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Kazem Mahdavi, Rebecca Culshaw & John Boucher The University of Texas at Tyler, USA
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0 Introduction

This paper gives infinitely many examples of unknot diagrams that are hard, in the sense that the diagrams need to be made more complicated by Reidemeister moves before they can be simplified. In order to construct these diagrams, we prove theorems characterizing when the numerator of the sum of two rational tangles is an unknot. The paper then uses these results in studying processive DNA recombination and finding minimal size unknot diagrams. This paper is a short version of a paper in which we include complete proofs of all statements. Many proofs are omitted in the present paper.

See Figure 2 for a diagram that we shall refer throughout this paper as the “Culprit.” This culprit is not the only culprit, but it is the exemplar that we shall use, and it is the example that started this investigation. The first author likes to use the Culprit as an example in introductory talks about knot theory. One draws the Culprit on the board and asks whether it is knotted or not. This gives rise to a discussion of easy and hard unknots, and how the existence of hard unknots makes us need a theory of knots in order to prove knottedness when it occurs. After using this example, we began to ask how to produce other examples that were hard and to wonder if our familiar culprit might be the smallest such example (size being the number of crossings, in this case 10).

We show that there are infinitely many examples of hard unknot diagrams, obtained by using the theory of rational tangles and their closures. In order to use the theory of rational tangles, one must become familiar with the notion of tangle and the notion of the fraction of a tangle. In Sections 1 and 2, we introduce the tangle analysis and assume that the reader knows about tangle fractions.

In Section 1 we see that the Culprit can be divided into two rational tangles whose fractions add up to a fraction whose numerator has absolute value equal to 1. It turns out that whenever the sum of the fractions of two rational tangles

...
has numerator equal to plus or minus one, then the closure of the sum of the two tangles will be an unknot. This result is Theorem 5 in Section 3. In Theorem 7 of Section 4 we take a further step and characterize fractions $P \over Q$ and $R \over S$ such that $P \over Q - R \over S = \pm 1 \over QS$ in terms of their associated continued fractions. It turns out that this last equation is satisfied if and only if one of the two continued fractions is a convergent of the other. This means that one continued fraction is a one-term truncate of the other. For example

$$3/2 = 1 + \frac{1}{2}$$

is a convergent of

$$10/7 = 1 + \frac{1}{2 + \frac{1}{3}}.$$  

Section 2 sets up the matrix representations for continued fractions that underpin the proof of the Theorems. This completely solves the question of when two fractions give rise to an unknot via the (numerator) closure of the sum of their associated tangles.

In Section 5 we use these results to construct many examples of hard unknots. The first example, $K = N([1, 4] - [1, 3])$, of this section is given in Figure 15 and its mirror image $H$ in Figure 19. This culprit $K$ is a hard unknot diagram with only 9 crossings. We then show how our original culprit (of 10 crossings) arises from a “tucking construction” applied to an unknot that is an easy diagram without the tuck (Figure 17). This section then discusses other applications of the tucking construct. In Section 6 we prove that the 9 crossing examples of Figure 19 and some relatives obtained by flyping and taking mirror images are the smallest hard unknot diagrams that can be made by taking the closure of the sum of two alternating rational tangles. In Section 7 we show the historically first hard unknot, due to Goeritz in 1934. The Goeritz diagram has 11 crossings.

In Section 8, we show how our unknots are related to the study of processive recombination of DNA. In the tangle model for DNA recombination, pioneered by DeWitt Sumners and Claus Ernst, the initial substrate of the DNA is represented as the closure of the sum of two rational tangles. It is usual to assume that the initial DNA substrate is unknotted. We have characterized such unknot configurations in this paper, and so are in a position to apply our results to the model. We show that processive recombination stabilizes, in the sense that the form of the resulting knotted or linked DNA is obtained by just adding twists in a single site on the closure of a certain tangle. This result helps to understand the form of the recombination process.
Acknowledgements. The first author thanks the National Science Foundation for support of this research under NSF Grant DMS-0245588. It gives both authors pleasure to acknowledge the hospitality of the Mathematisches Forschungsinstitut Oberwolfach, the University of Illinois at Chicago and the National Technical University of Athens, Greece, where much of this research was conducted. We particularly thank Slavik Jablan for conversations and for helping us, with his computer program LinKnot, to find some key omissions in our initial enumerations.

1 Culprits

Combinatorial knot theory got its start in the hands of Kurt Reidemeister [31] who discovered a set of moves on planar diagrams that capture the topology of knots and links embedded in three dimensional space. Reidemeister proved that the set of diagrammatic moves shown in Figure 1 generate isotopy of knots and links. That is, he showed that if we have two knots or links in three dimensional space, then they are ambient isotopic if and only if corresponding diagrams for them can be obtained, one from the other, by a sequence of moves of the types shown in Figure 1.

![Figure 1 - The Reidemeister Moves](image)

Here is an example of a knot diagram (originally due to Ken Millett [27]), in Figure 2. We like to call this diagram the “Culprit.” The Culprit is a knot diagram that represents the unknot, but as a diagram, and using only the Reidemeister moves, it must be made more complicated before it can be simplified...
to an unknotted circle. We measure the complexity of a knot or link diagram by the number of crossings in the diagram. Culprit has 10 crossings, and in order to be undone, we definitely have to increase the number of crossings before decreasing them to zero. The reader can verify this for himself by checking each region in the diagram of the Culprit. A simplifying Reidemeister II move can occur only on a two-sided region, but no two-sided region in the diagram admits such a move. Similarly on the Culprit diagram there are no simplifying Reidemeister I moves and there are no Reidemeister III moves (note that a III move does not change the complexity of the diagram). We view the diagram of the Culprit and other such examples as resting on the surface of the two-dimensional sphere. Thus the outer region of the diagram counts as much as any other region in this search for simplifying moves.

