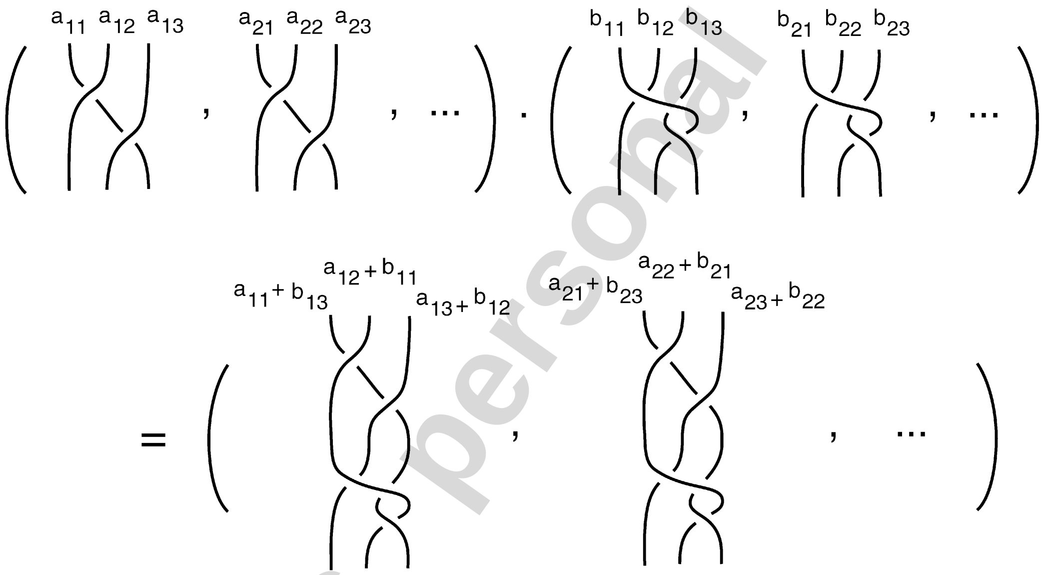

Fig. 6. Multiplication of p-adic framed braids in $\mathcal{F}_{\infty,n}$.

2.5. Geometric interpretations. By Definition 3 a p-adic framed braid is an infinite sequence of the same braid $\sigma \in B_n$, such that the rth braid of the sequence gets framed in the modular framed braid group $\mathcal{F}_{p^r,n}$ (recall Definition 2) with the framings $(a_{r1}, a_{r2}, \ldots, a_{rn}) \in (\mathbb{Z}/\mathbb{Z}_{p^r})^n$, where $\underset{\leftarrow}{a_i} = (a_{ri})_r$. By the isomorphism in Theorem 1, a p-adic framed braid can be identified with the element:

$$\mathbf{t}_1^{\underset{\leftarrow}{a_1}}\mathbf{t}_2^{\underset{\leftarrow}{a_2}} \ldots \mathbf{t}_n^{\underset{\leftarrow}{a_n}} \cdot \sigma \in \mathbb{Z}_p^n \rtimes B_n, \tag{2.27}$$

that is, the braid $\sigma \in B_n$ with each strand decorated with a p-adic integer. This in turn can be interpreted as an infinite framed cabling of a braid $\sigma \in B_n$. See Fig. 4. In particular, the element $\mathbf{t}_1^{\underset{\leftarrow}{a_1}}\mathbf{t}_2^{\underset{\leftarrow}{a_2}} \ldots \mathbf{t}_n^{\underset{\leftarrow}{a_n}} \in \mathbb{Z}_p^n$ can be viewed as the identity braid in B_n, having the p-adic framing $\underset{\leftarrow}{a_i}$ on the ith strand, see Fig. 5.

Remark 4. By (2.3) for classical framed braids, by Theorem 1 and by (2.27) a p-adic framed braid splits into the 'p-adic framing' part and the 'braiding' part.

The operation in $\mathcal{F}_{\infty,n}$ corresponds geometrically to concatenating in each position of the infinite sequence the two corresponding modular framed braids and collecting the total modular framings to the top (recall Section 2.1, (2.1) and Fig. 3). See Fig. 6 for an illustration.

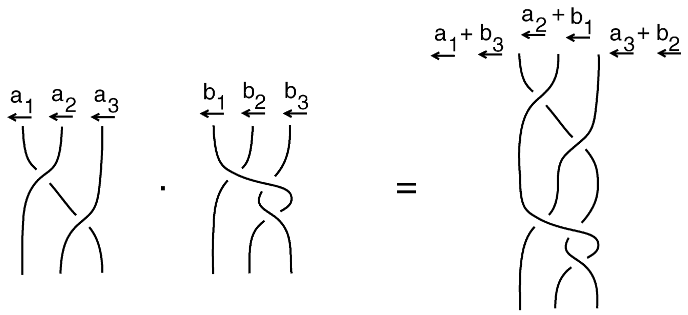

Fig. 7. Multiplication of p-adic framed braids in $\mathbb{Z}_p^n \rtimes B_n$.

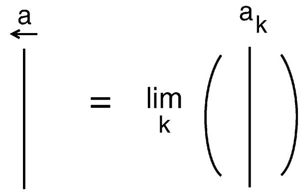

Fig. 8. The approximation of an one-strand p-adic framed braid.

On the other hand, by (2.9), the multiplication between two elements in $\mathbb{Z}_p^n \rtimes B_n$ is defined as follows:

$$\left(\mathbf{t}_1^{\underleftarrow{a_1}} \dots \mathbf{t}_n^{\underleftarrow{a_n}} \cdot \sigma\right) \cdot \left(\mathbf{t}_1^{\underleftarrow{b_1}} \dots \mathbf{t}_n^{\underleftarrow{b_n}} \cdot \tau\right) = \mathbf{t}_1^{\underleftarrow{a_1} + \underleftarrow{b_{\sigma(1)}}} \dots \mathbf{t}_n^{\underleftarrow{a_n} + \underleftarrow{b_{\sigma(n)}}} \cdot \sigma\tau \tag{2.28}$$

where $\underleftarrow{a_i} = (a_{ri})_r$ and $\underleftarrow{b_i} = (b_{ri})_r$. This corresponds geometrically to concatenating the two braids σ and τ with p-adic framings $(\underleftarrow{a_1}, \dots, \underleftarrow{a_n})$ and $(\underleftarrow{b_1}, \dots, \underleftarrow{b_n})$, respectively, and collecting the total p-adic framings to the top. The resulting braid will then have the p-adic framings $(\underleftarrow{a_1} + \underleftarrow{b_{\sigma(1)}}, \dots, \underleftarrow{a_n} + \underleftarrow{b_{\sigma(n)}})$, where $\underleftarrow{a_i} + \underleftarrow{b_{\sigma(i)}} = (a_{ri} + b_{r\sigma(i)})_r$, according to (2.9). See Fig. 7.

