Solid torus links and Hecke algebras of $\mathcal{B}$-type

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Abstract

We show how to construct a HOMFLY-PT type oriented link invariant for links inside a solid torus, following V.F.R. Jones's original approach, i.e via normalizing an Ocneanu-type linear trace function from the Hecke algebras of $\mathcal{B}$-type to the complex numbers. Before defining the invariant we set up the appropriate topological theory, then we find the braid groups related to the solid torus and observe that these can be represented by the Hecke algebras of $\mathcal{B}$-type. Finally we compare our invariant with a skein invariant for certain dichromatic links found by J. Hoste and M. Kidwell.

1 Introduction

The study of braids dates back to the turn of our century.[16],[3] In 1926 it was established by E. Artin[4] that the set of all braids on n strings forms a group, the well-known braid group B_n, with presentation:

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle.$$

As geometric objects, braids are related to oriented links via the *'closure'* operation (i.e. the joining of the corresponding end-points of a braid). Conversely, J.W. Alexander showed[1] in 1923 that there exists an algorithm for turning any oriented link into a braid with isotopic closure[1]. We study links up to *isotopy*, which on the diagram-level can be seen as classes of oriented link diagrams, any two elements of which differ by a finite sequence of the so-called Reidemeister moves.[2],[25]

In 1935 A.A. Markov gave the braid analogue of Reidemeister's theorem, by showing that isotopy classes of oriented links are in 1-1 correspondence with certain equivalence classes in the set of all braids.[22],[28] (A complete proof of Markov's theorem as well as a thorough study on braids can be found in J.S. Birman's book.[6]) On the group level, the braid equivalence in $\bigcup_{n=1}^{\infty} B_n$ can be formulated as follows:

[1]H. Brunn [8] in 1897 proved that any knot has a projection with a single multiple point; from which it follows immediately that we can braid any link diagram.

(i) Conjugation: If $\alpha, \beta \in B_n$ then $\alpha \sim \beta^{-1}\alpha\beta$
(ii) Markov moves: If $\alpha \in B_n$ then $\alpha \sim \alpha\sigma_n^{+1} \in B_{n+1}$ and $\alpha \sim \alpha\sigma_n^{-1} \in B_{n+1}$.

Hence, one can study links up to isotopy by studying braids up to Markov equivalence. This fact was first used for constructing link invariants by V.F.R. Jones[17],[18] in 1984. The way V. Jones reconstructed after Ocneanu the 2-variable generalization of his polynomial[2] (or HOMFLY-PT polynomial)[12],[24],[21] is based on the following ideas:[19]

We consider the Hecke algebra of $\mathcal{A}_n$-type, $\mathcal{H}_n(q), q \in \mathbb{C}$, which has the presentation

$$\langle g_1, ..., g_{n-1} \mid g_i g_j = g_j g_i \text{ for } |i-j| > 1,\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\ {g_i}^2 = (q-1)g_i + q\rangle.$$

We then observe that there is a natural epimorphism of the group algebra $\mathbb{C}B_n$ onto $\mathcal{H}_n(q)$: $\sigma_i \mapsto g_i$. There exists now a linear trace function (Ocneanu's trace) $\mathrm{tr} : \bigcup_{n=1}^{\infty} \mathcal{H}_n(q) \longrightarrow \mathbb{C}$, which is unique up to the following rules:

1) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for $a, b \in \mathcal{H}_n(q)$
2) $\mathrm{tr}(1) = 1$ for every $\mathcal{H}_n(q)$
3) $\mathrm{tr}(ag_n) = z\,\mathrm{tr}(a)$ for $a \in \mathcal{H}_n(q)$, $g_n \in \mathcal{H}_{n+1}(q)$ and $z \in \mathbb{C}$.

Using the epimorphism of $\mathbb{C}B_n$ onto $\mathcal{H}_n(q)$ and Ocneanu's trace we can assign to every braid a complex (Laurent) polynomial. We finally observe that rules 1) and 3) of the trace function resemble *(i)* and *(ii)* of the braid equivalence in S^3. Then, according to Markov's theorem, in order to obtain a link invariant this trace has to be normalized properly, so that the braids α , $\alpha\sigma_n$ and $\alpha{\sigma_n}^{-1}$ would be assigned the same label, namely a 2-variable (with variables q and z) Laurent-polynomial.

All the above take place in S^3. In [19] V. Jones asks whether other Hecke algebras, corresponding to general Artin groups, can be used in the same manner as the ones of $\mathcal{A}$-type. Here we show that the braid groups related to the solid torus can be represented by the Hecke algebras of $\mathcal{B}$-type; then, by extending the above topological set-up and by using analogous algebraic machinery and ideas, we obtain a HOMFLY-PT type isotopy invariant for links inside a solid torus. The paper is not self-contained, in the sense that it makes use of theorems, the proofs of which we only sketch. The full proofs of these theorems appear in [20].

2 In search of a Markov's theorem

It is known that a solid torus M may be seen as the complement in S^3 of another solid torus $\hat{I}$, say; i.e. $M = S^3 \backslash \widehat{I}$. So links in M may be seen as *mixed links* in

[2]V. Jones used initially quotients of the Hecke algebras – namely Temperley-Lieb algebras – to construct the original 1-variable Jones polynomial.

Figure 1:

S^3 containing the complementary solid torus. We avoid possible ambiguities by *fixing a projection* of $\hat{I}$ pointwise and also an orientation. Therefore, any link L in M is represented by a mixed link $\hat{I} \bigcup L$ in S^3, consisting of a *standard* link in S^3 linking with the *fixed* complementary *solid torus part* $\hat{I}$ (for an example see picture 1 below). A *mixed link diagram* is a projection $\hat{I} \bigcup \widetilde{L}$ of $\hat{I} \bigcup L$ on the plane of the projection of $\hat{I}$.

Next, we want to see how *isotopy* between links in M is reflected in S^3 : the description of M gives rise to the two additional local moves between mixed link diagrams shown in picture 2, where $\hat{I}$ also participates. Hence, **Reidemeister's theorem** is modified as follows:

'Two links in M are isotopic if and only if any two corresponding mixed link diagrams in S^3 differ by planar isotopy and a finite sequence of the above moves together with the three Reidemeister moves for the standard part of the mixed link.'