![The Culprit](image_url)

**Figure 2 - The Culprit**

\[
\frac{1}{1 + \frac{1}{2}} = \frac{2}{3}
\]

\[
\frac{-3}{4} + \frac{2}{3} = \frac{-1}{12}
\]

\[
F(A) = \frac{1}{-1 + \frac{1}{-3}} = \frac{-3}{4}
\]

\[
F(B) = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}
\]
Figure 3 - Cutting the Culprit into Rational Tangles

We shall call a diagram of the unknot *hard* if it has the following three properties:

1. There are no simplifying Type I moves on the diagram.

2. There are no simplifying Type II moves on the diagram.

3. There are no Type III moves on the diagram.

Hard unknot diagrams have to be made more complex before they will simplify to the unknot, if we use Reidemeister moves. It is an unsolved problem just how much complexity can be forced by a hard unknot.

One purpose of this paper is to give infinite classes of hard unknots by employing an insight about the structure of our Culprit, and generalizing this insight into results about the structure of tangles whose numerators are unknotted. These results are of interest in working with the tangle model of DNA recombination. See Section 8.

In order to see the Culprit in a way that allows us to generalize him, we shall use the language and technique of the theory of tangles. The next sections describe a bit of basic tangle theory, but we shall now analyze the Culprit using this language, to illustrate our approach. The reader familiar with the language of tangles will have no difficulty here. Other readers may wish to read the next section and then come back to this discussion.

2 Rational Tangles, Rational Knots and Continued Fractions

In this section we recall the subject of rational tangles and rational knots and their relationship with the theory of continued fractions. By the term “knots” we will refer to both knots and links, and whenever we really mean “knot” we shall emphasize it. Rational knots and links comprise the simplest class of links. They are also known in the literature as Viergeflechte, four-plats or 2-bridge knots depending on their geometric representation. The notion of a tangle was introduced in 1967 by Conway [5] in his work on enumerating and classifying knots and links.
A 2-tangle is a proper embedding of two unoriented arcs and a finite number of circles in a 3-ball $B^3$, so that the four endpoints lie in the boundary of $B^3$. A tangle diagram is a regular projection of the tangle on an equatorial disc of $B^3$. By “tangle” we will mean “tangle diagram”. A rational tangle is a special case of a 2-tangle obtained by applying consecutive twists on neighbouring endpoints of two trivial arcs. Such a pair of arcs comprise the [0] or $[\infty]$ tangles (see Figure 5), depending on their position in the plane. We shall say that the rational tangle is in twist form when it is obtained by such successive twists. For examples see Figure 8. Conway defined the rational knots as “numerator” or “denominator” closures of the rational tangles. See Figure 4. Conway [5] also defined the fraction of a rational tangle to be a rational number or $\infty$, obtained via a continued fraction that is associated with the tangle. We discuss this construction below.

![Figure 4 - A rational tangle and its closures to rational knots](image)

**Theorem 1 (Conway, 1975)** Two rational tangles are isotopic if and only if they have the same fraction.

More than one rational tangle can yield the same or isotopic rational knots, and the equivalence relation between the rational tangles is mapped into an arithmetic equivalence of their corresponding fractions. Indeed we have:
Theorem 2 (Schubert, 1956) Suppose that rational tangles with fractions \( \frac{p}{q} \) and \( \frac{p'}{q'} \) are given (\( p \) and \( q \) are relatively prime. Similarly for \( p' \) and \( q' \).) If \( K(\frac{p}{q}) \) and \( K(\frac{p'}{q'}) \) denote the corresponding rational knots obtained by taking numerator closures of these tangles, then \( K(\frac{p}{q}) \) and \( K(\frac{p'}{q'}) \) are isotopic if and only if

1. \( p = p' \) and
2. either \( q \equiv q' \mod p \) or \( qq' \equiv 1 \mod p \).

Different proofs of Theorem 2 are given in [34], [3], [23].

2.1 Rational Tangles and their Invariant Fractions

We shall now recall from [22] the main properties of rational tangles and of continued fractions, which illuminate the classification of rational tangles. The elementary rational tangles are displayed as either horizontal or vertical twists, and they are enumerated by integers or their inverses, see Figure 5.

The crossing types of 2-tangles (and of unoriented knots) follow the checkerboard rule: shade the regions of the tangle in two colors, starting from the left outside region with grey, and so that adjacent regions have different colors. Crossings in the tangle are said to be of “positive type” if they are arranged with respect to the shading as exemplified in Figure 5 by the tangle \([+1]\), i.e. they have the region on the right shaded as one walks towards the crossing along the over-arc. Crossings of the reverse type are said to be of “negative type” and they are exemplified in Figure 5 by the tangle \([-1]\).

![Figure 5 - The elementary rational tangles and the types of crossings](image)

In the class of 2-tangles we have the non-commutative operations addition and multiplication, as illustrated in Figure 6, which are denoted by “+” and
“∗” respectively. These operations are well-defined up to isotopy. A rational tangle in twist form is created inductively by consecutive additions of the tangles \([±1]\) on the right or on the left and multiplications by the tangles \([±1]\) at the bottom or at the top, starting from the tangle \([0]\) or \([∞]\). Since the very first crossing can be equally seen as horizontal or vertical, we may always assume that we start twisting from the tangle \([0]\). In order to read out a rational tangle we transcribe it as an algebraic sum using horizontal and vertical twists. For example, Figure 4 illustrates the tangle \(((3 ∗ \frac{1}{2}) + [2])\), see top of Figure 7, while Figure 8 illustrates a twist form of the same tangle: \([1] + ([1] ∗ [3] ∗ \frac{1}{-3}) + [1]\).