As we said, we consider $\mathcal{F}_{\infty,n} \doteq \mathbb{Z}_p^n \rtimes B_n$. So, the expression (2.27) and its corresponding geometric interpretation is what we will have in mind from now on. In this context, if $\underleftarrow{b} \in \mathbb{Z}_p^n \rtimes B_n$, such that all framings of $\underleftarrow{b}$ are constant sequences $(k_1), \dots, (k_n) \in \mathbb{Z}_p^n$ for $(k_i \in \mathbb{Z})$, then $\underleftarrow{b} \in \mathbb{Z}^n \rtimes B_n$ and it is a classical framed braid with framings $k_1, \dots, k_n$. Of course, a classical braid in B_n is meant as a p-adic framed braid with all framings 0.

2.6. Approximations. By Theorem 1, any element $w = (t_{r,1}^{a_{r1}}\, t_{r,2}^{a_{r2}} \dots t_{r,n}^{a_{rn}} \cdot \sigma)_r$ in $\mathcal{F}_{\infty,n}$ can be approximated as follows:

$$w = \lim_k (w_k), \tag{2.29}$$

where w_k is the constant sequence $(t_{r,1}^{a_{k1}}\, t_{r,2}^{a_{k2}} \dots t_{r,n}^{a_{kn}} \cdot \sigma)_r \in \mathcal{F}_{\infty,n}$. The product of two elements is approximated according to (2.29) and (2.23). Further, the fact that $\mathcal{F}_n$ is dense in $\mathbb{Z}_p^n \rtimes B_n \doteq \mathcal{F}_{\infty,n}$, means that any p-adic framed braid can be approximated by a sequence of classical framed braids. More precisely, let $\underleftarrow{\beta} = \mathbf{t}_1^{\underleftarrow{a_1}} \mathbf{t}_2^{\underleftarrow{a_2}} \dots \mathbf{t}_n^{\underleftarrow{a_n}} \cdot \sigma \in \mathbb{Z}_p^n \rtimes B_n$, where $\underleftarrow{a_i} = (a_{ri})_r$. Then, by (2.20), we have:

$$\underleftarrow{\beta} = \lim_k (\beta_k), \tag{2.30}$$

where $\beta_k = \mathbf{t}_1^{a_{k1}} \mathbf{t}_2^{a_{k2}} \dots \mathbf{t}_n^{a_{kn}} \cdot \sigma \in \mathcal{F}_n$, and where $a_{ki} = (a_{ki}, a_{ki}, \dots)$, the constant sequence in $\mathbb{Z} \subset \mathbb{Z}_p$. For example, the p-adic braid $\mathbf{t}^{\underleftarrow{a}}$ for $\underleftarrow{a} = (a_1, a_2, \dots)$, can be approximated as shown in Fig. 8, where $a_k = (a_k, a_k, \dots) \in \mathbb{Z} \subset \mathbb{Z}_p$. See Fig. 9 for a generic example. Of course, the product of two p-adic framed braids is approximated accordingly, by (2.30) and (2.21).

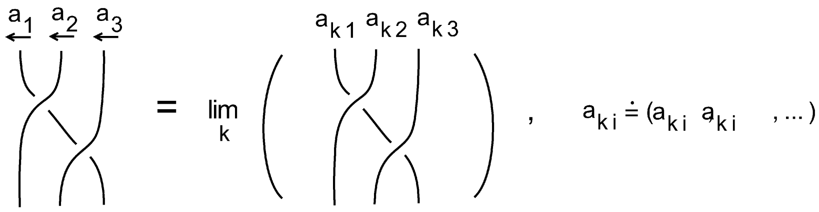

Fig. 9. The approximation of a p-adic framed braid.

3. Quotient algebras from p-adic framed braids

In this section we define the main algebra studied in the paper. This algebra arises as the inverse limit of an inverse system of so-called Yokonuma–Hecke algebras. In the sequel we fix an element u in $\mathbb{C}\backslash\{0,1\}$ and we shall denote $\mathbb{C}[G]$ (or simply $\mathbb{C}G$) the group algebra of a group G.

3.1. Let $H = \langle h \rangle$ be a finite cyclic group of order d. As in (2.11) we define the element h_i in $H^n := H \times \cdots \times H$ (n copies) as the element having h on the ith component and 1 elsewhere. So, for any element $(h^{a_1}, \ldots, h^{a_n}) \in H^n$ we can write

$$(h^{a_1}, \ldots, h^{a_n}) = h_1^{a_1} \ldots h_n^{a_n}.$$

For any i, j with $i \neq j$, we define the subgroups $H_{i,j}$ of H^n as follows:

$$H_{i,j} := \langle h_i h_j^{-1} \rangle. \tag{3.1}$$

Clearly, $H_{i,j}$ is isomorphic to the group H. In $\mathbb{C}[H^n] = \mathbb{C}H^n$ we define the following elements:

$$e_{d,i,j} := \frac{1}{d} \sum_{x \in H_{i,j}} x \in \mathbb{C}H^n$$

or, equivalently:

$$e_{d,i,j} = \frac{1}{d} \sum_{1 \leqslant m \leqslant d} h_i^m h_j^{-m}.$$

Lemma 4. *For any i, j with $i \neq j$ the elements $e_{d,i,j}$ are idempotents.*

Proof. It is enough to observe that $e_{d,i,j}$ is the average on the elements of the group $H_{i,j}$. Indeed,

$$(e_{d,i,j})^2 = \frac{1}{d} \sum_{y \in H_{i,j}} y \frac{1}{d} \sum_{x \in H_{i,j}} x = \frac{1}{d^2} \sum_{y \in H_{i,j}} \sum_{x \in H_{i,j}} yx = \frac{d}{d^2} \sum_{x' \in H_{i,j}} x' = e_{d,i,j}. \qquad \square$$

Remark 5. Notice that $H_{i,j} = H_{j,i}$. In the case $j = i+1$ we denote $H_{i,i+1}$ by H_i and $e_{d,i,i+1}$ by $e_{d,i}$.