2.1 Braiding

A mixed link is lying in S^3, so one could think of applying to it (any proof of)[23],[29],[27],[26],[20] Alexander's theorem in order to obtain a braid. The problem one would run into, then, is that, when braiding a mixed link diagram $\hat{I} \bigcup \widetilde{L}$ we may end up with more than one braid strings for the solid torus component $\hat{I}$ and so, after closing, $\hat{I}$ does not necessarily remain fixed. So, we need a braiding that guarantees that the braid we obtain from a mixed link is a *mixed braid*. A mixed braid is defined as a braid in S^3, $I \bigcup B$, with closure a mixed link and, in particular, the closure of the first string, I, is the fixed component $\hat{I}$ (see picture 3 below for an example). We number the strings of a mixed braid by numbering only the *standard* part of it. Indeed:

Theorem 1 (Alexander's theorem for mixed links). *Any oriented mixed link is isotopic to the closure of some mixed braid.*

Proof. Isotope the mixed link to one as in picture 4, by combing and sliding around the standard part of the link. Then apply to the right-hand side of the line l in picture 4 any braiding process that does not affect the fixed downward oriented part of the component $\hat{I}$ (most braiding processes have this property). Finally eliminate the upward oriented part of $\hat{I}$ at the left-hand side of l, by cutting it at some point and by pulling the two ends, so as to obtain a mixed braid.

The above braiding is good for providing a proof of Theorem 1, but it is not algorithmic enough to provide an easy proof of a Markov's theorem for mixed

Figure 2:

Figure 3:

braids. For this reason we shall describe a more sophisticated braiding process; this is a modification of a braiding algorithm for oriented links in S^3, the idea of which is the following (for a detailed exposition see [20]):

We start with an oriented link diagram and we mark with points the local maxima and minima. This set of points separates naturally the diagram into horizontal or downward arcs on one hand, and into 'opposite' arcs (i.e. arcs that go upwards) on the other hand. We want to eliminate the opposite arcs, as they go the 'wrong' way for a braid. We cut every opposite arc into smaller pieces that we call *little opposite arcs*, so that each little opposite arc contains crossings of only one type – if any – and also, any pair of little opposite arcs satisfies a certain overlapping condition. Finally, we eliminate a little opposite arc by cutting it at some point and by pulling and stretching its two ends both *over* or *under* the rest of the original diagram, according to whether the arc lies *over* or *under* other arcs of the diagram. Notice that the downward arcs of the original diagram remain unaltered throughout the braiding. In the example below we have two opposite arcs but three little opposite arcs.

Take now a mixed link diagram $\hat{I} \bigcup \widetilde{L}$. If we apply to it the above braiding algorithm, we will probably end up with more than one braid strings for the solid torus component $\hat{I}$, as there might be parts of $\widetilde{L}$ interfering with the upwards oriented part of $\hat{I}$. We overcome the problem by modifying the above algorithm as follows:

Step 1 We draw the vertical line l that passes through the maximum and minimum of $\hat{I}$ (see picture 6). By general position it does not pass through any crossings of $\widetilde{L}$.

Step 2 We apply our algorithm to the part of the mixed link that lies to the left of l, considering the points of $\widetilde{L}$ that intersect l as end-points of little opposite arcs. This will leave I unaltered, since this part of it goes already downwards. Then we close this *braided* part of $\widetilde{L}$ by applying *closure* on its left-hand side, and we enclose the 'closure' strings of $\widetilde{L}$ in a tube T_1 (see picture 7 above).

Figure 4:

Figure 5:

Step 3 Now we apply our algorithm on the right-hand side of l considering the orientation to be *reversed*. This will leave $\hat{I}$ unchanged. Then we also close this *braided* part of $\widetilde{L}$ by applying *closure* on its right-hand side, and we enclose the new 'closure' strings of $\widetilde{L}$ in a tube T_2 (see picture 7).
Step 4 By rotating around the back of the diagram, we bring T_1 to the very right of the diagram and then T_2 to the very left of the diagram, so that the resulting diagram goes around a central point P on l (see picture 8).

Step 5 If we are left with local *maxima/minima* in the *lower/upper* part of the diagram, these will have to be lying on l, as it follows from the braiding process (see picture 8 above). To complete the modified algorithm we eliminate these as follows:
We number with integers the *maxima/minima* according to their position with respect to P (which we label with 0), and we isolate them in neighbourhoods that contain no other parts of the diagram. Then we stretch the arcs one by one in order (starting from the ones with least absolute value) *over/under* the rest of the diagram and *above/below* P, so that the *maxima/minima* lie on l in inverse order of closeness to P (see picture 9).
Step 6 We *open* the braided diagram by cutting through a half-line starting from P, after possibly isotoping part of $\hat{I}$ so that it appears in the first position of the braid. Finally we isotope in $D^2 \times I$. □

Remark 1. If there is no part of $\widetilde{L}$ interfering with the upwards oriented part of $\hat{I}$, we only need to apply Steps 2, 4 and 6.

2.2 Markov's theorem for mixed braids

Using the braiding described above for links in S^3, we can give an elementary proof of the classical Markov's theorem[20], in which the downward arcs of the original diagram *do not participate*.[3] So we have the following *even stronger* result:

Theorem 2 (Relative version of Markov's theorem). *Two links containing the same braided part are isotopic if and only if any two corresponding braids, both containing the same braided part, differ by conjugation and Markov moves that do not affect the already braided part.*

Corollary 1. *In particular, all mixed links related to a solid torus contain the same braided part $\hat{I}$ and, by the modified braiding algorithm described above, all corresponding braids contain the same braided part I. Therefore, the (geometric) analogue of Markov's theorem for links inside a solid torus is a special case of Theorem 2.*

[3] Other proofs of the classical Markov's theorem can be found in [5], [23], [26].