Note that addition and multiplication do not, in general, preserve the class of rational tangles. For example, the 2-tangle \(\frac{1}{3} + \frac{1}{\bar{3}}\) is not rational. The sum (product) of two rational tangles is rational if and only if one of the two consists in a number of horizontal (vertical) twists.

\[\text{Figure 6 - Addition, multiplication and rotation of 2-tangles}\]

The mirror image of a tangle \(T\), denoted \(−T\), is \(T\) with all crossings switched. For example, \(−[n] = [−n]\) and \(−\frac{1}{[n]} = \frac{1}{[−n]}\). Then, the subtraction is defined as \(T − S := T + (−S)\). The rotation of \(T\), denoted \(T^{\text{rot}}\), is obtained by rotating \(T\) on its plane counterclockwise by \(90°\). The inverse of \(T\) is defined to be \(−T^{\text{rot}}\). Thus, inversion is accomplished by rotation and mirror image. Note that \(T^{\text{rot}}\) and the inverse of \(T\) are in general not isotopic to \(T\) and they are order 4 operations. But for rational tangles the inversion is an operation of order 2 (this follows from the flipping lemma discussed below). For this reason we shall denote the inverse of a rational tangle \(T\) by \(1/T\), and hence the rotation of the tangle \(T\) will be denoted by \(−1/T\). This explains the notation for the tangles \(\frac{1}{[n]}\).
There is a fraction associated to a rational tangle $R$ which characterizes its isotopy class (Theorem 1). In fact, the fraction is defined for any 2-tangle and always has the following three properties. These suffice for computing the fraction $F(R)$ inductively for rational tangles:
1. $F([\pm 1]) = \pm 1$.

2. $F(T + S) = F(T) + F(S)$.

3. $F(T^{\text{rot}}) = -1/F(T)$.

In Figure 7 we illustrate this process by using only these three rules to compute a specific tangle fraction. In the following discussion we discuss the fraction in more detail and how it is related to the continued fraction structure of the rational tangles.

We shall then say that the rational tangle as shown in Figure 8 is in standard form. In this Figure we illustrate how to convert a tangle that is in “twist form” to standard form and to the braided form discussed below. Twist form is obtained from two parallel strands by successive twisting at the top, bottom, right or left. In this sense twist form is the general picture of a rational tangle before any simplifications have been applied to it.

It is useful to use the braid form illustrated in Figure 8. This is the 3-strand-braid representation. As illustrated in Figure 8, the 3-strand-braid representation is obtained from the standard representation by planar rotations of the vertical sets of crossings, thus creating a lower row of horizontal crossings. Note that the type of crossings does not change by this planar rotation. Indeed the checkerboard coloring convention for the crossing signs identifies the signs as unchanged. Nevertheless, the crossings on the lower row of the braid representation appear to be of opposite sign, since when we rotate them to the vertical position we obtain crossings of the opposite type in the local tangles.

\[ [1]+([1] \cdot [3] \cdot \frac{1}{[-3]}) + [1] \]

\[ ([3] \cdot \frac{1}{[-2]} ) + [2] := [2, -2, 3] \]

Figure 8 - A rational tangle in twist form converted to its standard form and to its 3-strand-braid representation
One can associate to a rational tangle in standard form a vector of integers $(a_1, a_2, \ldots, a_n)$, where the first entry denotes the place where the tangle starts untwisting and the last entry where it begins to twist. For example the tangle of Figure 4 corresponds to the vector $(2, -2, 3)$.

Note that the set of twists of a rational tangle may be always assumed odd. Indeed, let $n$ be even and let the left-most twist $[a_1]$ be on the upper part of the braid representation. Then, the right-most crossing of the last twist $[a_n]$ may be assumed upper, so that $[a_n]$ can break into $a_n - 1$ lower crossings and one upper. Up to the ambiguity of the right-most crossing, the vector associated to a rational tangle is unique, i.e. $(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n - 1, 1)$, if $a_n > 0$, and $(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n + 1, -1)$, if $a_n < 0$. See Figure 9.

Another move that can be applied to a 2-tangle is a flip, its rotation in space by $180^\circ$. We denote $T^{h\text{flip}}$ a horizontal flip (rotation around a horizontal axis on the plane of $T$) and $T^{v\text{flip}}$ a vertical flip. See Figure 10 for illustrations. Note that a flip switches the endpoints of the tangle and, in general, a flipped tangle is not isotopic to the original one. Rational tangles have the remarkable property that they are isotopic to their horizontal or vertical flips. We shall refer to this as the Flipping Lemma.
A consequence of the Flipping Lemma is that addition and multiplication by $[\pm 1]$ are commutative. Another consequence of the Flipping Lemma is that rotation and inversion of rational tangles each have order two. In particular, rotation is defined via a ninety degree turn of the tangle either to the left or to the right. With this in mind the reader can easily deduce the formula below:

$$T * \frac{1}{[n]} = \frac{1}{[n] + \frac{1}{T}}$$

Indeed, rotate $T * \frac{1}{[n]}$ by ninety degrees and note that it becomes $-[n] - \frac{1}{T}$. Use this to deduce that the original tangle is the negative reciprocal of this tangle. This formula implies that the two operations: addition of $[+1]$ or $[-1]$ and inversion between rational tangles suffice for generating the whole class of rational tangles. As for the fraction, we have the corresponding formula

$$F(T * \frac{1}{[n]}) = \frac{1}{n + \frac{1}{F(T)}}.$$

The above equation for tangles leads to the fact that a rational tangle in standard form can be described algebraically by a continued fraction built from the integer tangles $[a_1], [a_2], \ldots, [a_n]$ with all numerators equal to 1, namely by an expression of the type:

$$[[a_1], [a_2], \ldots, [a_n]] := [a_1] + \frac{1}{[a_2] + \cdots + \frac{1}{[a_{n-1} + \frac{1}{[a_n]}}}}$$

for $a_2, \ldots, a_n \in \mathbb{Z} - \{0\}$ and $n$ even or odd. We allow $[a_1]$ to be the tangle $[0]$. Then, a rational tangle is said to be in continued fraction form.