3.2. Consider now the modular framed braid group $\mathcal{F}_{d,n}$ (Definition 2). The $\mathbb{C}$-algebra $\mathbb{C}H^n$ is a subalgebra of the group algebra $\mathbb{C}\mathcal{F}_{d,n}$ and the elements $e_{d,i,j}$ are still idempotents in $\mathbb{C}\mathcal{F}_{d,n}$. The main commutation relations among them and the elementary braids σ_i are given in the proposition below.

Proposition 5. *For any $i, j \in \{1, \ldots, n-1\}$ we have*:

(1) $\sigma_i^{\pm 1} e_{d,j} = e_{d,j} \sigma_i^{\pm 1}$, *for all $j \neq i-1, i+1$.*
(2) $\sigma_i^{\pm 1} e_{d,j} = e_{d,i,j} \sigma_i^{\pm 1}$, *for $|i-j| = 1$.*
(3) $e_{d,j} \sigma_i^{\pm 1} = \sigma_i^{\pm 1} e_{d,i,j}$, *for $|i-j| = 1$.*
(4) $e_{d,i}\, h_1^{a_1} \ldots h_n^{a_n} = e_{d,i}\, h_1^{a_1} \ldots h_{i-1}^{a_{i-1}} (h_i^{a_{i+1}} h_{i+1}^{a_i}) h_{i+2}^{a_{i+2}} \ldots h_n^{a_n}$.

Proof. (1) If $j \neq i, i \pm 1$, the claim follows from the fact that σ_i commutes with h_j. Let now $j = i$. We have $\sigma_i e_{d,i} = \sigma_i d^{-1} \sum_s h_i^s h_{i+1}^{-s}$. Note that $\sigma_i h_i^s h_{i+1}^{-s} = h_{i+1}^s \sigma_i h_{i+1}^{-s} = h_{i+1}^s h_i^{-s} \sigma_i$. Then

$$\sigma_i e_{d,i} = \frac{1}{d}\left(\sum_s h_{i+1}^s h_i^{-s}\right)\sigma_i = e_{d,i}\sigma_i.$$

(2) Let $j = i+1$. We have that $\sigma_i h_{i+1}^s h_{i+2}^{-s} = h_i^s \sigma_i h_{i+2}^{-s} = h_i^s f_{i+2}^{-s} \sigma_i$. So, we deduce: $\sigma_i e_{d,i+1} = d^{-1} \sum_s h_i^s h_{i+2}^{-s} \sigma_i$. Claim 3 follows similarly as Claim 2.

(4) Setting $c := h_1^{a_1} \dots h_n^{a_n}$ we have:

$$\begin{aligned} h_i^s h_{i+1}^{-s} c &= h_1^{a_1} \dots h_{i-1}^{a_{i-1}} h_i^{a_i+s} h_{i+1}^{a_{i+1}-s} h_{i+2}^{a_{i+2}} \dots h_n^{a_n} \\ &= h_1^{a_1} \dots h_{i-1}^{a_{i-1}} h_i^{(s+a_i-a_{i+1})+a_{i+1}} h_{i+1}^{-(s+a_i-a_{i+1})+a_i} h_{i+2}^{a_{i+2}} \dots h_n^{a_n} \\ &= h_1^{a_1} \dots h_{i-1}^{a_{i-1}} h_i^{(s+a_i-a_{i+1})} h_i^{a_{i+1}} h_{i+1}^{-(s+a_i-a_{i+1})} h_{i+1}^{a_i} h_{i+2}^{a_{i+2}} \dots h_n^{a_n} \\ &= \left(h_i^{(s+a_i-a_{i+1})} h_{i+1}^{-(s+a_i-a_{i+1})}\right) h_1^{a_1} \dots h_{i-1}^{a_{i-1}} h_i^{a_{i+1}} h_{i+1}^{a_i} h_{i+2}^{a_{i+2}} \dots h_n^{a_n}. \end{aligned}$$

Therefore,

$$\begin{aligned} e_{d,i} c &= \frac{1}{d} \sum_{0 \leqslant s \leqslant d-1} h_i^s h_{i+1}^{-s} c \\ &= \left(\frac{1}{d} \sum_s h_i^{(s+a_i-a_{i+1})} h_{i+1}^{-(s+a_i-a_{i+1})}\right) h_1^{a_1} \dots h_{i-1}^{a_{i-1}} \left(h_i^{a_{i+1}} h_{i+1}^{a_i}\right) h_{i+2}^{a_{i+2}} \dots h_n^{a_n} \\ &= e_{d,i} h_1^{a_1} \dots h_{i-1}^{a_{i-1}} \left(h_i^{a_{i+1}} h_{i+1}^{a_i}\right) h_{i+2}^{a_{i+2}} \dots h_n^{a_n}. \qquad \square \end{aligned}$$

Remark 6. The elements h_i correspond to the elementary framings f_i in the additive notation of Section 2.1 and, for $d = p^r$, to the elements $t_{r,i}$ defined in (2.11).

3.3. The Yokonuma–Hecke (Y-H) algebras were introduced by Yokonuma [13] in the context of Chevalley groups, as generalizations of the Iwahori–Hecke algebras. More precisely, the Iwahori–Hecke algebra associated to a finite Chevalley group G is the centralizer algebra associated to the permutation representation of G with respect to a *Borel subgroup* of G. The Y-H algebra is the centralizer algebra associated to the permutation representation of G with respect to a *unipotent subgroup* of G. So, the Y-H algebra can be also regarded as a particular case of a unipotent algebra. See [10] for the general definition of unipotent algebras.