Figure 6:

Next, we want to investigate the existence of algebraic structures in the sets of mixed braids, in order to formulate Markov's theorem algebraically (i.e. looking at mixed braids as algebraic rather than geometrical objects). Indeed, we observe that the set of all mixed braids on n standard strings, $B_{1,n}$, forms a group with concatenation as operation and generators the usual braid generators $\sigma_1, \ldots, \sigma_{n-1}$ for the standard strings and the mixed generator T pictured below: As a group, $B_{1,n}$ is the semidirect product of the usual braid group B_n and the free pure braid normal subgroup, $P_{1,n}$, generated by $T, T_1, \ldots, T_{n-1}$, where $T_i = \sigma_i \ldots \sigma_1 T {\sigma_1}^{-1} \ldots {\sigma_i}^{-1}$ is shown in picture 11. (Note that $P_{1,n}$ is *not* the corresponding to $B_{1,n}$ pure braid group, since it does not contain elements with pure braiding among the n standard strings.) Therefore we can put together a presentation for $B_{1,n}$ and obtain the following:[4]

$$\left\langle \begin{array}{c} \sigma_1, \ldots, \sigma_{n-1} \\ T, T_1, \ldots, T_{n-1} \end{array} \;\middle|\; \begin{array}{ll} \left.\begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1 \end{array}\right\} & \textit{braid relations} \\ \left.\begin{array}{ll} (1) & {\sigma_i}^{-1}T_{\lambda-1}\sigma_i = T_{\lambda-1} \text{ if } \lambda > i+1 \text{ or } \lambda < i \\ (2) & {\sigma_i}^{-1}T_{i-1}\sigma_i = T_{i-1}T_i{T_{i-1}}^{-1}, \; i = 1, \ldots, n-1 \\ (3) & {\sigma_i}^{-1}T_i\sigma_i = T_{i-1} \text{ for } i = 1, \ldots, n-1 \end{array}\right\} & \textit{`mixed' relations} \end{array} \right\rangle$$

Doing Tietze transformations we eliminate the T_i's and we obtain:

$$B_{1,n} = \left\langle T, \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } i \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j|>1 \\ T\sigma_i = \sigma_i T & \text{for } i > 1 \\ T\sigma_1 T\sigma_1 = \sigma_1 T \sigma_1 T & \end{array} \right\rangle$$

From the above and the geometric analogue of Markov's theorem (Corollary 1), we have the following:

Theorem 3 (Markov's theorem for solid torus links). *Let $M = S^3 \backslash \widehat{I}$ be a solid torus with $\widehat{I}$ also a solid torus; let L_1, L_2 be two oriented links in M and $I \bigcup B_1$, $I \bigcup B_2$ be mixed braids in $\bigcup_{n=1}^{\infty} B_{1,n}$ corresponding to L_1, L_2. then L_1 is isotopic to L_2 in M if and only if $I \bigcup B_1$ is equivalent to $I \bigcup B_2$ in $\bigcup_{n=1}^{\infty} B_{1,n}$, under equivalence generated by the augmented braid relations together with the following two moves:*

(i) Conjugation: If $\alpha, \beta \in B_{1,n}$ *then* $\alpha \sim \beta^{-1}\alpha\beta$
(ii) Markov moves: If $\alpha \in B_{1,n}$ *then* $\alpha \sim \alpha{\sigma_n}^{\pm 1} \in B_{1,n+1}$.

[4] This presentation appears also in [14] and in [9], where $B_{1,n}$ is seen as the subgroup of the usual braid group B_{n+1}, the elements of which fix the first string, and it is used to aid in finding a presentation for the usual pure braid group.

Figure 7:

Note 1. There is a natural inclusion of $B_{1,n}$ into $B_{1,n+1}$ adequately described by picture 12, and therefore, the direct limit $\bigcup_{n=1}^{\infty} B_{1,n}$ is well-defined.

3 $B_{1,n}$ and the Hecke algebras of $\mathcal{B}$-type

3.1 Algebraic definitions et cetera

A group G with a presentation

$$\langle w_1, ..., w_n \mid (w_i w_j)^{m_{ij}} = 1,\ m_{ii} = 1 \text{ for } i = 1, \ldots, n\rangle$$

is called a *Coxeter group*. All finite Coxeter groups have been classified. For example, the Coxeter group of $\mathcal{A}_n$-type is the symmetric group S_n, and it has the following presentation:

$$S_n = \langle s_1, ..., s_{n-1} \mid {s_i}^2 = 1,\ \ (s_i s_j)^2 = 1 \text{ for } |i-j| > 1,\ (s_i s_{i+1})^3 = 1\rangle,$$

where s_i corresponds to the transposition $(i, i+1)$.
If G is a Coxeter group with a presentation as above, then the corresponding *Artin group* B is given by

$$\langle \tau_1, \ldots, \tau_n \mid \tau_i \tau_j \tau_i \ldots = \tau_j \tau_i \tau_j \ldots, \textit{where the number of factors in either side is}\, m_{ij}\rangle.$$

For example, the usual braid group, B_n, is the Artin group of S_n.

Every Coxeter group is related to a *Hecke algebra* $\mathcal{H}$ (usually considered over the field $\mathbb{C}$), a presentation of which is obtained from the presentation of the corresponding Artin group given above, by adding the quadratic relation ${\tau_i}^2 = (q_i - 1) \cdot \tau_i + q_i \cdot 1$, where $q_i \neq 0 \in \mathbb{C}$ is seen as a fixed variable.
Note If $q_i = 1$ for all i or not a root of unity and we choose as field $\mathbb{C}$, then the Hecke algebra is *semisimple* and *isomorphic* to the group algebra $\mathbb{C}G$, where G is any Coxeter group (J. Tits, [7]). (For the dimension, existence and basic properties of arbitrary Hecke algebras corresponding to finite Coxeter groups see [10].)