We shall abbreviate the expression $[[a_1], [a_2], \ldots, [a_n]]$ by writing $[a_1, a_2, \ldots, a_n]$, and later will use the latter expression for a numerical continued fraction as well. There should be no ambiguity between the tangle and numerical interpretations, as these will be clear from context. Figure 4 illustrates the rational tangle $[2, -2, 3]$.

From the above discussion it makes sense to assign to a rational tangle in standard form, $T = [[a_1], [a_2], \ldots, [a_n]]$, for $a_1 \in \mathbb{Z}, a_2, \ldots, a_n \in \mathbb{Z} - \{0\}$ and $n$ even or odd, the numerical continued fraction

$$F(T) = F([[a_1], [a_2], \ldots, [a_n]]) = [a_1, a_2, \ldots, a_n] := a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}.$$

If a rational tangle $T$ changes by an isotopy, the associated continued fraction form may also change. However, the fraction is a topological invariant.
of $T$ and does not change. For example, $[2, -2, 3] = [1, 2, 2] = \frac{7}{5}$, see Figure 7. The fraction characterizes the isotopy class of $T$. For the isotopy type of a rational tangle $T$ with fraction $\frac{p}{q}$ we shall use the notation $[\frac{p}{q}]$. We have omitted here the proof of the invariance of the fraction. The interested reader can consult [5], [17], [22] for various proofs of this fact.

The key to the exact correspondence of fractions and rational tangles lies in the construction of a canonical alternating form for the rational tangle.

We shall say that the rational tangle $S = [\beta_1, \beta_2, \ldots, \beta_m]$ is in canonical form if $S$ is alternating and $m$ is odd. From the above, $S$ alternating implies that the $\beta_i$’s are all of the same sign. It turns out that the canonical form for $S$ is unique. In Figure 11 we bring our working rational tangle $T = [2, -2, 3]$ to its canonical form $S = [1, 2, 2]$. As noted above, $F(T) = F(S) = \frac{7}{5}$.

Figure 11 - Reducing to alternating form using the swing moves

By Euclid’s algorithm and keeping all remainders of the same sign, one can show that every continued fraction $[a_1, a_2, \ldots, a_n]$ can be transformed to a unique canonical form $[\beta_1, \beta_2, \ldots, \beta_m]$, where all $\beta_i$’s are positive or all negative integers and $m$ is odd. For example, $[2, -2] = [1, 1, 1] = \frac{3}{2}$. There is also an algorithm that can be applied directly to the initial continued fraction to obtain its canonical form, which works in parallel with the algorithm for the canonical form of rational tangles. Indeed, we have:

**Proposition 1** The following identity is true for continued fractions and it is also a topological equivalence of the corresponding tangles:

$$[\ldots, a, -b, c, d, e, \ldots] = [\ldots, (a - 1), 1, (b - 1), -c, -d, -e, \ldots].$$

This identity gives a specific inductive procedure for reducing a continued fraction to all positive or all negative terms. In the case of transforming to all negative terms, we can first flip all signs and work with the mirror image. Note also that

$$[\ldots, a, b, 0, c, d, e, \ldots] = [\ldots, a, b + c, d, e, \ldots]$$

will be used in these reductions.
Proof. The technique for the reduction is based on the formula
\[ a + 1/(-b) = (a - 1) + 1/(1 + 1/(b - 1)). \]
If \( a \) and \( b \) are positive, this formula allows the reduction of negative terms in a continued fraction. The identity in the Proposition follows immediately from this formula.

\[ \square \]

2.2 Rational Knots and Continued Fractions

By joining with simple arcs the two upper and the two lower endpoints of a 2-tangle \( T \), we obtain a knot called the Numerator of \( T \), denoted by \( N(T) \). A rational knot is defined to be the numerator of a rational tangle. Joining with simple arcs each pair of the corresponding top and bottom endpoints of \( T \) we obtain the Denominator of \( T \), denoted by \( D(T) \), see Figure 4. We have \( N(T) = D(T^{\text{rot}}) \) and \( D(T) = N(T^{\text{rot}}) \). As we shall see in the next section, the numerator closure of the sum of two rational tangles is still a rational knot. But the denominator closure of the sum of two rational tangles is not necessarily a rational knot, think for example of the sum \( \frac{1}{3} + \frac{1}{3} \).

Given two different rational tangle types \( \left[ \frac{p}{q} \right] \) and \( \left[ \frac{p'}{q'} \right] \), when do they close to isotopic rational knots? The answer is given in Theorem 2. Schubert classified rational knots by finding canonical forms via representing them as 2-bridge knots. In [23] we give a new combinatorial proof of Theorem 2, by posing the question: given a rational knot diagram, at which places may one cut it so that it opens to a rational tangle? We then pinpoint two distinct categories of cuts that represent the two cases of the arithmetic equivalence of Schubert’s theorem. The first case corresponds to the special cut, as illustrated in Figure 12. The two tangles \( T = [-3] \) and \( S = [1] + \frac{1}{[2]} \) are non-isotopic by the Conway Theorem, since \( F(T) = -3 = 3/ -1 \), while \( F(S) = 1 + 1/2 = 3/2 \). But they have isotopic numerators: \( N(T) \sim N(S) \), the left-handed trefoil. Now \(-1 \equiv 2 \mod 3\), confirming Theorem 2. See [23] for a complete analysis of the special cut.

\[ \begin{align*}
T &= [-3] \\
S &= [1] + \frac{1}{[2]} \\
\sim
\end{align*} \]
Figure 12 - An example of the special cut

The second case of Schubert’s equivalence corresponds to the *palindrome cut*, an example of which is illustrated in Figure 13. Here we see that the tangles

\[ T = [2, 3, 4] = [2] + \frac{1}{[3] + \frac{1}{[4]}} \]

and

\[ S = [4, 3, 2] = [4] + \frac{1}{[3] + \frac{1}{[2]}} \]

both have the same numerator closure. Their corresponding fractions are

\[ F(T) = 2 + \frac{1}{3 + \frac{1}{4}} = \frac{30}{13} \quad \text{and} \quad F(S) = 4 + \frac{1}{3 + \frac{1}{2}} = \frac{30}{7}. \]

Note that \( 7 \cdot 13 \equiv 1 \mod 30 \).