Definition 4. We define the *Yokonuma–Hecke algebra of type* A, $\mathrm{Y}_{d,n}(u)$, as the quotient of the group algebra of the modular framed braid group $\mathcal{F}_{d,n}$ under the quadratic relations:

$$g_i^2 = 1 + (u-1)\, e_{d,i}\, (1 - g_i) \quad (i = 1, \dots, n-1). \tag{3.2}$$

More precisely, $\mathrm{Y}_{d,n}(u)$ is defined as follows:

$$\mathrm{Y}_{d,n}(u) := \frac{\mathbb{C}\mathcal{F}_{d,n}}{\langle \sigma_i^2 - 1 - (u-1)e_{d,i}(1-\sigma_i),\ i = 1, \dots, n-1\rangle}.$$

Corresponding now $\sigma_i \in \mathbb{C}\mathcal{F}_{d,n}$ to $g_i \in \mathrm{Y}_{d,n}(u)$ and $f_i \in \mathcal{F}_{d,n}$ to $h_i \in \mathrm{Y}_{d,n}(u)$, we obtain from the above and from Proposition 2 a presentation of $\mathrm{Y}_{d,n}(u)$, by setting:

$$h_i = g_i \dots g_1 h_1 g_1^{-1} \dots g_i^{-1}. \tag{3.3}$$

Indeed, we have:

Theorem 2. *The algebra* $\mathrm{Y}_{d,n}(u)$ *can be presented with the generators* $h_1, g_1, \dots, g_{n-1}$ *and the following relations*:

(1) *Braid relations among the* g_i*'s.*

(2) $h_1 g_i = g_i h_1$, *for* $i \geqslant 2$.
(3) $h_1 g_1 h_1 g_1^{-1} = g_1 h_1 g_1^{-1} h_1$.
(4) $h_1^d = 1$.
(5) $g_i(g_{i-1} \dots g_1 h_1 g_1^{-1} \dots g_{i-1}^{-1}) g_i^{-1} = g_i^{-1}(g_{i-1} \dots g_1 h_1 g_1^{-1} \dots g_{i-1}^{-1}) g_i$.
(6) $g_i^2 = 1 + (u-1) e_{d,i}(1 - g_i)$ $(i = 1, \dots, n-1)$.

In this above notation, we may rewrite the elements $e_{d,i} \in \mathrm{Y}_{d,n}(u)$ as:

$$e_{d,i} = \frac{1}{d} \sum_{1 \leqslant m \leqslant d} \left(g_{i-1}^{-1} \dots g_1^{-1} h_1^m g_1 \dots g_{i-1}\right)\left(g_i \dots g_1 h_1^{-m} g_1^{-1} \dots g_i^{-1}\right).$$

Remark 7. The Y-H algebra $\mathrm{Y}_{d,n}(u)$ can be also thought of as a u-deformation of the group algebra $\mathbb{C}[H^n \rtimes S_n]$ in the following sense: The algebra $\mathbb{C}[H^n \rtimes S_n] = \mathbb{C}[H^n \rtimes S_n]$ contains $\mathbb{C}H^n$ as a subalgebra, so the elements $e_{d,i}$ are also in $\mathbb{C}[H^n \rtimes S_n]$. We correspond now the generator $s_i \in \mathbb{C}[H^n \rtimes S_n]$ to the generator $g_i \in \mathrm{Y}_{d,n}(u)$, the generator $h_1 \in \mathbb{C}[H^n \rtimes S_n]$ to the generator $h_1 \in \mathrm{Y}_{d,n}(u)$ and $e_{d,i} \in \mathbb{C}[H^n \rtimes S_n]$ to $e_{d,i} \in \mathrm{Y}_{d,n}(u)$ (we keep the same notation). Then, the canonical presentation of $\mathbb{C}[H^n \rtimes S_n]$ gives rise to a presentation of $\mathrm{Y}_{d,n}(u)$ (the same as in Theorem 2) by imposing the quadratic relations (3.2) instead of the relations $s_i^2 = 1$.

Remark 8. The fact that the element $e_{d,i}$ is an idempotent makes it possible to define in $\mathrm{Y}_{d,n}(u)$ the inverse of g_i. Indeed, multiplying relation (3.2) by g_i gives $g_i^3 = g_i + (u-1)\, e_{d,i} g_i - (u-1)\, e_{d,i}\, g_i^2$. Replacing now g_i^2 by its expression (3.2) and using the fact that $e_{d,i}$ is an idempotent, we obtain that $g_i^3 = g_i - (u^2 - u) e_{d,i} + (u^2 - u) e_{d,i} g_i$. Using again (3.2) we substitute $e_{d,i} g_i$ by $(u-1)^{-1}(1 + (u-1) e_{d,i} - g_i^2)$, so we have $g_i^3 = u + g_i - u g_i^2$. Multiplying the latter by g_i^{-1} we deduce $g_i^{-1} = u^{-1}(g_i^2 + u g_i - 1)$ and, using (3.2) once more, we finally obtain:

$$g_i^{-1} = g_i - (u^{-1} - 1)\, e_{d,i} + (u^{-1} - 1)\, e_{d,i}\, g_i. \tag{3.4}$$

3.4. In this part we give a diagrammatic interpretation of the elements $e_{d,i}$ and of the quadratic relations in $\mathrm{Y}_{d,n}(u)$. The elements $e_{d,i}$ seen as elements of $\mathbb{C}\mathcal{F}_{d,n}$ can be interpreted geometrically as the average of the sum of d identity framed braids with framings as shown in Fig. 10.

Similarly, the quadratic relations $g_i^2 = 1 + (u-1)\, e_{d,i} - (u-1) e_{d,i} g_i$ can be also considered as relations in $\mathbb{C}\mathcal{F}_{d,n}$. In Fig. 11 we illustrate the relation for g_1^2 in $\mathbb{C}\mathcal{F}_{d,3}$. Note that the effect of $e_{d,i}$ on the identity element or on g_i is to produce d copies and frame appropriately the ith and $(i+1)$st strand. Similar is the effect of $e_{d,i}$ on any braid. In Fig. 12 we illustrate the quadratic relation in a compact form. Finally, in Fig. 13 we illustrate the equation for g_1^{-1} in $\mathbb{C}\mathcal{F}_{d,3}$.