3.2 The Coxeter group of $\mathcal{B}_n$-type

We shall start by giving an intuitive 'pairs of shoes'-description of W_n, the Coxeter group of $\mathcal{B}_n$-type, given by G.D. James:
We think of n numbered shelves and n numbered and ordered pairs of shoes;

we put one pair on each shelf, not necessarily in the right order pairwise, and not necessarily on the right shelves. (The description can be made rigorous if we consider an ordered n-tuple of ordered pairs of objects.) We want to place the pairs of shoes correctly, but we are only allowed to swap over the shoes of the pair that is placed on the first shelf, and also to swap pairs that lie on consecutive shelves. If, for example, we want to arrange the word $\begin{pmatrix} 2 & \overline{2} \\ \overline{4} & 4 \\ 3 & \overline{3} \\ \overline{1} & 1 \end{pmatrix}$, where we use the notation $i\overline{i}$ for the pair of shoes with number i, and $\overline{i}$ is the left shoe, then one possible procedure is the following:

$$\begin{pmatrix} 2 & \overline{2} \\ \overline{4} & 4 \\ 3 & \overline{3} \\ \overline{1} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{4} & 4 \\ 2 & \overline{2} \\ 3 & \overline{3} \\ \overline{1} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & \overline{4} \\ 2 & \overline{2} \\ 3 & \overline{3} \\ \overline{1} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & \overline{4} \\ 2 & \overline{2} \\ \overline{1} & 1 \\ 3 & \overline{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & \overline{4} \\ \overline{1} & 1 \\ 2 & \overline{2} \\ 3 & \overline{3} \end{pmatrix} \longrightarrow$$
$$\begin{pmatrix} \overline{1} & 1 \\ 4 & \overline{4} \\ 2 & \overline{2} \\ 3 & \overline{3} \end{pmatrix} \longrightarrow$$
$$\begin{pmatrix} 1 & \overline{1} \\ 4 & \overline{4} \\ 2 & \overline{2} \\ 3 & \overline{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \overline{1} \\ 2 & \overline{2} \\ 4 & \overline{4} \\ 3 & \overline{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \overline{1} \\ 2 & \overline{2} \\ 3 & \overline{3} \\ 4 & \overline{4} \end{pmatrix}.$$

We can see that we have been making use of the symmetric group S_n to swap pairs on consecutive shelves, and of a cyclic group with two elements, C_2, for swapping shoes on the first shelf.

If now v_i means 'swap the shoes on the ith shelf', then, for $v_1 = t$ we have:

$$v_2 = s_1 t s_1,\ v_3 = s_2 s_1 t s_1 s_2 = s_2 v_2 s_2,\ \cdots,\ v_n = s_{n-1} \dots s_1 t s_1 \dots s_{n-1}\ .$$

The elements $v_1, \dots, v_n$ generate the group $2^n \cong C_2 \times \dots \times C_2$ (n copies), which is a normal subgroup of W_n. As a set, W_n is the cartesian product $2^n \times S_n$ and therefore, $|W_n| = 2^n \cdot n!$. As a group, W_n is the semidirect product of 2^n and S_n. So we have the following presentation:

$$W_n = \left\langle \begin{array}{c|l} s_1, \dots, s_{n-1},\ s_i \in S_n & {s_i}^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i \\ v_1, \dots, v_n,\ v_j \in 2^n & s_i s_j = s_j s_i \ \text{ for } |i-j| > 1 \\ & {v_i}^2 = 1,\ v_i v_j = v_j v_i \text{ for all } i,j \\ & {s_j}^{-1} v_i s_j = s_j v_i s_j = {v_i}^{s_j} = v_{s_j(i)} \\ & \text{where } s_j(i) \text{ is the image of } i \text{ under} \\ & \text{the transposition } s_j = (j, j+1) \end{array} \right\rangle,$$

Figure 8:

which, after Tietze transformations, yields the following reduced presentation:

$$W_n = \left\langle \; t, s_1, s_2, \ldots, s_{n-1} \;\middle|\; \begin{array}{l} (ts_1)^4 = 1 \quad \text{or} \quad ts_1ts_1 = s_1ts_1t \\ (ts_i)^2 = 1 \quad \text{or} \quad ts_i = s_it \;\; \text{for} \;\; i > 1 \\ t^2 = {s_i}^2 = 1 \;\; \text{for} \;\; i = 1, \ldots, n-1 \\ (s_is_{i+1})^3 = 1 \quad \text{or} \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad \text{for all } i \\ (s_is_j)^2 = 1 \quad \text{or} \quad s_is_j = s_js_i \;\; \text{for} \;\; |i-j| > 1 \end{array} \right\rangle .$$

The above presentation can be coded in the following *Dynkin* diagram:

where the single bonds of strength mean relations of degree 3 and the double bond a relation of degree 4. Also, if two generators are not connected by a bond, the relation between them is of degree 2, i.e. they commute.

Theorem 4. *$B_{1,n}$ is the Artin group of W_n.*

Proof. According to the definition of an Artin group, it follows by comparing the presentation of $B_{1,n}$ with the presentation of W_n given above. □

As a consequence, the following is a presentation for the Hecke algebra of $\mathcal{B}_n$-type, $\mathcal{H}_n(q, Q)$, $q, Q \in \mathbb{C}$, which corresponds to W_n:

$$\mathcal{H}_n(q,Q) = \left\langle \; t, g_1, \ldots, g_{n-1} \;\middle|\; \begin{array}{l} tg_1tg_1 = g_1tg_1t \\ tg_i = g_it \;\; \text{for} \;\; i > 1 \\ t^2 = (Q-1)t + Q \\ {g_i}^2 = (q-1)g_i + q \;\; \text{for} \;\; i = 1, \ldots, n-1 \\ g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1} \quad \text{for all } i \\ g_ig_j = g_jg_i \;\; \text{for} \;\; |i-j| > 1 \end{array} \right\rangle .$$

Remark 2. The Dynkin diagram of W_n indicates that there is a natural inclusion of W_n into W_{n+1} (by adding an extra node at the end), and this extends to a natural inclusion of $\mathcal{H}_n(q,Q)$ into $\mathcal{H}_{n+1}(q,Q)$. Thus the direct limit $\bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q)$ is well-defined, and therefore we may observe that there is a natural epimorphism, $\pi : \bigcup_{n=1}^{\infty} \mathbb{C}B_{1,n} \mapsto \bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q)$, such that $\pi(T) = t,\ \pi(\sigma_i) = g_i$.