Figure 13 - An instance of the palindrome equivalence

In the general case if \( T = [a_1, a_2, \ldots, a_n] \), we shall call the tangle \( S = [a_n, a_{n-1}, \ldots, a_1] \) the *palindrome of T*. Clearly these tangles have the same numerator. In order to check the arithmetic in the general case of the palindrome cut we need to generalize this pattern to arbitrary continued fractions and their palindromes (obtained by reversing the order of the terms).
The next Theorem is a known result about continued fractions. See [22], [35] or [24]. We shall omit our proof of this statement. For this we will first present a way of evaluating continued fractions via $2 \times 2$ matrices (compare with [15], [26]). This method of evaluation is crucially important for the rest of the paper. We define matrices $M(a)$ by the formula

$$M(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$  

These matrices $M(a)$ are said to be the generating matrices for continued fractions, as we have:

**Theorem 3 (The matrix product interpretation for continued fractions)**

Let $\{a_1, a_2, \ldots, a_n\}$ be a collection of $n$ integers, and let

$$\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$$

and

$$\frac{P'}{Q'} = [a_n, a_{n-1}, \ldots, a_1].$$

Then $P = P'$ and $QQ' \equiv (-1)^{n+1} \mod P$.

In fact, for any sequence of integers $\{a_1, a_2, \ldots, a_n\}$ the value of the corresponding continued fraction

$$\frac{P}{Q} = [a_1, a_2, \ldots, a_n]$$

is given through the following matrix product

$$M = M(a_1)M(a_2) \cdots M(a_n)$$

via the identity

$$M = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}$$

where this matrix also gives the evaluation of of the palindrome continued fraction

$$[a_n, a_{n-1}, \ldots, a_1] = \frac{P}{Q'}.$$  

**Proof.** We omit the proof of this Theorem.  

\[\Box\]
3 Sums of Two Rational Tangles

In this section we note that the numerator of the sum of two rational tangles is a rational knot or link. We characterize the knot or link that emerges from this process.

![Diagram](cut here!) (on the two-sphere)

\[ N([1,2,3] + [1,1,2]) = N([3,2,1+1,1,2]) \]

Figure 14 - The numerator of a sum of rational tangles is a rational link

**Theorem 4 (Addition of Rational Tangles)** Let \( \{a_1, a_2, \ldots, a_n\} \) be a collection of integers, so that

\[
\frac{P}{Q} = [a_1, a_2, \ldots, a_n].
\]

Let \( \{b_1, b_2, \ldots, b_m\} \) be another collection of integers, so that

\[
\frac{R}{S} = [b_1, b_2, \ldots, b_m].
\]

Let \( A = [\frac{P}{Q}] \) and \( B = [\frac{R}{S}] \) be the corresponding rational tangles. Then the knot or link \( N(A + B) \) is rational, and in fact

\[
N(A + B) = N([a_n, a_{n-1}, \ldots, a_2, a_1 + b_1, b_2, \ldots, b_m]).
\]

**Proof.** View Figure 14. In this figure we illustrate a special case of the Theorem. The geometry of reconnection in the general case should be clear from this illustration. \( \square \)
The next result tells us when we get the unknot.

**Definition 1** Given continued fractions \( \frac{P}{Q} = [a_1, \ldots, a_n] \) and \( \frac{R}{S} = [b_1, \ldots, b_m] \), let
\[
[a_1, \ldots, a_n] \# [b_1, \ldots, b_m] = [a_n, \ldots, a_2, a_1 + b_1, b_2, \ldots, b_m].
\]
If
\[
\frac{F}{G} = [a_n, \ldots, a_2, a_1 + b_1, b_2, \ldots, b_m],
\]
we shall write
\[
\frac{P}{Q} \# \frac{R}{S} = \frac{F}{G}.
\]
Note that \( \frac{F}{G} \) is a fraction such that \( N([\frac{F}{G}]) = N([\frac{P}{Q}] + [\frac{R}{S}]) \).

**Theorem 5** Let
\[
\frac{P}{Q} = [a_1, a_2, \ldots, a_n]
\]
and
\[
\frac{R}{S} = [b_1, b_2, \ldots, b_m]
\]
be as in the previous Theorem. Then
\[
N([\frac{P}{Q}] + [\frac{R}{S}])
\]
is unknotted if and only if \( PS + QR = \pm 1 \).

**Proof.** We omit the proof.

## 4 Continued Fractions, Convergents and Lots of Unknots

Consider a rational fraction, its corresponding continued fraction, and its matrix representation:
\[
P/Q = [a_1, \ldots, a_n]
\]
with
\[
M = M(\tilde{a}) = M(a_1) \cdots M(a_n) = \begin{pmatrix} P & Q' \\ Q & U \end{pmatrix}.
\]
Note that since the determinant of this matrix is \((-1)^n\), we have the formula

\[
PU - QQ' = (-1)^n
\]

from which it follows that

\[
P/Q - Q'/U = (-1)^n/QU.
\]

Hence, by Theorem 5, the diagram

\[
N([P/Q] - [Q'/U])
\]

is unknotted and, as we shall see, is a good candidate to produce a hard unknot. Furthermore, the fraction \(Q'/U\) has an interpretation as the truncation of our continued fraction \([a_1,\ldots,a_n]\):

\[
Q'/U = [a_1,\ldots,a_{n-1}].
\]

To see this formula, let

\[
N = M(a_1) \cdots M(a_{n-1}) = \begin{pmatrix} R & S' \\ S & V \end{pmatrix},
\]

so that

\[
R/S = [a_1,\ldots,a_{n-1}].
\]

Then

\[
\begin{pmatrix} P & Q' \\ Q & U \end{pmatrix} = M(a_1) \cdots M(a_{n-1})M(a_n) = NM(a_n)
\]

\[
= \begin{pmatrix} R & S' \\ S & V \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} Ra_n + S' & R \\ Sa_n + V & S \end{pmatrix}.
\]

This shows that \(Q'/U = R/S = [a_1,\ldots,a_{n-1}]\), as claimed.