Remark 9. Note the resemblance of relation (3.4) to the skein relations used for defining classical quantum link invariants. For $d = 1$ the relation gives rise to the skein relation of the 2-variable Jones polynomial (HOMFLYPT), that arises from the quadratic relation of the Hecke algebra of type A, see [3]. In fact, $\mathrm{Y}_{1,n}(u)$ coincides with the Hecke algebra of type A.

3.5. The p-adic Yokonuma–Hecke algebra. We shall now explain our construction of the p-adic Yokonuma–Hecke algebra $\mathrm{Y}_{\infty,n}(u)$. The $\mathbb{C}$-algebra $\mathrm{Y}_{\infty,n}(u)$ will be defined as the inverse limit of an inverse system of the Y-H algebras

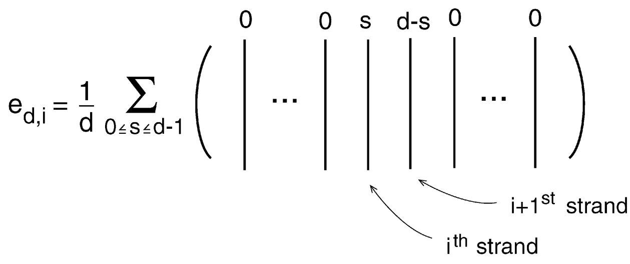

Fig. 10. The elements $e_{d,i}$.

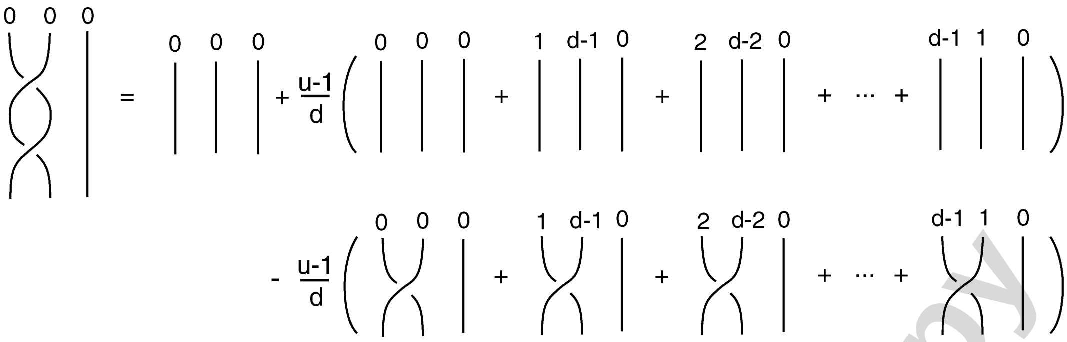

Fig. 11. Geometric interpretation of g_1^2.

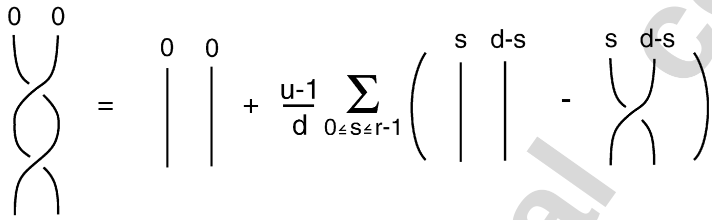

Fig. 12. $g_i^2 = 1 + (u-1)\, e_{d,i}\, (1 - g_i)$.

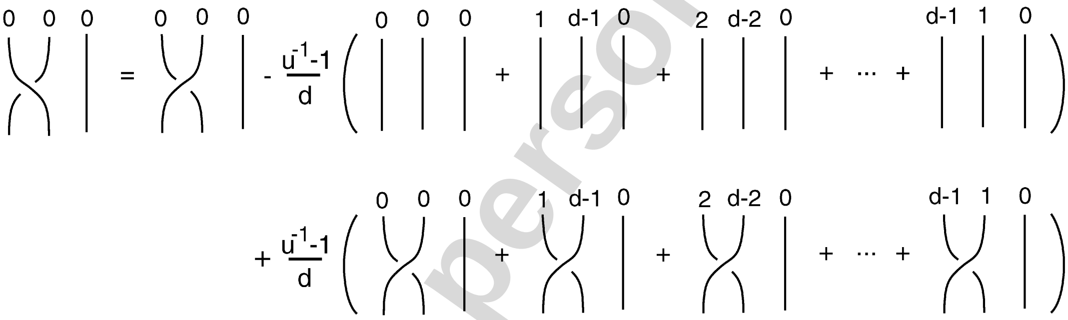

Fig. 13. Geometric interpretation of g_1^{-1}.

$\mathrm{Y}_{p^r,n}(u)$, $r \in \mathbb{N}$, where p is a fixed prime number. On this family of Y-H algebras we consider epimorphisms

$$\varphi_s^r : \mathrm{Y}_{p^r,n}(u) \to \mathrm{Y}_{p^s,n}(u) \quad (r \geqslant s),$$

induced from the group homomorphisms $\pi_s^r \cdot \mathrm{id}$ defined in (2.26). More precisely, extending $\pi_s^r \cdot \mathrm{id}$ linearly, yields a natural algebra epimorphism

$$\phi_s^r : \mathbb{C}\mathcal{F}_{p^r,n} \to \mathbb{C}\mathcal{F}_{p^s,n} \quad (r \geqslant s).$$

It is a routine to check the following lemma.

Lemma 5. $(\mathbb{C}\mathcal{F}_{p^r,n}, \phi_s^r)$ *is an inverse system of rings, indexed by* $\mathbb{N}$.