4 A trace function

The above now suggest that we look for a trace function from $\bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q)$ to $\mathbb{C}$ analogous to Ocneanu's trace, so as to attach to each braid in $B_{1,n}$ a complex polynomial. Indeed, we have the following theorem, which is joint work of the author with M. Geck.

Theorem 5. *Given z and s in $\mathbb{C}$, there exists a unique linear function*

$$tr : \mathcal{H} := \bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q) \longrightarrow \mathbb{C}$$

such that the following hold:

$$\begin{array}{ll} 1) & tr(ab) = tr(ba) \quad , \quad a,b \in \mathcal{H} \\ 2) & tr(1) = 1 \quad \textit{for all } \mathcal{H}_n(q,Q) \\ 3) & tr(ag_n) = z\, tr(a) \quad , \quad a \in \mathcal{H}_n(q,Q) \\ 4) & tr(at_n) = s\, tr(a) \quad , \quad a \in \mathcal{H}_n(q,Q) \\ & \quad \textit{where } t_n = g_n \ldots g_1 t {g_1}^{-1} \ldots {g_n}^{-1}. \end{array}$$

Proof. (Sketch of the proof) The proof[20] rests squarely on the proof of Ocneanu's trace function as given in [19]. The proof of existence relies on inductive arguments using the following information on the structure of $\mathcal{H}_n(q,Q)$:[11]

The group W_n is a subgroup of W_{n+1} of index $2(n+1)$. In [11], R. Dipper and G.D. James show that a complete set of right coset representatives of W_n in W_{n+1} is given by

$$J_{n+1} := \{1,\ s_n \ldots s_i \mid i = 1, \ldots, n\} \bigcup \{s_n \ldots s_1 s_0 s_1 \ldots s_i \mid i = 0, 1, \ldots, n \ \text{ for } \ s_0 = t\}$$

I.e. every element $w \in W_{n+1}$ can be written uniquely in the form

$$w \in W_n \ \text{ or } \ w = u \cdot x \ , \ \text{ where } \ u \in W_n \ \text{ and } \ x \in J_{n+1}\ .$$

Equivalently, every element $w \in W_{n+1}$ has one of the following forms:

$$\begin{array}{ll} (a) & w \in W_n \\ (b) & w = u s_n v\ , \quad u,v \in W_n \\ (c) & w = u s_n \ldots s_1 t s_1 \ldots s_n\ , \quad u \in W_n \end{array}$$

We also have analogous statements in the Hecke algebra $\mathcal{H}_{n+1}(q,Q)$. I.e. every element in $\mathcal{H}_{n+1}(q,Q)$ can be written as a linear combination of elements w, each of precisely one of the following forms[5]:

$$\begin{array}{ll} (a) & w \in \mathcal{H}_n(q,Q) \\ (b) & w = u g_n v\ , \quad u,v \in \mathcal{H}_n(q,Q) \\ (c) & w = u t_n'\ , \quad u \in \mathcal{H}_n(q,Q)\ ,\ t_n' = g_n \ldots g_1 t g_1 \ldots g_n\ . \end{array}$$

This canonical basis is equivalent to the following:

$$\begin{array}{ll} (a) & w \in \mathcal{H}_n(q,Q) \\ (b) & w = u g_n v\ , \quad u,v \in \mathcal{H}_n(q,Q) \\ (c) & w = u t_n\ , \quad u \in \mathcal{H}_n(q,Q)\ ,\ t_n = g_n \ldots g_1 t {g_1}^{-1} \ldots {g_n}^{-1}\ . \end{array}$$

[5] An algorithm for writing an arbitrary element as a linear combination of elements in the canonical basis is described in [13].

Conclusion There is a canonical basis for $\mathcal{H}_{n+1}(q,Q)$, such that the higher index elements g_n and t_n appear at most once in any word.

We shall also need the map $c_n : \mathcal{H}_n \oplus \mathcal{H}_n \oplus \mathcal{H}_n \otimes_{\mathcal{H}_{n-1}} \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}$ defined by

$c_n(a \oplus b \oplus c \otimes d) = a + bt_n + cg_nd$. c_n is an isomorphism of $(\mathcal{H}_n, \mathcal{H}_n)$-bimodules.

We can now define inductively a trace, tr, on $\mathcal{H} = \bigcup_{n\geq 1} \mathcal{H}_n(q,Q)$ as follows: Suppose tr is defined on $\mathcal{H}_n(q,Q)$ and let $x \in \mathcal{H}_{n+1}(q,Q)$ be arbitrary; then there exist $a,b,c,d \in \mathcal{H}_n(q,Q)$ such that $x = c_n(a \oplus b \oplus c \otimes d)$.

$$\text{Define } \; tr(x) := tr(a) + s\,tr(b) + z\,tr(cd).$$

This trace satisfies the rules 2), 3) and 4) of the statement of Theorem 5. It remains then to prove that property 1) is also satisfied for all $a, x \in \mathcal{H}$, and we do this case by case.

Finally note that, having proved existence, the uniqueness of the trace will follow immediately since, given $w \in \mathcal{H}_n(q,Q)$, it is clear that $tr(w)$ can be computed inductively from the above using rules 1), 2), 3), 4) and linearity. □

Remark 3. If a word $a \in \mathcal{H}_n(q,Q)$ does not contain any t_i's, then for calculating $tr(a)$ we only need to use rules 1), 2) and 3) of Theorem 5; so $tr(a)$ is the same as Ocneanu's trace applied on a.

Below we give an example of calculating the trace of a word in $\mathcal{H}_4(q,Q)$, in which we also demonstrate how to bring the word to the canonical form. To facilitate the reader, we underline the words on which we apply the algebra relations or the trace rules, and we indicate which rule we apply each time. Also, we make use of the relations:

$$g_i{}^{-1} = \frac{1}{q}\, g_i + \frac{1-q}{q} \cdot 1, \; g_{i+1}t_ig_{i+1}t_i = t_ig_{i+1}t_ig_{i+1},$$

which follow easily from the defining relations of $\mathcal{H}_n(q,Q)$.