**Definition 2** One says that \([a_1,\ldots,a_{n-1}]\) is a **convergent** of \([a_1,\ldots,a_{n-1},a_n]\).

We shall say that two fractions \(P/Q\) and \(R/S\) are **convergents** if the continued fraction of one of them is a convergent of the other.

We see from the above calculation that the two consecutive integers \(PU\) and \(QQ'\) produce two continued fractions \(P/Q = [a_1,\ldots,a_n]\) and \(Q'/U = [a_1,\ldots,a_{n-1}]\) so that the second fraction is a convergent of the first.

We have proved the following result.
Theorem 6 Let $P/Q$ and $Q'/U$ be fractions such that the continued fraction of $Q'/U$ is a convergent of the continued fraction of $P/Q$. Then

$$N([P/Q] - [Q'/U])$$

is an unknot.

Proof. The proof is given in the discussion above.

Remark 1 This Theorem applies to Figure 3, and our early discussion of the Culprit.

The property of one fraction being a convergent of the other is in fact, always a property of fractions produced from consecutive integers. We make this statement formally in the next Lemma (see also [10]).

Lemma 1 Let $P$ and $Q$ be relatively prime integers and let $s$ and $r$ be a pair of integers such that $Ps - Qr = \pm 1$. Let $R = r + tP$ and $S = s + tQ$ where $t$ is any integer. Then $\{R, S\}$ comprises the set of all solutions to the equation $PS - QR = \pm 1$. If $Ps - Qr = \pm 1$ and $PS - QR = \mp 1$, Then all solutions are given in the form $R = -r + tP$ and $S = -s + tQ$.

Proof. We omit the proof of this Lemma.

Theorem 7 Let $P$ and $Q$ be relatively prime integers and let $P/Q = [a_1, \ldots, a_n]$ be a continued fraction expansion for $P/Q$. Let $r/s = [a_1, \ldots, a_{n-1}]$ be the convergent for $[a_1, \ldots, a_n]$. Let $R = r + tP$ and $S = s + tQ$ where $t$ is any integer. Then $R/S = [a_1, \ldots, a_n, t]$. Thus $P/Q$ is a convergent of $R/S$. We conclude that if $P/Q$ and $R/S$ satisfy the condition that $N([P/Q] - [R/S])$ is an unknot, then one of $P/Q$ and $R/S$ is a convergent of the other.

Proof. We omit the proof of this Theorem.

5 Constructing Hard Unknots

In this section we indicate how to construct hard unknots by using positive alternating tangles $A$ and $B$ such that $N(A - B)$ is unknotted. By our main results we know how to construct infinitely many such pairs of tangles by taking a continued fraction and its convergent, with the corresponding tangles in reduced (alternating) form.
Let’s begin with the case of $5/4 = [1,4] = [1,3,1]$ and $4/3 = [1,3]$. In Figure 15 we show the standard representations of $[1,4]$ and $[1,3]$ as tangles, and the corresponding construction for the diagram of $K = N([1,4] - [1,3])$. The reader will note that this diagram is a hard unknot with 9 crossings, one less than our original Culprit of Figure 3. We give another version of it in Figure 19 (equivalent to its mirror image $H$). In Section 6 we show that $H$ is one of a small collection of minimal hard unknot diagrams having the form $N(A - B)$ for reduced positive rational tangle diagrams $A$ and $B$.

In most cases, if one takes the standard representations of the tangles $A$ and $B$, and forms the diagram for $N(A - B)$, the resulting unknot diagram will be hard. There are some exceptions however, and the next example illustrates this phenomenon.
In Figure 16 we show the standard representations of \([1, 3]\) and \([1, 2]\) as tangles, and the corresponding construction for the diagram of \(N([1, 3] - [1, 2])\). This diagram, while unknotted, is not a hard unknot diagram due to the three-sided outer region. This outer region allows a type III Reidemeister move on the surface of the two dimensional sphere. In this example, tucking an arc does not create a hard unknot from the given diagram (there is be a type III move available after the tuck).

\[
\begin{align*}
[1,3] & \quad -[1,2]^{vflip} \\
N([1,3] - [1,2]^{vflip}) & \quad \text{Figure 17 - } N([1, 3] - [1, 2]^{vflip})
\end{align*}
\]

**The Tucking Construct.** Figure 17 shows a way to remedy this situation. Here we have replaced \([1, 2]\) by \([1, 2]^{vflip}\), the 180 degree turn of the tangle \([1, 2]\) about the vertical direction in the page. Now we see that the literal diagram of \(N([1, 3] - [1, 2]^{vflip})\) is of course still unknotted and is also not a hard unknot diagram. However this diagram can be converted to an unknot diagram by tucking an arc as shown in the Figure. The resulting hard unknot is the same diagram of 10 crossings that we had in Figure 3 as our initial Culprit. Note that the other possibility of flipping both tangles in Figure 16 or flipping the first tangle do not lead to a hard unknotts. We call this strategem the **tucking construct**. Tucking is accompanied by the vertical flip of on one of the tangles to avoid the placement of a Reidemeister move of type III as a result of the tuck.
**The Culprit Revisited** Let’s consider the example in Figure 3 again. Here we have \( P/Q = F(A) = -3/4 \) and \( R/S = F(B) = 2/3 \). We have \( P/Q + R/S = -3/4 + 2/3 = -1/12 \). Thus \( N([-3/4] + [2/3]) \) is an unknot by Theorem 5. This is exactly the unknot \( C' \) illustrated in Figure 3.