Note that the natural embedding $\iota_r : \mathcal{F}_{p^r,n} \hookrightarrow \mathbb{C}\mathcal{F}_{p^r,n}$ induces a natural embedding $\varprojlim \iota_r : \mathcal{F}_{\infty,n} \hookrightarrow \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$. So, up to identifications, we have the inclusions:

$$\mathcal{F}_n \subseteq \mathcal{F}_{\infty,n} \subseteq \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}.$$

Recall now that $\mathbf{t}_1 := (\mathbf{t}, \mathbf{1}, \ldots, \mathbf{1})$ and $\sigma_i := (\sigma_i, \sigma_i, \ldots)$ in $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$. Then we have the following result:

Proposition 6. *The set $X = \{\mathbf{t}_1, \sigma_1, \ldots, \sigma_{n-1}\}$ is a set of topological generators of the algebra $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$. In particular, the subalgebra $\mathbb{C}\mathcal{F}_n$ is dense in $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$.*

Proof. By Proposition 1, the set X is a set of generators for the group $\mathcal{F}_n$, hence X spans the algebra $\mathbb{C}\mathcal{F}_n$. Now, the mapping $\sigma_i \mapsto \sigma_i$, $\mathbf{t}_1 \mapsto t_{r,1}$ defines an epimorphism $\eta_r : \mathbb{C}\mathcal{F}_n \to \mathbb{C}\mathcal{F}_{p^r,n}$, for any $r \in \mathbb{N}$. Notice now that η_r is surjective and that we have the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}\mathcal{F}_n \arrow[r] \arrow[d, "\eta_r"'] & \varprojlim \mathbb{C}\mathcal{F}_{p^r,n} \arrow[d, "\xi_r"'] \\
 & \mathbb{C}\mathcal{F}_{p^r,n} \arrow[r, equal] & \mathbb{C}\mathcal{F}_{p^r,n}
\end{tikzcd}$$

where ξ_r is the natural projection. Then the proof follows from Corollary 1. □

Recall now the subgroups $H_{i,j}$ defined in (3.1). With the notations of Section 2 for $H = C_r$ we denote these subgroups by $H_{r,i,j}$ and we have:

$$H_{r,i,j} = \langle t_{r,i} t_{r,i+1}^{-1} \rangle.$$

Hence $e_{p^r,i,j} \in \mathbb{C}C_r^n$. Recalling also that $\mathcal{F}_{p^r,n} = C_r^n \rtimes B_n$, we have the following.

Proposition 7. *For any i, j with $i \neq j$ and for $s \leqslant r$, we have*:

(1) *The homomorphism ϕ_s^r maps $H_{r,i,j}$ onto $H_{s,i,j}$.*
(2) *The kernel of the restriction of ϕ_s^r on $H_{r,i,j}$ has order p^{r-s}.*
(3) $\phi_s^r(e_{p^r,i,j}) = e_{p^s,i,j}$.

Proof. Since $\phi_s^r(t_{r,i} t_{r,j}^{-1}) = t_{s,i} t_{s,j}^{-1}$ Claim 1 follows. Claim 2 is clear by the fundamental theorem of homomorphisms for groups. Finally, Claim 3 follows directly from Claims 1 and 2. □

Defining now in $\mathbb{C}\mathcal{F}_{p^r,n}$ the elements:

$$\varepsilon_{r,i} := \sigma_i^2 - 1 - (u-1) e_{p^r,i}(1-\sigma_i) \in \mathbb{C}\mathcal{F}_{p^r,n} \quad (i = 1, \dots, n-1),$$

and the ideal

$$I_{p^r,n} = \langle \varepsilon_{r,i};\ i = 1, \dots, n-1 \rangle.$$

We have that

$$\mathrm{Y}_{p^r,n}(u) = \frac{\mathbb{C}\mathcal{F}_{p^r,n}}{I_{p^r,n}}.$$

Using (3) of Proposition 7 we obtain the following lemma.

Lemma 6. *For all i and for $s \leqslant r$, we have*: $\phi_s^r(I_{p^r,n}) = I_{p^s,n}$.

According to Lemma 6, we obtain the following commutative diagram of rings:

$$\begin{tikzcd}
\mathbb{C}\mathcal{F}_{p^r,n} \arrow[r, "\phi_s^r"] \arrow[d, "\rho_r"'] & \mathbb{C}\mathcal{F}_{p^s,n} \arrow[d, "\rho_s"'] \\
\mathrm{Y}_{p^r,n}(u) \arrow[r, "\varphi_s^r"] & \mathrm{Y}_{p^s,n}(u)
\end{tikzcd}$$

where ρ_r and ρ_s are the canonical epimorphisms and φ_s^r is defined via ϕ_s^r as:

$$\varphi_s^r(x + I_{p^r,n}) := \phi_s^r(x) + I_{p^s,n}. \tag{3.5}$$

Recall that $\mathrm{Ker}(\rho_r) = I_{p^r,n}$. Thus, the inverse system $(\mathbb{C}\mathcal{F}_{p^r,n}, \phi_s^r)$ induces the inverse system

$$\left(\mathrm{Y}_{p^r,n}(u), \varphi_s^r\right)$$

indexed by $\mathbb{N}$.

Definition 5. The *p-adic Yokonuma–Hecke algebra* $\mathrm{Y}_{\infty,n}(u)$ is defined as the inverse limit of this last inverse system.

$$\mathrm{Y}_{\infty,n}(u) := \varprojlim \mathrm{Y}_{p^r,n}(u).$$

The algebra $\mathrm{Y}_{\infty,n}(u)$ is equipped with canonical epimorphisms:

$$\Xi_r : \mathrm{Y}_{\infty,n}(u) \to \mathrm{Y}_{p^r,n}(u),$$

such that $\varphi_s^r \circ \Xi_r = \Xi_s$.

3.6. We shall now try to understand better the structure of $\mathrm{Y}_{\infty,n}(u)$. By Lemma 6 the restriction of ϕ_s^r to $I_{p^r,n}$ yields the inverse system $(I_{p^r,n}, \phi_s^r)$. Furthermore, for any r we have the following exact sequence:

$$0 \to I_{p^r,n} \xrightarrow{\iota_r} \mathbb{C}\mathcal{F}_{p^r,n} \xrightarrow{\rho_r} \mathrm{Y}_{p^r,n}(u) \to 0.$$

Then, by (1.2), we obtain the exact sequence:

$$0 \to \varprojlim I_{p^r,n} \xrightarrow{\iota} \varprojlim \mathbb{C}\mathcal{F}_{p^r,n} \xrightarrow{\rho} \mathrm{Y}_{\infty,n}(u),$$

where $\iota := \varprojlim \iota_r$ and $\rho := \varprojlim \rho_r$. Hence, and since $\varprojlim I_{p^r,n}$ is an ideal in $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$, we have:

$$\frac{\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}}{\varprojlim I_{p^r,n}} \cong \rho(\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}).$$

At this writing it is not clear whether the map ρ is a surjection or not. Yet, we have the following result.