Example $tr(g_2t_1g_2t_1g_2g_1\underline{g_3}^{-1}t_2) =$

$$\tfrac{1}{q}\, tr(g_2t_1g_2t_1g_2g_1\underline{g_3}t_2) + \tfrac{1-q}{q}\, tr(g_2t_1g_2t_1g_2g_1t_2) \overset{3)}{=}$$

$$\tfrac{z+1-q}{q}\, tr(\underline{g_2t_1g_2t_1}g_2g_1t_2) = \tfrac{z+1-q}{q}\, tr(t_1g_2t_1g_2{}^2\underline{g_1t_2}) =$$

$$\tfrac{z+1-q}{q}\, tr(t_1g_2t_1g_2{}^2\underline{t_2}g_1) = \tfrac{z+1-q}{q}\, tr(t_1g_2t_1\underline{g_2}^3t_1g_2{}^{-1}g_1) =$$

$$\tfrac{z+1-q}{q}(q^2-q+1)\, tr(t_1g_2\underline{t_1g_2t_1g_2}^{-1}g_1) + \tfrac{z+1-q}{q}q(q-1)\, tr(t_1g_2t_1t_1g_2{}^{-1}g_1) =$$

$\frac{(z+1-q)(q^2-q+1)}{q}\, tr(t_1 g_2 {g_2}^{-1} t_1 g_2 t_1 g_1) + (z+1-q)(q-1)\, tr(t_1 g_2 \underline{t_1}^2 {g_2}^{-1} g_1) =$

$\frac{(z+1-q)(q^2-q+1)}{q}\, tr(\underline{t_1}^2 g_2 t_1 g_1) +$

$(z+1-q)(q-1)Q\, tr(t_1 \underline{g_2 t_1 g_2}^{-1} g_1) + (z+1-q)(q-1)(Q-1)\, tr(t_1 g_2 {g_2}^{-1} g_1) =$

$\frac{(z+1-q)(q^2-q+1)}{q} Q\, tr(t_1 \underline{g_2} t_1 g_1) + \frac{(z+1-q)(q^2-q+1)}{q} (Q-1)\, tr(\underline{g_2} t_1 g_1) +$

$(z+1-q)(q-1)Q\, tr(t_1 \underline{t_2} g_1) + (z+1-q)(q-1)(Q-1)\, tr(t_1 g_1) \overset{3),4)}{=}$

$\frac{(z+1-q)(q^2-q+1)Q}{q} z\, tr(t_1 t_1 g_1) + \frac{(z+1-q)(q^2-q+1)(Q-1)}{q} z\, tr(t_1 g_1) +$

$(z+1-q)(q-1)Qs\, tr(t_1 g_1) + (z+1-q)(q-1)(Q-1)\, tr(t_1 g_1) =$

$\frac{(z+1-q)(q^2-q+1)Qz}{q}\, tr(\underline{t_1}^2 g_1) +$

$\Big[\frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + (z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\Big]\, tr(t_1 g_1) =$

$\frac{(z+1-q)(q^2-q+1)Qz}{q} Q\, tr(t_1 g_1) + \frac{(z+1-q)(q^2-q+1)Qz}{q} (Q-1)\, tr(\underline{g_1}) +$

$\Big[\frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} + (z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\Big]\, tr(t_1 g_1) \overset{3)}{=}$

$\frac{(z+1-q)(q^2-q+1)Qz(Q-1)}{q} z + \Big[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} +$

$(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\Big]\, tr(\underline{t_1} g_1) =$

$\frac{(z+1-q)(q^2-q+1)Q(Q-1)z^2}{q} + \Big[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} +$

$(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\Big]\, tr(\underline{g_1} t {g_1}^{-1} g_1) \overset{3)}{=}$

$\frac{(z+1-q)(q^2-q+1)Q(Q-1)z^2}{q} + \Big[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} +$

$(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\Big]\, z\, tr(\underline{t}) \overset{4)}{=}$

$\frac{(z+1-q)(q^2-q+1)Q(Q-1)z^2}{q} + \Big[\frac{(z+1-q)(q^2-q+1)Q^2 z}{q} + \frac{(z+1-q)(q^2-q+1)(Q-1)z}{q} +$

$(z+1-q)(q-1)Qs + (z+1-q)(q-1)(Q-1)\Big]\, zs.$

5 A trace-invariant for solid torus links

5.1 Construction of the invariant

The epimorphism $\pi : \bigcup_{n=1}^{\infty} \mathbb{C}B_{1,n} \mapsto \bigcup_{n=1}^{\infty} \mathcal{H}_n(q,Q)$ defined by sending $T \mapsto t$, $\sigma_i \mapsto g_i$ (and therefore $T_i \mapsto t_i$, since $T_i = \sigma_i \ldots \sigma_1 T {\sigma_1}^{-1} \ldots {\sigma_i}^{-1}$ and $t_i = g_i \ldots g_1 t {g_1}^{-1} \ldots {g_i}^{-1}$), together with the trace, result that to every mixed braid in $B_{1,n}$ we can assign an expression in the variables q, Q, z, s .