![Diagram](attachment)

**Figure 18 - The tucking construct**

We can make infinitely many examples of this type. View Figure 18. The pattern is as follows. Suppose that \( T = [P/Q] \) and \( T' = [R/S] \) are rational tangles such that \( PS - QR = \pm 1 \). Then we know that \( N(T - (T')^{vflip}) \) is an unknot. Furthermore we can assume that each of the tangles \( T \) and \( T' \) are in alternating form. The two tangle fractions have opposite sign and hence the alternation of the weaves in each tangle will be of opposite type. We create a new diagram for \( N(T - (T')^{vflip}) \) by putting an arc from the bottom of the
closure entirely underneath the diagram as shown in Figure 18. This is an example of a successful tucking construct. Note how in the example shown in Figure 18, the knot diagram resulting from the tucking construction is indeed our original hard unknot diagram. There are no simplifying Reidemeister moves and there are no moves of type III available on the diagram.

6 The Smallest Hard Unknots

Figure 19 illustrates two hard unknot diagrams $H$ and $J$ with 9 crossings.

**Conjecture 1** Up to mirror images and flyping tangles in the diagrams, the hard unknot diagrams $H$ and $J$ of 9 crossings, shown in Figure 19 ($K = -H$ appears earlier in Figure 15), have the least number of crossings among all hard unknot diagrams.

![Diagram](image-url)

**Figure 19 - $H$ and $J$ are hard unknots of 9 crossings**

Two equivalent versions of the diagram $H$ appear in Figure 19. The right-hand version of $H$ in this figure is of the form

$$H = N([1 + 1/3] - [1 + 1/4]) = N([1, 3] - [1, 4]) = N([4/3] - [5/4]).$$

Note that $[1, 3]$ and $[1, 4] = [1, 3, 1]$ are convergents. Note also that the diagram $K$ of Figure 15 is given by $K = N([1, 4] - [1, 3]) = -N([1, 3] - [1, 4]) = -H$. Thus $H$ and $K$ are mirror images of each other.
The diagram $J$ in Figure 19 is of the form
\[
N([1 + 1/3] - [1 + 1/(2 + 1/2)]) = N([1, 3] - [1, 2, 2]) = N([4/3] - [7/5]).
\]
Note that $[1, 3] = [1, 2, 1]$ and $[1, 2, 2] = [1, 2, 1, 1]$ are convergents.

Note also that the crossings in $J$ corresponding to $1$ in $[1, 3]$ and $-1$ in $-[1, 2, 2] = [-1, -2, -2]$ can be switched and we will obtain another diagram $J'$, arising as sum of two alternating rational tangles, that is also a hard unknot. This diagram can be obtained from the diagram $J$ without switching crossings by performing flypes (A flype is a turn of a tangle by $\pi$ that carries a crossing to the other side of the tangle.) on the subtangles $[1, 3]$ and $[1, 2, 2]$ of $J$, and then doing an isotopy of this new diagram on the two dimensional sphere. (We leave the verification of this statement to the reader.) Thus the diagram $J'$ can be obtained from $J$ by flyping. A similar remark applies to the diagram $H$, giving a corresponding diagram $H'$, but in this case $H'$ is easily seen to be equivalent to $H$ by an isotopy that does not involve any Reidemeister moves. Thus, up to these sorts of modifications, we have produced essentially two hard diagrams with 9 crossings. Other related hard unknot diagrams of 9 crossings can be obtained from these by taking mirror images.

We have the following result.

**Theorem 8** The diagrams $H$ and $J$ shown in Figure 19 are, up to flyping subtangle diagrams and taking mirror images, the smallest hard unknot diagrams in the form $N(A - B)$ where $A$ and $B$ are rational tangles in reduced positive alternating form.

**Proof.** It is easy to see that we can assume that $A = [P/Q]$ where $P$ and $Q$ are positive, relatively prime and $P$ is greater than $Q$. We leave the proof that one can choose $P$ greater than $Q$ to the reader, with the hint: Verify that the closure diagram in Figure 16 is equivalent to the diagram in Figure 19 on the surface of the two dimensional sphere, without using any Reidemeister moves.

We then know from Theorem 8 that $B = [-R/S]$ where one of $P/Q$ and $R/S$ is a convergent of the other. We can now enumerate small continued fractions. We know the total of all terms in $A$ and $B$ must be less than or equal to 9 since $H$ and $J$ each have nine crossings.

In order to make a 9 crossing unknot example of the form $N(A - B)$ where $A$ and $B$ are rational tangles in reduced positive alternating form, we must partition the number 9 into two parts corresponding to the number of crossings in each tangle. It is not hard to see that we need to use the partition...
$9 = 4 + 5$ in order to make a hard unknot of this form. Furthermore, 4 must correspond to the the continued fraction $[1, 3]$, as $[2, 2]$ will not produce a hard unknot when combined with another tangle. Thus, for producing 9 crossing examples we must take $A = [1, 3]$. Then, in order that $A$ and $B$ be convergents, and $B$ have 5 crossings, the only possibilities for $B$ are $B = [1, 4]$ and $B = [1, 2, 2]$. These choices produce the diagrams $H, H', J, J'$. It is easy to see that no diagrams with less than 9 crossings will suffice to produce hard unknots, due to the appearance of Reidemeister moves related to the smaller partitions. This completes the proof.

\[ \square \]

7 The Goeritz Unknot

The earliest appearance of a hard unknot is a 1934 paper of Goeritz [16]. In this paper Goeritz gives the hard unknot shown in Figure 20. As the reader can see (for example by twisting vertically the tangle $[-3]$ twice), this example is a variant on $N([4] + [-3])$ which is certainly unknotted. The Goeritz example $G$ has 11 crossings, due to the extra two twists that make it a hard unknot. It is part of an infinite family based on $N([n] + [-n + 1])$.