Proposition 8. *$\rho(\varprojlim \mathbb{C}\mathcal{F}_{p^r,n})$ is dense in $\mathrm{Y}_{\infty,n}(u)$.*

Proof. The proof is again an application of Corollary 1. Indeed, define the map $\theta : \rho(x) \mapsto (\rho_r \circ \xi_r)(x)$, for $x = (x_r) \in \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$. Clearly θ is a surjective map. Also, we have: $(\rho_r \circ \xi_r)(x) = \rho_r(\xi_r(x)) = \rho_r(x_r) = x_r + I_{p^r,n} = \Xi_r((x_r + I_{p^r,n})_{r\in\mathbb{N}}) = \Xi_r(\rho_r(x_r)) = (\Xi_r \circ \varprojlim \rho_r)(x)$. Hence the proposition follows. $\square$

Proposition 8 tells us that, although $\mathrm{Y}_{\infty,n}(u)$ may not arise as a quotient of $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$, yet it does contain a dense quotient. This means that, if we find a set of topological generators for $\rho(\varprojlim \mathbb{C}\mathcal{F}_{p^r,n})$ we will have a set of topological generators for $\mathrm{Y}_{\infty,n}(u)$. In order to do that, we define first certain idempotents $e_{i,j}$ in $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$ that play analogous role to the idempontent $e_{p^r,i,j}$. According to (3) in Proposition 7 we can define the following elements:

$$e_{i,j} := (e_{p,i,j}, e_{p^2,i,j}, \ldots) \in \varprojlim \mathbb{C}C_r^n \subseteq \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}, \tag{3.6}$$

where $i, j \in \{1, \ldots n-1\}$ and $i \neq j$. For $j = i + 1$ we shall denote:

$$e_i := e_{i,i+1}.$$

Notice that $e_{i,j} = e_{j,i}$. According to Remark 6 and Definition 4, $e_{p^r,i,j}$ is also an element in $\mathrm{Y}_{p^r,n}(u)$. So (3.6) defines an element in $\mathrm{Y}_{\infty,n}(u)$ (with same notation) and we have from the diagram below:

$$\begin{tikzcd}
\varprojlim \mathbb{C}C_r^n \arrow[r] \arrow[d] & \varprojlim \mathbb{C}\mathcal{F}_{p^r,n} \arrow[r, "\rho"] \arrow[d, "\xi_r"'] & \mathrm{Y}_{\infty,n}(u) \arrow[d, "\Xi_r"'] & \\
\mathbb{C}C_r^n \arrow[r] & \mathbb{C}\mathcal{F}_{p^r,n} \arrow[r, "\rho_r"] & \mathrm{Y}_{p^r,n}(u) \arrow[r] & 1
\end{tikzcd}$$

$$(\Xi_r \circ \rho)(e_{i,j}) = (\rho_r \circ \xi_r)(e_{i,j}) = e_{p^r,i,j} \quad \text{(for all } r\text{)}.$$

Lemma 7. *For any i, j with $i \neq j$, the elements $e_{i,j} \in \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$ are idempotents.*

Proof. The multiplication in $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$ is defined componentwise, so the proof follows directly from Lemma 4. □

Lemma 8. *In* $\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$, *we have*:

$$\sigma_i^2 = 1 + (u-1)e_i(1-\sigma_i) \bmod (\varprojlim I_{p^r,n}).$$

Proof. We must prove that $\sigma_i^2 - (1 + (u-1)e_i(1-\sigma_i)) \in \varprojlim I_{p^r,n}$. Recall that σ_i is the constant sequence $(\sigma_i, \sigma_i, \ldots)$, hence σ_i^2 is the constant sequence $(\sigma_i^2, \sigma_i^2, \ldots)$. Also, the rth component of the element $1 + (u-1)e_i(1-\sigma_i) \in \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$ is $1 + (u-1)e_{p^r,i}(1-\sigma_i) \in \mathbb{C}\mathcal{F}_{p^r,n}$. Therefore, the element $\sigma_i^2 - (1 + (u-1)e_i(1-\sigma_i))$ is the sequence $(\varepsilon_{1,r}, \varepsilon_{2,r}, \ldots)$, and $\varepsilon_{i,r} \in I_{p^r,n}$. Hence the lemma follows. □

Proposition 9. *Setting* $\boldsymbol{\varepsilon}_i := \sigma_i^2 - 1 - (u-1)e_i + (u-1)e_i\sigma_i \in \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$, *we have*:

$$\varprojlim I_{p^r,n} = \langle \boldsymbol{\varepsilon}_i;\ i = 1, \ldots, n-1 \rangle.$$

Proof. Recall that $\boldsymbol{\varepsilon}_i = (\varepsilon_{r,i})_{r\in\mathbb{N}}$. Now, for any i and for any $x = (x_r)$, $y = (y_r) \in \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$ we have that $x\boldsymbol{\varepsilon}_i y = (x_r \varepsilon_{r,i}\, y_r)$. Furthermore $\phi_s^r(x_r\varepsilon_{r,i}y_r) = \phi_s^r(x_r)\varepsilon_{s,i}\phi_s^r(y_r) \in I_{p^s,n}$. Thus, $x\boldsymbol{\varepsilon}_i y$ belongs to $\varprojlim I_{p^r,n}$ for all i. Hence, the ideal generated by the $\boldsymbol{\varepsilon}_i$'s is contained in $\varprojlim I_{p^r,n}$. Let now $w = (w_r)_{r\in\mathbb{N}} \in \varprojlim I_{p^r,n}$. Then $w_r = \sum_i y_{r,i}\varepsilon_{r,i}z_{r,i}$, where $y_{r,i}, z_{r,i} \in \mathbb{C}\mathcal{F}_{p^r,n}$. Thus, we can write:

$$w = \sum_i (y_{r,i})_r(\varepsilon_{r,i})_r(z_{r,i})_r \in \varprojlim I_{p^r,n}.$$

As $(y_{r,i})_r, (z_{r,i})_r \in \varprojlim \mathbb{C}\mathcal{F}_{p^r,n}$ we obtain $w \in \langle \boldsymbol{\varepsilon}_i;\ i = 1, \ldots, n-1 \rangle$. □