We observe now that the moves $(i), (ii)$ in Theorem 3 resemble the rules 1) and 3) of Theorem 5. So, we reason that, in order to obtain a HOMFLY-PT type link invariant we want to normalize the g_i's so that both Markov moves affect the trace in the same way. I.e. we want to normalize g_i to θg_i , $\theta \in \mathbb{C}$, so as to obtain

$$tr(a(\theta g_n)) = tr(a((\theta g_n)^{-1})) \quad \text{for} \quad a \in \mathcal{H}_n(q,Q).$$

(The normalization as well as the phrasing is the same as in [19], but for completeness we repeat it here adapted to our case). Then, for $z \neq 0$ we have:

$$\theta^2\, tr(a g_n) = tr(a {g_n}^{-1}) = \frac{1}{q}\, tr(a g_n) + \frac{1-q}{q}\, tr(a) \iff$$

$$\theta^2 z\, tr(a) = \frac{z+1-q}{q}\, tr(a) \iff \theta^2 = \frac{z+1-q}{qz} = \lambda\ .$$

Thus

$$tr(\sqrt{\lambda} g_i) = tr((\sqrt{\lambda} g_i)^{-1}) = \sqrt{\lambda}\, z = -\sqrt{\lambda}\, \frac{1-q}{1-\lambda q}\ .$$

It follows now that, if we represent $B_{1,n}$ by π_λ, where $\pi_\lambda(\sigma_i) = \sqrt{\lambda}\, g_i \in \mathcal{H}_n(q,Q)$ and $\pi_\lambda(T) = t \in \mathcal{H}_n(q,Q)$, (which also implies that $\pi_\lambda(T_i) = t_i \in \mathcal{H}_n(q,Q)$), then the function of q, λ, Q, s given by

$$\Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} tr(\pi_\lambda(\alpha)), \ \text{for} \ \alpha \in B_{1,n}\ ,$$

depends only on the mixed link $\widehat{\alpha}$ (the closure of α). The epimorphism π, though, has the advantage of only involving the variables q, Q; so we incorporate $\sqrt{\lambda}$ in the 'universal' coefficient and we define:

Definition 1. The 4-variable invariant $X_{\widehat{I} \cup L}(q,Q,\lambda,s)$ of the oriented mixed link $\widehat{I} \cup L$ that represents an oriented link inside the solid torus M, is the function:

$$X_\alpha = X_{\widehat{I} \cup L}(q,Q,\lambda,s) = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} (\sqrt{\lambda})^e\, tr(\pi(\alpha))$$

where $\alpha \in B_{1,n}$ is a word in the σ_i's and (t_i)'s such that $\widehat{\alpha} = \widehat{I} \cup L$, e is the exponent sum of the σ_i's that appear in α, and π the representation of $B_{1,n}$ in $\mathcal{H}_n(q,Q)$ such that $t \mapsto t, \sigma_i \mapsto g_i$.

Note 2. By their definition, the mixed braids T and T_i's do not affect the exponent sum e, so we can ignore them when we estimate e.

Examples 1. • *As it follows from Remark 3, if* α *does not contain any* (T_i)*, then* X_α *is the HOMFLY-PT polynomial of the link in* S^3 *obtained by removing from* α *the solid torus string* I*. So, if for instance,* $\alpha = 1 \in B_{1,1}$ *, then* $X_\alpha = 1$*; and if* $\alpha = 1 \in B_{1,n}$ *(corresponding to the* n*-component unlink), then*

$$X_\alpha = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1}.$$

• *If* $\alpha = T \in B_{1,1}$*, then* $X_\alpha = s$*; and if* $\alpha = t_i \in B_{1,n}$ *(corresponding to the* n*-component unlink, the* $(i+1)$*st string of which wraps once around* I *in a positive sense), then*

$$X_\alpha = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} s$$

whilst, if $\alpha = (t_i)^{-1} \in B_{1,n}$ *(corresponding to the* n*-component unlink, the* $(i+1)$*st string of which wraps once around* I *in a negative sense), then*

$$X_\alpha = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} \Big[\frac{1}{Q}\, s + \frac{1-Q}{Q}\Big] .$$

• *Similarly, if* $\alpha = (t_i)^2 \in B_{1,n}$ *(the* n*-component unlink, the* $(i+1)$*st component of which wraps twice around* I *in a positive sense), then*

$$X_\alpha = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\Big]^{n-1} \big[(Q-1)s + Q\big] .$$

• *Finally, if* $\alpha = {\sigma_1}^3 (t)^2 \in B_{1,2}$ *(a right-handed trefoil that wraps twice around* I *in a positive sense), then*

$$X_\alpha = -\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\, (\sqrt{\lambda})^3\, tr({g_1}^3 t^2) \ , \quad \textit{where}$$

$tr(\underline{g_1}^3 t^2) = (q^2-q+1)\, tr(g_1 \underline{t}^2) + q(q-1)\, tr(\underline{t}^2) =$

$(q^2-q+1)(Q-1)\, tr(g_1 t) + (q^2-q+1)Q\, tr(g_1) + q(q-1)(Q-1)\, tr(t) + q(q-1)Q\, tr(1) =$

$(q^2-q+1)(Q-1)\frac{q-1}{1-\lambda q} s + (q^2-q+1)Q\frac{q-1}{1-\lambda q} + q(q-1)(Q-1)s + q(q-1)Q$.

5.2 A note on skein relations

Let L_+, L_-, L_0 be oriented links that have identical diagrams, except in one crossing, where they are as depicted below:

Then, one can find a recursive linear formula in L_+, L_-, L_0 – known as *skein rule* – for defining the HOMFLY-PT polynomial.[12],[24],[21]

Figure 9:

Figure 10:

In [19] is explained a way of finding the skein rule of the 2-variable polynomial that derives from Ocneanu's trace function. Here we modify this way, in order to find the skein relations of the trace-invariant we defined above:

We consider a mixed link, which may be assumed to be the closure of a mixed braid, and we pick a crossing in it, which is not a mixed one. Using conjugation, this crossing appears in the end of the word, and – again by conjugation – we may assume that $L_+ = \widehat{\alpha\sigma_i{}^2}, L_- = \widehat{\alpha}$ and $L_0 = \widehat{\alpha\sigma_i}$, for some $\alpha \in B_{1,n}$. By the defining relations of $\mathcal{H}_n(q,Q)$ we have

$$tr(\pi(\alpha\sigma_i{}^2)) - q\, tr(\pi(\alpha)) = (q-1)\, tr(\pi(\alpha\sigma_i)).$$

Let e be the exponent sum of α with respect to the σ_i's, and multiply the above equation by $T\,\frac{(\sqrt{\lambda})^{e+1}}{\sqrt{q}}$, where

$$T = \Big[-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)} \Big]^{n-1}.$$