![Figure 20 - The Goeritz Hard Unknot](image)

8 Stability in Processive DNA Recombination

In this section we use the techniques of this paper to study properties of processive DNA recombination topology. Here we use the tangle model of DNA recombination [13, 14, 36] developed by C. Ernst and D.W. Sumners. In this model the DNA is divided into two regions corresponding to two tangles $O$ and $I$ and a recombination site that is associated with $I$. This division is a model of how the enzyme that performs the recombination traps a part of the DNA, thereby effectively dividing it into the tangles $O$ and $I$. The recombination site is represented by another tangle $R$. The entire arrangement is then a knot or link $K[R] = N(O + I + R)$. We then consider a single recombination in the form of starting with $R = [0]$, the zero tangle, and replacing $R$ with the
tangle $[1]$ or the tangle $[-1]$. Processive recombination consists in consecutively replacing again and again by $[1]$ or by $[-1]$ at the same site. Thus, in processive recombination we obtain the knots and links

$$K[n] = N(O + I + [n]).$$

The knot or link $K[0] = N(O + I)$ is called the DNA substrate, and the tangle $O + I$ is called the substrate tangle. It is of interest to obtain a uniform formula for knots and links $K[n]$ that result from the processive recombination.

In some cases the substrate tangle is quite simple and is represented as a single tangle $S = O + I$. For example, we illustrate processive recombination in Figure 21 with $S = [-1/3] = [0, -3]$ and $I = [0]$ with $n = 0, 1, 2, 3, 4$. Note that by Proposition 1 of Section 2.1,

$$K[n] = N(S + [n]) = N([0, -3] + [n]) = N([-3, 0 + n]) = N([-3, n])$$

$$= N([-3, n]) = N([-2, 1, n - 1]).$$

This formula gives the abstract form of all the knots and links that arise from this recombination process. We say that the formula

$$K[n] = N([-2, 1, n - 1])$$

for $n > 1$ is stabilized in the sense that all the terms in the continued fraction have the same sign and the $n$ is in one single place in the fraction. In general, a stabilized fraction will have the form

$$N(\pm[a_1, a_2, \ldots a_{k-1}, a_k + n, a_{k+1}, \ldots, a_n])$$

where all the terms $a_i$ are positive for $i \neq k$ and $a_k$ is non-negative.
Let’s see what the form of the processive recombination is for an arbitrary sequence of recombinations. We start with

\[ O = [a_1, a_2, \ldots, a_{r-1}, a_r] \]

\[ I = [b_1, b_2, \ldots, b_{s-1}, b_s] \]

Then

\[ K[n] = N(O + (I + [n])) = N([a_1, a_2, \ldots, a_{r-1}, a_r] + [n + b_1, b_2, \ldots, b_{s-1}, b_s]) \]
Unknots and DNA

\[ K[n] = N([a_r, a_{r-1}, \ldots, a_2, a_1 + n + b_1, b_2, \ldots, b_{s-1}, b_s]). \]

**Proposition 2** The formula

\[ K[n] = N([a_r, a_{r-1}, \ldots, a_2, a_1 + n + b_1, b_2, \ldots, b_{s-1}, b_s]) \]

can be simplified to yield a stable formula for the processive recombination when \( n \) is sufficiently large.

**Proof.** Apply Proposition 1 of Section 2.1. \( \square \)

Here is an example. Suppose we take \( O = [1, 1, 1] \) and \( I = [-1, -1, -1] \) so that the DNA substrate is an (Fibonacci) unknot. \( (I \) is the negative of the convergent of \( O. \)\) Then by the above calculation

\[ K[n] = N([1, 1, 1 + n + (-1), -1, -1]) = N([1, 1, 1, n, -1, -1]). \]

Suppose that \( n \) is positive. Applying the reduction formula of Proposition 1, we get

\[ K[n] = N([1, 1, 1, n, -1, -1]) = N([1, 1, 1, n - 1, 1, 0, 1]) = N([1, 1, 1, n - 1, 2]), \]

and this is a stabilized form for the processive recombination.

More generally, suppose that \( O = [a_1, a_2, \ldots, a_n] \) where all of the \( a_i \) are positive. Let \( I = [-a_1, -a_2, \ldots, -a_{n-1}] \). Then \( K[0] = N(O + I) \) is an unknotted substrate by our result about convergents. Consider \( K[n] \) for positive \( n \). We have

\[ K[n] = N([a_n, a_{n-1}, \ldots, a_2, a_1 + n - a_1, -a_2, \ldots, -a_{n-1}]) \]

\[ = N([a_n, a_{n-1}, \ldots, a_2, n - a_2, \ldots, -a_{n-1}]) \]

\[ = N([a_n, a_{n-1}, \ldots, a_2, (n - 1), 1, (a_2 - 1), a_3, \ldots, a_{n-1}]). \]

If \( a_2 - 1 \) is not zero, the process terminates immediately. Otherwise there is one more step. In this way the knots and links proceeding from the recombination process all have a uniform stabilized form. Further successive recombination just adds more twist in one entry in the continued fraction diagram whose closure is \( K[n] \).
The reader may be interested in watching a visual demonstration of these properties of DNA recombination. For this, we recommend the program Ginterface (TangleSolver) [37] of Mariel Vasquez. Her program can be downloaded from the internet as a Java applet, and it performs and displays DNA recombination. Figure 22 illustrates the form of display for this program. The reader should be warned that the program uses the reverse order from our convention when listing the terms in a continued fraction. Thus we say \([1, 2, 3, 4]\) while the program uses \([4, 3, 2, 1]\) for the same structure.

Figure 22 - Processive Recombination with
\[ S = [1, 1, 1, 1] + [-1, -1, -1] \]

References


L.H. Kauffman: Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 South Morgan St., Chicago IL 60607-7045, U.S.A.

S. Lambropoulou: Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-157 80 Athens, Greece.

E-mails: kauffman@math.uic.edu sofia@math.ntua.gr