Recall that, according to our inverse system, the element $\sigma_i \in B_n$ corresponds to the constant sequence $(g_i, g_i, \ldots)$ in $\mathrm{Y}_{\infty,n}(u)$. We denote this sequence by g_i. Similarly, the braid $\sigma_i^{-1} \in B_n$ corresponds to the constant sequence $(g_i^{-1}, g_i^{-1}, \ldots)$ in $\mathrm{Y}_{\infty,n}(u)$ and it shall be denoted by g_i^{-1}. Thus, in $\rho(\varprojlim \mathbb{C}\mathcal{F}_{p^r,n}) \subseteq \mathrm{Y}_{\infty,n}(u)$ the following quadratic relations holds:

$$g_i^2 = 1 + (u-1)e_i(1-g_i) \quad (i = 1, \ldots, n-1).$$

We define now $\mathbf{t}_i := \rho(\mathbf{t}_i)$ and $e_i := \rho(e_i)$. Then, from Theorem 2 and Proposition 8, we deduce the following theorem.

Theorem 3. $\{1, \mathbf{t}_1, g_1, \ldots, g_{n-1}\}$ *is a set of topological generators of* $\mathrm{Y}_{\infty,n}(u)$. *Moreover, these elements satisfy the following relations*:

(1) *Braid relations among the* g_i*'s.*
(2) $\mathbf{t}_1 g_i = g_i \mathbf{t}_1$, *for* $i \geqslant 2$.
(3) $\mathbf{t}_1 g_1 \mathbf{t}_1 g_1^{-1} = g_1 \mathbf{t}_1 g_1^{-1} \mathbf{t}_1$.
(4) $g_i g_{i-1} \ldots g_1 \mathbf{t}_1 g_1^{-1} \ldots g_{i-1}^{-1}) g_i^{-1} = g_i^{-1}(g_{i-1} \ldots g_1 \mathbf{t}_1 g_1^{-1} \ldots g_{i-1}^{-1}) g_i$.
(5) $g_i^2 = 1 + (u-1)e_i(1-g_i)$ $(i = 1, \ldots, n-1)$.

Moreover, as in Proposition 5, we can prove analogous commutation relations for e_i. More precisely we have:

Proposition 10. *In* $\mathrm{Y}_{\infty,n}(u)$ *we have*:

(1) $g_i^{\pm 1} e_j = e_j g_i^{\pm 1}$, *for* $j \neq i-1, i+1$.
(2) $g_i^{\pm 1} e_j = e_{ij} g_i^{\pm 1}$, *for* $|i-j| = 1$.
(3) $e_j g_i^{\pm 1} = g_i^{\pm 1} e_{ij}$, *for* $|i-j| = 1$.

Proof. The proofs follow directly from Lemma 7 and Proposition 5. □

Remark 10. It is worth observing that $\mathrm{Y}_{\infty,n}(u)$ can be regarded as a topological deformation of a quotient of the group algebra $\mathbb{C}\mathcal{F}_n$, recall Theorem 3. Roughly, the algebra $\mathrm{Y}_{\infty,n}(u)$ can be described in terms of topological generators, in the sense of Definition 1, and the same relations as the algebra $\mathrm{Y}_{d,n}(u)$ but where the relations $h_i^d = 1$ do not hold. Consequently, $\mathrm{Y}_{\infty,n}(u)$ has a set of topological generators which look like the canonical generators of the framed braid group $\mathcal{F}_n$ (recall Proposition 1), but with the addition of the quadratic relation.

3.7. As already noted in the introduction, the advantage of the classical and the p-adic Y-H algebras is that, by definition of the elements e_i, their quadratic relations involve the framing. Using the well-known Iwahori–Hecke quadratic relations we define the modular and classical *framed Hecke algebras*:

$$\mathrm{H}_{d,n}(q) := \mathbb{C}\mathcal{F}_{d,n}/\langle \sigma_i^2 - (q-1)\sigma_i - q;\ i = 1, \ldots, n-1\rangle$$

and

$$\mathrm{H}_{\infty,n}(q) := \mathbb{C}\mathcal{F}_n/\langle \sigma_i^2 - (q-1)\sigma_i - q;\ i = 1, \ldots, n-1\rangle.$$

The structure of these algebras is simpler than that of the Y-H algebras. Yet, the framed Hecke algebras are related to the cyclotomic and 'generalized' Hecke algebras of type B (see [7] and references therein) in a similar manner that the modular and classical framed braid groups are related to the B-type Artin braid group. So, the Markov traces and the link invariants for the solid torus constructed in [7] by the second author can be adapted here for obtaining invariants of framed links.

In a sequel paper we construct a p-adic linear Markov trace using the linear Markov traces in [4]. More precisely, we can prove the following result.

Theorem 4. *There exists a unique p-adic linear Markov trace defined as*

$$\tau := \varprojlim \tau_r : \mathrm{Y}_{\infty,n+1}(u) \to \varprojlim \mathbb{C}[\mathrm{X}_r],$$

where τ_r is the trace tr_k of [4] *for $k = p^r$ and where $\varprojlim \mathbb{C}[\mathrm{X}_r]$ is constructed via appropriate connecting epimorphisms: $\delta_s^r : \mathbb{C}[\mathrm{X}_r] \to \mathbb{C}[\mathrm{X}_s]$ (see* [5]*).*

Furthermore

$$\begin{aligned}
&\tau(ab) = \tau(ba),\\
&\tau(1) = 1,\\
&\tau(a g_n b) = (z)_r \tau(ab),\\
&\tau(a \mathbf{t}_{n+1}^m b) = (x_m)_r \tau(ab)
\end{aligned}$$

for any $a, b \in \mathrm{Y}_{\infty,n}(u)$ and $m \in \mathbb{Z}$.

Normalizing all these traces according to the Markov equivalence for classical framed and p-adic framed braids, we construct invariants of classical and p-adic oriented framed links.

We hope that this new concept of p-adic framed braids and p-adic framed links that we propose, as well as the use of the Yokonuma–Hecke algebras and our framed and p-adic framed link invariants, will lead to the construction of new 3-manifold invariants.

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