Then

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}\, T\,(\sqrt{\lambda})^{e+2}\, tr(\pi(\alpha\sigma_i{}^2)) - \sqrt{q}\sqrt{\lambda}\, T\,(\sqrt{\lambda})^{e}\, tr(\pi(\alpha))$$

$$= (\sqrt{q} - \frac{1}{\sqrt{q}})\, T\,(\sqrt{\lambda})^{e+1}\, tr(\pi(\alpha\sigma_i))\ ;$$

so from the definition of X we obtain the skein relation:

$$\frac{1}{\sqrt{q}\sqrt{\lambda}}\, X_{L_+} - \sqrt{q}\sqrt{\lambda}\, X_{L_-} = (\sqrt{q} - \frac{1}{\sqrt{q}})\, X_{L_0} \qquad \bullet$$

(The above relation together with the initial condition in S^3, $X(unknot) = 1$, define uniquely the HOMFLY-PT polynomial.) In the same manner, but with less difficulty, we obtain a second skein rule for the mixed braiding, that derives from the relation

$$t_i'^{-1} = \frac{1}{Q}\, t_i' + \frac{1-Q}{Q} \cdot 1$$

as follows: Let M_+, M_-, M_0 be oriented mixed links that have diagrams identical, except in the regions depicted below:

We consider a mixed link, which – as already mentioned – may be assumed to be the closure of a mixed braid, and we pick in it a positive mixed twist (as illustrated above). Note that, using conjugation in $B_{1,n}$, we can always create such a twist. Thus, by conjugation we may assume that $M_+ = \widehat{\alpha T_i}, M_- = \widehat{\alpha T_i{}^{-1}}$ and $M_0 = \widehat{\alpha}$, for some $\alpha \in B_{1,n}$. So we obtain:

$$tr(\pi(\alpha T_i{}^{-1})) = \frac{1}{Q} tr(\pi(\alpha T_i)) + \frac{1-Q}{Q} tr(\pi(\alpha)) \ ;$$

and, if we multiply the above equation by $T\,(\sqrt{\lambda})^e \sqrt{Q}$ we have

$$\sqrt{Q}\, T\,(\sqrt{\lambda})^e\, tr(\pi(\alpha T_i{}^{-1})) = \frac{1}{\sqrt{Q}} T\,(\sqrt{\lambda})^e\, tr(\pi(\alpha T_i)) + \frac{1-Q}{\sqrt{Q}} T\,(\sqrt{\lambda})^e\, tr(\pi(\alpha)) \,.$$

Hence, since the T_i's do not change the exponent sum of α, neither the number of its strings, we obtain the following skein rule:

$$\frac{1}{\sqrt{Q}} X_{M_+} - \sqrt{Q}\, X_{M_-} = (\sqrt{Q} - \frac{1}{\sqrt{Q}})\, X_{M_0} \quad \bullet\bullet$$

One can check that the two skein rules together with the initial conditions

$$X_1 = 1 \ , \ 1 \in B_{1,1} \quad \text{and} \quad X_T = s \ , \ T \in B_{1,1} \quad \bullet\bullet\bullet$$

suffice to calculate X inductively for any mixed link; but one would also have to prove that X defined this way is well-defined.

J. Hoste and M. Kidwell defined in [15] a 'new chromatic skein invariant for a special class of dichromatic links, which may be viewed as an invariant of oriented monochromatic links inside a solid torus; and this as such is the exact analogue of the HOMFLY-PT polynomial'. In their set-up, the solid torus string $\widehat{I}$ is perpendicular to the plane on which the rest of the link projects, and it is allowed to move by isotopy. The theorem they proved in the preliminary version of [15] (Theorem 2.1) is the following, (where, for convenience, we use our notation for expressing the different links):

Theorem 2.1 *There exists a unique invariant* $W^i \in \mathbb{Z}\big[v^{\pm 1}, z_j{}^{\pm 1}, \alpha, x^{\pm 1}, \lambda^{\pm 1}, h_+\big]$, $j \neq i$, *of Type* I_i *links satisfying the following properties:*

1. *Crossing Rule:* $v^{-1}\, W_{L_+} - v\, W_{L_-} = z_j\, X_{L_0}$

2. *Clasp Rule:* $x^{-1}\, W_{M_+} + x\, W_{M_-} = \alpha\, W_{M_0}$

3. *Connected Sum Rule:* $W_{K conn.sum_i J} = (v^{-1} - v)\, z_j{}^{-1}\, \lambda^{-1}\, W_J W_K$

4. *Initial Data:* $W_{\widehat{1}} = \lambda$, $1 \in B_{1,1}$ and $W_{\widehat{T'}} = h_+$, $T' \in B_{1,1}$

Figure 11:

where the i-coloured unknot corresponds to $\widehat{I}$, and the j-coloured components correspond to the rest of the mixed link.

We can observe now that the Crossing Rule is the same as the skein rule • above, if we set $v = \sqrt{q}\sqrt{\lambda}$ and $z_j = \sqrt{q} - \frac{1}{\sqrt{q}}$; whilst the Clasp Rule resembles the skein rule •• above, if we set $x = \sqrt{Q}$ and $\alpha = \sqrt{Q} - \frac{1}{\sqrt{Q}}$, but apparently the two rules still differ by a sign. As J. Przytycki pointed out, we can show that the two rules are essentially the same if we substitute $x = iy$ and $\alpha = -i\alpha'$. Also, the Initial Data is the same as in rule ••• , if we set $\lambda = 1$ and $h_+ = s$.

Notice that, in our set-up there does not appear any connected-sum rule for the component $\widehat{I}$ of two mixed links. The explanation lies in the fact that in our set-up, the component $\widehat{I}$ of a mixed link as well as the string I of a mixed braid remain always pointwise fixed.

Aside If $\widehat{I} \cup L_1, \widehat{I} \cup L_2$ are two mixed links, and $I \cup B_1, I \cup B_2$ are two corresponding mixed braids then, $I \cup (L_1 conn.sum L_2)$ corresponds to $I \cup (B_1 conn.sum B_2)$, as pictured below (compare with [19]):

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